Asymptotic Expansions for Feynman Integrals

Henrik J. Munch

University of Padova

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Work in collaboration with

Saiei J. Matsubara-Heo

Seva Chestnov

Nobuki Takayama

- Talk mainly based on *Restrictions of Pfaffian Systems for Feynman Integrals* [2305.01585]
- + WIP (HJM et al. [2312.?????])

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Lightning intro to Feynman integrals

From model to prediction

Quantum Field Theory model: $\mathcal{L} \sim -\frac{1}{2}(\partial\varphi)^2 + \frac{\lambda}{4!}\varphi^4$

$\downarrow$

Cross section: $\sigma \sim \int |\mathcal{A}|^2$

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Bottleneck: Multi-loop and multi-scale diagrams

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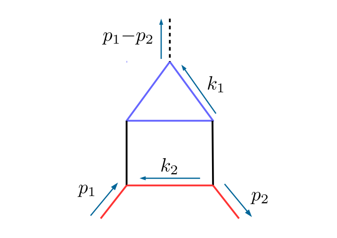
Cross section: $\sigma \sim \int |\mathcal{A}|^2$

Feynman integrals

Given a Feynman diagram



we associate a family of Feynman integrals

$$I_{\nu_1 \ldots \nu_n}(p_i, m_i) = \int \frac{\mathrm{d}^{\mathrm{D}} k_1 \wedge \cdots \wedge \mathrm{d}^{\mathrm{D}} k_L}{D_1^{\nu_1} \cdots D_n^{\nu_n}}$$

in terms of propagators

$$D_i = q_i(p,k)^2 - m_i^2\,, \quad q_i = \sum_a \pm k_a + \sum_b \pm p_b$$

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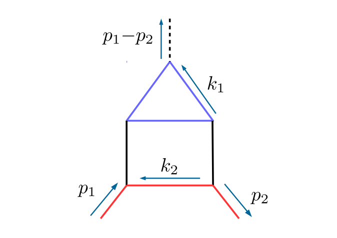

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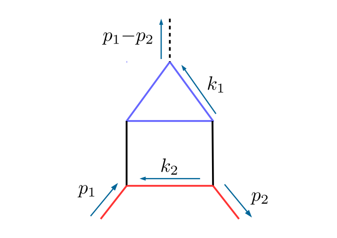

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Computing Feynman integrals via DEQs

- Consider some integral family $I_{\vec{\nu}}$ for $\vec{\nu} \in \mathbb{Z}^n$
- $\exists$ finite set of independent master integrals $\vec{I}$ with special values of $\vec{\nu}$
- Other integrals in the family related to $\vec{I}$ via integration-by-parts identities

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- Get master integrals $\vec{I}$ from solving a Pfaffian system

$$\partial_i \vec{I}(z) = P_i(z) \cdot \vec{I}(z)$$

Kinematic variables: $z = (z_1, \ldots, z_N)$

Rational matrices: $P_i(z)$

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Solving the DEQs

- Have powerful methods to solve the Pfaffian system

$$\partial_i \vec{I}(z) = P_i(z) \cdot \vec{I}(z)$$

when $\vec{I}$ can be expressed in terms of multiple polylogarithms

$$I(z) \sim \sum_{\vec{w}} \mathrm{rat}(z)\, G(\vec{w}|z)\,, \quad G(w_1, \ldots, w_k|z) = \int_0^z \frac{\mathrm{d}t}{t - w_1} G(w_2, \ldots, w_k|t)$$

- But, at $L > 1$ loops and several mass scales, we encounter

- Then it is much harder to solve the Pfaffian system!

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  - Modular forms
  - Integrals over Calabi-Yau manifolds

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Asymptotic expansions

- Alternative strategy: Asymptotic expansion

$$\vec{I} \sim \sum_{n,m} \vec{I}^{(n,m)}(z_2, \ldots, z_N)\; z_1^n \log^m(z_1)$$

given $z_1 \ll 1$ (small mass, threshold, collinear ...)

- Many previously succesful studies [Beneke, Davies, Harlander, Kudashkin, Lee, Mastrolia, Melnikov, Mishima, Passera, Pozzorini, Primo, Remiddi, Schonwald, Schubert, Seidensticker, Smirnov, Smirnov, Steinhauser, Wasow, Wever, Zhang ...] Book recommendation: [Haraoka '20]

- Our approach:

1. Solve *simpler* Pfaffian system for $\vec{I}^{(0,0)}(z_2, \ldots)$

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Holomorphic restriction of DEQs

What is a restriction?

Restriction $\leftrightarrow$ localizing PDEs to a specific region in the space of variables

- Terminology from the field of $\mathcal{D}$-modules (*algebraic* study of PDEs)
- In physics language: Study PDEs near $m^2 = 0$, $s = 4m^2$, $p^2 \to \infty \ldots$

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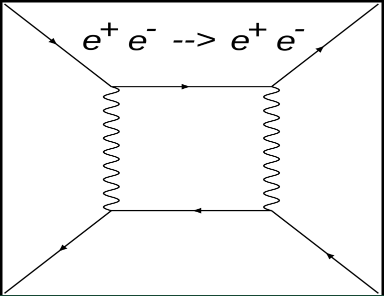
Example: PDE for 1-loop Bhabha scattering



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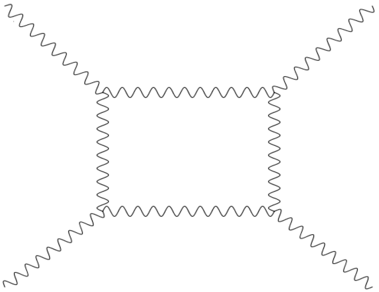
Example: Holomorphic solution at $m^2 = 0$



Two issues to adress

Consider $z_1 \to 0$ limit of a Pfaffian system

$$\partial_i \vec{I}(z) = P_i(z) \cdot \vec{I}(z) , \quad i = 1, 2 .$$

Suppose $P_1(z)$ has a pole at $z_1 = 0$.

Rank jumps

- The *number* of master integrals changes at $z_1 \to 0$
- New *relations* among master integrals must hence arise at $z_1 \to 0$
- What is the new rank? How to systematically find these relations?

Basis

- We suppose to know *some* basis $\vec{I}(z_1, z_2)$ for generic z
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Suppose $P_1(z)$ has a pole at $z_1 = 0$.

We assume that the system is in normal form

$$P_1(z) = \sum_{n=-1}^{\infty} P_{1,n}(z_2)\, z_1^n$$

$$P_2(z) = \sum_{n=0}^{\infty} P_{2,n}(z_2)\, z_1^n$$

Done via Moser reduction

- Completely algorithmic (gauge transformations)
- Not computationally costly w.r.t. one pole $z_1 = 0$

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Issue 1) Rank jump

Solutions $\vec{I}(z)$ to

$$\partial_i \vec{I}(z) = P_i(z) \cdot \vec{I}(z)\,, \quad i = 1, 2$$

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Recall

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Inserting all 3 expansions into the Pfaffian system:

PDE: $\partial_2\, \vec{I}^{(0)}(z_2) = P_{2,0}(z_2) \cdot \vec{I}^{(0)}(z_2)$

Rank jump: $P_{1,-1}(z_2) \cdot \vec{I}^{(0)}(z_2) = 0$

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The rank jump relations are IBPs that hold in the *integrand* limit $z_1 \to 0$

Example: Master integrals for 1-loop Bhabha box:

$$\vec{I}(z_1, z_2) = \left[I_{1000}\,, I_{0101}\,, I_{1010}\,, I_{0111}\,, I_{1111} \right]^T$$

Say $z_1 = m^2/(-s)$. As $z_1 \to 0$, the rank jump relation gives

$$I_{1000} = 0\,, \quad I_{0111} = (\text{IBP coefficient}) \times I_{0101}$$

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Let's re-write this system in a minimal basis.

Define

$$R = \texttt{RowReduce}\left[P_{1,-1}\right], \quad \vec{I}^{(0)} = \left[I_1, I_2, \ldots\right]^T$$

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$$\vec{J} = B \cdot \vec{I}^{(0)}\,, \quad B_{ij} \in \{0,\,1\}$$

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Recall the two matrices R and B from

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$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = m^2 \,, \quad s = p_{12}^2 \,, \quad t = p_{23}^2$$

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Logarithmic restriction of DEQs

Taking stock

- Have so far looked at holomorphic solutions $\vec{I}(z_1 \to 0, z_2)$ to

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- Let's generalize to logarithmically singular solutions at $z_1 \to 0$:

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- Computational strategy: Repeated use of the restriction method

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Recall the residue matrix $P_{1,-1}$ from

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Eigenvalues of $P_{1,-1}$:

$$\mathtt{Spec}[P_{1,-1}] = \Big\{ \underbrace{\lambda_1, \lambda_1, \ldots}_{\Lambda_1} \;,\; \underbrace{\lambda_2, \lambda_2, \ldots}_{\Lambda_2} \Big\}$$

where Λ_i is the multiplicity of eigenvalue λ_i

Fact: In dimensional regularization $\mathrm{D} = 4 - 2\epsilon$,

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General form of the asymptotic series

Asymptotic series decomposes into a sum over *unique* eigenvalues [Haraoka '20]:

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- For each λ, solve a Pfaffian system for $\vec{I}^{(\lambda,0,0)}(z_2)$ by the restriction method
- I.e. find matrices $\{R^{(\lambda)}, B^{(\lambda)}, M^{(\lambda)}\} \implies \partial_2 \vec{J}^{(\lambda)} = Q_2^{(\lambda)} \cdot \vec{J}^{(\lambda)}$
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Rank jump

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Fix an eigenvalue λ. Logarithmic case:

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Basis

Dummy symbols

$$\vec{I}^{(\lambda,0,0)} = \left[I_1^{(\lambda)},\, I_2^{(\lambda)},\, \ldots\right]^T$$

Basis found by solving

$$R^{(\lambda)} \cdot \vec{I}^{(\lambda,0,0)} = 0$$

for independent $\vec{J}^{(\lambda)} = \left[I_{i_1}^{(\lambda)},\, I_{i_2}^{(\lambda)},\, \ldots,\, I_{i_\Lambda}^{(\lambda)}\right]^T$

Rectangular basis matrix $B^{(\lambda)}$:

$$\vec{J}^{(\lambda)} = B^{(\lambda)} \cdot \vec{I}^{(\lambda,0,0)}, \quad B_{ij}^{(\lambda)} \in \{0,1\}$$

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Pfaffian system for $\vec{J}^{(\lambda)}$

How to get the $\Lambda \times \Lambda$ Pfaffian matrix $Q_2^{(\lambda)}(z_2)$ in

$$\partial_2 \vec{J}^{(\lambda)} = Q_2^{(\lambda)} \cdot \vec{J}^{(\lambda)} \ ?$$

Join rank jump and basis matrices into

$$M^{(\lambda)} = \left[\frac{B^{(\lambda)}}{R^{(\lambda)}}\right] \quad \Longrightarrow \quad M^{(\lambda)} \cdot \vec{I}^{(\lambda,0,0)} = \begin{bmatrix} \vec{J}^{(\lambda)} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

Gauge transformation of $P_{2,0}$:

$$\left(\partial_2 M^{(\lambda)} + M^{(\lambda)} \cdot P_{2,0}\right) \cdot \left(M^{(\lambda)}\right)^{-1} = \left[\begin{array}{c|c} Q_2^{(\lambda)} & \star \\ \hline \mathbf{0} & \star \end{array}\right]$$

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Summarizing the algorithm for computing $\vec{I}^{(\lambda,n,m)}$

Step 0: Bring P_1, P_2 to normal form.

For each unique $\lambda \in \mathrm{Spec}[P_{1,-1}]$, do

1. Construct $M^{(\lambda)} = \left[\frac{B^{(\lambda)}}{R^{(\lambda)}}\right]$.

2. Get Q_2 from $\left(\partial_2 M^{(\lambda)} + M^{(\lambda)} \cdot P_{2,0}\right) \cdot \left(M^{(\lambda)}\right)^{-1} = \left[\begin{array}{c|c} Q_2^{(\lambda)} & \star \\ \hline \mathbf{0} & \star \end{array}\right]$

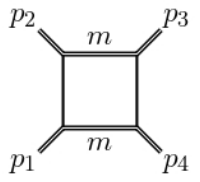
3. Solve $\partial_2 \vec{J}^{(\lambda)} = Q_2 \cdot \vec{J}^{(\lambda)}$

4. Insert $\vec{J}^{(\lambda)}$ into $\vec{I}^{(\lambda,0,0)} = \left(M^{(\lambda)}\right)^{-1} \cdot \left[\frac{\vec{J}^{(\lambda)}}{\mathbf{0}}\right]$

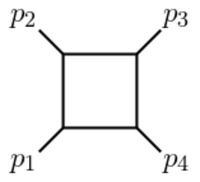
5. Get $\vec{I}^{(\lambda,n,m)}$ from recursion relations (see extra slides)

Example: Bhabha scattering

1-loop: Eigenvalues



$\xrightarrow{m\to 0}$



Eigenvalues:

$$\mathrm{Spec}\left[P_{1,-1}\right] \quad = \quad \Big\{ \underbrace{0,0,0}_{\Lambda_1 \,=\, 3} \;,\; \underbrace{-\epsilon,-\epsilon}_{\Lambda_2 \,=\, 2} \Big\}$$

- $\lambda_1 = 0$: 3×3 Pfaffian system for the massless box
- $\lambda_2 = -\epsilon$: 2×2 Pfaffian system contributing with $\log$s

1-loop: Pfaffian sub-systems

$\lambda_1 = 0$

$$Q_2^{(\lambda_1)} = \begin{bmatrix} -\frac{\epsilon}{z_2} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2(1-2\epsilon)}{z_2^2(z_2+1)} & \frac{2(2\epsilon-1)}{z_2(z_2+1)} & \frac{-z_2-\epsilon-1}{z_2(z_2+1)} \end{bmatrix}$$

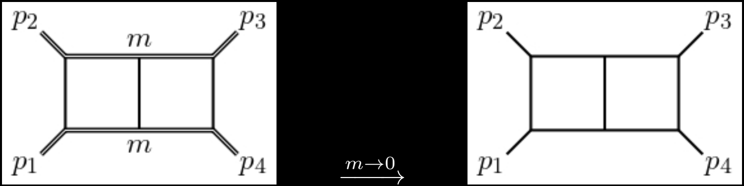
Can easily be ϵ-factorized [Henn]

$\lambda_1 = -\epsilon$

$$Q_2^{(\lambda_2)} = \begin{bmatrix} 0 & 0 \\ \frac{\epsilon-1}{z_2^2} & \frac{-1}{z_2} \end{bmatrix}$$

Can be solved exactly in z_2

WIP: 2-loops [Henn, Smirnov]



Eigenvalues:

$$\mathrm{Spec}\left[P_{1,-1}\right] \quad = \quad \Big\{ \underbrace{0,\ldots,0}_{\Lambda_1 \;=\; 8} \;,\; \underbrace{\lambda_2}_{\Lambda_2 \;=\; 1} \;,\; \underbrace{\lambda_3,\ldots,\lambda_3}_{\Lambda_3 \;=\; 7} \;,\; \underbrace{\lambda_4,\ldots,\lambda_4}_{\Lambda_4 \;=\; 7} \Big\}$$

- All 4 Pfaffian sub-systems are easily ϵ-factorized
- HPL letters: $\{z_2,\, z_2+1,\, z_2-1\}$

Conclusion and outlook

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- Presented a computationally cheap method for obtaining asymptotic series

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Outlook

- Boundary constants?
- Momentum space representation of $\vec{J}^{(\lambda)}$?
- Do all eigenvalues λ contribute to the result?
- Try many more examples:
  - 2-loop non-planar Bhabha scattering
  - 2-loop μ-e scattering with massive e
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Extra 1) Recursion relations

Shift power of $\log^m(z_1)$.

$$\vec{I}^{(\lambda,\,0,\,m)} = \left[\frac{P_{1,-1} - \lambda\mathbf{1}}{m}\right] \cdot \vec{I}^{(\lambda,\,0,\,m-1)}, \quad 1 \leq m \leq M_\lambda \tag{1}$$

with terminating condition

$$\vec{I}^{(\lambda,\,0,\,M_\lambda+1)} = 0$$

Shift power of z_1^n. Set $\Pi_{\lambda,n} = \left[P_{1,-1} - (\lambda+n)\mathbf{1}\right]^{-1}$.

$$\vec{I}^{(\lambda,\,n,\,m)} = \Pi_{\lambda,n} \cdot \left((m+1)\vec{I}^{(\lambda,\,n,\,m+1)} - \sum_{i=0}^{n-1} P_{1,n-i-1} \cdot \vec{I}^{(\lambda,\,i,\,m)}\right) \tag{2}$$

$$\vec{I}^{(\lambda,\,n,\,M_\lambda)} = -\Pi_{\lambda,n} \cdot \sum_{i=0}^{n-1} P_{1,n-i-1} \cdot \vec{I}^{(\lambda,\,i,\,M_\lambda)} \tag{3}$$

Extra 2) Recursion flowchart [Haraoka '20]

N_λ: Max power of z_1^n. M_λ: Max power of $\log^m(z_1)$

$$
\begin{array}{ccccccccc}
(0,0) & \xrightarrow{(1)} & (0,1) & \xrightarrow{(1)} & \cdots & \xrightarrow{(1)} & (0,M_\lambda-1) & \xrightarrow{(1)} & (0,M_\lambda) \\
 & & & & & & & & \big\downarrow{\scriptstyle (3)} \\
(1,0) & \xleftarrow{(2)} & (1,1) & \xleftarrow{(2)} & \cdots & \xleftarrow{(2)} & (1,M_\lambda-1) & \xleftarrow{(2)} & (1,M_\lambda) \\
 & & & & & & & & \big\downarrow{\scriptstyle (3)} \\
 & & & & & & & & \vdots \\
 & & & & & & & & \big\downarrow{\scriptstyle (3)} \\
(N_\lambda,0) & \xleftarrow{(2)} & (N_\lambda,1) & \xleftarrow{(2)} & \cdots & \xleftarrow{(2)} & (N_\lambda,M_\lambda-1) & \xleftarrow{(2)} & (N_\lambda,M_\lambda)
\end{array}
$$