Loop calculations with graphical functions

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Setup

The graphical functions method works for

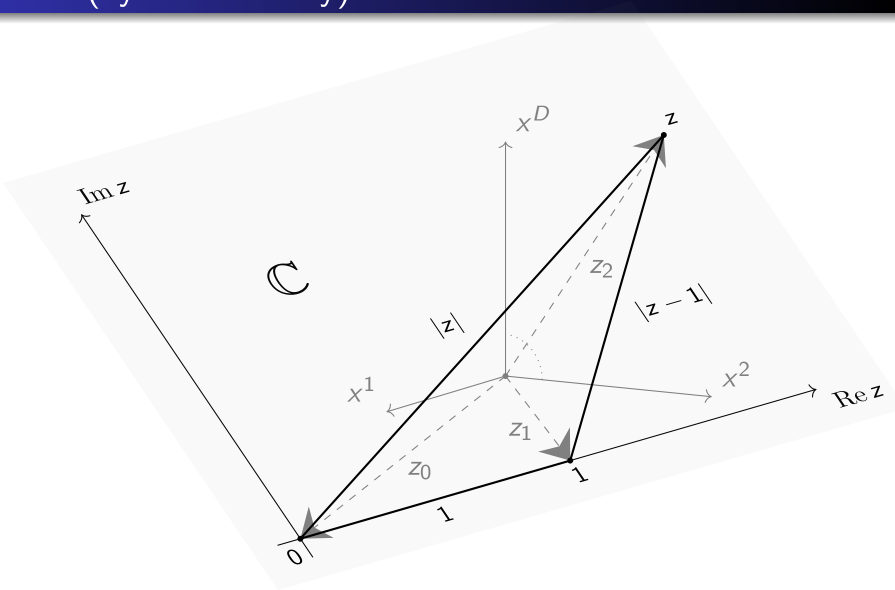
- massless,
- 2pt, 3pt, or convergent (conformal) 4pt amplitudes
- in even dimensions ≥ 4.

In this setup, high loop orders are possible.
Ideal playground: renormalization functions $\beta(g)$, $\gamma(g)$, $\gamma_m(g)$.

Idea

- Massless 2pt amplitudes are scalars (periods). Add a third point for more structure.
- Massless 3pt integrals (or 4pt conformal) are the simplest functions in QFT (two-scale).
- Construct a given Feynman integral by an increasing sequence of 3pt subgraphs.
- Use position space. Three points span a plane in $\mathbb{R}^D$. Consider this plane as $\mathbb{C}$.
- Study the 3pt integrals as functions on $\mathbb{C}$ using the theory of complex functions.
- Add edges by solving the Laplace equation.

Picture (by M. Borinsky)



Definition

Consider a Feynman graph G with three external vertices $z_0, z_1, z_2 \in \mathbb{R}^D$ and $v = V_G^{\mathrm{int}}$ internal vertices $x_1, \ldots, x_v \in \mathbb{R}^D$. In D-dimensional position space an edge $e = y_1 y_2$ in G has the propagator

$$p_e(y_1, y_2) = \frac{1}{||y_1 - y_2||^{D-2}}.$$

The vertices y_1 and y_2 can be internal or external. We generalize the propagator by allowing edge weights $\nu_e \in \mathbb{R}$,

$$p_{e,\nu_e}(y_1, y_2) = \frac{1}{||y_1 - y_2||^{2\lambda\nu_e}},$$

where $\lambda = D/2 - 1$. The Feynman integral of the graph G is

$$A_G(z_0, z_1, z_2) = \int \frac{\mathrm{d}x_1}{\pi^{D/2}} \cdots \int \frac{\mathrm{d}x_v}{\pi^{D/2}} \prod_{e \in E_G} p_{e,\nu_e}(x, z).$$

Definition

The graphical functions $f_G(z)$ is defined by

$$f_G(z) = A_G(z_0, z_1, z_2)$$

for the external vectors

$$z_0 = 0,\ z_1 = (1, 0, 0, \ldots, 0)^T,\ z_2 = (\mathrm{Re}\, z, \mathrm{Im}\, z, 0, \ldots, 0)^T.$$

Definition

For general z_0, z_1, z_2 one has the relation

$$A_G(z_0, z_1, z_2) = ||z_1 - z_0||^{-2\lambda N_G} f_G(z),$$

with invariants

$$\frac{||z_2 - z_0||^2}{||z_1 - z_0||^2} = z\overline{z}, \quad \frac{||z_2 - z_1||^2}{||z_1 - z_0||^2} = (z-1)(\overline{z}-1),$$

and the scaling weight (superficial degree of divergence)

$$N_G = \Big(\sum_{e \in E_G} \nu_e \Big) - \frac{(\lambda+1)v}{\lambda}.$$

General properties

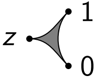
- Reflection symmetry $f_G(z) = f_G(\overline{z})$.
- f_G is a real-analytic single-valued function on $\mathbb{C}\backslash\{0,1\}$ (with M. Golz, E. Panzer).
- There exist single-valued log-Laurent expansions for the ϵ^k coefficients of $f_G(z)$ at the singular points $s = 0, 1$ and at ∞.

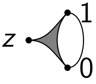
$$\sum_{\ell\geq 0}\sum_{m,n=M_s}^{\infty} c_{\ell,m,n}^{s,k}[\log(z-s)(\overline{z}-s)]^\ell (z-s)^m(\overline{z}-s)^n \quad \text{if } |z-s|<1,$$

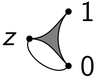
$$\sum_{\ell\geq 0}\sum_{m,n=-\infty}^{M_\infty} c_{\ell,m,n}^{\infty,k}(\log z\overline{z})^\ell z^m \overline{z}^n \quad \text{if } |z|>1,$$

with $c_{\ell,m,n}^{\bullet,k} = c_{\ell,n,m}^{\bullet,k} \in \mathbb{R}$.

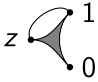
- Add edges between external vertices

$\Big[$ 

$\Big] = \Big[$ 

$\Big] = (z\overline{z})^{\lambda\nu_e} \Big[$ 

$\Big]$

$= [(z-1)(\overline{z}-1)]^{\lambda\nu_e} \Big[$ 

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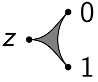
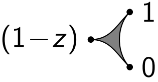
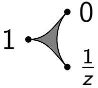
- Permute external vertices

$\Big[$ 

$\Big] = \Big[$ 

$\Big] = (z\overline{z})^{-\lambda N_G} \Big[$ 

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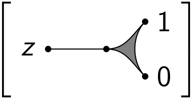
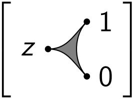
Appending edges

- Invert the effective Laplace operator $\Box_D$ for an isolated edge of weight 1 at vertex z,

$$\left(\Delta_n + \frac{\varepsilon/2}{z-\overline{z}}(\partial_z - \partial_{\overline{z}})\right)$$



$$= -\frac{1}{\Gamma(\lambda)}$$



$$\text{with} \quad \Delta_n = \frac{1}{(z-\overline{z})^{n+1}}\partial_z\partial_{\overline{z}}(z-\overline{z})^{n+1} + \frac{n(n+1)}{(z-\overline{z})^2},$$

$$\text{where} \quad D = 2n+4-\epsilon.$$

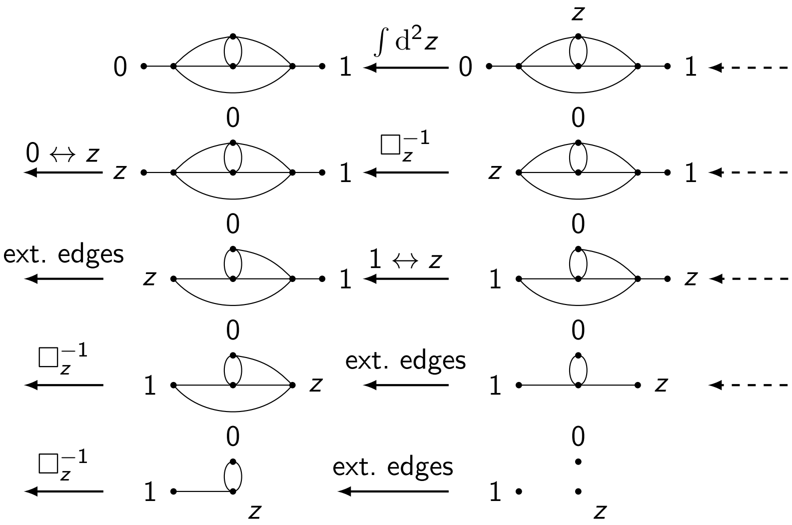
Vertex integration

- In the last step one may want to integrate over z to pass from a 3pt function to a 2pt function using

$$\frac{1}{(2\mathrm{i})^{2\lambda}\sqrt{\pi}\Gamma(\lambda+1/2)}\int_{\mathbb{C}} f_G(z)(z-\overline{z})^{2\lambda}\mathrm{d}^2z.$$

  In even integer dimensions one can use a residue theorem to do the integral.
  In non-integer dimensions we add an edge between 0 and z of weight -1, append an edge of weight 1 to z, and set $z=0$.

Picture (by M. Borinsky)



The five miracles of graphical functions

- For even integer D there exists a closed solution for the effective Laplace equation by taking single-valued primitives (with M. Borinsky). This is trivial in $D = 4$ dimensions.
- The solution is unique in the space of graphical functions.
- Generalized single-valued hyperlogarithms (GSVHs) are closed under solving the effective Laplace equation. The algorithm is efficient for GSVHs.
- The solution generalizes to non-integer dimensions $2n + 4 - \epsilon$.
- Spin k in D dimensions (QED, Yang-Mills) makes the effective Laplace equation a coupled system with triangular matrix whose diagonal is populated by (copies of) $\Box_D, \Box_{D+2}, \ldots, \Box_{D+2k}$.

GSVHs

Generalized single-valued hyperlogarithms (GSVHs) are iterated single-valued primitives of differential forms

$$\frac{\mathrm{d}z}{az\overline{z} + bz + c\overline{z} + d}, \qquad a, b, c, d \in \mathbb{C},$$

on the punctured (!) Riemann sphere $\mathbb{C}\backslash\{0, s_1, \ldots, s_n\}$.
Example (C. Duhr et al.).

$$\int_{\mathrm{sv}} \frac{D(z)\mathrm{d}z}{z - \overline{z}},$$

where $D(z)$ is the Bloch-Wigner dilogarithm,

$$D(z) = \mathrm{Im}\,(\mathrm{Li}_2(z) + \log(1 - z)\log|z|).$$

The commutative hexagon

GSVHs can be constructed with a commutative hexagon:

$$
\begin{tikzcd}
& \mathcal{G} & \\
\partial_z\mathcal{G} \arrow[ur, "\int_{\mathrm{sv}} \mathrm{d}z"] & & \partial_{\overline{z}}\mathcal{G} \arrow[ul, "\int_{\mathrm{sv}} \mathrm{d}\overline{z}"'] \\
\partial_z\mathcal{G} \arrow[u, "\pi_{\partial_z}"] \arrow[dr, "\partial_{\overline{z}}"'] & & \partial_{\overline{z}}\mathcal{G} \arrow[u, "\pi_{\partial_{\overline{z}}}"'] \\
& \partial_{\overline{z}}\partial_z\mathcal{G} \arrow[ur, "\int_{\mathrm{sv}} \mathrm{d}z"'] &
\end{tikzcd}
$$

where $\mathcal{G}$ is the $\mathbb{C}$-algebra of GSVHs and π_{∂_z} $(\pi_{\partial_{\overline{z}}})$ kills (anti-)residues in $\partial_z\mathcal{G}$ $(\partial_{\overline{z}}\mathcal{G})$.

$2n + 4 - \epsilon$ dimensions

- Taylor coefficients of convergent graphical functions in non-integer dimensions are obtained by a straight forward expansion method.
- For singular graphical functions a sophisticated subtraction method is necessary to obtain the Laurent coefficients. Problem: inversion of the effective Laplace equation. Example: bottom line in the cat eye calculation,

$$\frac{1}{(z\overline{z})^{2\lambda}((z-1)(\overline{z}-1))^{\lambda}}.$$

After inverting the effective Laplace operator, the graphical function has a singular part which is annihilated by Δ_0,

$$\frac{1}{z-\overline{z}}\partial_z\partial_{\overline{z}}(z-\overline{z})\,\frac{2}{\epsilon z\overline{z}} = 0.$$

Subtraction of subdivergences

Solution: Subtract (logarithmic) subdivergences:

$$\left(\frac{1}{(z\overline{z})^{2\lambda}((z-1)(\overline{z}-1))^{\lambda}} - \frac{1}{(z\overline{z})^{2\lambda}}\right) + \frac{1}{(z\overline{z})^{2\lambda}}.$$

- The first term is sufficiently regular at $z = 0$: The effective Laplace equation can be inverted uniquely.
- The inversion of the second term is a convolution:

$$\frac{1}{\pi^{D/2}} \int_{\mathbb{R}^D} \frac{1}{||x||^{4\lambda}||x - z_2(z)||^{2\lambda}} \mathrm{d}x.$$

- The general situation is fully algorithmic.
- Quadratic subdivergences are mere 2pt insertions.
- No a priori analysis or extra orders in ϵ necessary.

The graphical function toolbox

There exists a large toolbox for calculating low order Laurent coefficients of (singular) graphical functions.

- Completion: conformal symmetry.
- Approximation: replace a subgraph with a sum of simpler graphs with the same low order ϵ expansion.
- Rerouting: subtraction of subdivergences with simpler graphs to reduce the pole order in ϵ (F. Brown, D. Kreimer).
- Integration by parts (in particular spin > 0 or dimension ≥ 6).
- Algebraic identities (in particular spin > 0).
- Special identities: Twist, planar duals...
- Parametric integration: HyperInt (F. Brown, E. Panzer).
- ...

Comparison with classical techniques

- Momentum space techniques are more general (masses, Npt functions).
- Momentum space techniques can also be applied to graphical functions (master integrals).
- The theory of graphical functions performs integrations.
- The large set of constructible graphs is always computable with graphical functions (to sensible orders in ϵ).
- It is not necessary to solve large systems of linear equations.
- One always obtains a reduction of complexity by integrating out some vertices of the Feynman graph.

Results

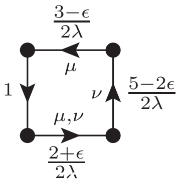
- Calculation of many primitive ϕ^4 periods up to 11 loops (and primitive ϕ^3 periods up to 9 loops) which lead to the discovery of the connection between motivic Galois theory and QFT (the coaction principle, the cosmic Galois group).
- ϕ^4 theory (4 dim.): 8 loops field anomalous dimension γ.
  7 loops β, mass anomalous dimension γ_m, self-energy Σ.
- ϕ^3 theory (6 dim.): 6 loops field anomalous dimension γ, β, mass anomalous dimension γ_m.
  5 loops self-energy Σ.

Six loops ϕ^3

$$\begin{aligned}
\beta_6^{\phi^3} &= \frac{245045}{144}\zeta(9) + 37\zeta(3)^3 + \frac{3357}{40}\zeta(5,3) - \frac{11}{3}\zeta(5)\zeta(3) \\
&- \frac{81733}{2016000}\pi^8 - \frac{456443}{1152}\zeta(7) + \frac{99}{800}\pi^4\zeta(3) - \frac{2425}{384}\zeta(3)^2 \\
&+ \frac{176425}{2612736}\pi^6 - \frac{24878747}{34560}\zeta(5) + \frac{42654751}{74649600}\pi^4 \\
&- \frac{85523425}{186624}\zeta(3) - \frac{173655397121}{3224862720} \\
&= -241.455497609497\ldots
\end{aligned}$$

$\zeta(5,3) = \sum_{k_1>k_2\geq 1} \frac{1}{k_1^5 k_2^3}$ (May 19, 2023).

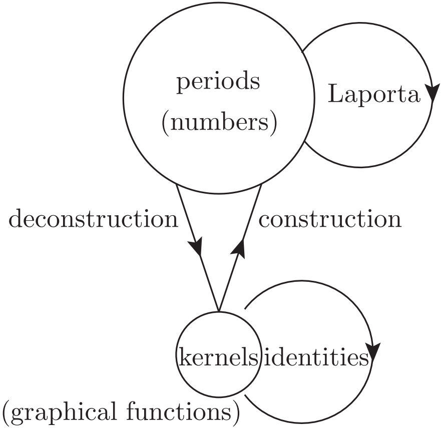
Express all integrals in terms of Feynman periods: scalar (fully contracted) integrals with no external vertices. Each Feynman period can be expressed by an unlabeled vacuum graph. Any choice of two vertices 0 and 1 give the same Feynman integral. Example:



Reduction of complexity

- A sizable subset of Feynman periods can be calculated immediately.
- One can increase the number of known Feynman periods by calculating kernel graphical functions.
- One can use IBP identities to reduce an unknown Feynman period to known Feynman periods.
- A combination of both techniques can reduce the complexity. For six loop primitive graphs in ϕ^3 theory:
  M. Borinsky, O. Schnetz, *Recursive computation of Feynman periods*, JHEP No. 08, 291 (2022).

(De-)construction



- HyperlogProcedures is a Maple package that performs calculations using graphical functions and GSVHs.
- It is also a toolbox to handle multiple zeta values (MZVs) including extensions to second (Euler sums), third, fourth, and sixth roots of unity.
- A large number of manipulations for hyperlogarithms (Goncharov polylogs) are implemented in HyperlogProcedures.
- HyperlogProcedures has the results for the renormalization functions in ϕ^4 and ϕ^3 with a large number of extra data.
- HyperlogProcedures is available for free download from my homepage.
  https://www.math.fau.de/person/oliver-schnetz/