IBP reduction with modular arithmetic and Rational Tracer

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Integration-By-Parts reduction

An *IBP integral family* with L *loop momenta* l_i, and E *external momenta* p_i, is the set of Feynman integrals

$$\underbrace{I_{\nu_1,\nu_2,\ldots,\nu_N}}_{\text{“indices”}} \equiv \int \underbrace{\frac{\mathrm{d}^d l_1 \cdots \mathrm{d}^d l_L}{D_1^{\nu_1} \cdots D_N^{\nu_N}}}_{\text{“denominators”}}, \qquad \begin{cases} D_i \equiv \left(l_j \pm p_k \pm \ldots\right)^2 - m_i^2 + i0, \\ N \equiv L\,(L+1)\,/2 + LE. \end{cases}$$

The idea: shifting any l_k by any vector v should not change I:

$$\lim_{\alpha\to 0} \frac{\partial}{\partial\alpha} I(l_k \to l_k + \alpha\, v) = \int \mathrm{d}^d l_1 \cdots \mathrm{d}^d l_L \frac{\partial}{\partial l_k^\mu} \frac{v^\mu}{D_1^{\nu_1} \ldots D_N^{\nu_N}} \overset{!}{=} 0, \qquad \forall\, k, v.$$

These are the *IBP relations*. We use them to:

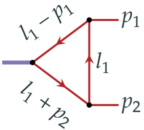
- reduce $I_{\nu_1 \ldots \nu_N}$ to combinations of *master integrals*, [Chetyrkin, Tkachov ’81]
  - the number of master integrals is finite; [Smirnov, Petukhov ’04]
- evaluate the master integrals via
  - *differential equations* (analytically, numerically), [Kotikov ’91, ’91; Henn ’13]
  - or *dimensional recurrence relations*. [Tarasov ’96; Lee ’10]

Solving IBP relations is a *major bottleneck* in cutting edge calculations.

IBP relations example

Consider a *massless triangle* topology:

$$I_{a,b,c} \equiv$$



$$\equiv \int \frac{\mathrm{d}^d\, l}{\left(l_1^2\right)^a \left((l_1 - p_1)^2\right)^b \left((l_1 + p_2)^2\right)^c},$$

where $p_1^2 = p_2^2 = 0,$ and $p_1 \cdot p_2 = s/2.$

Choosing $k = 1$ and $v = \{p_1, p_2, p_3\}$ we get linear relations between $I_{a,b,c}$:

$$(b-a)\, I_{a,b,c} - csI_{a,b,c+1} - cI_{a-1,b,c+1} - bI_{a-1,b+1,c} + cI_{a,b-1,c+1} + aI_{a+1,b-1,c} = 0,$$

$$(a-c)\, I_{a,b,c} + bsI_{a,b+1,c} + cI_{a-1,b,c+1} + bI_{a-1,b+1,c} - bI_{a,b+1,c-1} - aI_{a+1,b,c-1} = 0,$$

$$(d - 2a - b - c)\, I_{a,b,c} - cI_{a-1,b,c+1} - bI_{a-1,b+1,c} = 0,$$

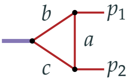
$$\forall\, a, b, c.$$

Laporta algorithm

Solving IBP relations “by hand” (with indices as symbolic variables) can be done in simpler cases. For more complicated problems use the *Laporta algorithm*: [Laporta ’00]

1. Substitute integer values for the indices ν_i into the IBP relations, obtaining a large linear system with many different $I_{\nu_1 \ldots \nu_N}$.
2. Define an ordering on $I_{\nu_1 \ldots \nu_N}$ from “simple” to “complex” integrals.
   - E.g. $I_{0,1,1} < I_{1,1,0} < I_{1,1,1} < I_{1,2,1} < I_{2,1,1}$, etc.
3. Perform *Gaussian elimination* on the linear system, eliminating the most “complex” integrals first.
4. A small number of “simple” integrals will remain uneliminated.
   $\Rightarrow$ These are the master integrals. The rest will be expressed as their linear combinations.

IBP reduction example

$$I_{a,b,c} \equiv$$ 

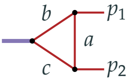
$$, \qquad p_1^2 = p_2^2 = 0, \qquad p_1 \cdot p_2 = s/2.$$

$$\begin{gathered}
(d-6)\, I_{2,1,1} - I_{1,2,1} - I_{1,1,2} = 0 \\
s\, I_{1,2,1} - s\, I_{1,1,2} = 0 \\
s\, I_{1,1,2} + I_{0,2,1} + I_{0,1,2} = 0 \\
(d-4)\, I_{1,1,1} - I_{0,2,1} - I_{0,1,2} = 0 \\
-2\, I_{-1,3,1} - I_{-1,2,2} + (d-3)\, I_{0,2,1} = 0 \\
-2\, I_{-1,1,1} - s\, I_{0,1,1} = 0 \\
-I_{0,3,1} + I_{0,1,3} = 0 \\
s\, I_{-1,2,1} - s\, I_{-1,1,2} = 0 \\
-I_{-1,2,1} - I_{-1,1,2} + (d-2)\, I_{0,1,1} = 0 \\
-I_{-1,2,1} - I_{-1,1,2} - s\, I_{0,1,2} + I_{0,1,1} = 0 \\
-I_{-1,2,2} - 2 I_{-1,1,3} + (d-3)\, I_{0,1,2} = 0 \\
s\, I_{-1,2,2} - 2s\, I_{-1,1,3} + s\, I_{0,1,2} = 0 \\
-2\, I_{-1,2,2} - s\, I_{0,2,2} + I_{0,2,1} + I_{0,1,2} = 0 \\
-I_{-1,2,2} - 2\, I_{-1,1,3} - 2s\, I_{0,1,3} + I_{0,1,2} = 0 \\
-I_{0,2,1} + I_{0,1,2} = 0
\end{gathered}
\iff
\begin{pmatrix}
d-6 & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & d-4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -2 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2s & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & -2s & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix} = 0$$

After Gaussian elimination (2 operations):

$$\begin{pmatrix}
1 & 1/(6-d) & 1/(6-d) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & d-4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -2 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2s & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & -2s & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix} = 0$$

IBP reduction example

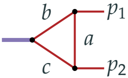
$$I_{a,b,c} \equiv$$  $$, \qquad p_1^2 = p_2^2 = 0, \qquad p_1 \cdot p_2 = s/2.$$

$$
\begin{gathered}
(d-6)\, I_{2,1,1} - I_{1,2,1} - I_{1,1,2} = 0 \\
s\, I_{1,2,1} - s\, I_{1,1,2} = 0 \\
s\, I_{1,1,2} + I_{0,2,1} + I_{0,1,2} = 0 \\
(d-4)\, I_{1,1,1} - I_{0,2,1} - I_{0,1,2} = 0 \\
-2\, I_{-1,3,1} - I_{-1,2,2} + (d-3)\, I_{0,2,1} = 0 \\
-2\, I_{-1,1,1} - s\, I_{0,1,1} = 0 \\
-I_{0,3,1} + I_{0,1,3} = 0 \\
s\, I_{-1,2,1} - s\, I_{-1,1,2} = 0 \\
-I_{-1,2,1} - I_{-1,1,2} + (d-2)\, I_{0,1,1} = 0 \\
-I_{-1,2,1} - I_{-1,1,2} - s\, I_{0,1,2} + I_{0,1,1} = 0 \\
-I_{-1,2,2} - 2I_{-1,1,3} + (d-3)\, I_{0,1,2} = 0 \\
s\, I_{-1,2,2} - 2s\, I_{-1,1,3} + s\, I_{0,1,2} = 0 \\
-2\, I_{-1,2,2} - s\, I_{0,2,2} + I_{0,2,1} + I_{0,1,2} = 0 \\
-I_{-1,2,2} - 2\, I_{-1,1,3} - 2s\, I_{0,1,3} + I_{0,1,2} = 0 \\
-I_{0,2,1} + I_{0,1,2} = 0
\end{gathered}
\quad \Longleftrightarrow \quad
\begin{pmatrix}
d-6 & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & d-4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -2 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2s & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & -2s & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix}
= 0
$$

After Gaussian elimination (5 operations):

$$
\begin{pmatrix}
1 & 0 & -2/(d-6) & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & d-4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -2 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2s & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & -2s & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix}
= 0
$$

IBP reduction example

$$I_{a,b,c} \equiv$$ 

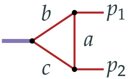
$$, \qquad p_1^2 = p_2^2 = 0, \qquad p_1 \cdot p_2 = s/2.$$

$$\begin{gathered}
(d-6)\, I_{2,1,1} - I_{1,2,1} - I_{1,1,2} = 0 \\
s\, I_{1,2,1} - s\, I_{1,1,2} = 0 \\
s\, I_{1,1,2} + I_{0,2,1} + I_{0,1,2} = 0 \\
(d-4)\, I_{1,1,1} - I_{0,2,1} - I_{0,1,2} = 0 \\
-2\, I_{-1,3,1} - I_{-1,2,2} + (d-3)\, I_{0,2,1} = 0 \\
-2\, I_{-1,1,1} - s\, I_{0,1,1} = 0 \\
-I_{0,3,1} + I_{0,1,3} = 0 \\
s\, I_{-1,2,1} - s\, I_{-1,1,2} = 0 \\
-I_{-1,2,1} - I_{-1,1,2} + (d-2)\, I_{0,1,1} = 0 \\
-I_{-1,2,1} - I_{-1,1,2} - s\, I_{0,1,2} + I_{0,1,1} = 0 \\
-I_{-1,2,2} - 2I_{-1,1,3} + (d-3)\, I_{0,1,2} = 0 \\
s\, I_{-1,2,2} - 2s\, I_{-1,1,3} + s\, I_{0,1,2} = 0 \\
-2\, I_{-1,2,2} - s\, I_{0,2,2} + I_{0,2,1} + I_{0,1,2} = 0 \\
-I_{-1,2,2} - 2\, I_{-1,1,3} - 2s\, I_{0,1,3} + I_{0,1,2} = 0 \\
-I_{0,2,1} + I_{0,1,2} = 0
\end{gathered}
\quad \Longleftrightarrow \quad
\begin{pmatrix}
d-6 & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & d-4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -2 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2s & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & -2s & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix} = 0$$

After Gaussian elimination (11 operations):

$$\begin{pmatrix}
1 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2/(ds-6s) & 2/(ds-6s) & \cdot \\
\cdot & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/s & 1/s & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/s & 1/s & \cdot \\
\cdot & \cdot & \cdot & d-4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -2 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2s & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & -2s & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix} = 0$$

IBP reduction example

$$I_{a,b,c} \equiv$$ 

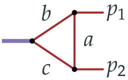
$$, \qquad p_1^2 = p_2^2 = 0, \qquad p_1 \cdot p_2 = s/2.$$

$$\begin{gathered}
(d-6)\, I_{2,1,1} - I_{1,2,1} - I_{1,1,2} = 0 \\
s\, I_{1,2,1} - s\, I_{1,1,2} = 0 \\
s\, I_{1,1,2} + I_{0,2,1} + I_{0,1,2} = 0 \\
(d-4)\, I_{1,1,1} - I_{0,2,1} - I_{0,1,2} = 0 \\
-2\, I_{-1,3,1} - I_{-1,2,2} + (d-3)\, I_{0,2,1} = 0 \\
-2\, I_{-1,1,1} - s\, I_{0,1,1} = 0 \\
-I_{0,3,1} + I_{0,1,3} = 0 \\
s\, I_{-1,2,1} - s\, I_{-1,1,2} = 0 \\
-I_{-1,2,1} - I_{-1,1,2} + (d-2)\, I_{0,1,1} = 0 \\
-I_{-1,2,1} - I_{-1,1,2} - s\, I_{0,1,2} + I_{0,1,1} = 0 \\
-I_{-1,2,2} - 2I_{-1,1,3} + (d-3)\, I_{0,1,2} = 0 \\
s\, I_{-1,2,2} - 2s\, I_{-1,1,3} + s\, I_{0,1,2} = 0 \\
-2\, I_{-1,2,2} - s\, I_{0,2,2} + I_{0,2,1} + I_{0,1,2} = 0 \\
-I_{-1,2,2} - 2\, I_{-1,1,3} - 2s\, I_{0,1,3} + I_{0,1,2} = 0 \\
-I_{0,2,1} + I_{0,1,2} = 0
\end{gathered}
\quad \Longleftrightarrow \quad
\begin{pmatrix}
d-6 & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & d-4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -2 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2s & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & -2s & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix} = 0$$

After Gaussian elimination (62 operations):

$$\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2/(ds-6s) & 2/(ds-6s) & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/s & 1/s & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/s & 1/s & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/(4-d) & 1/(4-d) & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (d-4)/4 & (3-d)/2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & s/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 & \cdot & (d-4)/(2s) & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (2-d)/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (2-d)/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 3-d \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (2-d)/4 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (4-d)/2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & (d-5)/s & -1/s & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & (d-4)/(2s) & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix} = 0$$

IBP reduction example

$$I_{a,b,c} \equiv$$ 

$$, \qquad p_1^2 = p_2^2 = 0, \qquad p_1 \cdot p_2 = s/2.$$

$$
\begin{aligned}
(d-6)\, I_{2,1,1} - I_{1,2,1} - I_{1,1,2} &= 0 \\
s\, I_{1,2,1} - s\, I_{1,1,2} &= 0 \\
s\, I_{1,1,2} + I_{0,2,1} + I_{0,1,2} &= 0 \\
(d-4)\, I_{1,1,1} - I_{0,2,1} - I_{0,1,2} &= 0 \\
-2\, I_{-1,3,1} - I_{-1,2,2} + (d-3)\, I_{0,2,1} &= 0 \\
-2\, I_{-1,1,1} - s\, I_{0,1,1} &= 0 \\
-I_{0,3,1} + I_{0,1,3} &= 0 \\
s\, I_{-1,2,1} - s\, I_{-1,1,2} &= 0 \\
-I_{-1,2,1} - I_{-1,1,2} + (d-2)\, I_{0,1,1} &= 0 \\
-I_{-1,2,1} - I_{-1,1,2} - s\, I_{0,1,2} + I_{0,1,1} &= 0 \\
-I_{-1,2,2} - 2I_{-1,1,3} + (d-3)\, I_{0,1,2} &= 0 \\
s\, I_{-1,2,2} - 2s\, I_{-1,1,3} + s\, I_{0,1,2} &= 0 \\
-2\, I_{-1,2,2} - s\, I_{0,2,2} + I_{0,2,1} + I_{0,1,2} &= 0 \\
-I_{-1,2,2} - 2\, I_{-1,1,3} - 2s\, I_{0,1,3} + I_{0,1,2} &= 0 \\
-I_{0,2,1} + I_{0,1,2} &= 0
\end{aligned}
\quad \Longleftrightarrow \quad
\begin{pmatrix}
d-6 & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & d-4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & -2 & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & -s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -s & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d-3 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2s & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & s & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -2 & \cdot & \cdot & \cdot & \cdot & \cdot & -s & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -2 & -1 & \cdot & \cdot & \cdot & \cdot & -2s & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix}
= 0
$$

After Gaussian elimination (108 operations, $\sim N^2_{\text{integrals}}$):

$$
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (12-4d)/(ds^2-6s^2) \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (6-2d)/(s^2) \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (6-2d)/(s^2) \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (2d-6)/(ds-4s) \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (d^2-5d+6)/(4s) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & s/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & (-d^2+7d-12)/(2s^2) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (2-d)/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (2-d)/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & (d-3)/s \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (d^2-5d+6)/(4s) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (d^2-7d+12)/(2s) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & (-d^2+9d-18)/(s^2) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & (-d^2+7d-12)/(2s^2) \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & (d-3)/s
\end{pmatrix}
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1} \\ I_{0,1,1}
\end{pmatrix}
= 0
\quad \Longleftrightarrow \quad
\begin{pmatrix}
I_{2,1,1} \\ I_{1,2,1} \\ I_{1,1,2} \\ I_{1,1,1} \\ I_{-1,3,1} \\ I_{-1,1,3} \\ I_{-1,2,2} \\ I_{-1,2,1} \\ I_{-1,1,2} \\ I_{-1,1,1} \\ I_{0,3,1} \\ I_{0,1,3} \\ I_{0,2,2} \\ I_{0,1,2} \\ I_{0,2,1}
\end{pmatrix}
=
\begin{pmatrix}
4(d-3)/((d-6)\, s^2) \\
2(d-3)/s^2 \\
2(d-3)/s^2 \\
-2(d-3)/((d-4)\, s) \\
-(d-3)(d-2)/(4s) \\
-(d-3)(d-2)/(4s) \\
-(d-4)(d-3)/(2s) \\
-(2-d)/2 \\
-(2-d)/2 \\
-s/2 \\
(d-4)(d-3)/(2s^2) \\
(d-4)(d-3)/(2s^2) \\
(d-6)(d-3)/s^2 \\
-(d-3)/s \\
-(d-3)/s
\end{pmatrix}
I_{0,1,1}
$$

Rational function arithmetic, I

$$f(x,y) = \frac{2xy - y^2}{x - y} + \frac{y^3 - 3xy^2}{x^2 - y^2} = ?$$

Rational arithmetic symbolically:

Common denominator:	$((2xy - y^2)(x + y) + y^3 - 3xy^2)/(x^2 - y^2)$
Expand the numerator:	$(2x^2y - xy^2 + 2xy^2 - y^3 + y^3 - 3xy^2)/(x^2 - y^2)$
Combine alike terms:	$(2x^2y - 2xy^2)/(x^2 - y^2)$
Cancel common factors:	$2xy/(x + y)$
Runtime:	$\mathcal{O}(N^2_{\text{initial monomials}} N_{\text{digits per monomial}})$
Peak memory needed:	$\mathcal{O}(N^2_{\text{initial monomials}} N_{\text{digits per monomial}})$

IBP reduction as a bottleneck

Optimistic *reduction time estimate* (symbolic reduction):

$$T \sim \frac{1}{\text{performance}} N_{\text{families}} \underbrace{N^2_{\text{integrals per family}}}_{(N_{\text{denominators}})^{N_{\text{lines}}}} \underbrace{N^2_{\text{monomials per term}}}_{\exp(N_{\text{scales}})} N^{T_{\text{operation}}}_{\text{digits per monomial}}$$

To match LHC experimental precision the theory requires 2-loop corrections.

For future colliders: 3-loop corrections. [Freitas '21]

	1 loop	2 loops	3 loops
4 legs	~100 diagrams 3 families 4 denominators 2+ scales	~2K diagrams 24 families 7+2 denominators 2+ scales	~50K diagrams 219 families 10+5 denominators 2+ scales
5 legs	~1K diagrams 12 families 5 denominators 5+ scales	~30K diagrams 180 families 8+3 denominators 5+ scales	~800K diagrams 2355 families 11+7 denominators 5+ scales

- Lines in a Feynman diagram: $N_{\text{lines}} = 3L + E - 2$.
- Denominators per family: $N_{\text{denominators}} = L(L+1)/2 + LE$.
- Mass scales per family: $N_{\text{scales}} = E(E-1)/2 - 1 + N_{\text{massive legs + masses}}$.

Rational function arithmetic, II

Rational arithmetic via an anzatz-based *interpolation*:

Prepare an ansatz:	$f(x,y) = c_1 xy/(x + c_2 y)$
Evaluate:	$f(1,1) = 1,\ f(1,2) = 4/3$
Solve for c_i:	$c_1 = 2,\ c_2 = 1$
Runtime, evaluation:	$N_{\text{final monomials}} \times \mathscr{O}(N_{\text{initial monomials}} N_{\text{digits per monomial}})$
Runtime, interpolation:	$\mathscr{O}(N^2_{\text{final monomials}} N_{\text{digits per final monomial}})$
Peak memory, needed:	$\mathscr{O}(N^2_{\text{final monomials}} N_{\text{digits per final monomial}})$

Same interpolation, but using *modular arithmetic*:

- Interpolate keeping the values as integers modulo a prime numer P_1.
- Use *rational number reconstruction* to upgrade c_i from integers to rationals modulo P_1. [Wang '81; Monagan '04]
- Repeat the same with primes P_2, P_3, ...
- Use the *Chinese remainder theorem* to get c_i modulo $P_1 P_2 P_3 \cdots$.
- Stop when c_i no longer change.
- Runtime: same, but $N_{\text{digits per monomial}} \to N_{\text{digits per final monomial}}$, and *faster on a computer*: all operations are on small integers!

Function reconstruction algorithms

If an anzatz is unknown, multiple *reconstruction algorithms* are available:

- *Univariate* case:
  - Newton interpolation for dense polynomials. [Newton 1675; Peraro '16]
    - Number of evaluations $\sim N_{\text{maximal degree}}$.
  - Ben-Or/Tiwari for sparse polynomials. [Ben-Or, Tiwari '88]
    - Number of evaluations $\sim 2N_{\text{monomials}}$.
  - Thiele interpolation for dense rationals.
    - Number of evaluations $\sim 2N_{\text{maximal degree}}$.
- *Multivariate* case:
  - Newton applied recursively in each variable for dense polynomials.
    - Number of evaluations $\sim (N_{\text{maximal degree}})^{N_{\text{scales}}}$.
  - Zippel ($\sim$ recursive Newton with prunning) + early termination for sparse polynomials. [Zippel '90; Kaltofen, Lee '03]
    - Number of evaluations $\lesssim N_{\text{scales}}\, N_{\text{maximal degree}}\, N_{\text{monomials}}$.
  - Multivariate Ben-Or/Tiwari for sparse polynomials. [Go '06]
    - Number of evaluations $\sim 2N_{\text{monomials}}$.
  - First Thiele, then Zippel and/or Ben-Or/Tiwari for multivariate rationals (the FIREFLY library). [Klappert, Lange '19; Klappert, Klein, Lange '20]

Performance of IBP via modular reconstruction

Total time for modular *reconstruction* of the IBP system solution:

$$\frac{1}{\text{performance}} \underbrace{N_{\text{families}} \; \underbrace{\cancel{N^2_{\text{integrals per family}}}}_{\to N_{\text{amplitudes}} \; N_{\text{masters}}}}_{\text{expressions to reconstruct}} \; N^2_{\text{monomials per term}} \; N_{\text{digits per monomial}}$$

- To minimize the reconstruction time, only reduce expressions for whole amplitudes, not individual integrals.

Total time for *evaluation* of the IBP system solution:

$$\overbrace{\frac{1}{\text{performance}} \; N_{\text{families}} \; N^2_{\text{integrals per family}}}^{T_{\text{single evaluation}}} \; \overbrace{\underbrace{N^{\cancel{2}}_{\text{monomials per term}}}_{\text{Best case; depends on the algorithm.}} \; N_{\text{digits per monomial}}}^{N_{\text{evaluations}}}$$

- The evaluation can be naturally parallelized.

Available software

IBP solvers not using modular arithmetic:

- LiteRed (useful Mathematica functions, required by Fire). [Lee '13]
- Forcer (for massless 2-point functions). [Ruijl, Ueda, Vermaseren '17]

Status unknown:

- Crusher (a private implementation). [Marquard, Seidel]

IBP solvers that use modular arithmetic:

- FinRed (a private implementation). [von Manteuffel et al]
- Fire6. [Smirnov, Chuharev '19]
  - Does not provide multivariate reconstruction.
- Kira when used with FireFly. [Klappert, Lange, Maierhöfer, Usovitsch '20; Klappert, Klein, Lange '20]
- FiniteFlow (a library for arbitrary computations). [Peraro '19]
- Caravel (a library for amplitude computations). [Cordero, Sotnikov et al '20]

... and others.

Now also introducing: Ratracer (with Kira and FireFly). [V.M. '22]

Improving IBP reduction time

Strategies to improve the reduction time:

1. Use modular arithmetic & rational function reconstruction methods.
2. *Make the result smaller*:
   - 2.1 Reduce whole amplitudes (not individual integrals).
   - 2.2 Choose master integrals that minimize the result size.
     - ⋆ Use *d-factorizing bases* that ensure the factorization of d in the denominators of IBP coefficients. [Usovitsch '20; Smirnov, Smirnov '20]
     - ⋆ Consider *quasi-finite bases*. [von Manteuffel, Panzer, Schabinger '14]
     - ⋆ Consider *uniform transcendentality bases*, if possible. [Bendle et al '19]
   - 2.3 Construct a smaller ansatz for the result. [Abreu et al '19; De Laurentis, Page '22]
   - 2.4 Set some of the variables to fixed numbers.
     - ⋆ E.g. reduce with m_H^2/m_t^2 set to $12/23$.
     - ⋆ Or perform IBP reduction separately for each phase-space point, and interpolate in between. [Jones, Kerner et al '18; Chen, Heinrich et al '19, '20]
3. Improve the *evaluation performance*:
   - 3.1 Combine IBP relations (using syzygies) to eliminate integrals with raised (or lowered) indices. [Gluza, Kajda, Kosower '10; Scahbinger '11]
   - 3.2 *Just solve the equations faster?*

Optimizing the modular Gaussian elimination

When performing Gaussian elimination one needs to:

- Represent the equations as a sparse matrix data structure.
  - Keep the equations sorted.
  - Keep terms in each equation sorted.
  - Adjust the layout (and maybe reallocate memory) after each operation.
    - This is not much work, but so is modular arithmetic!

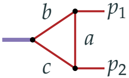
IBP solvers using modular arithmetics will:

- Recreate the same data structures, same memory allocations, in the same order during each evaluation, many times.
  - Only the modular values change between evaluations.
- Spend relatively little time on actual modular arithmetic.
  - Because it is so fast!

How to speed this up? Eliminate the data structure overhead:

- *Record the list of arithmetic operations* performed during the first evaluation (“*a trace*”).
- Simply *replay this list* for subsequent evaluations.

Rational traces

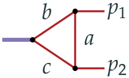
For $I_{a,b,c}(s,d) \equiv$ 

the *trace* of the IBP solution might look like:

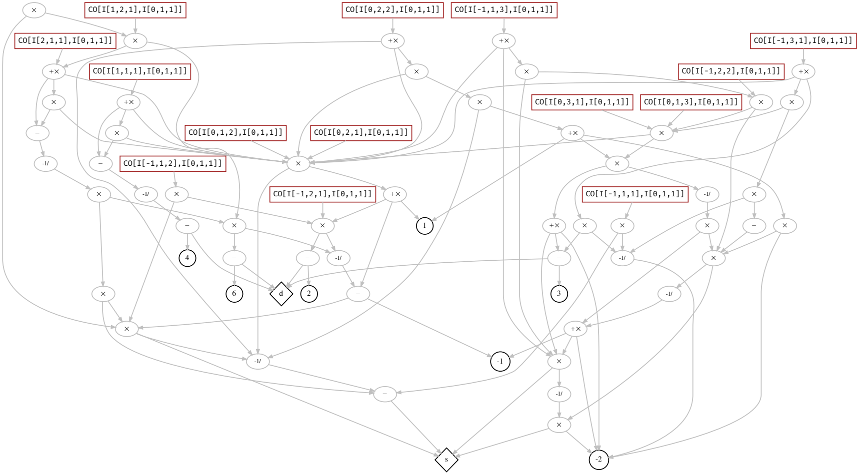
```
t0 = var 'd'
t1 = int 4
t2 = sub t0 t1
t3 = int 1
t4 = var 's'
t5 = neg t4
t6 = int 6
t7 = sub t0 t6
t8 = int -1
t9 = int 2
t10 = int -2
t11 = sub t0 t9
t12 = int 3
t13 = sub t0 t12
t14 = mul t4 t10
t15 = neginv t5
t16 = mul t4 t15
t17 = sub t8 t16
t18 = mul t5 t16
t19 = neginv t17
t20 = mul t7 t19
[...]
```

```
t54 = addmul t53 t27 t44
t55 = mul t25 t44
t56 = addmul t55 t25 t44
t57 = mul t23 t44
t58 = addmul t57 t23 t44
t59 = mul t20 t58
t60 = mul t16 t59
save t60 as C0[I[1,1,2],I[0,1,1]]
save t59 as C0[I[1,2,1],I[0,1,1]]
save t58 as C0[I[2,1,1],I[0,1,1]]
save t56 as C0[I[1,1,1],I[0,1,1]]
save t54 as C0[I[-1,1,3],I[0,1,1]
save t52 as C0[I[-1,2,2],I[0,1,1]
save t51 as C0[I[-1,3,1],I[0,1,1]
save t46 as C0[I[0,1,3],I[0,1,1]]
save t49 as C0[I[0,2,2],I[0,1,1]]
save t47 as C0[I[-1,1,2],I[0,1,1]
save t46 as C0[I[0,3,1],I[0,1,1]]
save t42 as C0[I[-1,2,1],I[0,1,1]
save t44 as C0[I[0,1,2],I[0,1,1]]
save t44 as C0[I[0,2,1],I[0,1,1]]
save t45 as C0[I[-1,1,1],I[0,1,1]
```

Rational traces

For $I_{a,b,c}(s,d) \equiv$ 

the *trace* of the IBP solution might look like:



RATRACER overview

RATRACER ("Rational Tracer"): a program for *solving systems of linear equations* using modular arithmetic based on rational traces. [V.M. '22]

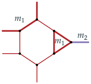
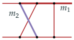
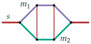
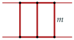
- Can trace the solution of arbitrary systems of linear equations: IBP relations, dimensional recurrence relations, amplitude definitions, etc.
- Can trace arbitrary rational expressions.
- Can optimize and transform traces.
- Uses FIREFLY for reconstruction. [Klappert, Klein, Lange '20, '19]
- Initially created for solving IBPs for massive 5-point 2-loop diagrams.
- Available at github.com/magv/ratracer.

Intended usage:

1. Use KIRA (or LITERED, or custom code) to export IBP relations.
2. Use RATRACER to load them and solve them.

RATRACER benchmarks

For IBP reduction of every integral (i.e. not single amplitudes):

	Evaluation speedup vs. KIRA+FIREFLY	$\frac{t_{\text{reconstruction}}}{t_{\text{evaluation}}}$	Total speedup vs. KIRA+FIREFLY	Total speedup vs. KIRA+FERMAT	Total speedup vs. FIRE6
	20	3.3	5.2	1.2	∞?
	7.8	1/3.3	6.0	37	∞?
	26	25	1.7	1/3.3	1.8
	9.6	8.8	5.2	2.6	8.8

[github.com/magv/ibp-benchmark]

Resulting performance:

- Consistent ~10x speedup in modular evaluation over KIRA+FIREFLY.
- Up to *~5x speedup in total reduction time* over KIRA+FIREFLY for complicated examples, 1x-30x over KIRA+FERMAT, ∞x over FIRE6.

Trace optimizations

Given a trace, RATRACER can optimize it using:

- *Constant propagation:*

$$\begin{cases}\texttt{t11 = int 2}\\ \texttt{t12 = int 3}\\ \texttt{t13 = mul t11 t12}\end{cases} \Rightarrow \begin{cases}\texttt{t11 = int 2}\\ \texttt{t12 = int 3}\\ \texttt{t13 = int 6}\end{cases}$$

- *Trivial operation simplification:*

$$\begin{cases}\texttt{t11 = int -1}\\ \texttt{t12 = mul t11 t7}\end{cases} \Rightarrow \begin{cases}\texttt{t11 = int -1}\\ \texttt{t12 = neg t7}\end{cases}$$

- *Common subexpression elimination:*

$$\begin{cases}\texttt{t11 = add t5 t7}\\ \texttt{t12 = add t5 t7}\end{cases} \Rightarrow \begin{cases}\texttt{t11 = add t5 t7}\\ \texttt{t12 = t11}\end{cases}$$

- *Dead code elimination:*

$$\begin{cases}\texttt{t11 = add t5 t7}\\ \texttt{[..., t11 is unused]}\end{cases} \Rightarrow \begin{cases}\texttt{nop}\\ \texttt{[...]}\end{cases}$$

  - Especially useful if a user wants to select a subset of the outputs.
- *"Finalization":*

$$\begin{cases}\texttt{t11 = add t5 t6}\\ \texttt{t12 = add t11 t7}\\ \texttt{[..., t11 is unused]}\end{cases} \Rightarrow \begin{cases}\texttt{t11 = add t5 t6}\\ \texttt{t11 = add t11 t7}\\ \texttt{[...]}\end{cases}$$

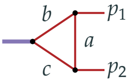
  - Needed to minimize the temporary memory needed for the evaluation.

Trace transformations

Given a trace, RATRACER can:

- *Set some of the variables to expressions* or numbers.
  - E.g. set `mh2` to "`12/23*mt2`", `d` to "`4-2*eps`", or `s` to "`13600`".
  - No need to remake the IBP system just to set a variable to a number.
- *Select any subset of the outputs*, and drop operations that don't contribute to them (via dead code elimination).
  - Can be used to split the trace into parts.
    - Each part can be reconstructed separately (e.g. on a different machine).
  - See master-wise and sector-wise reduction in other solvers.
- *Expand the result into a series* in any variable.
  - By evaluating the trace while treating each value as a series, and saving the trace of that evaluation.
  - Done before the reconstruction, so one less variable to reconstruct in, but potentially more expressions (depending on the truncation order).
  - In practice only few leading orders in ε are needed, so expand in ε up to e.g. $\mathscr{O}\left(\varepsilon^0\right)$, and *don't waste time on reconstructing the higher orders*.

Truncated series expansion

For $I_{a,b,c}(s,d) \equiv$ 

before expansion:

- Variables to reconstruct in: s and d.
- Trace outputs: "`CO[I[1,1,1],I[0,1,1]]`", etc:

$$I_{1,1,1} = \texttt{CO[I[1,1,1],I[0,1,1]]}\, I_{0,1,1}.$$

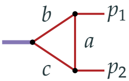
After expansion in ε to $\mathscr{O}\left(\varepsilon^0\right)$:

- Variables to reconstruct in: only s.
- Trace outputs: "`ORDER[CO[I[1,1,1],I[0,1,1],eps^-1]`", etc:

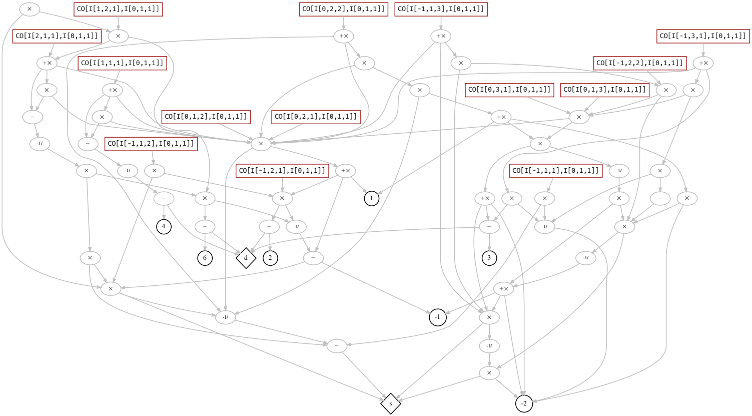
$$\begin{aligned} I_{1,1,1} &= \texttt{ORDER[CO[I[1,1,1],I[0,1,1],eps\^{}-1]}\, \varepsilon^{-1}\, I_{0,1,1} \\ &+ \texttt{ORDER[CO[I[1,1,1],I[0,1,1],eps\^{}0]}\, \varepsilon^{0}\, I_{0,1,1}. \end{aligned}$$

- Might be slower to evaluate, but fewer evaluations are needed.
  - $\Longrightarrow$ The more complicated the problem, the higher the speedup.

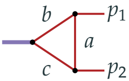
Truncated series expansion: the graph



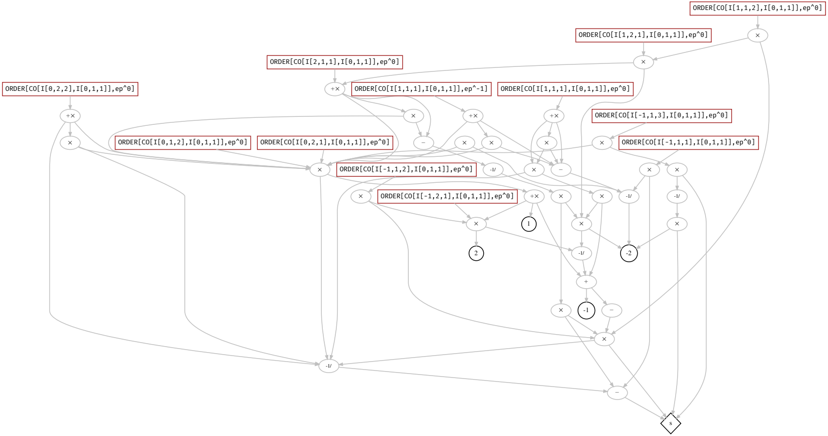
Expansion:	–	$\mathscr{O}\left(\varepsilon^0\right)$	$\mathscr{O}\left(\varepsilon^1\right)$	$\mathscr{O}\left(\varepsilon^2\right)$
Operations:	61	40	73	87
Outputs:	15	13	26	33



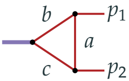
Truncated series expansion: the graph



Expansion:	–	$\mathcal{O}(\varepsilon^0)$	$\mathcal{O}(\varepsilon^1)$	$\mathcal{O}(\varepsilon^2)$
Operations:	61	40	73	87
Outputs:	15	13	26	33



Truncated series expansion: the graph



Expansion:	–	$\mathcal{O}(\varepsilon^0)$	$\mathcal{O}(\varepsilon^1)$	$\mathcal{O}(\varepsilon^2)$
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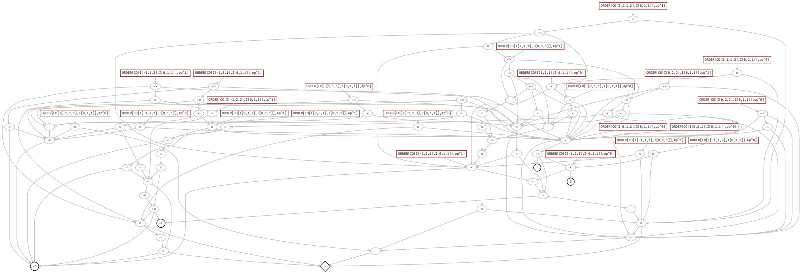
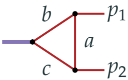
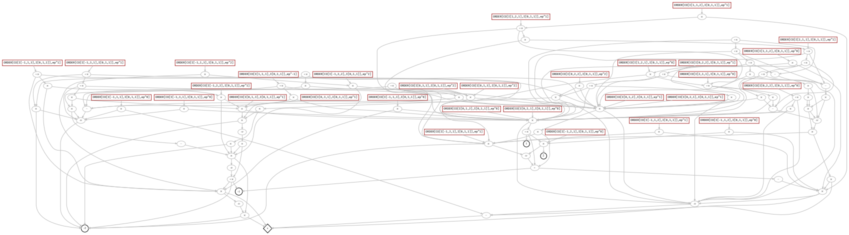


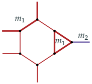
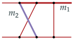
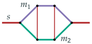
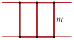
Truncated series expansion: the graph



Expansion:	–	$\mathcal{O}\left(\varepsilon^0\right)$	$\mathcal{O}\left(\varepsilon^1\right)$	$\mathcal{O}\left(\varepsilon^2\right)$
Operations:	61	40	73	87
Outputs:	15	13	26	33



RATRACER + series expansion benchmarks

	probe time speedup			total time speedup		
	$\mathcal{O}(\varepsilon^0)$	$\mathcal{O}(\varepsilon^1)$	$\mathcal{O}(\varepsilon^2)$	$\mathcal{O}(\varepsilon^0)$	$\mathcal{O}(\varepsilon^1)$	$\mathcal{O}(\varepsilon^2)$
	1/1.3	1/1.5	1/1.8	3.2	2.4	1.9
	1/2.0	1/2.5	1/3.0	2.7	1.4	1/1.3
	1/1.4	1/2.4	1/2.9	2.3	1.7	1.4
	1/1.0	1/1.6	1/2.1	4.3	2.3	1.6

[github.com/magv/ibp-benchmark]

Resulting performance:

- A *~3x speedup* with ε expansion up to $\mathcal{O}\left(\varepsilon^0\right)$.
- The higher the expansion, the less the benefit.

Guessing the denominators

The *denominators of IBP coefficients factorize* into few unique factors.

If some candidate factors are known, then we can find the powers of those factors in each coefficient: [Abreu et al '18; Heller, von Manteuffel '21]

1. Choose a factor to search for, e.g. $(d-6)$.
2. Set all variables to random values, e.g. $d = 95988281$, $s = 75579811$.
   $\Rightarrow$ $(d-6) = 95988275 = 5^2 \cdot 103 \cdot 37277$.
3. Evaluate the IBP solution using these numbers.
   * E.g. $\mathtt{CO}\left[I_{2,1,1}, I_{0,1,1}\right] = \frac{383953112}{548314574947073136171275} = \frac{2^3 \cdot 1117 \cdot 42967}{5^2 \cdot 103 \cdot 37277 \cdot 75579811^2}$.
4. Find common prime factors, identify their powers.
   * $\mathtt{CO}\left[I_{2,1,1}, I_{0,1,1}\right] \sim (d-6)^{-1}\, s^{-2}$.

Automated implementation: `toos/guessfactors` from Ratracer.

To find the set of possible factors:

* Reconstruct a simpler subset of the coefficients. (A few per sector).

$\Rightarrow$ Easy with Ratracer, just select individual outputs.

Once the factors are found, *speedup the reconstruction* by dividing them out from the expressions.

* I.e. reconstruct $\mathtt{CO}\left[I_{2,1,1}, I_{0,1,1}\right] / (d-6) / s^2$, not just $\mathtt{CO}\left[I_{2,1,1}, I_{0,1,1}\right]$.

Usage for IBP reduction

1. Use KIRA to generate the IBP equations.

```
$ cat >config/integralfamilies.yaml <<EOF
integralfamilies:
  - name: "I"
    loop_momenta: [l]
    top_level_sectors: [b111]
    propagators:
      - ["l", 0]
      - ["l-p1", 0]
      - ["l+p2", 0]
EOF
$ cat >config/kinematics.yaml <<EOF
kinematics:
 outgoing_momenta: [p1, p2]
 kinematic_invariants: [[s, 2]]
 scalarproduct_rules:
  - [[p1,p1], 0]
  - [[p2,p2], 0]
  - [[p1,p2], "s/2"]
# symbol_to_replace_by_one: s
EOF
```

```
$ cat >export-equations.yaml <<EOF
jobs:
 - reduce_sectors:
    reduce:
     - {sectors: [b111], r: 4, s: 1}
    select_integrals:
     select_mandatory_recursively:
      - {sectors: [b111], r: 4, s: 1}
    run_symmetries: true
    run_initiate: input
EOF
$ kira export-equations.yaml
```

2. Use RATRACER to create a trace with the solution.

```
$ ratracer \
    load-equations input_kira/I/SYSTEM_I_0000000007.kira.gz \
    load-equations input_kira/I/SYSTEM_I_0000000006.kira.gz \
    solve-equations choose-equation-outputs --maxr=4 --maxs=1 \
    optimize finalize save-trace I.trace.gz
```

3. Optionally expand the outputs into a series in ε.

```
$ ratracer \
    set d '4-2*eps' load-trace I.trace.gz \
    to-series eps 0 \
    optimize finalize save-trace I.eps0.trace.gz
```

4. Use RATRACER (+FIREFLY) to reconstruct the solution.

```
$ ratracer \
    load-trace I.eps0.trace.gz \
    reconstruct --to=I.solution.txt --threads=8 --inmem
```

Usage as a library

RATRACER is built to support custom user-defined traces.

Any rational algorithm can be turned into a trace (via the C++ API).

Usage:

```
#include <ratracer.h>
int main() {
    Tracer tr = tracer_init();

    Value x = tr.var(tr.input("x"));
    Value y = tr.var(tr.input("y"));

    Value x_sqr =
        tr.pow(x, 2);
    Value expr =
        tr.add(x_sqr, tr.mulint(y, 3));

    /* expr = x^2 + 3y */
    tr.add_output(expr, "expr");

    tr.save("example.trace.gz");
    return 0;
}
```

API:

```
struct Value { uint64_t id; uint64_t val; };
struct Tracer {
    Value var(size_t idx);
    Value of_int(int64_t x);
    Value of_fmpz(const fmpz_t x);
    bool is_zero(const Value &a);
    bool is_minus1(const Value &a);
    Value mul(const Value &a, const Value &b);
    Value mulint(const Value &a, int64_t b);
    Value add(const Value &a, const Value &b);
    Value addint(const Value &a, int64_t b);
    Value sub(const Value &a, const Value &b);
    Value addmul(const Value &a,
                 const Value &b1,
                 const Value &b2);
    Value inv(const Value &a);
    Value neginv(const Value &a);
    Value neg(const Value &a);
    Value pow(const Value &base, long exp);
    Value div(const Value &a, const Value &b);
    void assert_int(const Value &a, int64_t n);
    void add_output(const Value &src, const char *name);
    size_t input(const char *name, size_t len);
    size_t input(const char *name);
    int save(const char *path);
    void clear();
};
Tracer tracer_init();
```

Problems with big traces

Problem:

- The size of a trace is proportional to the number of operations.
  - I.e. $\sim N^2_{\text{integrals}}$ for sparse IBP systems.
  - $\Rightarrow$ Megabytes to gigabytes for 2-loop 5-point massive problems.
- Computer memory is expensive and limited.

Solution:

1. Always keep the trace on disk, never load it fully into memory.
   - Compress it on disk for storage (via Zstd, Gzip, Bzip2, or Lzma).
2. During the evaluation read the trace piece by piece.
3. During the optimization make sure the algorithms have bounded memory usage.

$\Rightarrow$ Multi-GB traces are supported easily in Ratracer.

$\Rightarrow$ Multi-TB traces—less so.

RATRACER future plans

For very large examples:

- The *main memory speed becomes the bottleneck* for the modular evaluation:
  - ⟹ Investigate optimizing traces to improve memory access patterns.
- The initial trace contrsuction and optimization is slow:
  - ⟹ Work more on benchmarking & optimization of the trace construction.

For other examples RATRACER speeds up the evaluation enough that the *modular reconstruction in FIREFLY becomes the bottleneck*:

- Reduce the overhead in FIREFLY to speed up simpler examples.
- Improve paralelizability in FIREFLY to help with complicated examples.

⟹ Ongoing collaboration with FIREFLY authors.

Beyond guessing the denominators:

- Investigate smaller ansätze for the results (via e.g. partial fractioning).

[Abreu et al '19; De Laurentis, Page '22]

Summary

Solving IBP equation systems faster:

- Use modular arithmetic & rational reconstruction methods.
- Spend time on choosing better master basis.
- IBP-reduce amplitudes, not individual integrals.
- Expand the coefficients into a series in ε up to the needed order.
- Guess the denominators of the coefficients.

RATRACER provides:

- Practical & fast modular reconstruction of solutions to linear systems.
  - Reconstruction of arbitrary rational expressions too.
- Trace optimization, transformation, slicing and dicing.
- Coefficient expansion in ε (and not only).
- Denominator guessing.
- Source code & bug tracker at github.com/magv/ratracer.
  - Benchmark code & results at github.com/magv/ibp-benchmark.
- TODO: faster evaluation, faster reconstruction, more tricks.