Reclassifying Feynman Integrals as Special Functions

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Outline

I. Introduction to Feynman integrals

II. Analytical computation

III. Semi-analytical computation

IV. Feynman integrals as special functions

V. Summary and outlook

Quantum field theory

- **The only way to combine quantum and relativity**
  - **Uncertainty principle: Large energy probed at short time**
  - **Relativity: Energy can produce mass and particles, thus multi-particle system**

- **The foundational theoretical framework of physics**
  - **Particle physics, nuclear physics**
  - **Many-body quantum system, cold atom physics**
  - **Gravitational waves**
  - **…**

Path integral formula for QFT

➢ **Green functions:**

$$\langle \Omega | T\mathcal{O}(\hat{\phi}) | \Omega \rangle = \frac{\int \mathcal{D}\phi \mathcal{O}(\phi) e^{\mathrm{i}S[\phi]}}{\int \mathcal{D}\phi e^{\mathrm{i}S[\phi]}}$$

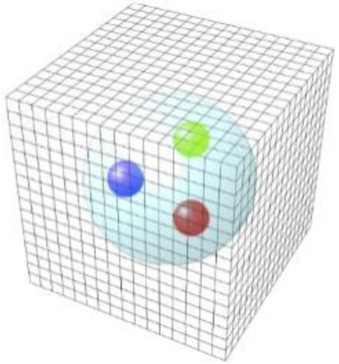
➢ **Definition of path integral**

$$\mathcal{D}\phi \sim \prod_x \int d\phi_x$$

- Integrate over fields at each spacetime point
- Divergent due to infinite number of spacetime points

Two ways to compute QFT (path integrals)

- **Numerical computation via Monte Carlo**
  - Nonperturbative lattice QFT



- **Expanding to asymptotic series**
  - Perturbative QFT

$$\int \mathcal{D}\phi \mathcal{O}(\phi) e^{iS[\phi]}$$

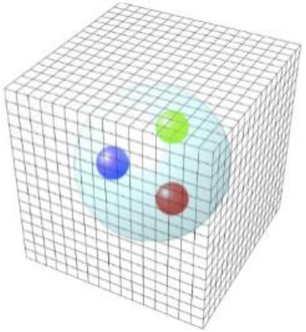
Complementary to each other

Solving QFT nonperturbatively

- **Discretize spacetime: reduction to finite degrees of freedom**
- **Imaginary time ($t \to i\,\tau$): avoidance of oscillatory behavior**

  - **Computing path integrals via Monte Carlo simulation**

K. G. Wilson



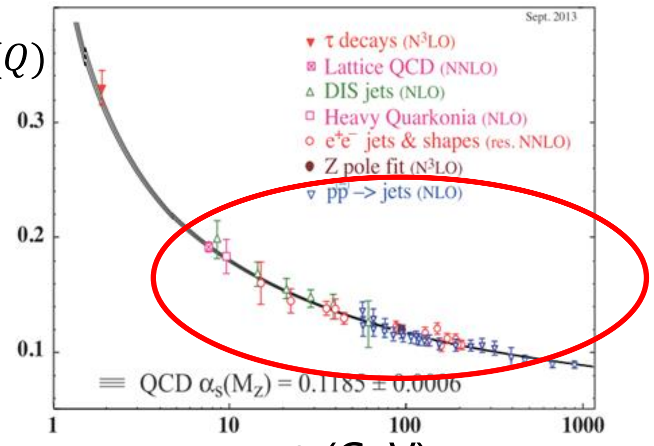
Lattice

Super computer

Solving QFT nonperturbatively

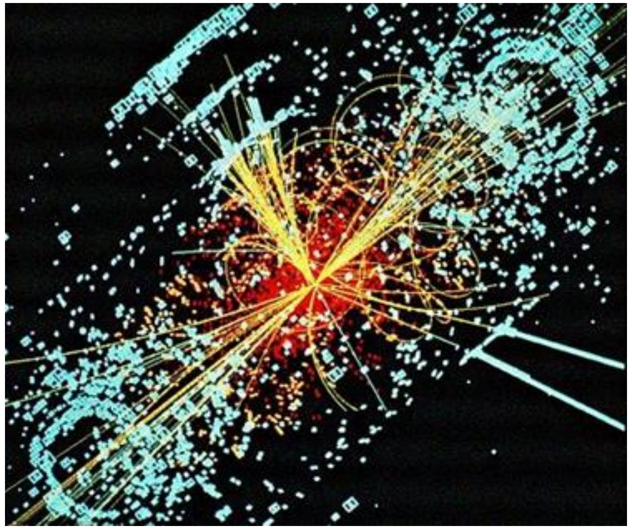
➢ Difficulties

- Computational complexity scales as $O(Q^4)$, hard for high energy Q physics
- Hard for time-dependent observables, like scattering processes



$\alpha_s(Q)$

Q (GeV)



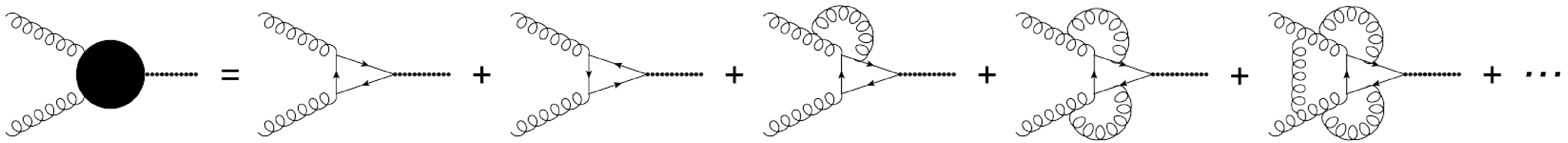
Solving QFT perturbatively

➢ Perturbation: expanding interacting terms as small numbers

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\int \mathcal{D}\phi \Big(1 + \mathrm{i} \int \mathrm{d}^4 x \mathcal{L}_I(\phi) + \cdots \Big) e^{\mathrm{i} \int \mathrm{d}^4 x (\mathcal{L}_0 + J\phi)}$$

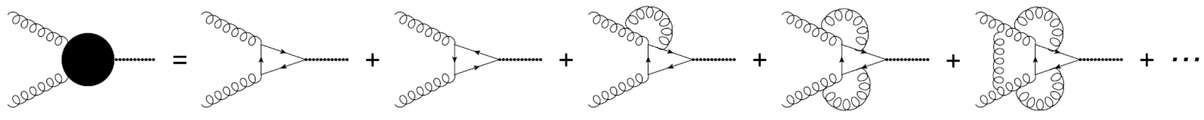
- Valid only if coupling is small!
- Integration over fields: Gaussian integrals, can be worked out
- Result in lots of Feynman diagrams



Perturbative QFT computation

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules



2. Calculate Feynman loop integrals (FIs)

- Amplitudes: linear combinations of FIs with rational coefficients

3. Perform phase-space integrations

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals via reverse unitarity

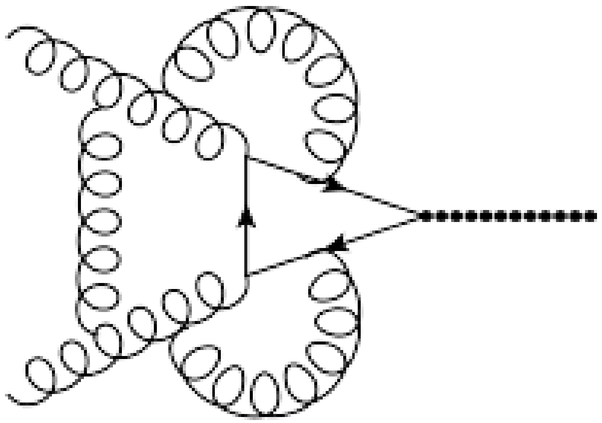
$$\int \frac{\mathrm{d}^D p}{(2\pi)^D}(2\pi)\delta_+(p^2) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\left(\frac{\mathrm{i}}{p^2+\mathrm{i}0^+} + \frac{-\mathrm{i}}{p^2-\mathrm{i}0^+}\right)$$

Definition of Feynman integrals

➢ A family of Feynman integrals

$$I_{\vec{\nu}}(D,\vec{s}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_\alpha = A_{\alpha ij} \ell_i \cdot \ell_j + B_{\alpha ij} \ell_i \cdot p_j + C_\alpha$$



- $\ell_1, \ldots, \ell_L$: loop momenta; $p_1, \ldots, p_E$: external momenta;
- A, B: integers; C: linear combination of $\vec{s}$ (including masses)
- $\mathcal{D}_1, \ldots, \mathcal{D}_K$: inverse propagators; $\nu_1, \ldots, \nu_K$: integers
- $\mathcal{D}_{K+1}, \ldots, \mathcal{D}_N$: irreducible scalar products; $\nu_{K+1}, \ldots, \nu_N$: non-negative integers
- D: dimensional regularization, avoid infinities (originated from path integrals)

Nature of FIs

- **Remnants of path integrals**
  - After integrated out fields, integration over the position/momentum of fields remains
- **Complexity of path integrals ⇒ complexity of FIs**
  - Resurgence theory: the perturbative series can eventually recover all nonperturbative information of quantum field theory
  - Simpler for small number of loops
  - Should be extremely hard as the number of loops increasing

Challenges of computing FIs

➢ Long-standing challenging problem

- One-loop computation: satisfactory approach existed as early as 1970s

  ’t Hooft, Veltman, NPB (1979)

- Multi-loop computation: challenging the field for more than 40 years

➢ Difficulties of computing FIs

- Analytical: known special functions are insufficient to express multi-loop FIs
- Numerical: (from path integrals) UV, IR, integrable singularities, …

Outline

Integration-by-parts: example

- **A family of FIs:** $F(n) = \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{1}{(\ell^2 - \Delta)^n}$

➢ Vanishing on the big hypersphere with radius R

Lagrange, Gauss, Green, Ostrogradski, 1760s-1830s

't Hooft, Veltman, NPB (1972)

$$\int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \frac{\partial}{\partial \ell^\mu} \left[\frac{\ell^\mu}{(\ell^2 - \Delta)^n} \right] \overset{\Downarrow}{=} \int_\partial \frac{\mathrm{d}^{D-1} S_\mu}{(2\pi)^D} \left[\frac{\ell^\mu}{(\ell^2 - \Delta)^n} \right] \overset{\Downarrow}{=} 0$$

- **Integrand: fixed power in R; Measure: R^{D-1}**
- **Thus vanishing in dimensional regularization**

➢ Relations between FIs

$$0 = \int_\ell \left[\frac{D}{(\ell^2 - \Delta)^n} - n \int_\ell \frac{2(\ell^2 - \Delta) + 2\Delta}{(\ell^2 - \Delta)^{n+1}} \right] = (D - 2n)F(n) - 2n\Delta F(n+1)$$

$$F(n+1) = \frac{1}{-\Delta} \frac{n - \frac{D}{2}}{n} F(n)$$

- **All FIs in this family can be expressed by $F(1)$**

General IBP equations

➢ Dimensional regularization: vanish at boundary

't Hooft, Veltman, NPB (1972)
Chetyrkin, Tkachov, NPB (1981)

$$\int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\partial}{\partial \ell_j^\mu} \left(q_k^\mu \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_1^{\nu_1} \cdots \mathcal{D}_K^{\nu_K}} \right) = 0, \qquad \forall \vec{\nu}, j, k$$

$$\overrightarrow{q^\mu} = (\ell_1^\mu, \cdots, \ell_L^\mu, p_1^\mu, \cdots, p_E^\mu)$$

$$\Downarrow$$

- **Linear equation:** $\sum_{\vec{\nu}'} Q_{\vec{\nu}'}^{\vec{\nu} jk}(D, \vec{s}) I_{\vec{\nu}'}(D, \vec{s}) = 0$
- Q: **polynomials in** $D, \vec{s}$
- **Plenty of linear equations can be easily obtained by varying:** $\vec{v}, j, k$

IBP reduction

➢ A family of FIs form a FINITE-dim. linear space

Proved by: Smirnov, Petukhov, 1004.4199

- Bases of the linear space called master integrals (MIs)
- IBPs reduce plenty of FIs to much less MIs

$$I_{\vec{\nu}} = \sum_{i=1}^{M} c_i I_i$$

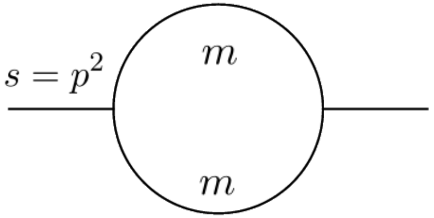
➢ Solving IBP eqs. based on Laporta’s algorithm:

Laporta, 0102033

- Automatic, any-loop order
- Public codes: AIR, FIRE, LiteRed, Reduze, Kira, FiniteFlow, NeatIBP, Blade…
- Many more private codes

Differential equations: example

- **Due to IBP: DEs of MIs**



$$I_{\nu_1\nu_2} = \int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{1}{(\ell^2-m^2)^{\nu_1}[(\ell+p)^2-m^2]^{\nu_2}}$$

$$\begin{cases} \dfrac{\partial}{\partial m^2} I_{11} = I_{21} + I_{12} \overset{\text{IBP}}{=} \dfrac{2(D-3)}{4m^2-s} I_{11} - \dfrac{D-2}{m^2(4m^2-s)} I_{10} \\ \dfrac{\partial}{\partial m^2} I_{10} = I_{20} \overset{\text{IBP}}{=} \dfrac{D-2}{2m^2} I_{10} \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial s} I_{11} = \dfrac{p^\mu}{2s}\dfrac{\partial}{\partial p^\mu} I_{11} = -\dfrac{1}{2s}\displaystyle\int \frac{\mathrm{d}^D\ell}{\mathrm{i}\pi^{D/2}} \frac{2(\ell+p)\cdot p}{(\ell^2-m^2)[(\ell+p)^2-m^2]^2} \\ \qquad = -\dfrac{sI_{12}+I_{11}-I_{02}}{2s} \overset{\text{IBP}}{=} a_{11}I_{11} + a_{10}I_{10} \\ \dfrac{\partial}{\partial s} I_{10} = 0 \end{cases}$$

- **Boundary Condition**

$$\begin{cases} I_{11}|_{m^2\to 0} = (-s)^{D/2-2}\Gamma(2-D/2)\dfrac{\Gamma(D/2-1)^2}{\Gamma(D-2)} \\ I_{10} \end{cases}$$

DEs method

➢ Step 1: Set up $\vec{s}$-DEs of MIs

Kotikov, PLB(1991)

- Differentiate MIs w.r.t. invariants $\vec{s}$, such as $m^2, p_i \cdot p_j$
- IBP relations result in:

$$\frac{\partial}{\partial s_i}\boldsymbol{I}(\epsilon, \vec{s}\,) = A_i(\epsilon, \vec{s}\,)\boldsymbol{I}(\epsilon, \vec{s}\,)$$

- A_i: matrix with rational elements

➢ Step 2: Calculate boundary condition at a given value of $\vec{s}$

➢ Step 3: Solve DEs either analytically or numerically

Canonical form

- In some cases, choosing proper basis

Henn, 1304.1806

$$\frac{\partial}{\partial s_i}\boldsymbol{I}'(\epsilon,\vec{s}\,) = \epsilon A'_i(\vec{s}\,)\boldsymbol{I}'(\epsilon,\vec{s}\,)$$

- Solution after expanding ϵ: Multiple Polylogarithms

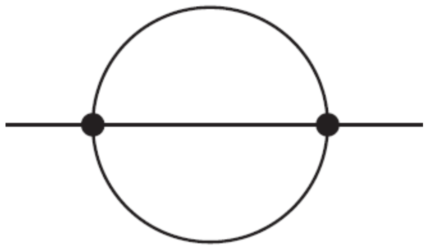
$$G(a_1, a_2, \cdots, a_n; z) := \int_0^z \frac{\mathrm{d}t}{t - a_1} G(a_2, \cdots, a_n; t),$$

$$G(\overbrace{0, \cdots, 0}^{n}; z) := \frac{1}{n!}\log^n z, \quad G(; z) := 1.$$

- Properties well-known, easy to obtain numerical values
- Many cutting-edge problems have been solved in this way!

Beyond Multiple Polylogarithms

- Elliptic functions



  - Appear as early as equal mass sunrise diagram!

- More complicated functions exist, like defined by Calabi-Yau manifold
  - Not well-studied mathematical objects
  - Hard to obtain numerical values

Studying these mathematical objects: a hot topic

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Auxiliary mass terms

X. Liu, YQM, C. Y. Wang, 1711.09572

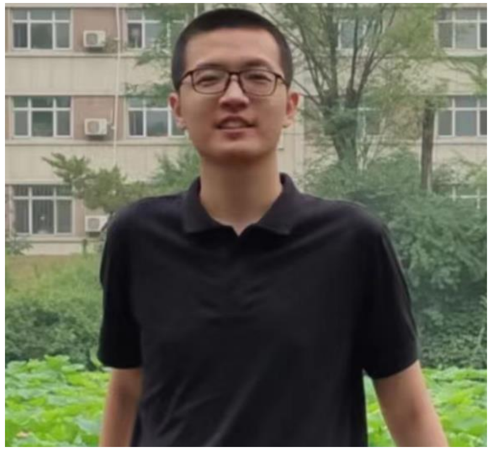
➢ Auxiliary FIs

$$I_{\vec{\nu}}^{\mathrm{aux}}(D,\vec{s},\eta) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \lambda_1 \eta + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K - \lambda_K \eta + \mathrm{i}0^+)^{\nu_K}}$$

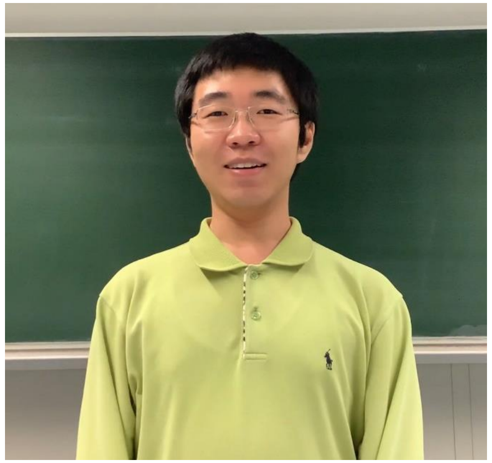
- $\lambda_i \geq 0$ (typically 0 or 1), an auxiliary mass if $\lambda_i > 0$
- Analytical function of η
- Physical result obtained by (causality)

$$I_{\vec{\nu}}(D,\vec{s}) \equiv \lim_{\eta \to \mathrm{i}0^-} I_{\vec{\nu}}^{\mathrm{aux}}(D,\vec{s},\eta)$$

- 1) Setup η-DEs; 2) Calculate boundary conditions; 3) Solve η-DEs



Xiao Liu, Oxford U.



Chen-Yu Wang, MPP

➢ η-DEs for MIs in auxiliary family using IBP

$$\frac{\partial}{\partial \eta} \vec{I}^{\mathrm{aux}}(D,\vec{s},\eta) = A(D,\vec{s},\eta) \vec{I}^{\mathrm{aux}}(D,\vec{s},\eta)$$

Boundary values at $\eta \to \infty$

➢ Nonzero integration regions as $\eta \to \infty$

- Linear combinations of loop momenta: $\mathcal{O}(\sqrt{|\eta|})$ or $\mathcal{O}(1)$

Beneke, Smirnov, 9711391
Smirnov, 9907471

➢ Simplify propagators at $\eta \to \infty$

- ℓ_L is the $\mathcal{O}(\sqrt{|\eta|})$ part of loop momenta
- ℓ_S is the $\mathcal{O}(1)$ part of loop momenta
- p is linear combination of external momenta

$$\frac{1}{(\ell_{\rm L}+\ell_{\rm S}+p)^2-m^2-\kappa\,\eta}\sim\frac{1}{\ell_{\rm L}^2-\kappa\,\eta}$$

- Unchange if $\ell_L = 0$ and $\kappa = 0$

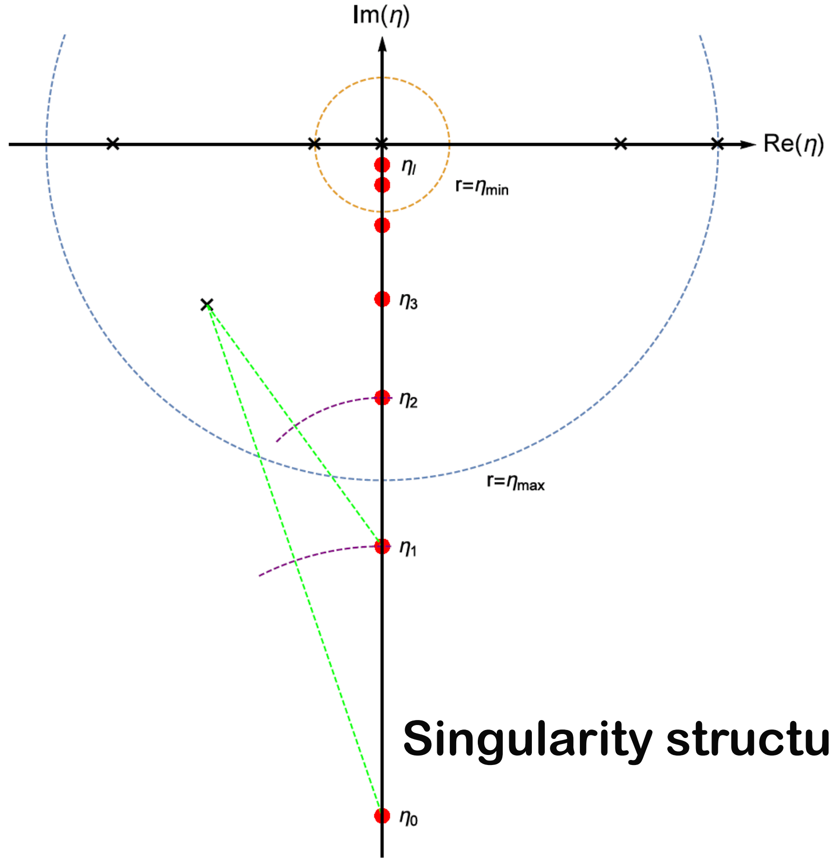
Tips: *Strategy of regions* is very powerful and useful. It is the rationale of effective field theory.

➢ Boundary FIs are simpler

1. Vacuum integrals
2. Simpler FIs with less denominators, if all loop momenta are $\mathcal{O}(1)$

Flow of auxiliary mass

➢ Solve ODEs of MIs



Singularity structure

$$\frac{\partial}{\partial \eta}\vec{I}^{\rm aux}(D,\vec{s},\eta)=A(D,\vec{s},\eta)\vec{I}^{\rm aux}(D,\vec{s},\eta)$$

- If $\vec{I}^{aux}(D,\vec{s},\infty)$ is known , solving ODEs numerically to obtain $\vec{I}^{aux}(D,\vec{s},\mathrm{i}0^-)$
- A well-studied mathematical problem

Step1: Asymptotic expansion at $\eta=\infty$
Step2: Taylor expansion at analytical points
Step3: Asymptotic expansion at $\eta=0$

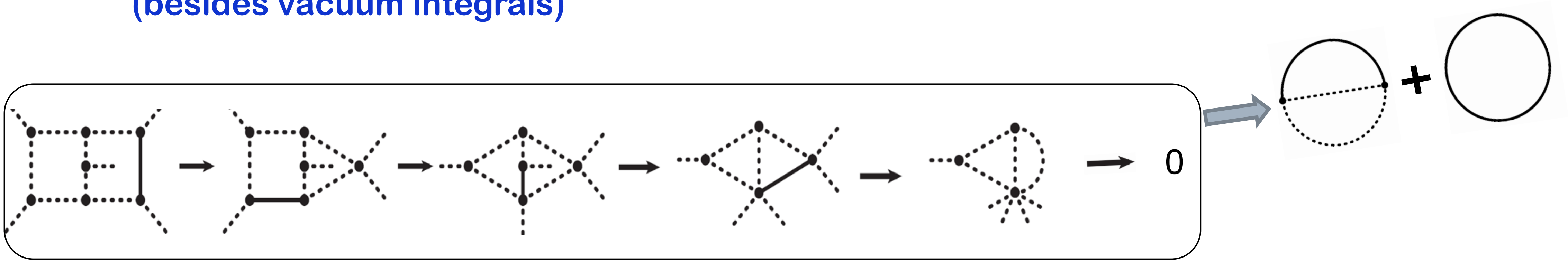
- Efficient to get high precision :
  ODEs, known singularity structure

Iterative strategy: FIs with less denominators

➢ **For boundary FIs with less denominators:**

X. Liu, YQM, 2107.01864

- Calculate them again use AMF method, even simpler boundary FIs as input (besides vacuum integrals)



- Eventually, leaving only (single-mass) vacuum integrals as input

Iterative strategy: vacuum integrals

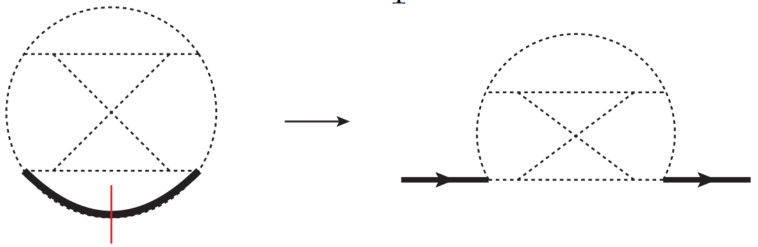
➢ A family of single-mass vacuum integrals

Z.F.Liu, YQM, 2201.11637

$$I_{\vec{\nu}}(D,m^2) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D\ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}}\cdots\mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1+\mathrm{i}0^+)^{\nu_1}\cdots(\mathcal{D}_K+\mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_1 = \ell_1^2 - m^2 + \mathrm{i}0^+$$

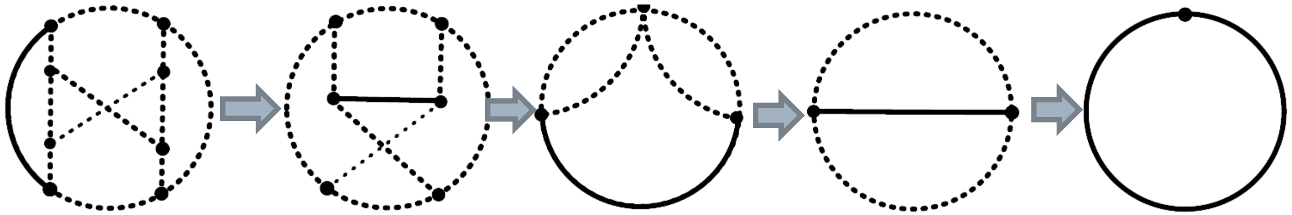
$$\widehat{I}_{\vec{\nu}'}(\ell_1^2) = \int \left(\prod_{i=2}^{L} \frac{\mathrm{d}^D\ell_i}{\mathrm{i}\pi^{D/2}}\right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}}\cdots\mathcal{D}_N^{-\nu_N}}{\mathcal{D}_2^{\nu_2}\cdots\mathcal{D}_K^{\nu_K}}$$



Zhi-Feng Liu, Zhejiang U.

$$I_{\vec{\nu}} = \int \frac{\mathrm{d}^D\ell_1}{\mathrm{i}\pi^{D/2}} \frac{(-\ell_1^2)^{\frac{(L-1)D}{2}-\nu+\nu_1}}{(\ell_1^2-1+\mathrm{i}0^+)^{\nu_1}} \widehat{I}_{\vec{\nu}'}(-1) = \frac{\Gamma(\nu-LD/2)\Gamma(LD/2-\nu+\nu_1)}{(-1)^{\nu_1}\Gamma(\nu_1)\Gamma(D/2)} \widehat{I}_{\vec{\nu}'}(-1)$$

- L-loop vacuum integrals expressed by $(L-1)$-loop p-integrals
- Using AMFLow: L-loop vacuum integrals reduced to $(L-1)$-loop vacuum integrals



Zero input; valid to any loop

AMFlow: Package

➢ Download

Liu, YQM, 2201.11669

Link: https://gitlab.com/multiloop-pku/amflow

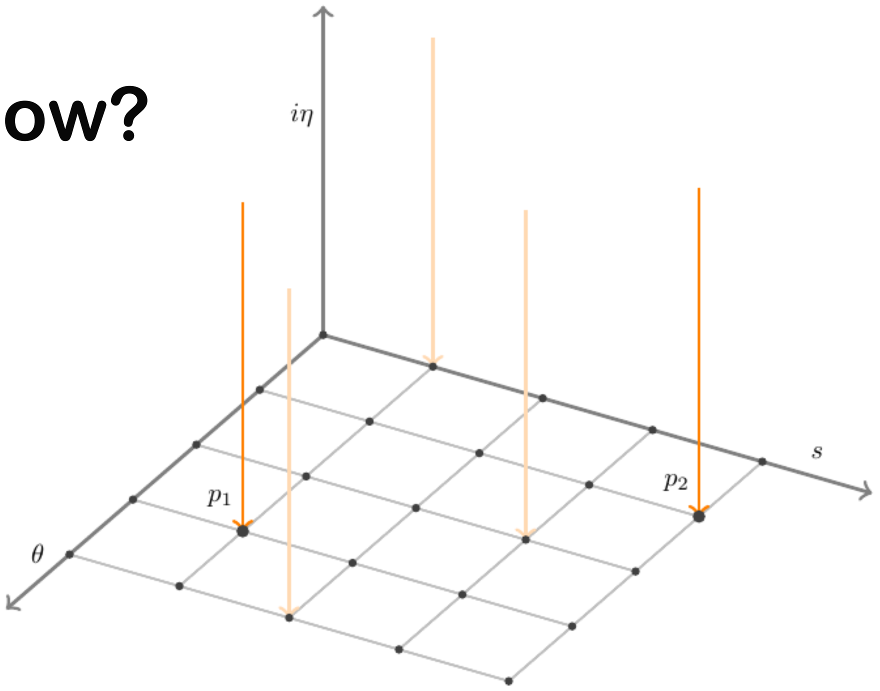
Name	Last commit	Last update
diffeq_solver	update	5 months ago
examples	update	3 months ago
ibp_interface	fix_a_bug_for_mpi_version	1 week ago
AMFlow.m	fix mass mode	2 months ago
CHANGELOG.md	update changelog	1 week ago
FAQ.md	update	6 months ago
LICENSE.md	test	7 months ago
README.md	update	3 months ago
options_summary	update	3 months ago

➢ Feature

- **The first package that can calculate any FI (with any number of loops, any D and $\vec{s}$) to arbitrary precision,** ***given sufficient resource***

Phenomenological applications

➢ **Compute FIs point by point using AMFlow?**



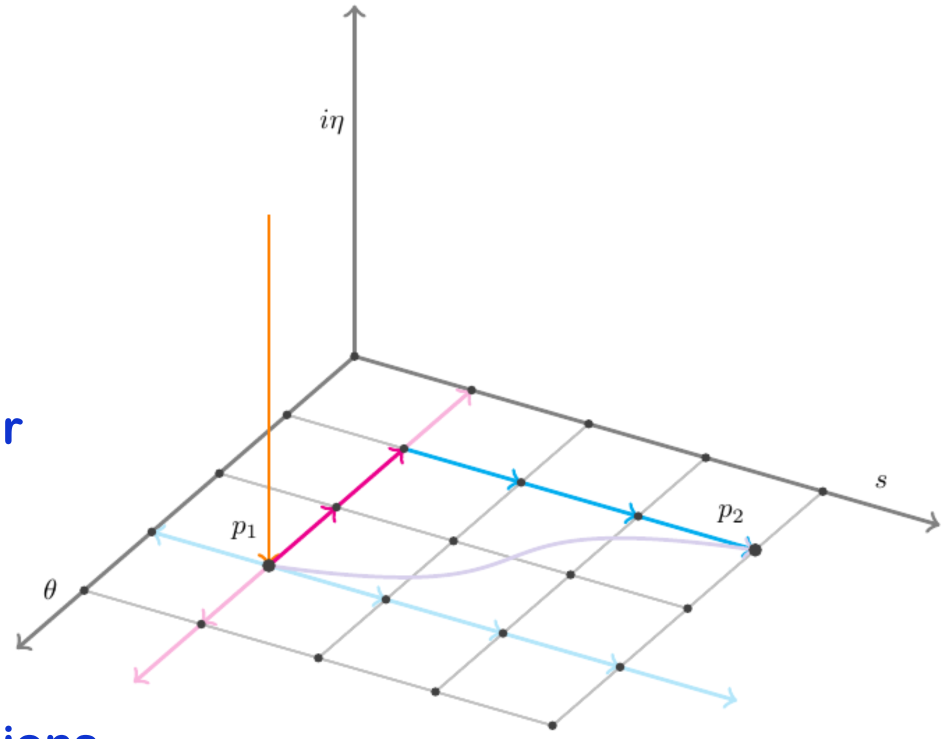
- **Easy to implement and to parallelize**
- **But ignoring the fact that the value of FIs at two nearby points have small difference**
- **Not an efficient way**

AMFlow+kinematic DEs

➢ Information of kinematic DEs

$$\frac{\partial}{\partial s_i}\boldsymbol{I}(\epsilon,\vec{s}\,)=A_i(\epsilon,\vec{s}\,)\boldsymbol{I}(\epsilon,\vec{s}\,)$$

- Tell how FIs change in the kinematic space
- Very efficient when two points are close to each other



➢ Tips

- Zero-dim.: a number computed by AMFlow
- One-dim.: series solutions to cover all interested regions
- Low dim.: a grid (of series solutions)
- High dim.: importance sampling

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Short summary

- **Analytical computation**
  - In general involving not well-studied special functions, which are hard to obtain numerical values
- **Semi-analytical computation**
  - General enough to deal with any FI
  - Can obtain numerical values to arbitrary precision
- **Can we define MIs as special functions?**
  - What are still missing for this purpose?

Special functions

- Typically require the following conditions:

  1. Having both integral and differential representations
  2. Clear singularities and branching cuts
  3. Availability of expansions to Taylor series or asymptotic series

- Facilitate the exploration of global and local properties, as well as efficient evaluation

1. Integral and differential representations

- **Integral representation: Yes**

- **Differential representation: Yes**

  - Have DEs w.r.t. kinematic variables, boundary conditions provided by AMFlow
  - Note: no differential equations w.r.t. ϵ, it should be thought as a parameter

- **Wish list**

  - Choosing better MIs, so that DEs are simple, no spurious poles; the method must be systematic, applicable to general cases beyond MPLs

2. Singularities and branching cuts

➢ Singularities: Yes

- Determined by Landau equations, but are hard to solve
- Subset of poles in DEs; spurious poles can be checked by solving DEs going around it

➢ Branching cuts: maybe

- Clear for simple cases, by studying the Feynman prescription $i\,0^+$
- No good method for general case, especially when there are cut propagators $1/\mathcal{D} \to \delta(\mathcal{D})$
- Bottom line: compute many points around a singularity using AMFlow, comparing with running using DEs

➢ Wish list

- A better way to determine singularities: solving Landau equations or other ways
- A better way to identify branching cuts

3. Taylor or asymptotic series

- **Yes**
  - Can do the expansion at any points using DEs
- **Efficiency**
  - Depending on the complexity of DEs, helpful to have better MIs

Bonus

- **Exhausting relations among coefficients of ϵ expansion**

$$\boldsymbol{I}(\epsilon, \vec{s}\,) = \sum_i \boldsymbol{I}^{(i)}(\vec{s}\,)\epsilon^i$$

  - More relations exists after expansion, how to systematically find these relations?
  - A famous example, one-loop 5-point function expressed by 4-point functions
  - Bottom line: PSLQ fit with high-precision input using AMFlow

- **Relations to not well-studied special functions**

  - Instead of using other special functions to study FIs, using FIs to study these special functions

Summary

- Solving QFT perturbatively: important for many fields
- One of the main challenges: FIs computation
- Analytical computation of FIs: fruitful, but has clear obstacles
- Semi-analytical computation of FIs: general enough
- Constructive to define FIs as special functions

Outlook

- **Plenty things to do**

  ① Systematical way to choose better MIs

  ② Systematical way to determine singularities

  ③ Systematical way to identify branching cuts

  ④ Systematical way to find relations after ϵ expansion

  ⑤ Improve the efficiency of IBP reduction: the main bottleneck for many problems

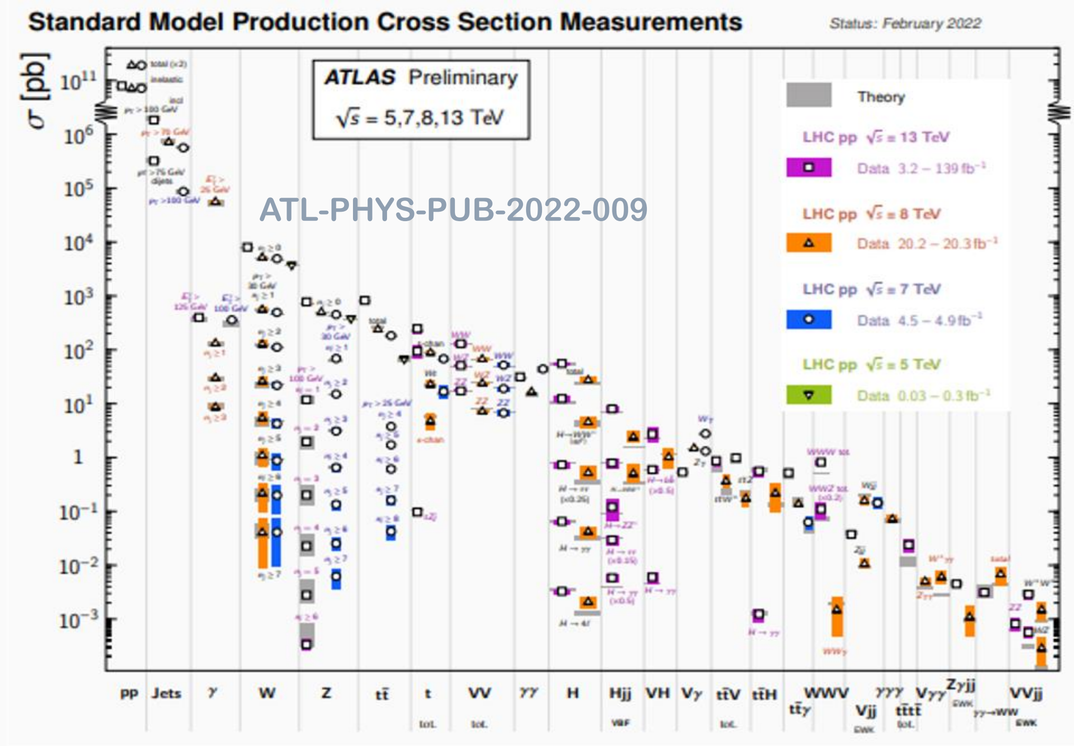
Thank you!

Era of precision physics at the LHC

➢ High-precision data

- Many observables probed at precent-level precision

- At least NNLO QCD corrections generally required (plus NLO EW, parton shower, resummation, etc.)



Automatic higher order perturbative calculation is highly demanded

Note: Automatic NLO correction obtained 15 years ago: MadGraph, Helac, etc