A semi-analytic approach to one-scale Feynman integrals

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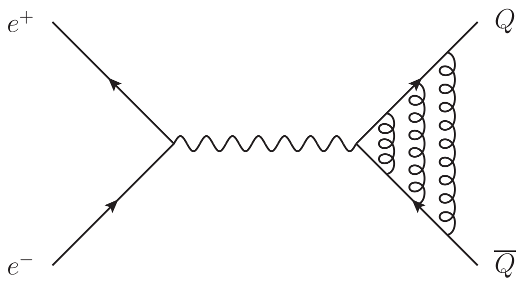
MARIE CURIE ACTIONS

Funded by the European Union

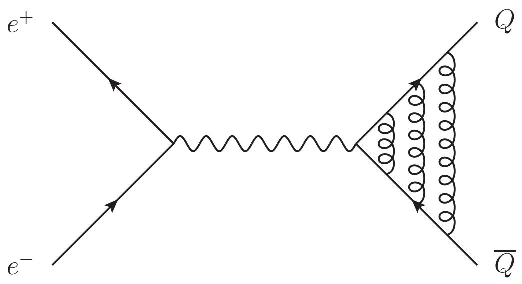
- **The method**
- **Application I: QCD massive form factors at 3 loops**
- **Application II: B-meson decays**

Solving Feynman integrals via differential equations



qgraf

Solving Feynman integrals via differential equations



qgraf

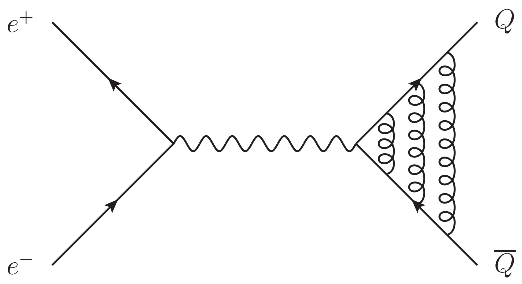
```
Symbol x;
Local E = f(1,2,x,3,4);

id f(?a,x,?b) = f(?b,?a);

Print;
.end
```

FORM

Solving Feynman integrals via differential equations



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FORM

AIR, Kira, FIRE, Reduze
Blade, NeatIBP, FiniteFlow

- $d = 4 - 2\epsilon$
- $\vec{x} = \{x_1, \ldots, x_m\}$ with e.g. $x_i = m/M, x = s/M^2$
- $\vec{j}(\vec{x}, \epsilon) = \{j_1(\vec{x}, \epsilon), \ldots, j_N(\vec{x}, \epsilon)\}$
- $M_i(\vec{x}, \epsilon)$ from IBP and dimensional shift relations

$$\frac{\partial \vec{j}}{\partial x_i} = M_i(\vec{x}, \epsilon)\, \vec{j}$$

Kotikov, Phys.Lett.B 254 (1991) 158
Gehrmann, Remiddi, Nucl.Phys.B 580 (2000) 485

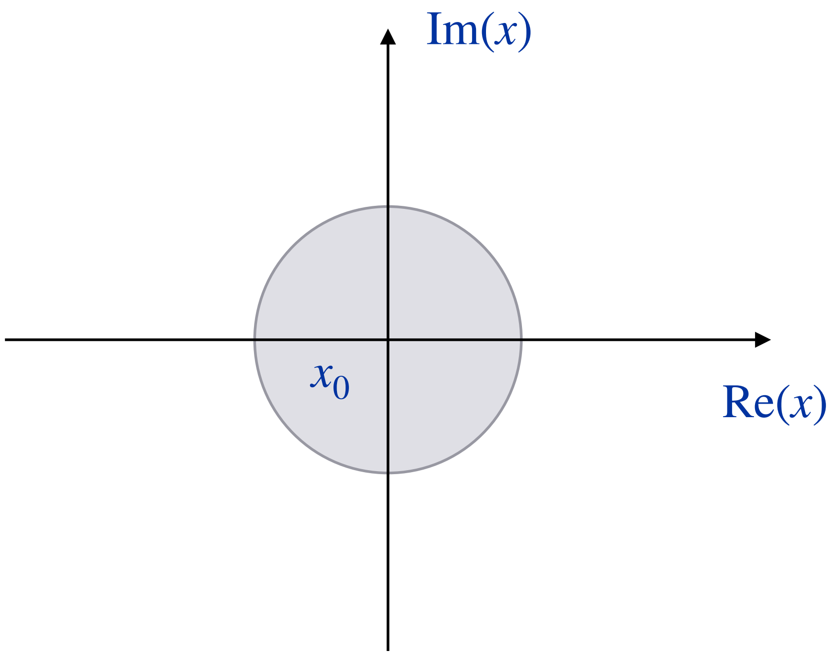
- **Analytic solution**
  - Solve in terms of known constants/functions
  - Function properties well understood
  - Known analytic structures and series expansions
  - Fast and generic numerical evaluation tools
- **Numerical solution**
  - Oriented to phenomenological studies
  - Applicable to larger class of problems
  - Finite numerical accuracy

A semi-analytic method for one-scale problem

$$\frac{\partial \vec{j}}{\partial x} = M(x, \epsilon)\, \vec{j}$$

- We consider one dimensionless variable x
- We can compute **boundary conditions for** $x = x_0$
- **GOAL**: construct in the complex plane a series expansion around some point x_0 (and ϵ)

$$j_a(x, \epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{n_{\max}} c_{a,mn} \epsilon^m (x - x_0)^n$$

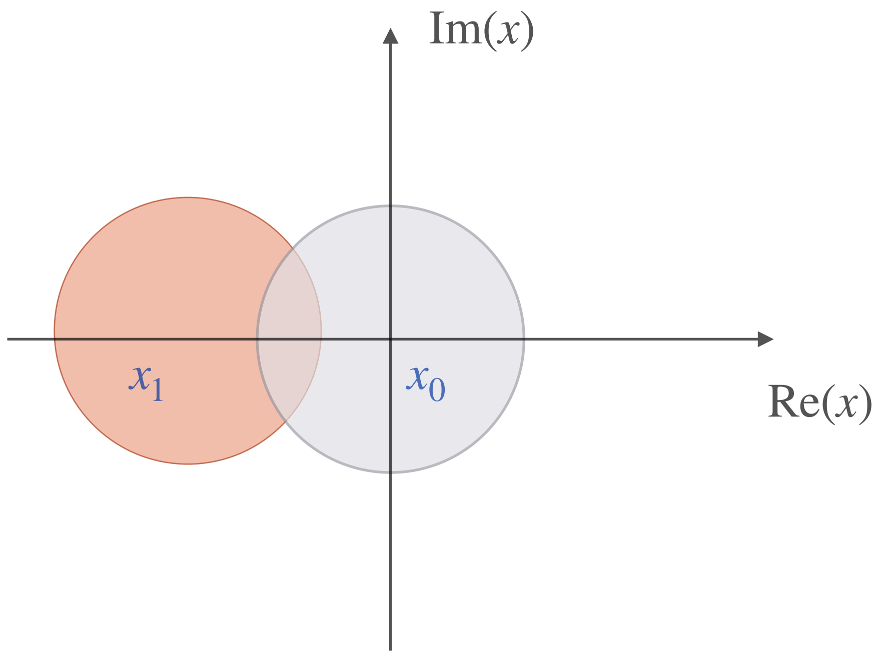


A semi-analytic method for one-scale problem

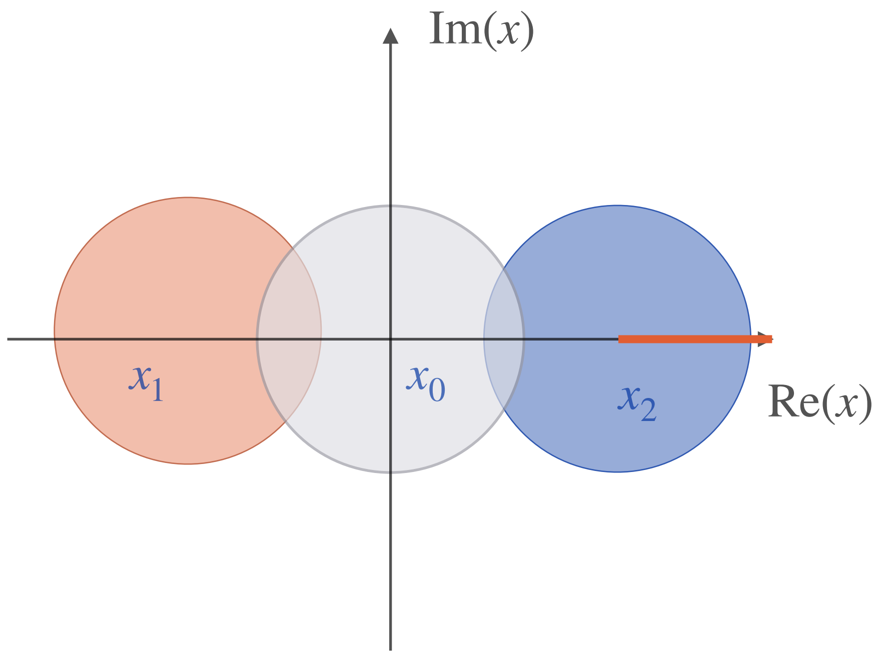
- Inserting the ansatz into the differential equation

$$\underbrace{\sum_{m}\sum_{n=1} n c_{a,mn}\epsilon^m (x-x_0)^{n-1}}_{\partial j_a/\partial x} = \sum_b M_{ab}(x,\epsilon) \underbrace{\sum_m \sum_{n=0} c_{b,mn}\epsilon^m (x-x_0)^n}_{j_b}$$

- Establish a **linear system of equations** for the expansion coefficients $c_{k,mn}$
- Solve the liner system in term of a **minimal set of coefficients** $\tilde{c}_{a,mn}$
- The minimal set of undetermined coefficients are **fixed from boundary conditions**



- Proceeds with a **new expansion around** $x = x_1$
- Match new expansion to the previous one (with finite accuracy)
- **Iterate** until all range of x is covered



- Proceeds with a **new expansion around** $x = x_1$
- Match new expansion to the previous one (with finite accuracy)
- **Iterate** until all range of x is covered

- Expansion around **singular points** (poles and thresholds)

$$j_a(x,\epsilon) = \sum_{m=m_{\min}}^{m_{\max}} \sum_{n=0}^{n_{\max}} \sum_{l\geq 0} c_{a,mnl}\, \epsilon^m \, (x_2 - x)^{\alpha n - \beta} \, \log^l(x_2 - x)$$

Features of our implementation

Fael, Lange, Schönwald, Steinhauser *JHEP* 09 (2021) 152

- **GOAL**: cover physical range of x with series expansions.
- **No special form** of the differential equations
- Well suited for fast numerical evaluation
- Precision systematic improvable:
    - more expansion points
    - deeper expansion in x
    - variable transformation

    (Möbius transformation)
- **Bottleneck**
    - Problems with $O(10^2)$ masters
    - Solve linear system with $O(10^6)$ equations
    - Match expansion in numerically stable way

Similar approaches

- SYS

  Laporta, Int.J.Mod.Phys.A 15 (2000) 5087
- SolveCoupledSystems.m

  Blümlein, Schneider, Phys.Lett.B 771 (2017) 31
- DESS

  Lee, Smirnov, Smirnov, JHEP 03 (2018) 008
- DiffExp

  Hidding, Comput.Phys.Commun. 269 (2021) 108125
- SeaSide

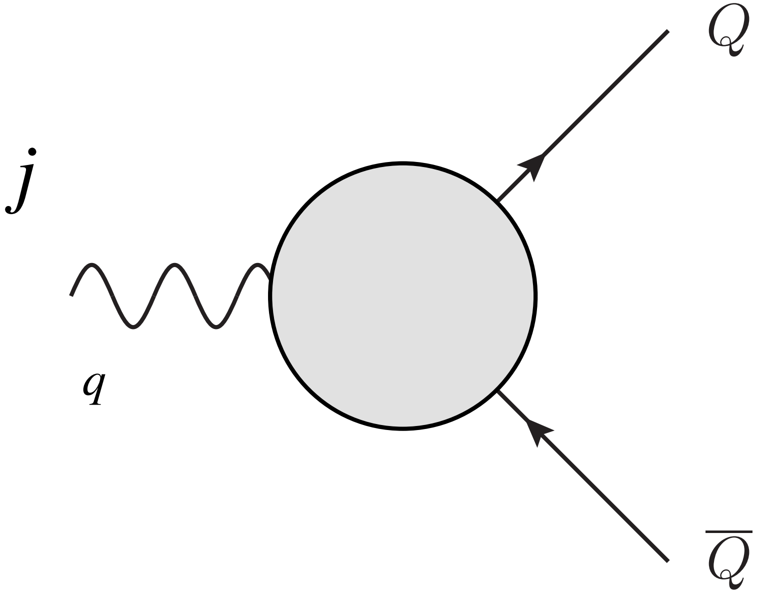
  Armadillo, Bonciani, Devoto, Rana, Vicini, Comput.Phys.Commun. 282 (2023) 108545
- AMFlow

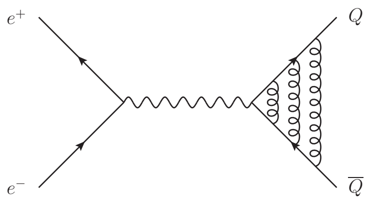
  Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565

Application I:
QCD massive form factors at 3 loops

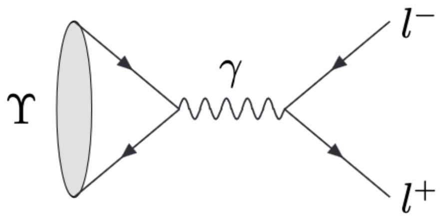
Phys.Rev.Lett. 128 (2022), Phys.Rev.D 106 (2023), Phys.Rev.D 107 (2023)

Massive form factors

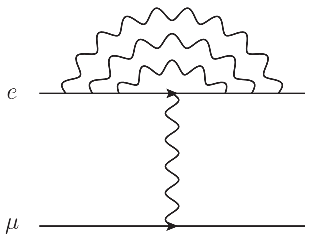




$t\bar{t}$ at e^+e^- collider



Quarkonium decay



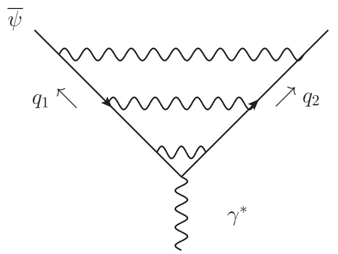
μe scattering

$$V(q_1, q_2) = \bar{u}(q_2)\Gamma(s)v(q_1)$$

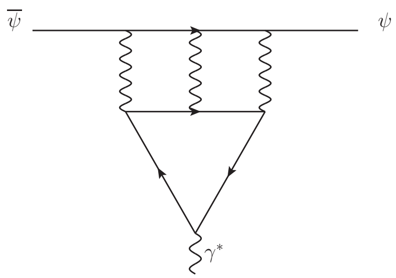
with $q_1^2 = q_2^2 = m^2$ and $s = (q_1 + q_2)^2$

- 2 loop QED
  Mastrolia, Remiddi, Nucl. Phys. B 664 (2003)
  Bonciani, Mastrolia, Remiddi, Nucl. Phys. B 676 (2004) 399
- 2 loop QCD
  Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi, Nucl. Phys. B 706 (2005) 245
  Gluza, Mitov, Moch, Riemann, JHEP 07 (2009), 001
  Ahmed, Henn, Steinhauser, JHEP 06 (2017), 125. Ablinger, et al, Phys.Rev. D 97 (2018), 094022
- 3 loop planar
  Henn, Smirnov, Smirnov, Steinhauser, JHEP 01 (2017), 074.
  Ablinger, Blümlein, Marquard, Rana, Schneider, Phys. Lett. B 782 (2018), 528
- 3 loop fermions
  Lee, Smirnov, Smirnov and M. Steinhauser, JHEP 03 (2018), 136.
  Blümlein, Marquard, Rana, Schneider, Nucl. Phys. B 949 (2019), 114751, Phys.Rev.D 108 (2023) 094003

	non singlet	n_h singlet	n_l singlet
diagrams	271	66	66
families	34	17	13
masters	422	316	158



Non-singlet



Singlet

- Before complete IBP reduction, we search for a **good basis** of master integrals

In the coefficients in front of the master integrals, ϵ and the kinematic variables factories in the denominators

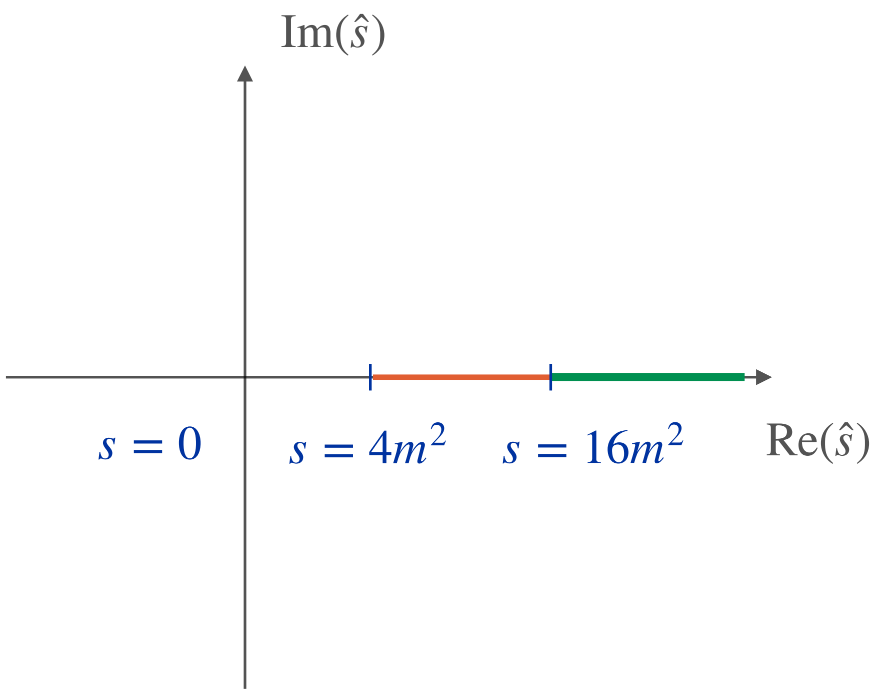
ImproveMasters.m A. & V. Smirnov, Nucl.Phys.B 960 (2020) 115213, Usovitsch, hep-ph/2002.08173.

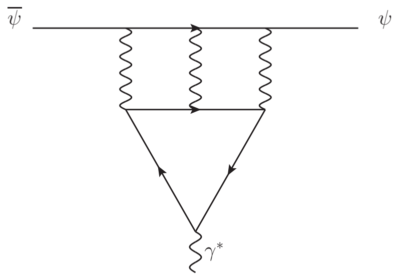
- Reduce each integral family to good basis
- Reduce masters across families to minimal set

Kira A. & V. Smirnov, Nucl.Phys.B 960 (2020) 115213, Usovitsch, hep-ph/2002.08173.

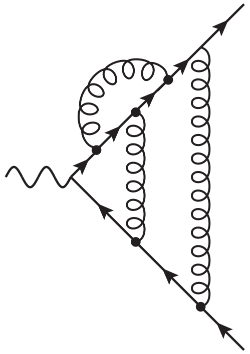
	Current	Form factors
vector	$j^v_\mu = \bar{\psi}\gamma_\mu\psi$	$\Gamma^v_\mu(s) = F^v_1(s)\gamma_\mu - \frac{i}{2m}F^v_2(s)\sigma_{\mu\nu}q^\nu$
axial-vector	$j^a_\mu = \bar{\psi}\gamma_\mu\gamma_5\psi$	$\Gamma^a_\mu(s) = F^a_1(s)\gamma_\mu\gamma_5 - \frac{1}{2m}F^a_2(s)\gamma_5 q_\mu$
scalar	$j_s = m\bar{\psi}\psi$	$\Gamma^s(s) = mF^s(s)$
pseudo-scalar	$j_p = im\bar{\psi}\gamma_5\psi$	$\Gamma^p(s) = imF^p(s)$

$\hat{s} = s/m^2$





Singlet



$\hat{s} = 4$: two-particle threshold

$$j_n = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+3} c_{n,ijk} \epsilon^i \left[4 - \hat{s}\right]^{j/2} \log^k(4 - \hat{s})$$

$\hat{s} = 16$: four-particle threshold

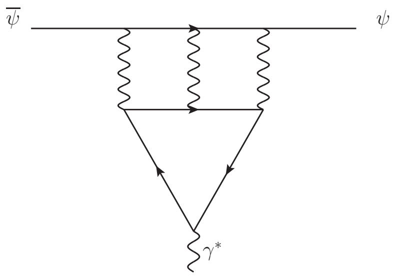
$$j_n = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+3} c_{n,ijk} \epsilon^i \left[16 - \hat{s}\right]^{j/2} \log^k(16 - \hat{s})$$

$\hat{s} = \infty$: high energy/massless limit

$$j_n = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{50} \sum_{k=0}^{i+6} c_{n,ijk} \left(\frac{1}{\hat{s}}\right)^j \log^k(-\hat{s})$$

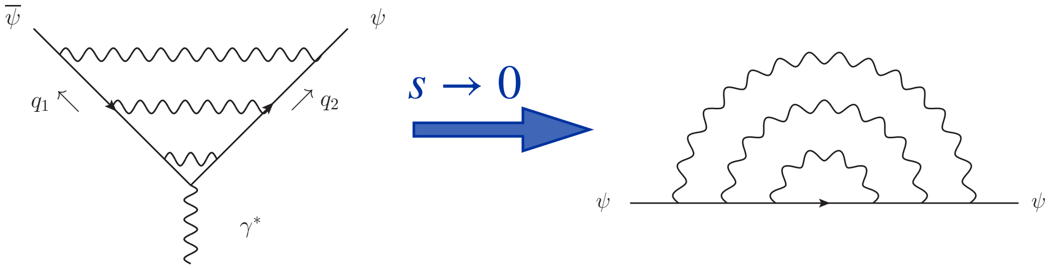
also $\hat{s} = 0$ for singlet diagrams

Boundary conditions



Non-singlet

- **Analytic results at $s = 0$**
- Simple Taylor expansion



Melnikov, van Ritbergen, Phys.Lett.B 482 (2000) 99
Lee, Smirnov, JHEP 02 (2011) 102

- However we need higher orders in ϵ (up to weight 9)
- Use SummerTime.m and PSLQ

Lee, Mingulov, Comput.Phys.Commun. 203 (2016) 255

Singlet

- Asymptotic expansion for singlet at $s = 0$
- n_h-singlet we have **analytic boundary cond**.
  - asy.m — Jantzen, Smirnov, Smirnov, Eur. Phys. J. C 72 (2012), 2139
  - HyperInt — Panzer, Comput. Phys. Commun. 188 (2015), 148
- n_l-singlet: **numerical boundary conditions**
  - AMFlow, high-precision evaluation at $\hat{s} = -1$

Liu, Ma, Comput.Phys.Commun. 283 (2023) 108565

Computational challenges

- Generation linear equations with Mathematica — ~ few hours
- Interface to Kira and solution via `reduce_user_defined_system` — ~ 1d per expansion point
- Singular points: finite field methods and rational reconstruction: Kira+FireFly — ~ 15 d

von Manteuffel, Schabinger, Phys.Lett.B 744 (2015) 101
Peraro, *JHEP* 12 (2016) 030
Klappert, Klein, Lange, Comput. Phys. Commun. 264 (2021), 107968

- Better more matching points than deeper expansions

e.g. for the non-singlet

$$\hat{s}_0 = \{\infty, -32, -28, -24, -16, -12, -8, -4, 0, 1, 2, 5/2, 3, 7/2, 4, 9/2, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 22, 28, 40, 52\}$$

- Numerical instabilities in the matching

Renormalization and IR subtraction

- UV renormalisation in the on-shell scheme

Melnikov, van Ritbergen, Phys.Lett.B 482 (2000) 99;
Chetyrkin, Steinhauser, Nucl.Phys.B 573 (2000) 617-651

- Structure of IR poles is universal
- Minimal subtraction

$$F_i^{\mathrm{UV\,ren}}(s) = Z_{\mathrm{IR}} F_i^f(s)$$

with Z_{IR} given by Γ_{cusp}

$$\log Z_{\mathrm{IR}} = -\frac{1}{2\epsilon}\frac{\alpha_s}{\pi}\Gamma^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\left[\frac{\beta_0\Gamma^{(1)}}{16\epsilon^2} - \frac{\Gamma^{(2)}}{4\epsilon}\right] + \left(\frac{\alpha_s}{\pi}\right)^3\left[-\frac{\beta_0^2\Gamma^{(1)}}{96\epsilon^3} + \frac{\beta_1\Gamma^{(1)} + 4\beta_0\Gamma^{(2)}}{96\epsilon^2} - \frac{\Gamma^{(3)}}{6\epsilon}\right]$$

Grozin, Henn, Korchemsky, Marquard, Phys.Rev.Lett. 114 (2015) 6, 062006; JHEP 01 (2016) 140.

Results

- Fortran library for Monte Carlo implementation

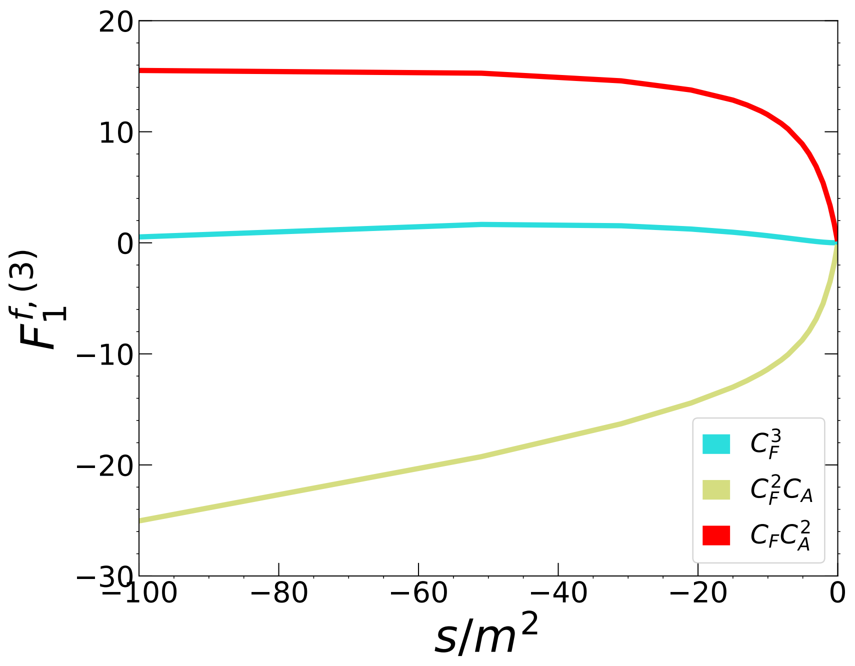
https://gitlab.com/formfactors3l/FF3l

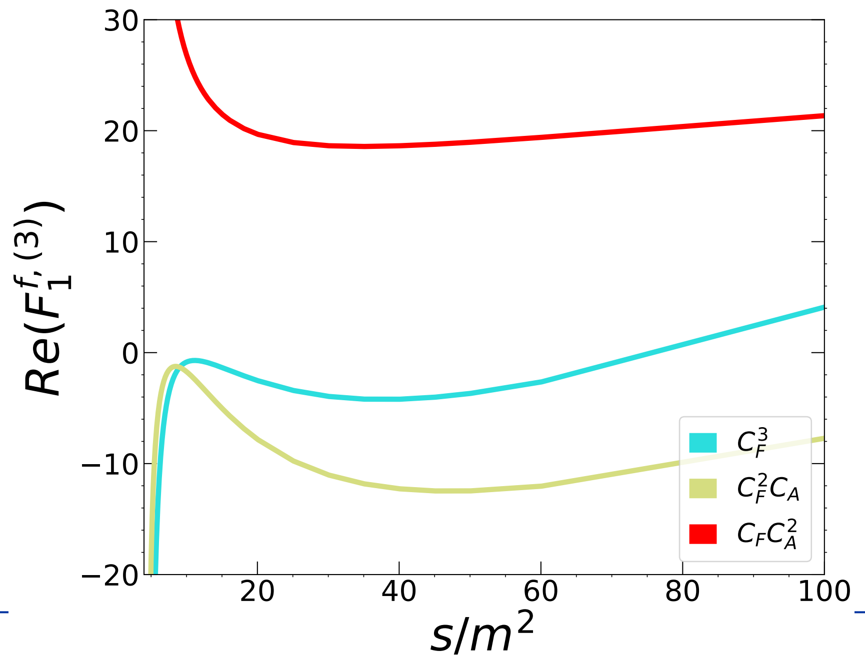
- UV renormalised
- No IR subtraction, other schemes beyond minimal can be applied
- Chebyschev interpolation grids for

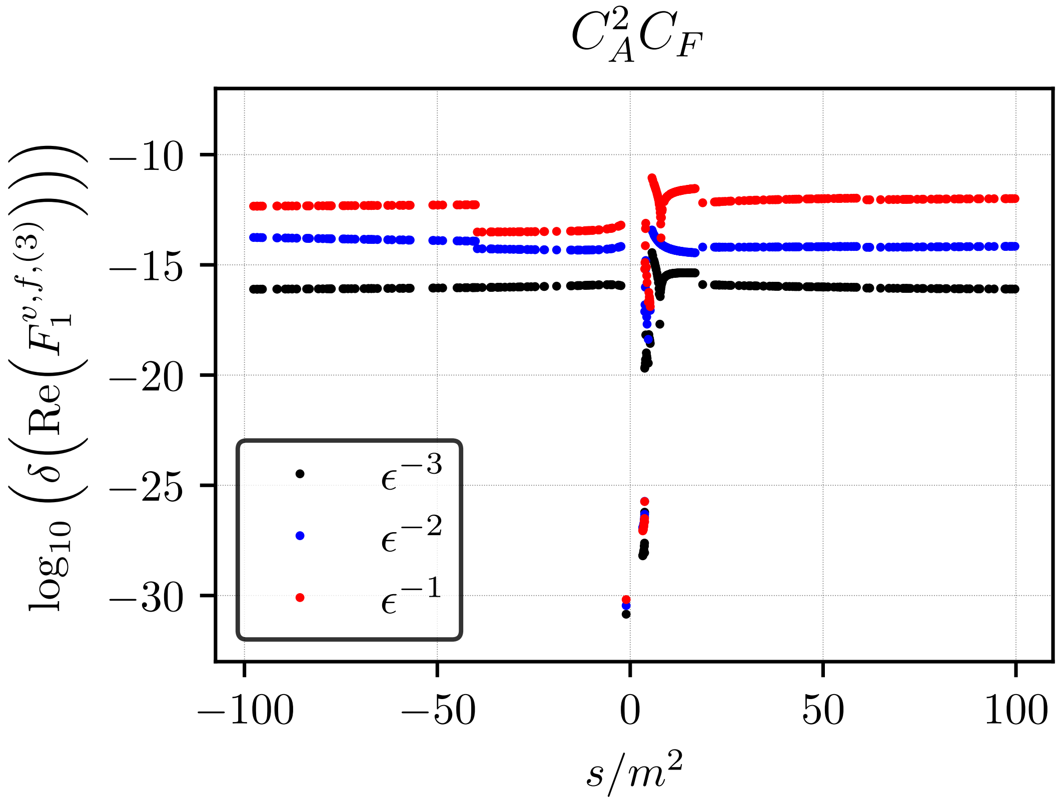
$-40 < \hat{s} < 3.75$ and $4.25 < \hat{s} < 60$

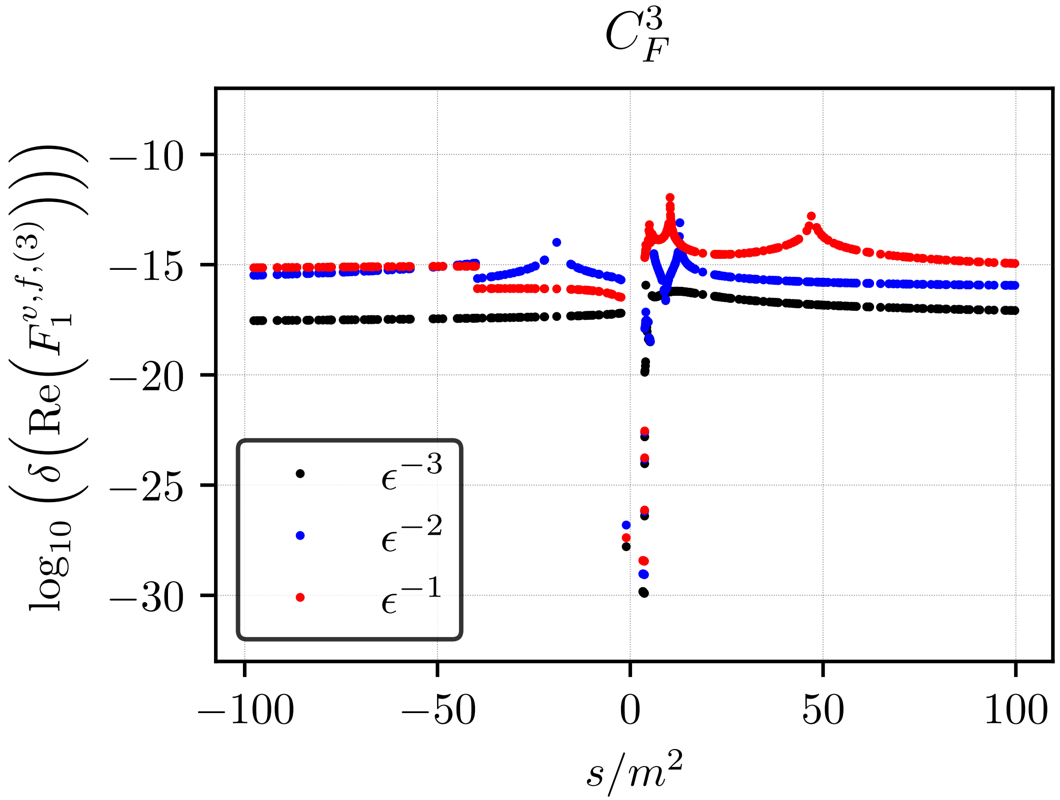
- Series expansions for

$s = \pm\infty, s = 4m^2$ (also $s = 0$ for singlet)









$$\delta(F^{f,(3)}|_{\epsilon^i}) = \frac{F^{(3)}|_{\epsilon^i} + F^{(\mathrm{CT+Z})}|_{\epsilon^i}}{F^{(\mathrm{CT+Z})}|_{\epsilon^i}}$$

Quarkonium contributions:

good agreement $O(10^{-10})$ with independent calculation

Blümlein, Marquard, Rana, Schneider, Phys.Rev.D 108 (2023) 094003

Facts of life with γ_5

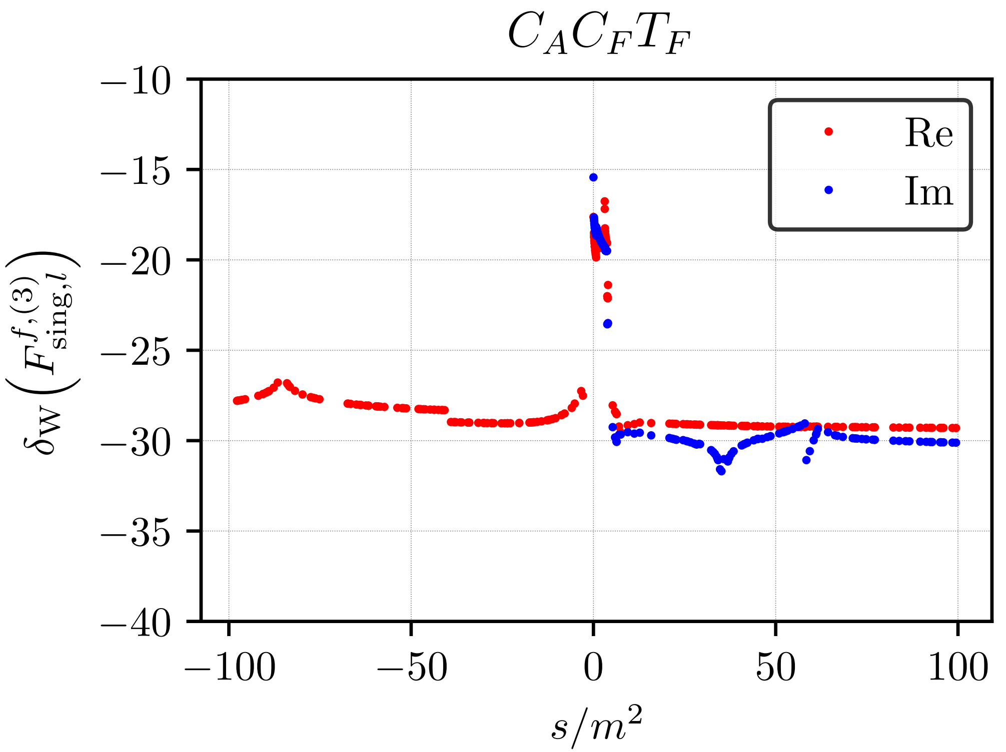
- For singlet diagrams we use the Larin prescription

$$\gamma^\mu \gamma_5 \to \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} (\gamma^\nu \gamma^\rho \gamma^\sigma - \gamma^\sigma \gamma^\rho \gamma^\nu)$$

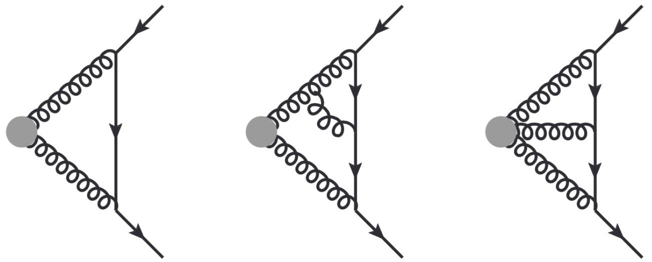
Larin, Phys.Lett.B 303 (1993) 113

Larin, Vermaseren, Phys. Lett. B 259 (1991), 345

- **Finite renormalization** constants for $j_a^\mu(x)$ and $j_p(x)$
- Only the sum of singlet and non-singlet diagrams renormalizes multiplicative
- Non-singlet must be calculated in the Larin scheme
- We check the **Chiral Ward Identity**



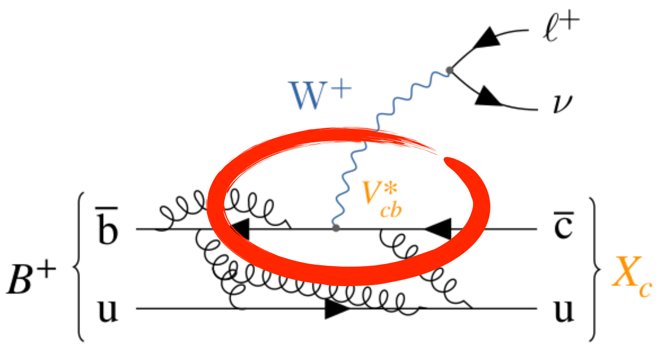
$$F^{a,f}_{1,\text{sing}} + \frac{s}{4m^2} F^{a,f}_{2,\text{sing}} = F^{p,f}_{\text{sing}} + \frac{\alpha_s}{4\pi} T_F F^f_{G\tilde{G}}$$



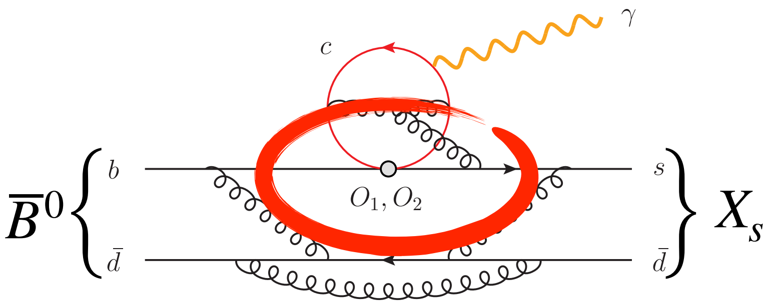
Application II: B-meson decays

JHEP 09 (2023) 112, hep-ph 2309.14706, hep-ph 2310.03685

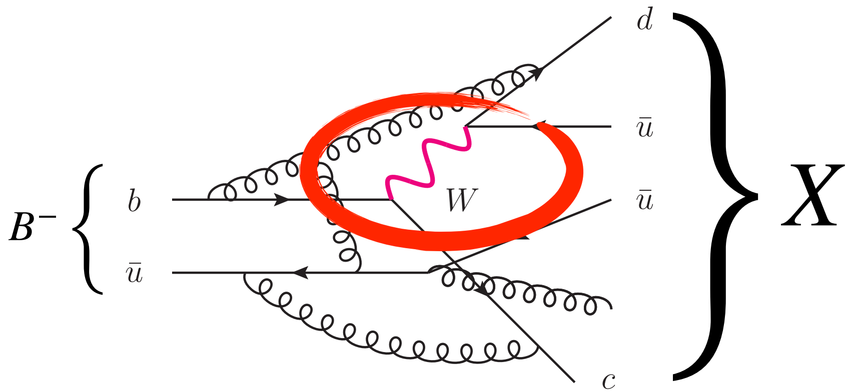
Inclusive decays of B mesons



Semileptonic $B \to X_c l \bar{\nu}_l$



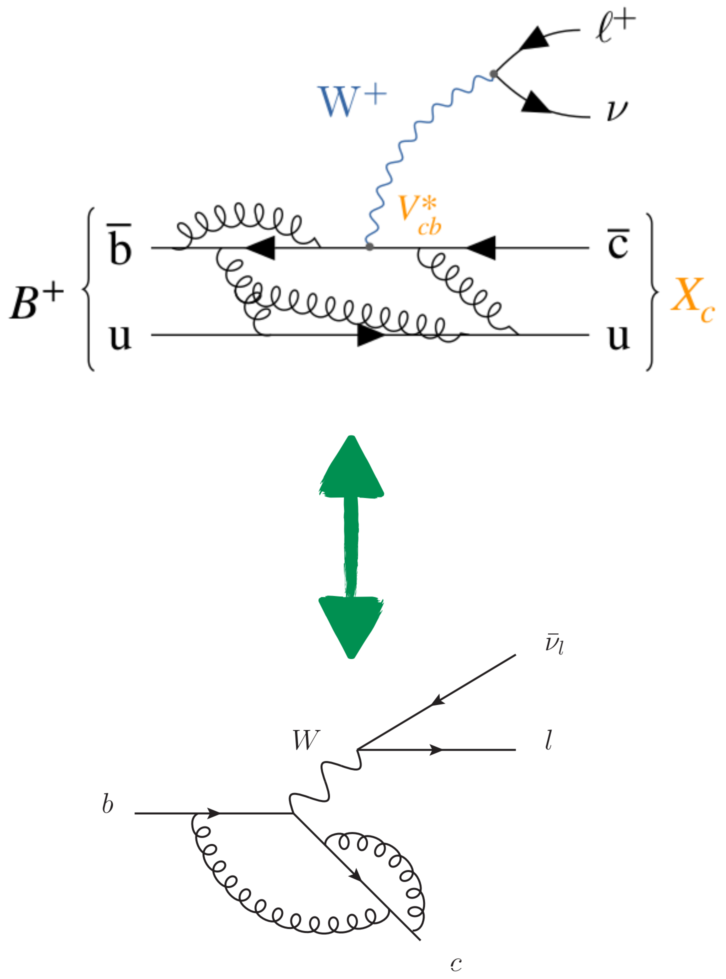
Rare decay $B \to X_s \gamma$



Lifetimes of the B meson

We need precise predictions in the SM, often at the 1% level!

The Heavy Quark Expansion

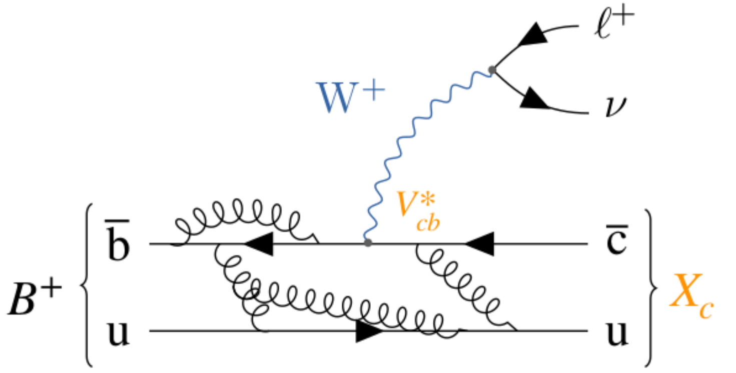


$$\Gamma_{\mathrm{sl}} = \frac{1}{2m_B} \sum_X \left| \langle X | \mathscr{H}_{\mathrm{eff}} | B \rangle \right|^2$$

$$= C_3 + \frac{C_5}{m_b^2} \langle B | O_5 | B \rangle + \frac{C_6}{m_b^3} \langle B | O_6 | B \rangle + \dots$$

No non-perturbative matrix element at leading power!
In a first approximation we can consider the **decay of a free bottom quark**

Inclusive decays of B mesons

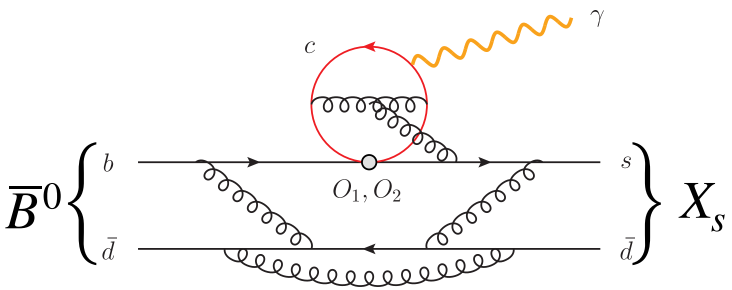


- Inclusive B-meson decays admit an **OPE**
- The ratio $m_c/m_b \simeq 0.25$ all over the place
- Several short-distance mass schemes are used

$$m_b^{\mathrm{OS}} : m_c^{\mathrm{OS}} \qquad \Gamma(B \to X_c \ell \bar{\nu}_\ell) \sim 1 - 1.78\left(\frac{\alpha_s}{\pi}\right) - 13.1\left(\frac{\alpha_s}{\pi}\right)^2 - 163.3\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{\mathrm{kin}}(1\text{ GeV}) : \overline{m}_c(2\text{ GeV}) \qquad \Gamma(B \to X_c \ell \bar{\nu}_\ell) \sim 1 - 1.24\left(\frac{\alpha_s}{\pi}\right) - 3.65\left(\frac{\alpha_s}{\pi}\right)^2 - 1.0\left(\frac{\alpha_s}{\pi}\right)^3$$

$$m_b^{1S} : m_c \text{ via HQET} \qquad \Gamma(B \to X_c \ell \bar{\nu}_\ell) \sim 1 - 1.38\left(\frac{\alpha_s}{\pi}\right) - 6.32\left(\frac{\alpha_s}{\pi}\right)^2 - 33.1\left(\frac{\alpha_s}{\pi}\right)^3$$



- Estimate **theoretical uncertainties** with scale variations

$$\overline{m}_c(\mu_c), \overline{m}_b(\mu_b), m_b^{\mathrm{kin}}(\mu_{WC}), \ldots$$

$B \to X_s\gamma$

$$\mathrm{Br}^{\mathrm{exp}}(B \to X_s\gamma, E_\gamma > 1.6\,\mathrm{GeV}) = (3.49 \pm 0.19) \times 10^{-4}$$

HFLAV, Phys. Rev. D 107 (2023), 052008

$$\mathrm{Br}^{\mathrm{th}}(B \to X_s\gamma, E_\gamma > 1.6\,\mathrm{GeV}) = (3.40 \pm 0.17) \times 10^{-4}$$

Misiak et al, Phys. Rev. Lett. 114
Misiak, Rehman, Steinhauser, JHEP 06 (2020), 175

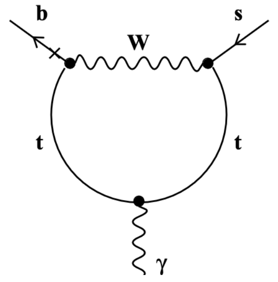
- Includes NNLO QCD corrections
- Charm mass interpolation responsible for 3% uncertainty
- Unknown higher-order correction (3%)
- Input and non-perturbative parameters (2.5%)

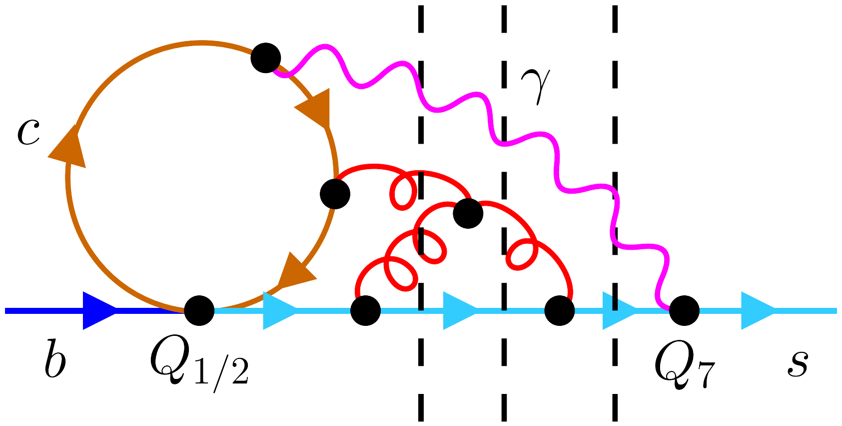
$$\mathscr{H}_{\mathrm{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ts}^\star V_{tb} \sum_i C(\mu_b) Q_i$$

$$Q_1 = (\bar{s}_L \gamma_\mu T^a c_L)(\bar{c}_L \gamma^\mu T^a b_L)$$

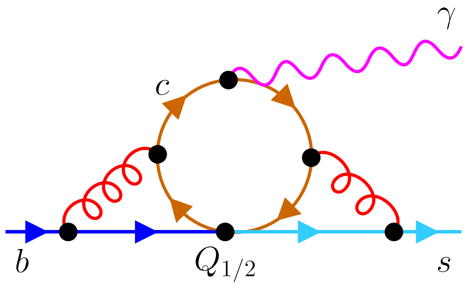
$$Q_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L)$$

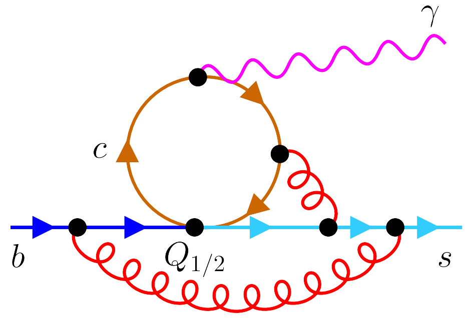
$$Q_7 = \frac{e m_b}{16\pi^2} (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu}$$





Three-loop corrections to $b \to s\gamma$ vertex





$$\mathcal{M}(b \to s\gamma) = \frac{4G_F m_b^2}{\sqrt{2}} V_{ts}^\star V_{tb} \varepsilon_\mu(q_\gamma)\, \bar{u}_s(p_s) P_R \left(t_1 \frac{q_\gamma}{m_b} + t_2 \frac{q_b^\mu}{m_b} + t_3 \gamma^\mu \right) u_b(p_b)$$

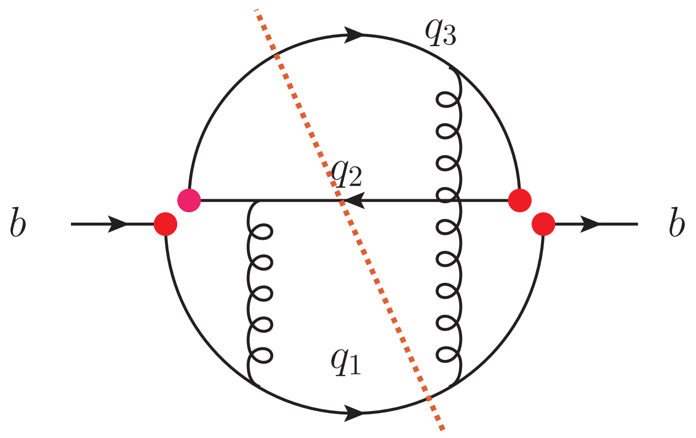
- Differential equations for 479 master integrals w.r.t. $x = m_c/m_b$
- Apply semi-analytic method
- Boundary conditions at $x_0 = m_c/m_b = 1/5$ with `AMFlow`
- Taylor expansions at $x_0 = 1/5, 1/10$ and power-log expansion at $x_0 = 0$

$$\mathrm{Re}(t_2^{Q_1}) = n_l \Big\{ -\frac{0.643804}{\epsilon^2} - \frac{6.31123}{\epsilon} - 27.9137$$
$$+ x^2 \frac{1}{\epsilon} \left(2.107 \log^3(x) + 3.16049 \log^2(x) - 27.8263 \log(x) \right) - 11.7523) + \ldots \Big\}$$

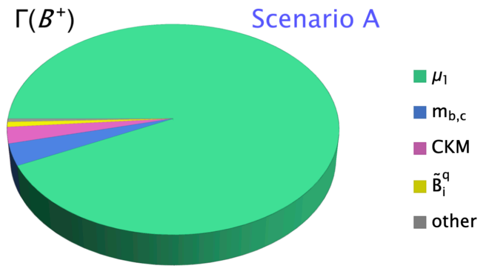
Fael, Lange, Schönwald, Steinhauser, 2309.14706
Misiak et al, 2309.14707

Lifetime of B mesons

$$\Gamma(B_q) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \ldots$$



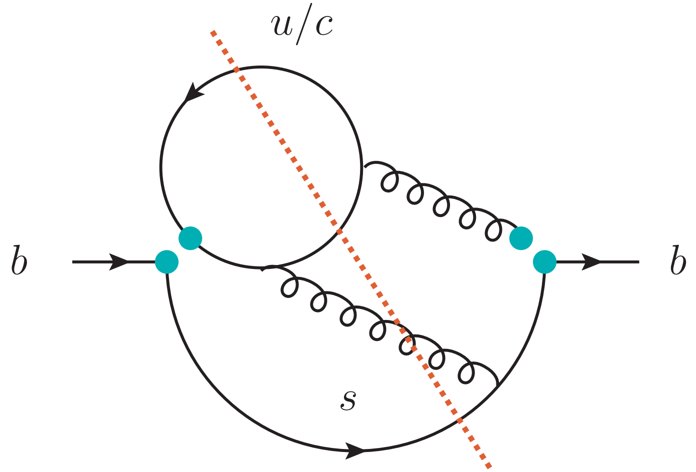
$$\Gamma(B^+) = (0.59^{+0.11}_{-0.07})\,\text{ps}^{-1}$$
$$\Gamma(B_d) = (0.63^{+0.11}_{-0.07})\,\text{ps}^{-1}$$
$$\Gamma(B_s) = (0.63^{+0.11}_{-0.07})\,\text{ps}^{-1}$$



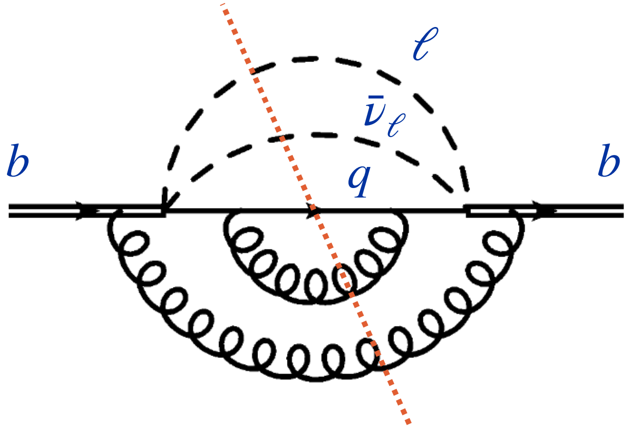
Lenz, Piscopo, Rusov, JHEP 01 (2023) 004

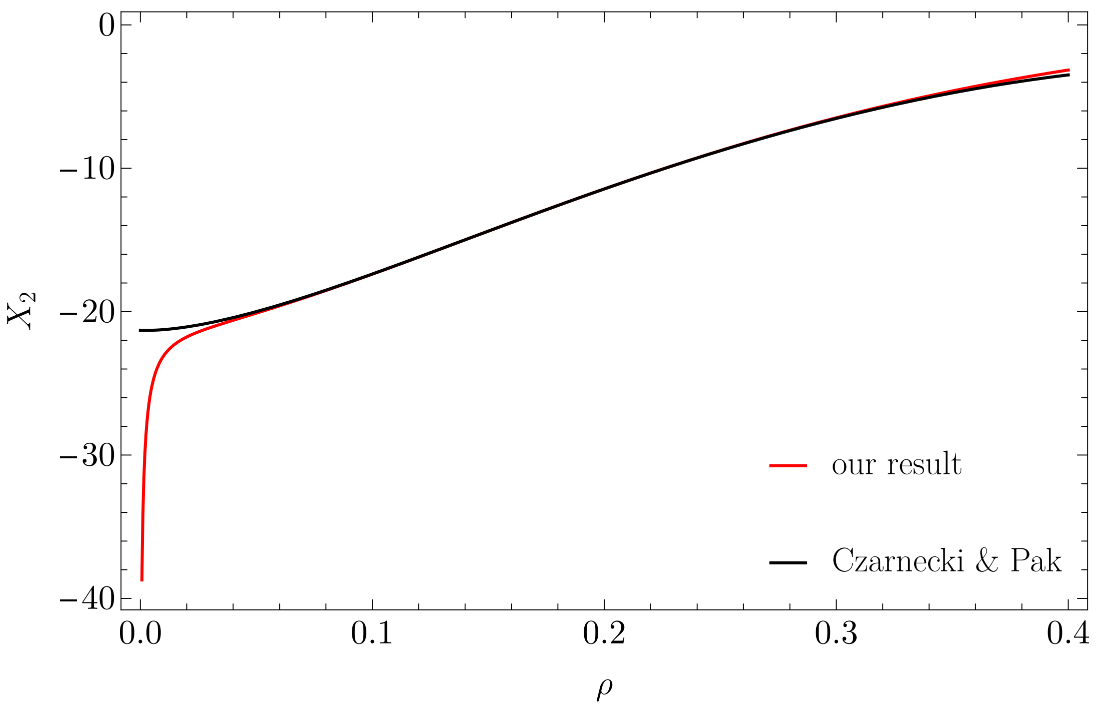
- NLO QCD corrections to non-leptonic decay
  Bagan, Patricia Ball, Braun, Gosdzinsky, Nucl.Phys.B 432 (1994) 3
  Krinner, Lenz, Rauh, Nucl.Phys.B 876 (2013) 31

- $\Gamma(B) = \Gamma(b \to c\ell\bar{\nu}_\ell) + \Gamma(b \to c\bar{u}d) + \Gamma(b \to c\bar{c}s) + \ldots$



$$\Gamma(B \to X_q \ell \bar{\nu}_\ell) = \frac{G_F^2 m_b^5 |V_{qb}|^2}{192\pi^3} \left[X_0(\rho) + \frac{\alpha_s}{\pi} X_1(\rho) + \left(\frac{\alpha_s}{\pi}\right)^2 X_2(\rho) + \ldots \right]$$

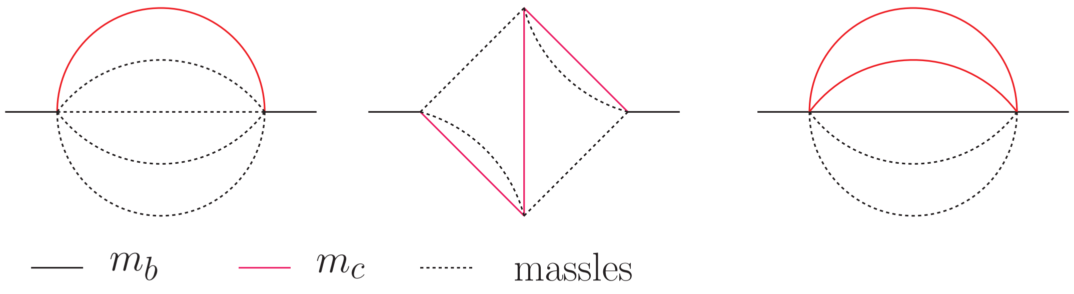




Egner, Fael, Schönwald, Steinhauser, *JHEP* 09 (2023) 112

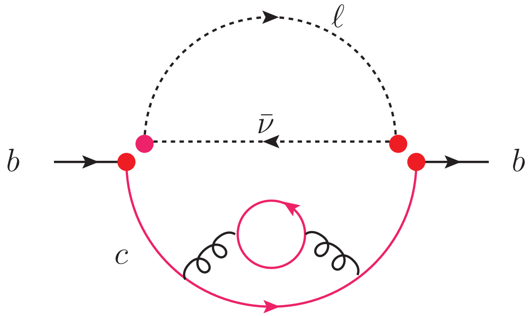
asymptotic expansion for $\rho = 0$ Czarnecki, Pak, Phys.Rev.D 78 (2008) 114015

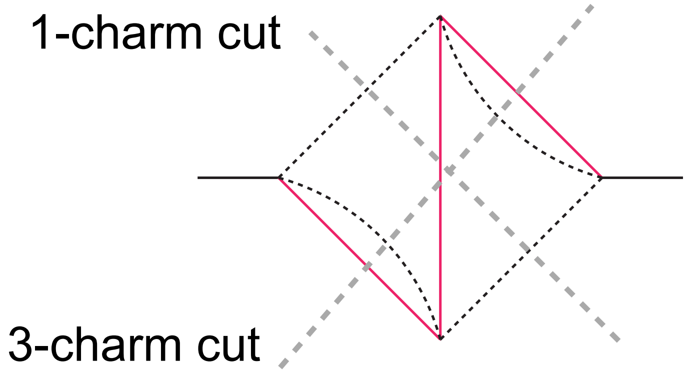
- Establish differential equations w.r.t. $\rho = m_c/m_b$
- Consider only the imaginary part of masters

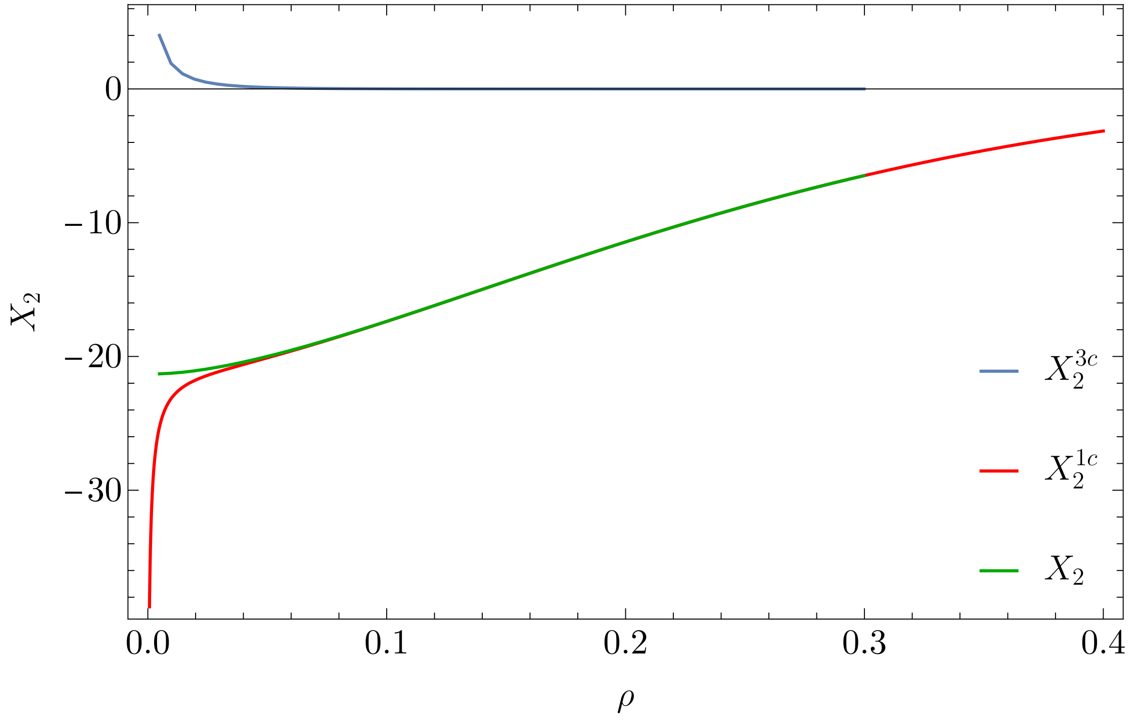


- Analytic boundary conditions in the limit $\rho \to 1$
- Continue the solution to $\rho = 0$

- Semi-analytic method should be applied at the level of Feynman integrals
- Boundary conditions at $\rho = 1$ contains final states with **only one charm quark**
- New threshold at $\rho = 1/3$







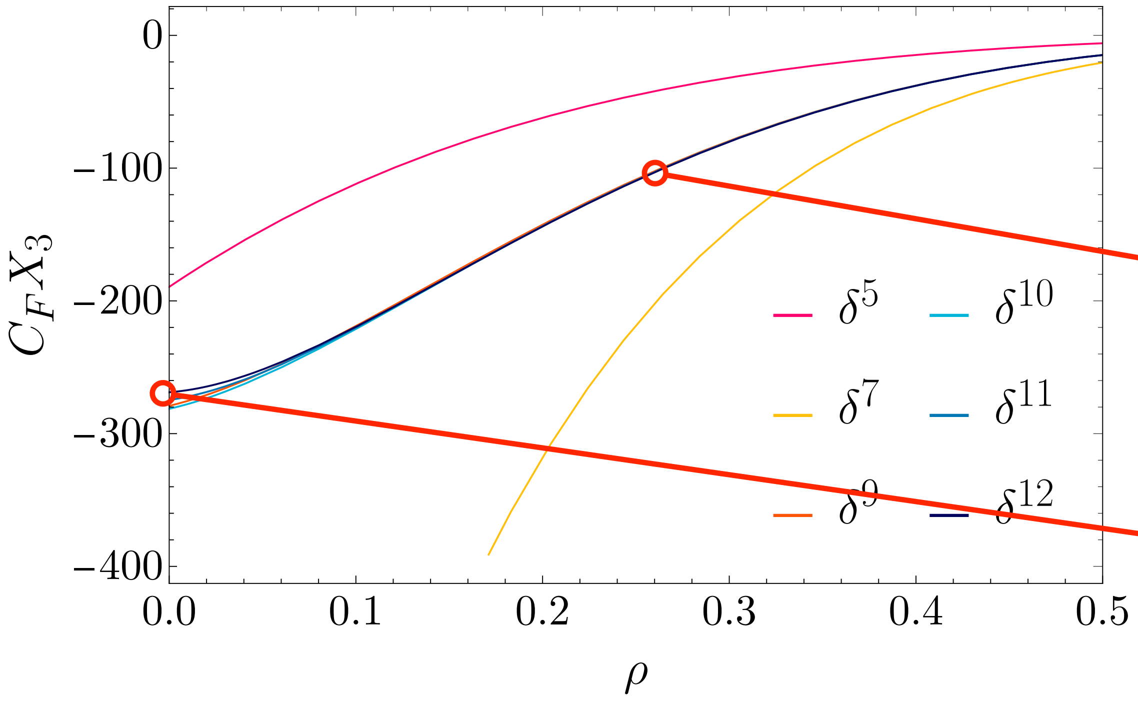
- Solving for complete real + imaginary part correctly reproduce the massless limit

Egner, Fael, Schönwald, Steinhauser, *JHEP* 09 (2023) 112

Third order corrections to $B \to X_u l\bar{\nu}_l$ decay

MF, Usovitsch, hep-ph 2310.03685

$$\Gamma_{\rm sl} = \frac{G_F^2 m_b^5 A_{\rm ew}}{192\pi^3} |V_{cb}|^2 \left(X_0(\rho) + C_F \sum_n \left(\frac{\alpha_s}{\pi}\right)^n X_n(\rho) \right)$$



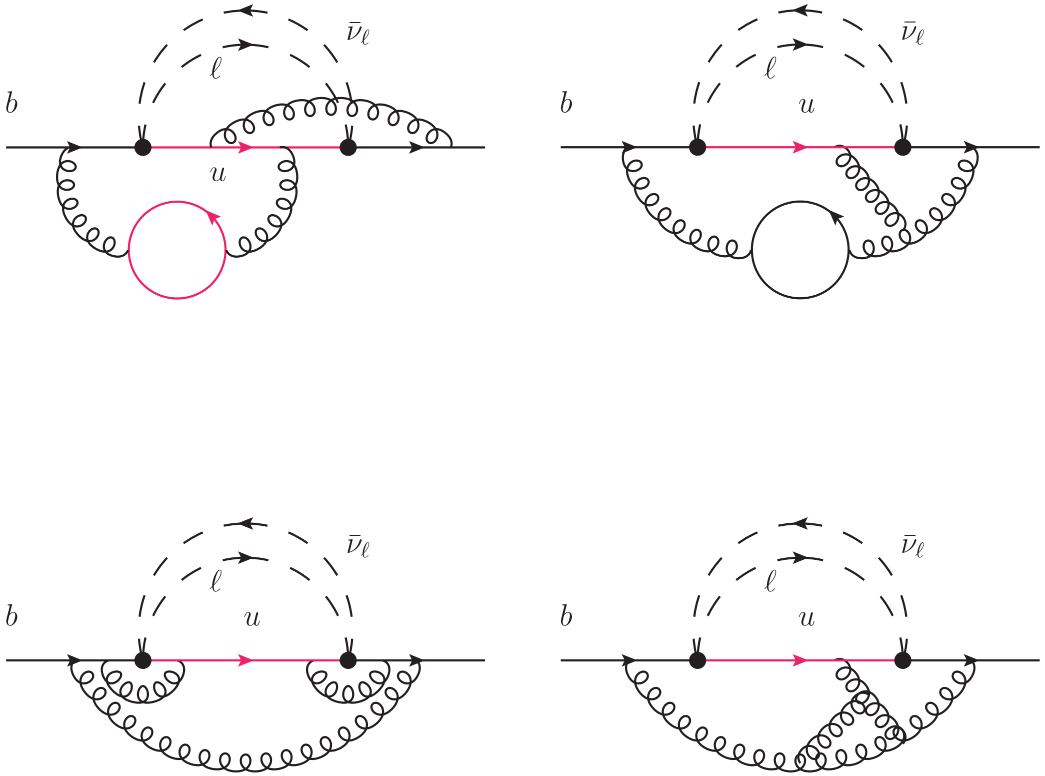
$$C_F X_3(\rho = 0.28) = -91.2 \pm 0.4\,(0.4\%)$$

MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

$$C_F X_3(\rho = 0) = -269 \pm 27\,(10\%)$$

MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

$$m_b^{\rm kin}(1\ \text{GeV}) : \overline{m}_c(3\ \text{GeV}) \quad \Gamma_{\rm sl} \simeq 1 - 0.019\,|_{\alpha_s} + 0.019\,|_{\alpha_s^2} + 0.032\,(9)\,|_{\alpha_s^3}$$

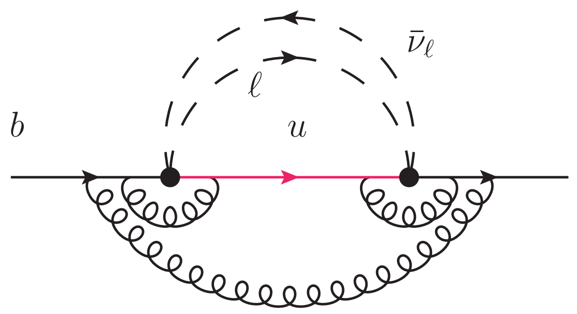


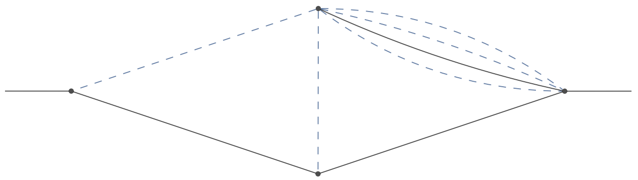
Fermionic corrections

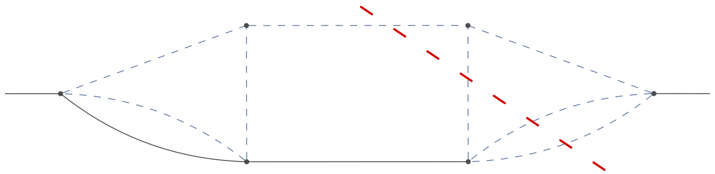
$$
\begin{aligned}
X_3 = {} & N_L^2 T_F^2 X_{N_L^2} + N_H^2 T_F^2 X_{N_H^2} + N_H N_L T_F^2 X_{N_H N_L} \\
& + N_L T_F (C_F X_{N_L C_F} + C_A X_{N_L C_A}) \\
& + N_H T_F (C_F X_{N_H C_F} + C_A X_{N_H C_A}) \\
& + C_F^2 X_{C_F^2} + C_F C_A X_{C_F C_A} + C_A^2 X_{C_A^2}
\end{aligned}
$$

Bosonic corrections

IBP reduction at 5 loops







Challenges

- 5loop integral families with 12 propagators and 8 numerators

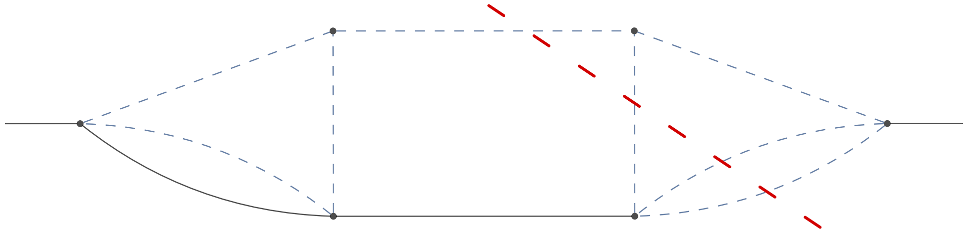
$$\int d^d p \frac{p^{\mu_1} \dots p^{\mu_N}}{(-p^2)[-(p-q)^2]} = \frac{i\pi^{2-\epsilon}}{(-q^2)^\epsilon} \sum_{i=0}^{[N/2]} f(\epsilon, i, N) \left(\frac{q^2}{2}\right)^i \{[g]^i [q]^{N-2i}\}^{\mu_1 \dots \mu_N}$$

- Rewrite 5loop families into 4loop ones with one denominator raised to symbolic power

$$I_5(n_1, n_2, \dots, n_{20}) \leftrightarrow \sum_{\vec{m} \in M} f_{\vec{m}}(\epsilon) \, J_{4\epsilon}(m_1, m_2, \dots, m_{14})$$

- Eliminate non-trivial sectors without physical cuts

Numerical evaluation of 5 loop integrals



- 1369 master integrals
- Numerical evaluation with 40 digits with **AMFlow**
  Xiao Liu, Yan-Qing Ma, Comput.Phys.Commun. 283 (2023) 108565
- Implement our own interface to Kira
- Parallel calculation large-N_c contributions
  Chen, Li, Li, Wang, Wand, Wu, hep-ph/2309.00762

	This work	Ref. [28]	Difference
$T_F^2 N_L^2$	-6.9195	-6.34 (42)	8.3%
$T_F^2 N_H^2$	-1.8768×10^{-2}	$-1.97\,(42) \times 10^{-2}$	5.0%
$T_F^2 N_H N_L$	-1.2881×10^{-2}	$-1.1\,(1.1) \times 10^{-2}$	12%
$C_F T_F N_L$	-7.1876	$-5.65\,(55)$	22%
$C_A T_F N_L$	42.717	39.7 (2.1)	7%
$C_F T_F N_H$	2.1098	2.056 (64)	2.5%
$C_A T_F N_H$	-0.45059	$-0.449\,(18)$	0.4%

$$
\begin{aligned}
C_F X_3 &= 280.2 && \text{fermionic} \\
&\quad -536.4 && \text{bosonic, large } N_c \\
&\quad -11.6\,(2.7) && \text{bosonic, subleading } N_c \\
&= -267.8\,(2.7)
\end{aligned}
$$

MF, Usovitsch, hep-ph 2310.03685

previous estimate:

$$C_F X_3(\rho = 0) = -269 \pm 27\,(10\%)$$

MF, Schönwald, Steinhauser, *Phys.Rev.D* 104 (2021) 016003, JHEP 08 (2022) 039.

Conclusions

- Numerical and semi-analytic methods offer very powerful tools for phenomenology
- Our implementation can deal with difficult problems with one-scale Feynman integrals
- We can study also interesting singular limits, e.g. threshold production or high-energy limits
- Extend to two-scale problems via construction of interpolation grids

Backup

- Flavour non-singlet

$$J^a_{\mathrm{NS},\mu} = \sum_{i=1}^{n_f} a_i \overline{\psi}_i \gamma_\mu \gamma_5 \psi_i$$

$$F_{i,\mathrm{NS}} = F_{i,\mathrm{non-sing}} + F_{i,\mathrm{nh-sing}} - F_{i,\mathrm{nl-sing}}$$

- Flavour singlet

$$J^a_{\mathrm{S},\mu} = \sum_{i=1}^{n_f} \overline{\psi}_i \gamma_\mu \gamma_5 \psi_i$$

$$F_{i,\mathrm{S}} = F_{i,\mathrm{non-sing}} + F_{i,\mathrm{nh-sing}} + \sum_{j=1}^{n_l} F_{i,\mathrm{nl-sing}}$$

- Flavour singlet and non-singlet renormalize multiplicatively

$$F_{i,\mathrm{NS}} = Z_{\mathrm{NS}} Z_2^{\mathrm{OS}} F_{i,\mathrm{NS}}^{\mathrm{bare}}$$
$$F_{i,\mathrm{S}} = Z_{\mathrm{S}} Z_2^{\mathrm{OS}} F_{i,\mathrm{S}}^{\mathrm{bare}}$$

- with

$$Z_{\mathrm{NS}} = Z_{\mathrm{NS}}^{\overline{\mathrm{MS}}} Z_{\mathrm{NS}}^{\mathrm{fin}} \qquad Z_{\mathrm{S}} = Z_{\mathrm{S}}^{\overline{\mathrm{MS}}} Z_{\mathrm{S}}^{\mathrm{fin}}$$

- Consistency relations:

$$F_{i,\mathrm{non-sing}} = Z_{\mathrm{NS}} Z_2^{\mathrm{OS}} F_{i,\mathrm{non-sing}}^{\mathrm{bare}}$$
$$F_{i,\mathrm{nX-sing}} = Z_{\mathrm{NS}} Z_2^{\mathrm{OS}} F_{i,\mathrm{nX-sing}}^{\mathrm{bare}} + \frac{1}{n_f}(Z_{\mathrm{S}} - Z_{\mathrm{NS}}) Z_2^{\mathrm{OS}} \left(F_{i,\mathrm{non-sing}}^{\mathrm{bare}} + \sum_{k=1}^{n_f} F_{i,\mathrm{k-sing}}^{\mathrm{bare}} \right)$$

N^{3}LO kick-off workstop / thinkstart

Aug 3 – 5, 2022
IPPP
Europe/London timezone

Enter your search term

- Overview
- Call for Abstracts
- Timetable
- Contribution List
- Book of Abstracts
- Registration
- Participant List
- Code of Conduct

Contact

yannick.ulrich@durham....

Recent progress at NNLO suggests that a fully differential N^{3}LO calculation of $\gamma^* \to ll$ should be possible. IPPP will host a three-day in-person kick-off workstop to try and organise the different contributions. We will cover

- VVV: the three-loop heavy-quark / heavy-lepton form factor (massification, exact, semi-numerical, etc.)
- RVV: adapting $\gamma^* \to qqg$, new calculations, mass dependence
- RRV: automation, numerical stability
- assembly & dirty tricks: Monte Carlo, massification, jettification, NTS stabilisation etc.
- open discussion

Each area will be given half a day, starting with an open 1h seminar followed by a lengthy discussion.

Just like previous workstops, we try to gather a small number of theorists who actively work on this topic to make very concrete progress. It should not just be about giving talks, but to actually learn from each other and put together the jigsaw pieces.

Starts Aug 3, 2022, 9:00 AM
Ends Aug 5, 2022, 10:00 PM
Europe/London

IPPP
OC218
Go to map

Yannick Ulrich

conference-photo.jpg
program.pdf

5th Workstop / Thinkstart: Radiative corrections and Monte Carlo tools for Strong 2020

Jun 5 – 9, 2023
University of Zurich
Europe/Zurich timezone

Enter your search term

Contact
yannick.ulrich@durham....

In this workstop, we will discuss radiative corrections and Monte Carlo tools for low-energy hadronic cross sections in e^+e^- collisions. This is to be seen as part of the Strong 2020 effort. We will cover

- leptonic processes at NNLO and beyond
- processes with hadrons
- parton shower
- experimental input

Each area will be given at least half a day, starting with an open 1h seminar followed by a lengthy discussion.

Just like previous workstops, this is an in-person event. We try to gather a small number of theorists who actively work on this topic to make very concrete progress. It should not just be about giving talks, but to actually learn from each other and put together the jigsaw pieces.

Additionally to the workstop that is only by-invite only, there is a broader conference directly following the workstop.

Starts Jun 5, 2023, 9:00 AM
Ends Jun 9, 2023, 5:00 PM
Europe/Zurich

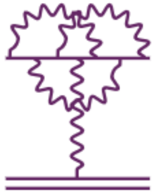
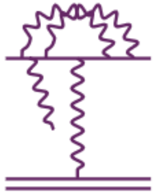
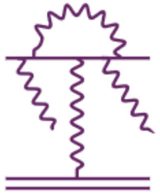
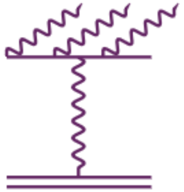
University of Zurich
Y36 K08

Yannick Ulrich
Adrian Signer
Andrzej Kupsc
Graziano Venanzoni

conference-photo.jpg
Location of the dinner

This event is by invite only!

anatomy of an N^3 LO calculation

-  $\sim$ three-loop heavy quark form factor $\rightarrow$ Matteo's talk
-  $\sim$ really difficult, see later $m_e = 0$: Badger, Kryś, Moodie, Zoia, JHEP 11 (2023) 041
-  $\sim$ OpenLoops$\oplus$NTS stabilisation $\rightarrow$ nasty but doable
-  $\sim$ tree level but difficult phase space $\rightarrow$ (hopefully) easy enough
- $\int \mathrm{d}\Phi_5 \Big| \cdots \Big| \sim \mathrm{FKS}^3$ subtraction