A tale of tails via generalized unitarity

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with

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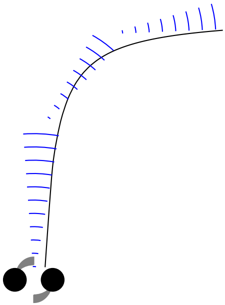
Road map

What are tails?



Scattering of gravitational radiation from a system off of its *own potential*

- Modifies emitted energy and binding energy
- Time asymmetric process: requires knowledge/assumptions about *entire history* of system
- Scale-mixing phenomenon: interplay between radiation-mode and potential-mode gravitons

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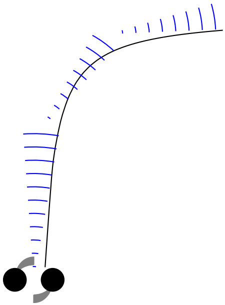

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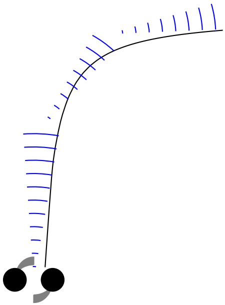

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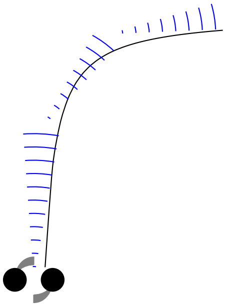

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Known methods for calculating tails

Traditional GR

- First principles
- Orbit agnostic
- Laborious iterations
- Coordinate dependent
- Regularization difficulties

Self-force

- Deep perturbative reach
- Strong field and relativistic
- Near-circular orbits
- Probe mass limit

Effective field theory

- Separation of scales
- Momentum space regularization
- Feynman diagrams
- Proliferation of Feynman diagrams
- Gauge dependence/fixing, BFG

Goal: Bring together modern Amplitudes methods with EFT insights for a method that is manifestly gauge invariant, minimally scaling, nearly trivially iterative.

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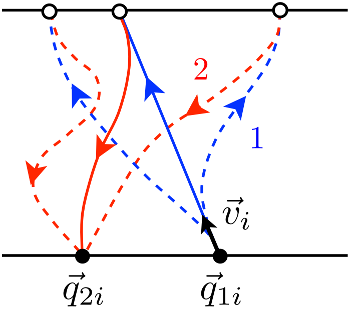
Dissipative effects via Closed Time Path

Galley et al. 1210.2745 1412.3082

Closed Time Path: action/variation principles for systematically handling non-conservative systems

1. Double all degrees of freedom $(-,+)$, "causal" $(- \to +)$ and "anti-causal" $(+ \to -)$ branches
2. "Integrate out" inaccessible DoF
3. Conservative piece, L: $(-,+)$ symmetric; Non-conservative, K: asymmetric
4. Calculus of variations with "$-$" variables

Image from Galley 1210.2745



NB: Enters momentum-space calculations by changing $i0$ prescription, analytic continuations

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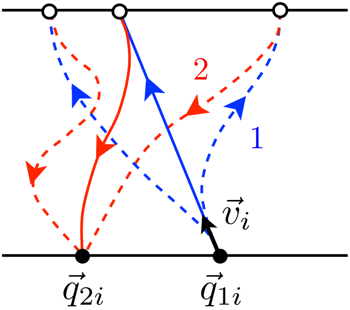

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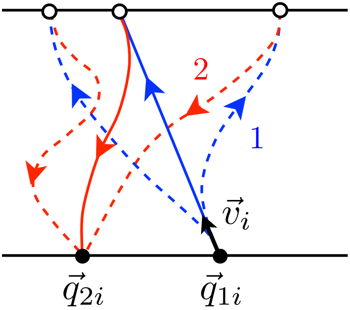

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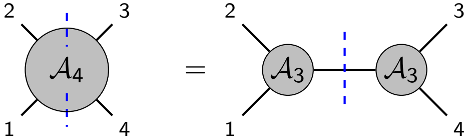
Tree amplitudes

Key Properties:

- Gauge-invariant description of particle interactions
  Contains all physical information required to extract observables
- Manifest locality through *factorization*

$$\operatorname*{Res}_{s_{12}} \mathcal{A}(1,2,3,4) = \sum_{\text{states of } i} \mathcal{A}(1,2,i)\mathcal{A}(i,3,4)$$

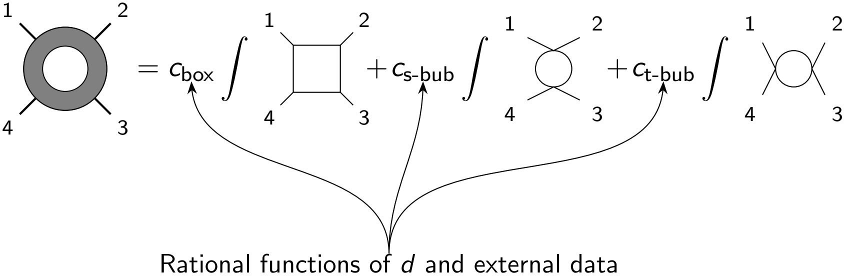


- Many methods for construction: Feynman rules, recursion relations, Cachazo-He-Yuan formula, color-kinematics duality

Unitarity: loops from trees

Two imporant facts:

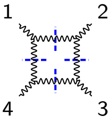
- Feynman integrals (including numerators) can be reduced to a basis of *scalar* integrals using integration-by-parts relations (IBPs)

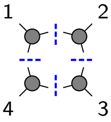


Unitarity: loops from trees

- Basis coefficients c_X can be determined by matching *generalized unitarity cuts*, constructed via repeated application of the QFT optical theorem

$$\mathrm{Cut}_G = \sum_{\substack{\text{states}\\ \text{of } E(G)}} \prod_{v \in V(G)} A_{\text{tree}}(v)$$

$\mathrm{Cut}_{\text{box}} =$ 

$\equiv \sum_{\text{states}}$ 

$\xrightarrow{IBPs} c_{\text{box}}$

$$\sum_{\text{states}} \varepsilon_k^{\mu\nu} \varepsilon_k^{\alpha\beta} \equiv \mathcal{P}_k^{\mu\nu;\alpha\beta} = \frac{1}{2}\left(P_k^{\mu\alpha} P_k^{\nu\beta} + P_k^{\mu\beta} P_k^{\nu\alpha} - \frac{1}{D-2} P_k^{\mu\nu} P_k^{\alpha\beta}\right)$$

$$P_k^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu q^\nu + k^\nu q^\mu}{k \cdot q}$$

Gauge invariance $\Leftrightarrow$ Cut is independent of q

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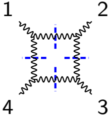

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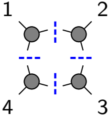

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Benefits of unitarity-based methods

- Don't have to think about gauge choices – turn gauge invariant input (tree amplitudes) into gauge invariant output (observables)
- Internal consistency checks through factorization – cuts have overlapping information
- Find hidden patterns and structures – decoupling integration makes it easier to see iteration and recursion relations

Road map

Unitarity for tails

1. Identify good basis of momentum invariants: propagators $(\ell^2 - \omega^2)$ and things that look like propagators
2. Determine relevant basis integrals via IBP (e.g. FIRE6, Kira)
3. Construct cuts for each basis integral
4. Reduce cuts to basis

Departures from "standard" unitarity:

- Only the frequency ω is taken as external data $\Rightarrow$ integration over remaining *Euclidean* spatial momenta
- Potential-mode gravitons are space-like
- Macroscopic masses suggest large mass expansions

Advantages over "standard" EFT:

- No mode-mixing or background-field gauge problems
- Complete control over contributing diagrams

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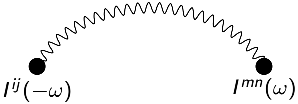
Radiation reaction: Preliminaries



Coupling of $SO(3)$ tensor to a graviton:

$$\begin{aligned}\mathcal{M}_{Qg} &\equiv \lambda_Q J^{\mu\nu}\varepsilon_{\mu\nu} \equiv \lambda_Q J^{\mu\nu}\varepsilon_\mu\varepsilon_\nu \\ &= -\lambda_Q I^{ij} \times \left(k_0 k_i \varepsilon_0 \varepsilon_j + k_0 k_j \varepsilon_0 \varepsilon_i - k_i k_j \varepsilon_0 \varepsilon_0 - k_0 k_0 \varepsilon_i \varepsilon_j\right) , \\ \lambda_Q &= \sqrt{2\pi G_N}\end{aligned}$$

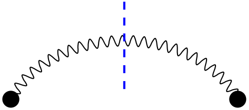
Basis integral:

$$F^{(1)}(1;\omega^2) = \int \frac{\mathrm{d}^d \ell_E}{(2\pi)^d} \frac{1}{(-\ell_E^2 + \omega^2)} = -\frac{\Gamma(1-d/2)(-\omega^2)^{d/2-1}}{(4\pi)^{d/2}}$$

Effective Action:

$$S_{\mathrm{RR}} = \int \frac{\mathrm{d}\omega}{2\pi} c_{\mathrm{RR}} F^{(1)}(1;\omega^2)$$

Radiation reaction: Unitarity



c_{RR} determined via unitarity cut: $c_{\mathrm{RR}} \sim \frac{\mathrm{Cut}_{\mathrm{RR}}}{S_{\mathrm{RR}}}$

$$\mathrm{Cut}_{\mathrm{RR}} = \lambda_Q^2 J_1^{\mu\nu}\ \mathcal{P}^{\mu\nu;\alpha\beta} J_2^{\alpha\beta}\delta(\ell_E^2-\omega^2) = \delta(\ell_E^2-\omega^2)\lambda_Q^2\left(J_1^{\mu\nu}J_2^{\mu\nu} - \frac{J_1^{\mu\mu}J_2^{\nu\nu}}{d-1}\right)$$

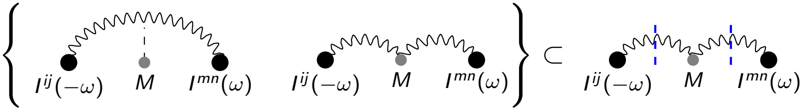
$$= \delta(P_\ell)\lambda_Q^2\frac{(d+1)(d-2)}{(d+2)(d-1)}\omega^4\underbrace{\kappa_{ab}(\omega)}_{I_a^{ij}(-\omega)I_{ij,b}(\omega)} \quad \Rightarrow c_{\mathrm{RR}} = \frac{2\pi G_N}{5}\omega^4\kappa_{ab}(\omega)$$

Effective action (after CTP sum):

$$S_{\mathrm{RR}} = -i\frac{G_N}{5}\int\frac{\mathrm{d}\omega}{2\pi}\,\omega^5\kappa_{-+}(\omega)$$

Well known result (in time domain)

Leading tail via unitarity



New building blocks: graviton-potential coupling
Want a *covariant* and *gauge invariant* form of $V \sim Eh_{00}$
Scalar-graviton interaction is natural choice, $m_s^2 = p_s^2$

$$\mathcal{M}_{sgs} \equiv \frac{\lambda_E}{m_s^2} p_s^\mu p_s^\nu \varepsilon^{\mu\nu} \xrightarrow[\text{rest frame}]{\text{Scalar}} \lambda_E \varepsilon^{00}$$

Easy generalization beyond three-point (multi-graviton contact)

$$\mathcal{M}_{sggs}(m_s \to \infty) \equiv \frac{\lambda_g \lambda_E}{\omega_{k_2}^2} \frac{1}{2(k_2 k_3)} \Big[(k_2 k_3)\varepsilon_2^0 \varepsilon_3^0 + \omega_{k_2}((\varepsilon_3 k_2)\varepsilon_2^0 - (\varepsilon_2 k_3)\varepsilon_3^0) - \omega_{k_2}^2 (\varepsilon_2 \varepsilon_3) \Big]^2 + \mathcal{O}(m_s^{-1})$$

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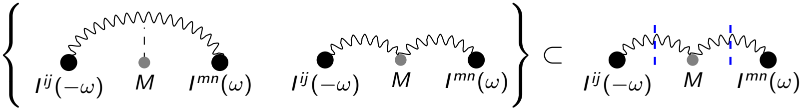

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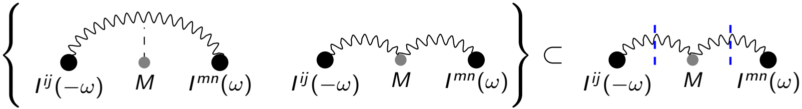

Basis integral:

$$F^{(2)}(1,1,0) \equiv \int \frac{\mathrm{d}^d\ell_1\,\mathrm{d}^d\ell_2}{(2\pi)^{2d}} \frac{1}{(-\ell_1^2+\omega^2)(-\ell_2^2+\omega^2)} = F^{(1)}(1;\omega^2)^2$$

Cut that we need to calculate

$$\begin{aligned}\mathrm{Cut}_{\mathrm{tail}} &= \sum_{\text{states}} \mathcal{M}_{Qg(-\omega)}\mathcal{M}_{sggs}\mathcal{M}_{Qg(\omega)}\Big|_{\substack{P_{\ell_1}=0,P_{\ell_2}=0\\ m_s\to\infty}} \\ &= \lambda_Q^2\,\delta(P_{\ell_1})\delta(P_{\ell_2}) J^{\mu\nu}_{I(-\omega)} P^{\mu\nu;\alpha\beta}\mathcal{M}_{sggs}^{\alpha\beta;\gamma\sigma} P^{\gamma\sigma;\rho\tau} J^{\rho\tau}_{I(\omega)}\Big|_{m_s\to\infty}\end{aligned}$$

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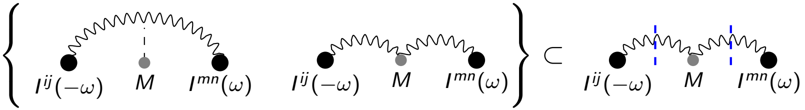

After integral reduction, CTP sum, and DimReg ($d \to 3 + \epsilon_d$)

$$S_{\mathrm{T}} = \frac{2}{5} G_N^2 E \int \frac{\mathrm{d}\omega}{2\pi}\, \omega^6 \kappa_{-+}(\omega) \left[\frac{1}{\epsilon_d} + \log\left(\frac{\omega^2}{\mu^2}\right) - i\pi\,\mathrm{sgn}(\omega) \right],$$

Goldberger & Ross 2009, Galley et.al. 2016

But what about the patterns I alluded to?

$$S_{\mathrm{T}} = \int \frac{\mathrm{d}\omega}{2\pi} \frac{(2\pi G_N)(d+1)(d-2)}{(d+2)(d-1)} \omega^4 \kappa_{ab}(\omega) \longleftarrow \text{Repeated from RR}$$

$$\text{New in T} \longrightarrow \left(-(16\pi G_N E) \frac{12 - 2d + 5d^2 - 4d^3 + d^4}{2(d-3)(d-1)d(d+1)} \right) F^{(1)}(1; \omega_{ab}^2)^2$$

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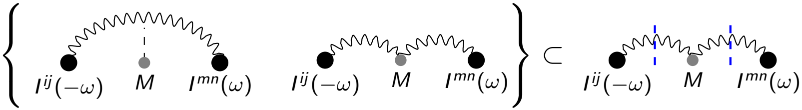

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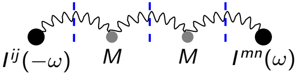
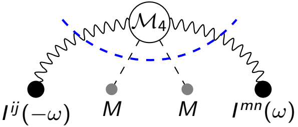
Tail-of-tail





vs $\sim$ 5 Feynman diagrams

Related to Goldberger & Ross 2009

$$S_{\mathrm{TT}} = \frac{107}{175}\, G_N^3 E^2 \int \frac{\mathrm{d}\omega}{2\pi}\, \omega^7 \kappa_{-+}(\omega) \left[\pi\, \mathrm{sgn}(\omega) + i \left[\frac{2}{3\,\epsilon_d} + \log\left(\frac{\omega^2}{\mu_1^2}\right)\right]\right]$$

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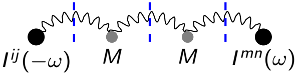

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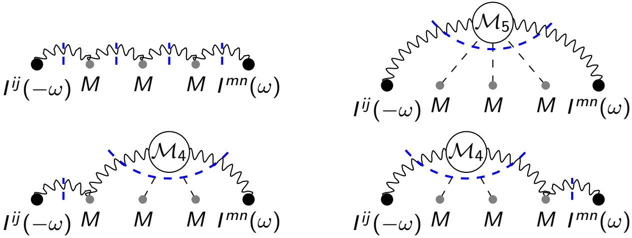
Tail-of-tail-of-tail

Never approached via EFT – GR via Blanchet 2017

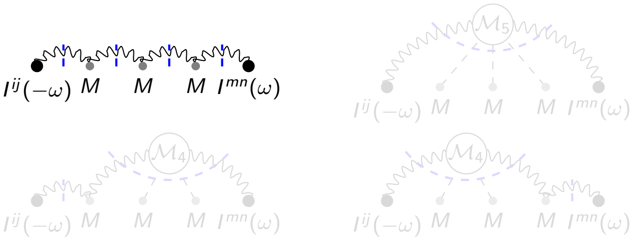


vs many Feynman diagrams

$$S_{\mathrm{TTT}} = -\frac{4}{525} G_N^4 E^3 \int \frac{\mathrm{d}\omega}{2\pi} \omega^8 \kappa_{-+}(\omega) \left[\frac{107}{2\epsilon_d^2} + \frac{107}{\epsilon_d} \log\left(\frac{\omega^2}{\mu_2^2}\right) + 107 \log^2\left(\frac{\omega^2}{\mu_2^2}\right) + \frac{20707426967}{60399360} - \frac{3103}{4}\zeta_2 - 420\,\zeta_3 - i\pi\,\mathrm{sgn}(\omega) \left[\frac{107}{\epsilon_d} + 214 \log\left(\frac{\omega^2}{\mu_2^2}\right) \right] \right],$$

Tail-of-tail-of-tail

Never approached via EFT – GR via Blanchet 2017



vs many Feynman diagrams

$$S^{\text{top-left}}_{\text{TTT}} = \int \frac{\mathrm{d}\omega}{2\pi} \frac{(2\pi G_N)(d+1)(d-2)}{(d+2)(d-1)} \omega^4 \kappa_{ab}(\omega)$$

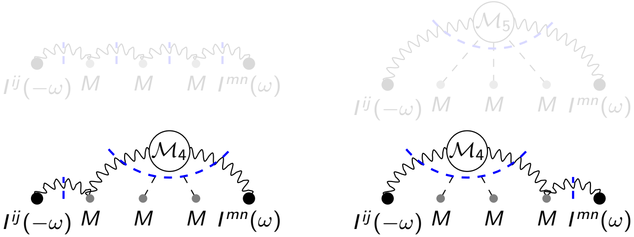
← Repeated from RR

Iteration of T! ⟶

$$\left(-(16\pi G_N E)\frac{12-2d+5d^2-4d^3+d^4}{2(d-3)(d-1)d(d+1)}\right)^3 F^{(1)}(1;\omega_{ab}^2)^4$$

Tail-of-tail-of-tail

Never approached via EFT – GR via Blanchet 2017



vs many Feynman diagrams

$$S^{\text{bottom}}_{\text{TTT}} = \int \frac{\mathrm{d}\omega}{2\pi} \text{Term from } \mathcal{M}_4$$

$$\times \left(-(16\pi G_N E) \frac{12 - 2d + 5d^2 - 4d^3 + d^4}{2(d-3)(d-1)d(d+1)} \right) F^{(1)}(1; \omega_{ab}^2)$$

Road map

Extracting energy spectra from CTP actions

CTP extension to Noether theorem:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\partial L}{\partial t} + \dot{q}^J \left[\frac{\partial K}{\partial q_-^J}\right]_{\mathsf{PL}} + \ddot{q}^J \left[\frac{\partial K}{\partial \dot{q}_-^J}\right]_{\mathsf{PL}} + \frac{\partial}{\partial \begin{smallmatrix}\text{higher}\\ \text{time}\\ \text{derivs}\end{smallmatrix}}$$

$$\int \frac{\mathrm{d}E}{\mathrm{d}t}\,\mathrm{d}t = \Delta E = \int \frac{\mathrm{d}E}{\mathrm{d}\omega}\,\mathrm{d}\omega = \int \mathrm{d}\omega\,\omega q^J(\omega) \left[\frac{\partial K}{\partial q_-^J(\omega)}\right]_{\mathsf{PL}}$$

Generic tail actions:

$$S_{\mathsf{T}^x} = \int \mathrm{d}\omega f(\omega) I_-^{ij}(-\omega) I_{ij,+}(\omega)$$

$$\Rightarrow \Delta E = \int \mathrm{d}\omega \kappa(\omega)\omega \,\mathrm{Im}\, f_{\text{odd}}(\omega)$$

Extracting energy spectra from CTP actions

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Raw energy spectra

$$(\Delta E)_{\text{RR}} = -\frac{G_N}{5\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\omega^6 \left\{1 - \frac{\epsilon_d}{2}\left[\frac{9}{10} - \log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right)\right] + \dots\right\}$$

$$(\Delta E)_{\text{T}} = -\frac{2}{5} G_N^2 E \int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\omega^7 \left\{1 + \epsilon_d\left[\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{41}{30}\right] + \dots\right\}$$

$$(\Delta E)_{\text{TT}} = \frac{214 G_N^3 E^2}{525\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\omega^8 \left\{\frac{1}{\epsilon_d} + \left[\frac{3}{2}\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{420\zeta_2}{107} - \frac{675359}{89880}\right]\right\}$$

$$(\Delta E)_{\text{TTT}} = \frac{428}{525} G_N^4 E^3 \int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega)\omega^9 \left\{\frac{1}{\epsilon_d} + \left[2\log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2\pi}\right) - \frac{252583}{29960}\right]\right\}$$

Need to deal with the DimReg divergences: introduce renormalized quadrupole coupling

$$\kappa(\omega) \to \kappa'(\omega) \equiv \kappa(\omega,\mu)\left(1 + \frac{214}{105}\omega^2\frac{(G_N E)^2}{\epsilon} + \dots\right)$$

The renormalized energy spectrum and RG flow

Wheeler-Throne Term

Blanchet & Damour Tail

$$(\Delta E)^{\text{inclusive}}_{\text{TTT}} = \int_{-\infty}^{\infty} \mathrm{d}\omega \kappa(\omega,\mu) \Bigg[-\frac{\omega^6 G_N}{5\pi} - \frac{2}{5}\omega^7 G_N^2 E$$

Goldberger & Ross 2009, Bini & Geralico 2021 (FT of Blanchet 1997)

$$+ \frac{1}{\pi} G_N^3 E^2 \omega^8 \left(\frac{214}{525} \log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2 \pi} \right) - \frac{634913}{220500} - \frac{8\zeta_2}{5} \right)$$

New for generic quadrupole!

$$+ G_N^4 E^3 \omega^9 \left(\frac{428}{525} \log\left(\frac{\omega^2 e^{\gamma_E}}{\mu^2 \pi} \right) - \frac{634913}{110250} \right) + \mathcal{O}(G_N^5) \Bigg]$$

Logs skip orders!

Renormalized coupling includes a scale dependence to balance log scaling

$$\frac{\mathrm{d}}{\mathrm{d}\mu}(\Delta E)^{\text{inclusive}}_{\text{TTT}} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\log\mu}\kappa(\omega,\mu) = -\frac{428}{105}(G_N \omega E)^2 \kappa(\omega,\mu) + \mathcal{O}(G_N^4)$$

The renormalized energy spectrum and RG flow

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Goldberger & Ross 2009,
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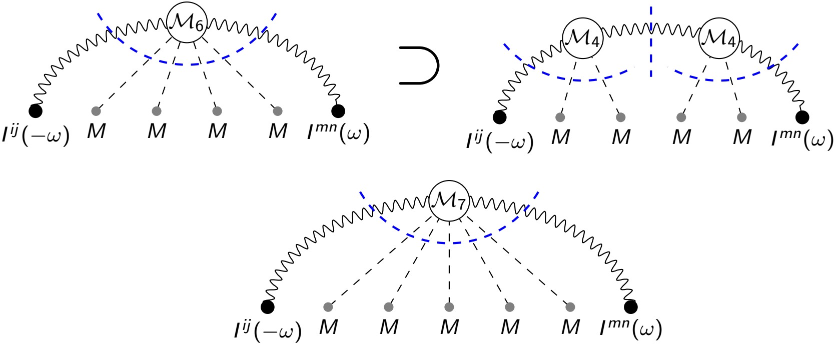
Road map

T^4 and T^5 in progress



Will contribute new counterterms and RG flow terms via subleading logs!

Conclusions

New theoretical approach to tails!

- Synthesis of insights from amplitudes and EFT
- Manifest gauge invariance, internal consistency checks
- Beginning to uncover hidden patterns and iterations
- Access to both conservative and non-conservative contributions via CTP
- Stay tuned for T^4 and T^5, with new RG flows

Thanks for your attention!