Series expansion of multivariate hypergeometric functions about its parameters

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Outline

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Definitions

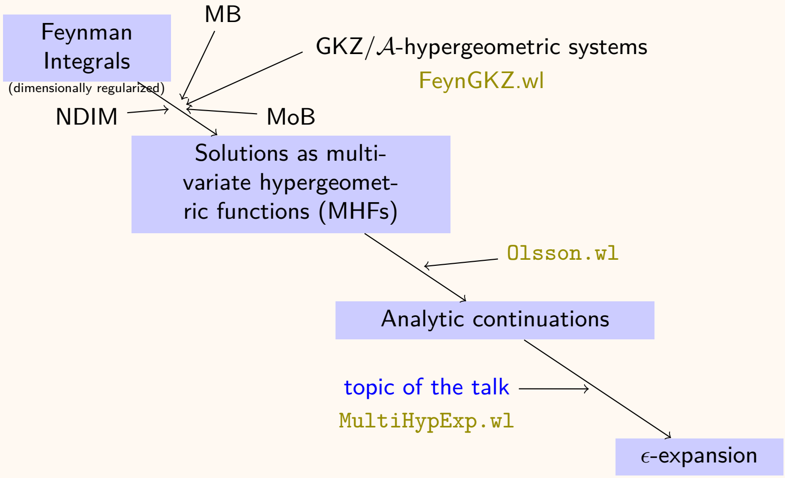
Feynman integrals and hypergeometric functions

Series expansion

Algorithm

Mathematica package : `MultiHypExp`

The Big Picture



Multivariate Hypergeometric functions

Definitions

- **Pochhammer symbol** :

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- General hypergeometric functions

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{}_pF_{p-1}\left(a_1, \dots, a_p; b_1, \dots, b_q; z\right) = \sum_{n=0}^{\infty} \frac{\left(a_1\right)_n \cdots \left(a_p\right)_n}{\left(b_1\right)_n \cdots \left(b_{p-1}\right)_n} \frac{z^n}{n!}, \qquad |z| < 1
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- $$F_4\left(a, b; c_1, c_2; x, y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{\left(c_1\right)_m\left(c_2\right)_n} \frac{x^m y^n}{m!n!}$$

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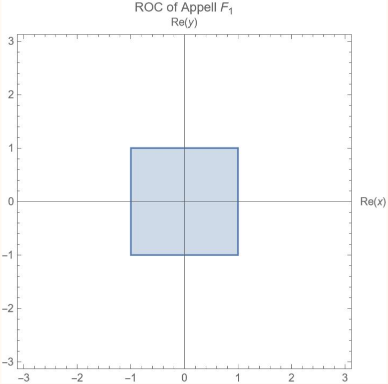
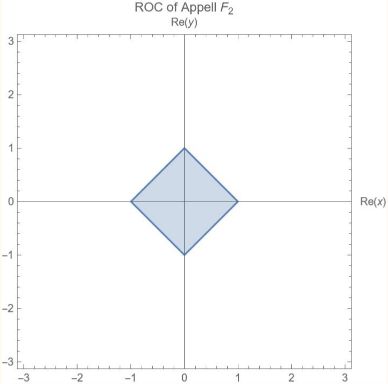
Region of Convergences

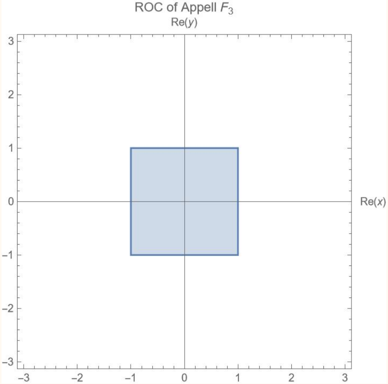
Appell F_1, F_2

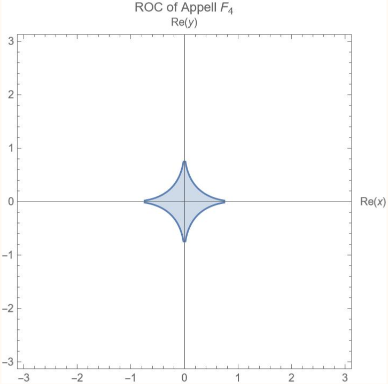




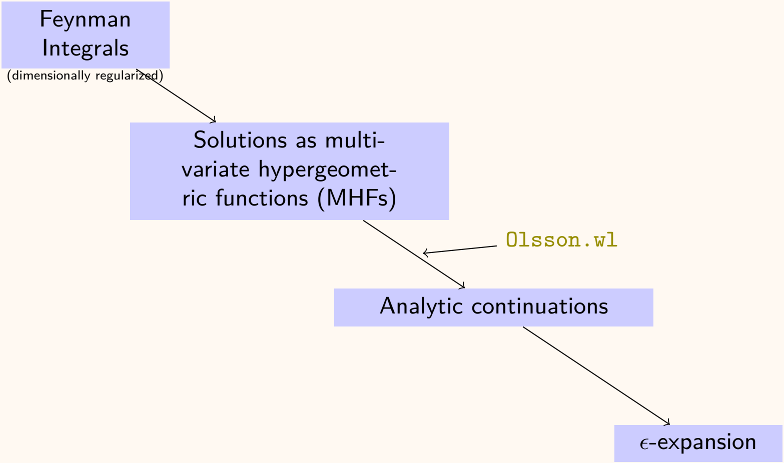
Region of Convergences

Appell F_3, F_4





The Big Picture



Analytic Continuation

Appell F_2

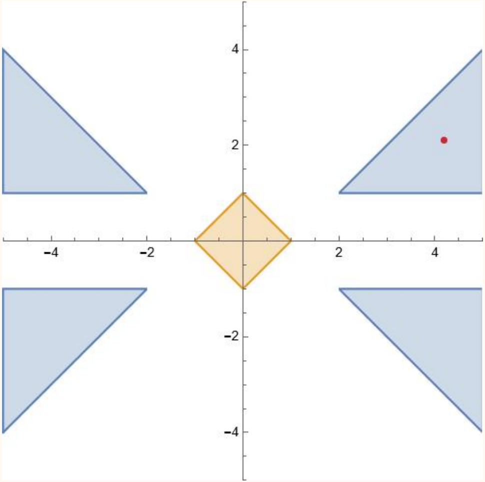


Figure: The defining region of convergence of Appell F_2 (in orange), and an analytic continuation of the same function that contains the red point (in blue) are plotted in real x-y plane

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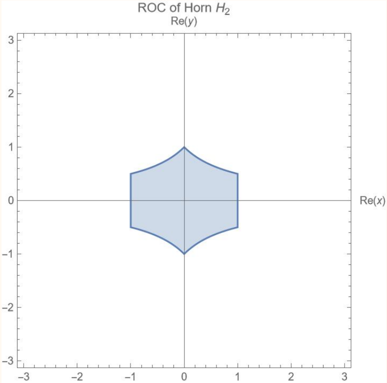
valid for $|x| < 1 \wedge |xy| + |y| < 1$

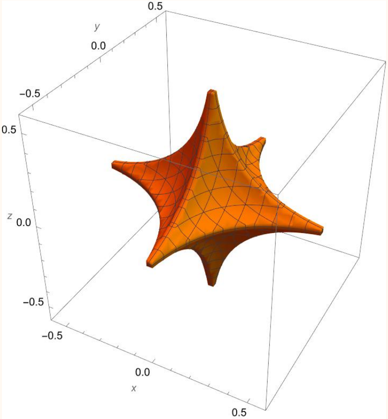


Definitions

The Lauricella $F_C^{(3)}$ function

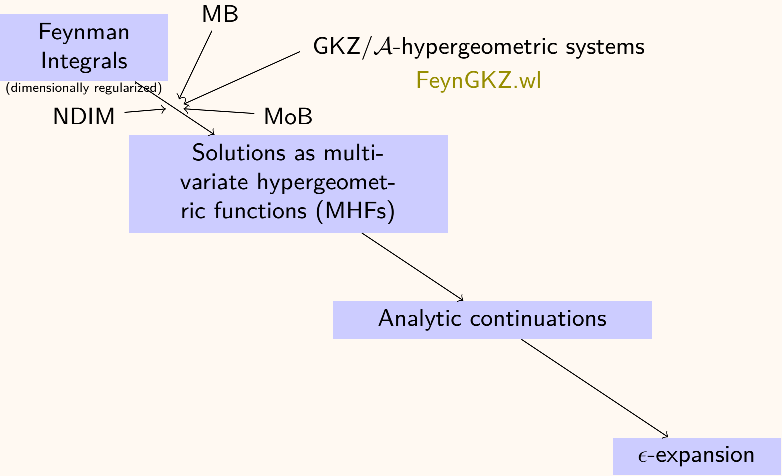
- $$F_C^{(3)} = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(a_1)_{n_1+n_2+n_3}(a_2)_{n_1+n_2+n_3}}{(c_1)_{n_1}(c_2)_{n_3}(c_3)_{n_2}} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{n_1! n_2! n_3!}$$

with domain of convergence : $\sqrt{|z_1|} + \sqrt{|z_2|} + \sqrt{|z_3|} < 1$



Feynman Integrals
&
Multivariate Hypergeometric functions

The Big Picture



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- One loop three-point function :

$$F_2(\epsilon + 1, 1, 1; \epsilon + 1, 2 - \epsilon; x, y),$$
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- The sunset integral with unequal masses : Berends et. al. [2]

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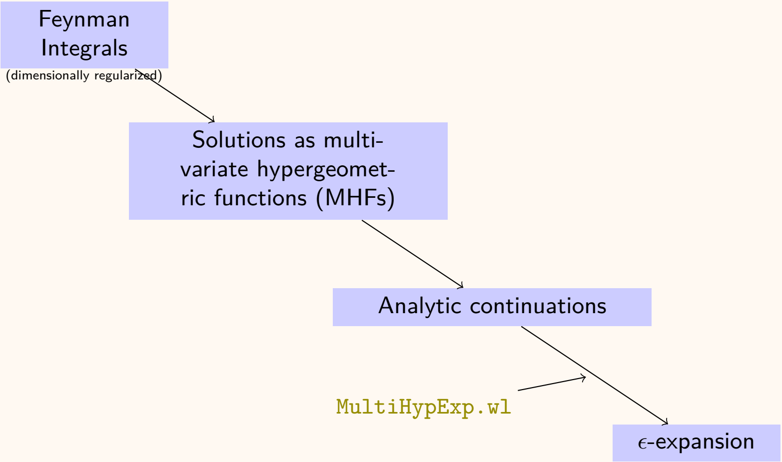
- ϵ-expansion of the multivariate hypergeometric functions (MHFs) are needed

The Big Picture



Series Expansion of Multivariate Hypergeometric functions

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- Each of the representations of the MHF can be used
  - Series
  - Integral and Mellin-Barnes representation
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- Available packages:
  - **Analytical** :
    - `HypExp` , `HypExp2` (Huber et. al. [3, 4])
    - `XSummer` (Moch et. al. [5])
    - `nestedsums` (Weinzierl [6, 7])
  - **Numerical** : `NumExp` (Huang et. al. [8])

Demonstration

Example 0

- Case I :

$$ {}_2F_1(1, 1; \epsilon + 1; x) = \sum_{m=0}^{\infty} \frac{(1)_m (1)_m}{(\epsilon + 1)_m} \frac{x^m}{m!} = \sum_{m=0}^{\infty} x^m + O(\epsilon) = \frac{1}{1 - x} + O(\epsilon) $$

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- MHFs with *singular* parameters may have Laurent series expansion

Algorithm

- *Step 1*: Check if the Pochhammer parameters of the given function ($F(\epsilon)$) are *singular* or not

- *Step 2*: If those are non-singular, find the Taylor expansion of $F(\epsilon)$

- *Step 3*: If any of the Pochhammer parameter of $F(\epsilon)$ is *singular* then find a new function ($G(\epsilon)$) by replacing

  singular Pochhammer $\longrightarrow$ non-*singular* Pochhammer

- *Step 4*: Relate them using a differential operator (H)

$$F(\epsilon) = H(\epsilon) \bullet G(\epsilon)$$
$$\left[\sum_{i=-n}^{\infty} \epsilon^i H_i\right] \bullet \left[\sum_{j=0}^{\infty} \epsilon^j G_j\right]$$

Step 1 & 3 : Checking the Pochhammers

- *Singular* Pochhammers :

1. When one or more lower Pochhammer parameters (i.e., Pochhammer parameters in the denominator) are of the form

$$(n + q\epsilon)_p$$

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- Example 1 :

$$A_1 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon-1)_m} \frac{x^m}{m!} \quad , \quad A_2 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon+1)_m} \frac{x^m}{m!}$$

singular Pochhammer in A_1 : $(-1+\epsilon)_m$
non-*singular* Pochhammer in A_2 : $(1+\epsilon)_m$

Step 2 :Taylor Expansion

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$$F(\epsilon) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \frac{d^i}{d\epsilon^i} F(\epsilon)\Big|_{\epsilon=0}$$

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  3. Bring it to the canonical form

Step 2 :Taylor Expansion

- From the definition of Taylor expansion

$$F(\epsilon) = \sum_{i=0}^{\infty} \frac{\epsilon^i}{i!} \frac{d^i}{d\epsilon^i} F(\epsilon)\Big|_{\epsilon=0}$$

- The series coefficients are MHFs
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Expansion around integer parameters

- The series coefficients are multiple polylogarithms
- The Taylor expansion can be found as follows :
  1. Find the differential equation
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  5. Solve the system order by order in ϵ

Step 2 :Taylor Expansion

1. Obtaining DE

- Continuing with Example 1

$$ {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} = \sum_{n=0}^{\infty} A(n) x^n $$

Step 2 :Taylor Expansion

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- Consider

$$ \frac{A_{n+1}}{A_n} = \frac{(a+n)(b+n)}{(n+1)(c+n)} = \frac{g(n)}{h(n)} $$

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- The annihilator

$$ L = \left[h(\theta) \frac{1}{x} - g(\theta) \right] $$

where $\theta = x\partial_x$

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Step 2 :Taylor Expansion

2. Pfaff System, 3. Canonical form

- The ODE can be brought to a Pfaff system.

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$$dg = \Omega g$$

where

$$\Omega = \begin{pmatrix} 0 & \frac{1}{x} \\ -\frac{ab}{x-1} & \frac{-a-b+c-1}{x-1} + \frac{1-c}{x} \end{pmatrix}$$

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- The length of the vector g = Holonomic rank of the system
- Find a transformation T to bring the system into *canonical* form (Henn [9])

$$dg' = \epsilon \Omega' g'$$

with

$$g = Tg' \qquad , \qquad \Omega' = T^{-1}\Omega T - T^{-1} dT$$

Step 2 :Taylor Expansion

Example 1 : 2. Pfaff System, 3. Canonical form, 4. Boundary condition, 5. Solution

- For our example of $A_2 = {}_2F_1(\epsilon, -\epsilon; 1+\epsilon; x)$

$$\Omega = \begin{pmatrix} 0 & \frac{1}{x} \\ \frac{\epsilon^2}{x-1} & \frac{\epsilon}{x-1} - \frac{\epsilon}{x} \end{pmatrix}$$

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- In this case,

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad , \quad \Omega' = \begin{pmatrix} 0 & \frac{1}{x} \\ \frac{1}{x-1} & \frac{1}{x-1} - \frac{1}{x} \end{pmatrix}$$

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- **Solution :**

$$A_2 = 1 + \epsilon^2 G(0,1;x) + \epsilon^3(-G(0,0,1;x) + G(0,1,1;x)) + O\left(\epsilon^4\right)$$

Differential Equations for MHF

- A HF with r number of variables

$$F(\mathbf{a},\mathbf{b},\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{N}_0^r} \frac{\Gamma(\mathbf{a}+\mu.\mathbf{m})/\Gamma(\mathbf{a})}{\Gamma(\mathbf{b}+\nu.\mathbf{m})/\Gamma(\mathbf{b})} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} = \sum_{\mathbf{m}\in\mathbb{N}_0^r} \frac{(\mathbf{a})_{\mu.\mathbf{m}}}{(\mathbf{b})_{\nu.\mathbf{m}}} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} = \sum_{\mathbf{m}\in\mathbb{N}_0^r} A(\mathbf{m})\mathbf{x}^{\mathbf{m}}$$

- Let

$$P_i = \frac{A(\mathbf{m}+\mathbf{e_i})}{A(\mathbf{m})} = \frac{g_i(\mathbf{m})}{h_i(\mathbf{m})} \qquad , \; i = 1,\ldots,r$$

- $\mathbf{e_i}$ is unit vector with 1 in its $i-$th entry
- The operators annihilate $F(\mathbf{a},\mathbf{b},\mathbf{x})$

$$L_i = \left[h_i(\boldsymbol{\theta})\frac{1}{x_i} - g_i(\boldsymbol{\theta})\right]$$

- where $\boldsymbol{\theta} = \{\theta_1,\ldots,\theta_r\}$ is a vector containing Euler operators $\theta_i = x_i\partial_{x_i}$

Step 4 : Differential operator

Contiguous relations

There exist contiguous relations that relate

$$ {}_2F_1(a \pm 1, b; c; x) \ , \ {}_2F_1(a, b \pm 1; c; x) \ , \ {}_2F_1(a, b; c \pm 1; x) $$

These can be obtained by applying differential operators

- *Example*:
  The unit step down operator for the GHS is given by $H(c) = \frac{1}{c}(\theta + c)$, i.e.,

$$ {}_2F_1(a, b; c; x) = H(c) \bullet \ {}_2F_1(a, b; c + 1; x) $$

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- Another example

$$ {}_2F_1(a + 1, b; c; x) = \frac{1}{a} (\theta + a) \bullet \ {}_2F_1(a, b; c; x) $$

Step 4 : Differential operator

Step Down Operators

- If needed, apply the step down operator multiple times

$$ {}_2F_1(a, b; c-1; x) = H(c-1)H(c) \bullet {}_2F_1(a, b; c+1; x) $$

Step 4 : Differential operator

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- Take quotient by the annihilator L

$$ L \bullet \ {}_2F_1(a, b; c+1, x) = 0 $$
$$ L = \left[-ab + (-x(a+b+1) + c + 1)\,\partial_x - (x-1)x\partial_x^2\right] $$

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- The step down operator :

$$ H = H(c-1)H(c) $$
$$ = \left(1 - \frac{abx}{(c-1)c(x-1)}\right) - \frac{x(a+b+1)-2cx+c-1}{(c-1)c(x-1)}\theta $$

Step Down Operators for MHF

- MHF with r variables

$$F(\mathbf{a},\mathbf{b},\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{N}_0^r} \frac{\Gamma(\mathbf{a}+\mu.\mathbf{m})/\Gamma(\mathbf{a})}{\Gamma(\mathbf{b}+\nu.\mathbf{m})/\Gamma(\mathbf{b})} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} = \sum_{\mathbf{m}\in\mathbb{N}_0^r} \frac{(\mathbf{a})_{\mu.\mathbf{m}}}{(\mathbf{b})_{\nu.\mathbf{m}}} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} = \sum_{\mathbf{m}\in\mathbb{N}_0^r} A(\mathbf{m})\mathbf{x}^{\mathbf{m}}$$

- The unit step down operator for the lower Pochhammer parameter

$$F(\mathbf{a},\mathbf{b},\mathbf{x}) = \frac{1}{b_i}\left(\sum_{j=1}^{r} \nu_{ij}\theta_{x_j} + b_i\right) \bullet F(\mathbf{a},\mathbf{b}+\mathbf{e_i},\mathbf{x}) = H(b_i) \bullet F(\mathbf{a},\mathbf{b}+\mathbf{e_i},\mathbf{x})$$

- The step down operator :

$$H = \left[\prod_{i=1} H(b_i)\right] / \langle L_1, \ldots, L_r \rangle$$

Example 1

- $$A_1 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon-1)_m} \frac{x^m}{m!} \quad , \quad A_2 := \sum_{m=0}^{\infty} \frac{(\epsilon)_m(-\epsilon)_m}{(\epsilon+1)_m} \frac{x^m}{m!}$$

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- So $A_1 = H \bullet A_2$

$$H = \frac{(\epsilon(2x-1)-x+1)}{(\epsilon-1)\epsilon(x-1)}\theta + \frac{\epsilon(2x-1)-x+1}{(\epsilon-1)(x-1)}$$

$$= \frac{1}{\epsilon}\theta + \left(1 - \frac{x}{x-1}\theta\right) + \epsilon\left(-\frac{x\,(\theta+1)}{x-1}\right) + \epsilon^2\left(-\frac{x\,(\theta+1)}{x-1}\right) + O(\epsilon^3)$$

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- The Taylor expansion of A_2 :

$$A_2 = 1 + \epsilon^2 G(0,1;x) + \epsilon^3(-G(0,0,1;x) + G(0,1,1;x)) + O\left(\epsilon^4\right)$$

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- The series expansion of A_1 :

$$A_1 = 1 + \epsilon\left[G(1;x) - \frac{x}{x-1}\right] + \epsilon^2\left[-\frac{x}{x-1}G(1;x) + G(1,1;x) - \frac{x}{x-1}\right] + O(\epsilon^3)$$

Example 1

- The Taylor expansion of A_2 :

$$A_2 = 1 - \epsilon^2 x \ {}_3F_2(1,1,1;2,2;x) + \epsilon^3 \left(\frac{2}{3} x \, {}_3F_2(1,1,1;2,2;x) \right.$$

$$\left. + \frac{x}{3} \ \mathrm{KdF}^{2:1;2}_{2:0;1} \left[\begin{matrix} 1,1 : \ 1\,;1,1 \\ 2,2 : -; \ \ 2 \end{matrix} \middle| \, x, x \right] + \frac{x^2}{12} \ \mathrm{KdF}^{2:1;2}_{2:0;1} \left[\begin{matrix} 2,2 : \ 1\,;1,2 \\ 3,3 : -; \ \ 3 \end{matrix} \middle| \, x, x \right] \right)$$

$$+ O(\epsilon^4)$$

Example 1

- The Taylor expansion of A_2 :

$$A_2 = 1 - \epsilon^2 x \ {}_3F_2(1,1,1;2,2;x) + \epsilon^3 \Bigg(\frac{2}{3} x \, {}_3F_2(1,1,1;2,2;x) + \frac{x}{3} \ \mathrm{KdF}^{2:1;2}_{2:0;1} \left[\begin{matrix} 1,1 : 1\,;1,1 \\ 2,2 : -;\ \ 2 \end{matrix} \middle| \, x, x \right] + \frac{x^2}{12} \ \mathrm{KdF}^{2:1;2}_{2:0;1} \left[\begin{matrix} 2,2 : 1\,;1,2 \\ 3,3 : -;\ \ 3 \end{matrix} \middle| \, x, x \right] \Bigg) + O(\epsilon^4)$$

- The Kampé de Fériet function :

$$\mathrm{KdF}^{2:1;2}_{2:0;1} \left[\begin{matrix} a_1, a_2 : b\,; c_1, c_2 \\ d_1, d_2 : -;\ \ e \end{matrix} \middle| \, x, y \right] = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}(b)_m(c_1)_n(c_2)_n x^m y^n}{(d_1)_{m+n}(d_2)_{m+n}(e)_n m! n!}$$

Example 1

- The ϵ-expansion of A_1:

$$
\begin{aligned}
A_1 = 1 &+ \epsilon \left(-\frac{x}{x-1} - x\,{}_3F_2(1,1,1;2,2;x) - \frac{x^2}{4}\,{}_3F_2(2,2,2;3,3;x) \right) \\
&+ \epsilon^2 \Bigg[-\frac{x}{x-1} + \frac{(x(2x+1))}{3(x-1)}\,{}_3F_2(1,1,1;2,2;x) \\
&+ \frac{\left(x^2(5x-2)\right)}{12(x-1)}\,{}_3F_2(2,2,2;3,3;x) + \frac{x}{3}\ \mathrm{KdF}\left[\begin{matrix} 1,1 :\ 1;1,1 \\ 2,2 : -;\ \ 2 \end{matrix} \middle| x, x \right] \\
&+ \frac{x^2}{12}\ \mathrm{KdF}\left[\begin{matrix} 2,2 :\ 2;1,1 \\ 3,3 : -;\ \ 2 \end{matrix} \middle| x, x \right] + \frac{x^2}{6}\ \mathrm{KdF}\left[\begin{matrix} 2,2 :\ 1;1,2 \\ 3,3 : -;\ \ 3 \end{matrix} \middle| x, x \right] \\
&+ \frac{x^2}{24}\ \mathrm{KdF}\left[\begin{matrix} 2,2 :\ 1;2,2 \\ 3,3 : -;\ \ 3 \end{matrix} \middle| x, x \right] + \frac{x^3}{27}\ \mathrm{KdF}\left[\begin{matrix} 3,3 :\ 2;1,2 \\ 4,4 : -;\ \ 3 \end{matrix} \middle| x, x \right] \\
&+ \frac{2x^3}{81}\ \mathrm{KdF}\left[\begin{matrix} 3,3 :\ 1;2,3 \\ 4,4 : -;\ \ 4 \end{matrix} \middle| x, x \right] \Bigg] + O(\epsilon^3)
\end{aligned}
$$

In [10], the series expansion coefficients are expressed in terms of multi-summation fold MHF with same domain of convergence

Advantage :

- The series expansion can be obtained around any values of Pochhammer parameters

Disadvantages :

- Valid for certain domain of convergence
- MHF with higher summation fold are slow for numerical evaluation

Series expansion around integer values of Pochhammer parameters of most of the functions can be expressed in terms of MPLs

Mathematica Package

MultiHypExp

MultiHypExp

Available at GitHub [11]

Dependencies :

- RISC`HolonomicFunctions (Koutschan [12, 13]) : To find the PDE associated with the given MHF and to form the Pfaff system
- HYPERDIRE (Bytev et.al. [14, 15, 16]) : For step up/down operations
- CANONICA (Meyer [17]) : To bring the Pfaff system into canonical form
- PolyLogTools (Duhr et. al. [18]) : To handle MPLs

Mathematica Package

MultiHypExp

The package is able to expand the following series

- **One variable** : ${}_pF_{p-1}$
- **Two variables** : Appell F_1, F_2, F_3, F_4, Horn $G_1, G_2, G_3, H_1, H_2, H_3, H_4, H_6$ and H_7 and certain KdF functions
- **Three variables** : Lauricella Saran $F_A, F_B, F_D, F_K, F_M, F_N$ and F_S
- Apart from Appell F_1, F_2, F_3 and Horn H_2, other Appell-Horn series are expanded using their relation to the former functions
- Series expansion of Appell F_4 and Horn H_1 is possible with certain restriction on the Pochhammer parameters

MultiHypExp

Commands for one variable

To obtain the series expansion ${}_2F_1(\epsilon, -\epsilon; \epsilon - 1; x)$

In[1]:=
```
SeriesExpand[{{e, -e}, {e - 1}}, {x}, e, 3]
```

Out[1]=
```
1 + (-(x/(-1+x))+G[1,x]) e + (-(x/(-1+x))-(x G[1,x])
/(-1+x)+G[1,1,x]) e^2 + O[e]^3
```

Alternatively,

In[2]:=
```
SeriesExpand[{n},(Pochhammer[e,n]Pochhammer[-e,n]x^n)
/(Pochhammer[e-1,n]n!),{x},e,3]
```

yields the same result.

MultiHypExp

Commands for bi- and tri-variate HS

In[3]:=
```
SeriesExpand[F2,{1,1,e,1+e,1-e},{x,y},e,3]
```

Out[3]=
```
 -(1/(-1+x))+((-2 G[1,x]+G[1,y]+G[1-y,x]) e)/(-1+x)
+(1/(-1+x))(2 G[1,x] G[1,y]-2 G[1,y] G[1-y,x]
+2 G[0,1,x]+G[0,1,y]-G[0,1-y,x]-4 G[1,1,x]-2 G[1,1,y]
+2 G[1,1-y,x]+2 G[1-y,1,x]-G[1-y,1-y,x]) e^2+O[e]^3
```

yields the first four series expansion coefficients of Appell $F_2(1, 1, e; 1+e, 1-e; x, y)$ with respect to e in terms of MPLs.

In[4]:=
```
SeriesExpand[{m,n}, exp, {x, y}, e, 4]
```

exp must be a series presentation of a MHF with summation indices m and n.

MultiHypExp

Commands for obtaining reduction formulae

To find reduction formulae of MHF

In[5]:=
```
ReduceFunction[F2,{3,2,1,3,2},{x,y}]
```

Out[5]=
```
1/((-1+x) x (-1+x+y))-G[1,x]/(x^2 y)+G[1-y,x]/(x^2 y)
```

In terms of logarithms

$$F_2(3,2,1;3,2;x,y) = -\frac{\log(1-x)}{x^2 y} + \frac{\log\left(1-\frac{x}{1-y}\right)}{x^2 y} + \frac{1}{(x-1)x(x+y-1)}$$

This command can find reduction formulae of Appell F_1, F_2, F_3, F_4 and Lauricella-Saran $F_D^{(3)}$ and $F_S^{(3)}$

MultiHypExp

Conclusions

- Applicable when the parameter ϵ appears linearly inside the Pochhammer symbols
- Series coefficients are not valid at the singular curves of MHF
- The package can find the expansion of most of the MHF around integer values of Pochhammer parameters
- It can find at most first 6 coefficients
- Takes 3-4 hours to find series expansion of a three variable HF in an ordinary personal computer

Thank You

References I

[1] C. Anastasiou, E. W. Nigel Glover, and C. Oleari. *Nucl. Phys. B* 572 (2000), pp. 307–360. arXiv: hep-ph/9907494.

[2] Frits A. Berends et al. *Z. Phys. C* 63 (1994), pp. 227–234.

[3] T. Huber and Daniel Maitre. *Comput. Phys. Commun.* 175 (2006), pp. 122–144. arXiv: hep-ph/0507094.

[4] Tobias Huber and Daniel Maitre. *Comput. Phys. Commun.* 178 (2008), pp. 755–776. arXiv: 0708.2443 [hep-ph].

[5] S. Moch and P. Uwer. *Comput. Phys. Commun.* 174 (2006), pp. 759–770. arXiv: math-ph/0508008.

[6] Stefan Weinzierl. *Comput. Phys. Commun.* 145 (2002), pp. 357–370. arXiv: math-ph/0201011.

[7] Stefan Weinzierl. *J. Math. Phys.* 45 (2004), pp. 2656–2673. arXiv: hep-ph/0402131.

[8] Zhi-Wei Huang and Jueping Liu. *Comput. Phys. Commun.* 184 (2013), pp. 1973–1980. arXiv: 1209.3971 [physics.comp-ph].

[9] Johannes M. Henn. *Phys. Rev. Lett.* 110 (2013), p. 251601. arXiv: 1304.1806 [hep-th].

References II

[10] Souvik Bera. *Nucl. Phys. B* 989 (2023), p. 116145. arXiv: 2208.01000 [math-ph].

[11] Souvik Bera. (June 2023). arXiv: 2306.11718 [hep-th].

[12] Christoph Koutschan. *ACM Commun. Comput. Algebra* 43.3/4 (June 2010), p. 119. ISSN: 1932-2240. URL: https://doi.org/10.1145/1823931.1823954.

[13] Christoph Koutschan. *Mathematics in Computer Science* 4.2 (2010), pp. 259–266.

[14] Vladimir V. Bytev, Mikhail Yu. Kalmykov, and Bernd A. Kniehl. *Comput. Phys. Commun.* 184 (2013), pp. 2332–2342. arXiv: 1105.3565 [math-ph].

[15] Vladimir V. Bytev, Mikhail Yu. Kalmykov, and Sven-Olaf Moch. *Comput. Phys. Commun.* 185 (2014), pp. 3041–3058. arXiv: 1312.5777 [math-ph].

[16] Vladimir V. Bytev and Bernd A. Kniehl. *Comput. Phys. Commun.* 189 (2015), pp. 128–154. arXiv: 1309.2806 [math-ph].

[17] Christoph Meyer. *Comput. Phys. Commun.* 222 (2018), pp. 295–312. arXiv: 1705.06252 [hep-ph].

References III

[18] Claude Duhr and Falko Dulat. *JHEP* 08 (2019), p. 135. arXiv: 1904.07279 [hep-th].

Backup Slides

Multiple Polylogarithms

- Multiple polylogarithms ;

$$G(a_1,\ldots,a_n;z)=\int_0^z \frac{dt}{t-a_1}G(a_2,\ldots,a_n;t)$$

  with $G(z)=1$ and a_i and z are complex-valued variables.
- The vector $\vec{a}=(a_1,\ldots,a_n)$ is called the weight vector, and its length is called the weight.
-

$$\begin{aligned}\mathrm{Li}_{m_1,\ldots,m_k}(z_1,\ldots,z_k) &= \sum_{0<n_1<n_2<\cdots<n_k} \frac{z_1^{n_1} z_2^{n_2}\cdots z_k^{n_k}}{n_1^{m_1} n_2^{m_2}\cdots n_k^{m_k}} \\ &= \sum_{n_k=1}^{\infty}\frac{z_k^{n_k}}{n_k^{m_k}}\sum_{n_{k-1}=1}^{n_k-1}\cdots\sum_{n_1=1}^{n_2-1}\frac{z_1^{n_1}}{n_1^{m_1}},\end{aligned}$$

- These are related

$$G\left(\vec{0}_{m_1-1},a_1,\ldots,\vec{0}_{m_k-1},a_k;z\right)=(-1)^k\,\mathrm{Li}_{m_k,\ldots,m_1}\left(\frac{a_{k-1}}{a_k},\ldots,\frac{a_1}{a_2},\frac{z}{a_1}\right)$$

$$G\left(\vec{a}_n;z\right)=\frac{1}{n!}\log^n\left(1-\frac{z}{a}\right),$$

$$G\left(\vec{0}_{n-1},a;z\right)=-\mathrm{Li}_n\left(\frac{z}{a}\right),$$

Taylor Expansion

- **Polygamma function** : $\psi^{(n)}(a) = \frac{d^{n+1}}{da^{n+1}} \ln \Gamma(a)$
- $\psi^{(0)}(a) = \frac{d}{da} \ln \Gamma(a) = \frac{\Gamma'(a)}{\Gamma(a)}$
- **Pochhammer derivative**:

$$\frac{d(a)_n}{da} = (a)_n \left(\psi^{(0)}(a+n) - \psi^{(0)}(a)\right) = (a)_n \sum_{k=0}^{n-1} \frac{1}{a+k} = (a)_n \frac{1}{a} \sum_{k=0}^{n-1} \frac{(a)_k}{(a+1)_k}$$

- Consider a MHF : $\sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} (a)_n$
- Derivative wrt a

$$\sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} \frac{d}{da} (a)_n = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} (a)_n \frac{1}{a} \sum_{k=0}^{n-1} \frac{(a)_k}{(a+1)_k}$$

- shifting $n \to n+1$,

$$\sum_{n=0}^{\infty} B(n) \frac{x^n}{n!} \frac{d}{da} (a)_n = \sum_{n=0}^{\infty} B(n+1) \frac{x^{n+1}}{(n+1)!} (a)_{n+1} \frac{1}{a} \sum_{k=0}^{n} \frac{(a)_k}{(a+1)_k}$$

Taylor Expansion

- Reshuffling the summation indices

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(n,k) \to \sum_{n,k=0}^{\infty} A(n+k,k)$$

- One gets

$$\sum_{n=0}^{\infty} B(n)\frac{x^n}{n!}\frac{d}{da}(a)_n = \sum_{n,k=0}^{\infty} B(n+k+1)\frac{x^{n+k+1}}{(n+k+1)!}(a)_{n+k+1}\frac{1}{a}\frac{(a)_k}{(a+1)_k}$$

$$= x\sum_{n,k=0}^{\infty} B(n+k+1)\frac{(1)_k(1)_n(a)_k(a+1)_{k+n}}{(a+1)_k(2)_{k+n}}\frac{x^n x^k}{k!n!}$$

- Thus, Pochhammer derivative introduces one extra summation index
- The domain of convergence remains same