The Structure of RG Functions

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Based on work with J. Davies, F. Herren, and C. Poole

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Renormalization group flow

Callan–Symanzik equation for renormalized *n*-point functions:

$$\left(\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + n\gamma\right) G^{(n)}(\{p\}) = 0, \qquad \beta_I(g_I) = \frac{\mathrm{d}}{\mathrm{d}t} g_I \equiv \frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_I$$

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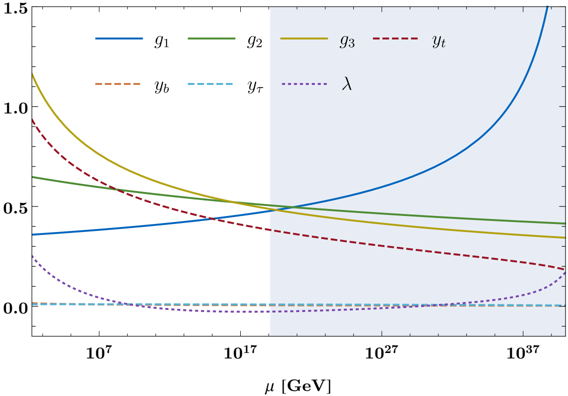
SM RG flow with 3rd generation Yukawa couplings:



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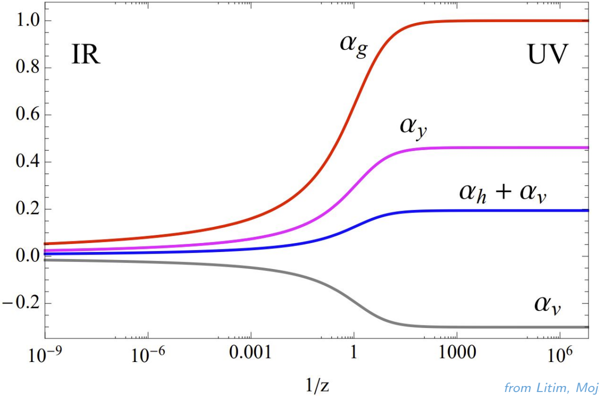
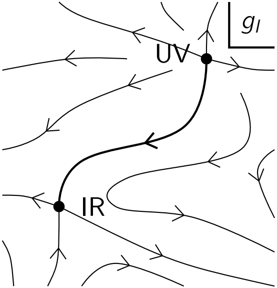
Perturbative asymptotic safety in the Litim–Sannino model:



from Litim, Mojaza, Sannino [1501.03061]

Fixed points

Fixed Points



FPs are CFTs:

$$T^\mu{}_\mu = \beta_I \mathcal{O}^I = 0$$

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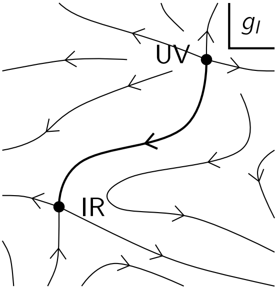

FPs are CFTs:

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The *A*-function of the Weyl anomaly imposes order on theory space:

- *The weak A-theorem*: Komargodski, Schwimmer [1107.3987]

$$A_{\text{UV}} - A_{\text{IR}} \geq 0$$

- *The strong A-theorem*:

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} A \geq 0$$

- *Gradient flow*: Osborn '89; Jack, Osborn '90

$$\partial_I A \equiv \frac{\partial A}{\partial g^I} = T_{IJ}\beta^J$$

Weyl Consistency Conditions

An underlying structure guiding the RG

Four-dimensional QFT

Generic renormalizable 4D theory (ignoring relevant couplings):

$$\begin{aligned}\mathcal{L} =& + \tfrac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a + i\psi_i^\dagger\bar{\sigma}^\mu(D_\mu\psi)^i + \mathcal{L}_{\mathrm{gh}} + \mathcal{L}_{\mathrm{gf}} \\ & - \tfrac{1}{4}a^{-1}_{AB}F^A_{\mu\nu}F^{B\mu\nu} - \tfrac{1}{2}\left(y_{aij}\psi^i\psi^j + \mathrm{H.c.}\right)\phi_a - \tfrac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d\end{aligned}$$

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Compactly, the action is

$$S = S_{\text{kin}}[\Phi] + \int \mathrm{d}^d x\, \big(\underset{\text{set of all marginal couplings}}{g_I}\mathcal{O}^I(x) + \overset{\text{all field sources}}{\mathcal{J}_\alpha}\Phi^\alpha\big)$$

The vacuum functional

$$e^{i\mathcal{W}[\mathcal{J}]} = \int [\mathcal{D}\Phi]\, e^{iS[\Phi,\,\mathcal{J}]}$$

generates all the connected n-point functions

There are no parameters, only sources

LRG probes the trace anomaly by introducing new sources:
Shore '87; Osborn 89'; Jack, Osborn '90; Osborn '91; Fortin, Grinstein, Stergiou [1208.3674]; Jack, Osborn [1312.0428]; Baume *et al.* [1401.5983]

$$[\underbrace{T}_{\text{stress-energy tensor}}{}^{\mu}{}_{\mu}] = \beta_I[\mathcal{O}^I] + \upsilon \cdot \partial_\mu[\underbrace{J^\mu_F}_{\text{flavor current; } J^\mu_F \in \mathfrak{g}_F}] \qquad \begin{cases} T_{\mu\nu}: & \eta_{\mu\nu} \to \gamma_{\mu\nu}(x) \\ \mathcal{O}^I: & g_I \to g_I(x) \\ J^\mu_F: & D_\mu \to D_\mu - a_\mu(x) \end{cases}$$

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Renormalization is source renormalization,

$$S = S_{\text{kin}}[\Phi,\, \gamma,\, a_0] + \int \mathrm{d}^d x \sqrt{\gamma} \left(g_{0,I} \mathcal{O}^I + \mathcal{J}_{0,\alpha} \Phi^\alpha \right) + S_{\text{ct}}[\gamma,\, g_0,\, a_0]$$

Weyl symmetry

The generator of Weyl symmetry—local scale invariance

$$\Delta^W_\sigma = \int \mathrm{d}^d x \left(2\overset{\text{infinitesimal parameter}}{\sigma} \gamma^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}} \right.$$

Weyl symmetry

The generator of Weyl symmetry—local scale invariance

$$\Delta_\sigma^W = \int \mathrm{d}^d x \left(2\overset{\text{infinitesimal parameter}}{\sigma} \gamma^{\mu\nu} \frac{\delta}{\delta \gamma^{\mu\nu}} - \sigma \underset{\beta\text{-function}}{\beta_I} \frac{\delta}{\delta g_I} \right.$$

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$$+ \big[\partial_\mu\sigma\,\upsilon - \sigma\,D_\mu g_I\,\rho^I\big]\cdot\frac{\delta}{\delta a_\mu}\bigg)$$

RG functions of the G_F current; $\upsilon, \rho^I \in \mathfrak{g}_F$

Weyl symmetry

The generator of Weyl symmetry—local scale invariance

infinitesimal parameter

field anomalous dimension

$$\Delta_\sigma^W = \int \mathrm{d}^d x \bigg(2\sigma \gamma^{\mu\nu} \frac{\delta}{\delta \gamma^{\mu\nu}} - \sigma \beta_I \frac{\delta}{\delta g_I} + \sigma \mathcal{J}_\beta \big[(d - \Delta_\alpha) \delta^\beta{}_\alpha - \gamma^\beta{}_\alpha \big] \frac{\delta}{\delta \mathcal{J}_\alpha}$$

β-function

$$+ \big[\partial_\mu \sigma\, \upsilon - \sigma\, D_\mu g_I\, \rho^I \big] \cdot \frac{\delta}{\delta a_\mu} \bigg)$$

RG functions of the G_F current; $\upsilon, \rho^I \in \mathfrak{g}_F$

The classic symmetry is anomalous ($\Delta_\sigma^W S_{\mathrm{ct}} \neq 0$)

$$\Delta_\sigma^W \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}_\sigma^W(\gamma,\, g,\, a)$$

Δ_σ^W contains the trace anomaly equation

$$[T^\mu{}_\mu] = \beta_I [\mathcal{O}^I] + \upsilon \cdot \partial_\mu [J_F^\mu] \qquad \text{(FSCC)}$$

Flat-space constant-coupling limit:
$\gamma_{\mu\nu}(x) = \eta_{\mu\nu},\ g_I(x) = g_I,\ a_\mu = 0$

The Weyl anomaly

The Weyl anomaly is parametrized as

$$\begin{aligned}
\Delta^W_\sigma \mathcal{W} = \int \mathrm{d}^4x \sqrt{\gamma}\,\sigma \Big(& -C\,W^2 + \tfrac{1}{4} A\, E_d + \tfrac{1}{8} B\, H^2 + G_{IJ} G^{\mu\nu} D_\mu g^I D^\mu g^J \\
& + \tfrac{1}{2} E_I H\, D^2 g^I + \tfrac{1}{2} F_{IJ} H\, D_\mu g^I D^\mu g^J - \tfrac{1}{2} A_{IJ} D^2 g^I D^2 g^J \\
& - B_{IJK} D^2 g^I D_\mu g^J D^\mu g^K - \tfrac{1}{2} C_{IJKL} D_\mu g^I D^\mu g^J D_\nu g^K D^\nu g^L \\
& - P_{IJ} \cdot f_{\mu\nu} D^\mu g^I D^\nu g^J - \tfrac{1}{4} f_{\mu\nu} \cdot \beta_f \cdot f^{\mu\nu} \Big) \\
+ \int \mathrm{d}^4x \sqrt{\gamma}\, 2\partial_\mu \sigma \Big(& W_I G^{\mu\nu} D_\nu g^I + \tfrac{1}{2} H_I H\, D^\mu g^I + S_{IJ} D^\mu g^I D^2 g^J \\
& + T_{IJK} D^\mu g^I D_\nu g^J D^\nu g^K + Q_I \cdot f^{\mu\nu} D_\nu g^I \Big) \\
+ \int \mathrm{d}^4x \sqrt{\gamma}\, \nabla^2 \sigma \Big(& \tfrac{1}{2} D\, H + U_I D^2 g^I + V_{IJ} D_\mu g^I D^\mu g^J \Big)
\end{aligned}$$

Osborn's equation

The Weyl anomaly satisfies Wess–Zumino consistency condition

$$[\Delta_\sigma,\, \Delta_{\sigma'}]\,\mathcal{W} = 0$$

Osborn's equation

Osborn '89, '91; Jack and Osborn '90, '13; Baume *et al.* [1401.5983]

$$\partial^I \hat{A} \equiv \frac{\partial \hat{A}}{\partial g_I} = \hat{T}^{IJ} B_J$$

- Proposed A-function: $\hat{A}$
- Would-be metric: $\hat{T}^{IJ}$
- Flavor-improved β-function: $B_I = \beta_I - (\upsilon\, g)_I$

Parametrizing β-functions

Osborn's Equation, now what?

$$\partial^I \hat{A} = \hat{T}^{IJ} B_J$$

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Osborn's Equation, now what?

Largely unknown

$$\partial^I \hat{A}(g) = \hat{T}^{IJ}(g)\,[\beta_J(g) - (\upsilon(g)\,g)_J]$$

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$$\partial^I \hat{A}(g) = \hat{T}^{IJ}(g)\,[\beta_J(g) - (\upsilon(g)\,g)_J]$$

Most general 4D renormalizable theory (ignoring relevant couplings):

$$\begin{aligned}\mathcal{L} = &+ \tfrac{1}{2}(\partial^\mu \phi - i A^A_\mu T^A_\phi \phi)^2_a + i\psi^\dagger_i \bar{\sigma}^\mu (\partial_\mu \psi - i A^A_\mu T^A_\psi \psi)^i \\ &- \tfrac{1}{4} a^{-1}_{AB} F^A_{\mu\nu} F^{B\mu\nu} - \tfrac{1}{2}\left(y_{aij}\psi^i\psi^j + \text{H.c.}\right)\phi_a - \tfrac{1}{24}\lambda_{abcd}\phi_a\phi_b\phi_c\phi_d\end{aligned}$$

Parametrize RG functions with monomials of the couplings, e.g.,

$$\beta_{aij} = \frac{\mathrm{d}y_{aij}}{\mathrm{d}t}, \qquad \beta^{(\ell)}_{aij} = \sum_n \mathbf{y}^{(\ell)}_n \,[Y^{(\ell)}_n(a,\, y,\, \lambda,\, T_\psi,\, T_\phi)]_{aij}$$

real-valued coefficients

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real-valued coefficients

Weyl Consistency Conditions on the β-functions

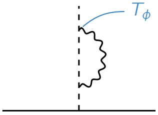
$$\text{O. Eq.} \implies \text{constraints on the } \beta_I \text{ coefficients, e.g., } \mathbf{y}^{(\ell)}_n$$

Jack, Osborn '90; Antipin *et al.* [1306.3234]; Jack, Poole [1411.1301]; ...

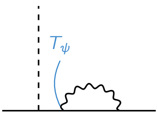
Weyl consistency conditions: 3-loop example

Parametrization of 1-loop Yukawa β-functions:

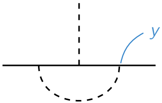
$$\beta^{(1)}_{aij} = \mathbf{y}^{(1)}_1 y_b [T^A_\phi a_{AB} T^B_\phi]_{ba} + \mathbf{y}^{(1)}_2 y_a [T^A_\psi a_{AB} T^B_\psi] + \mathbf{y}^{(1)}_3 y_b y^*_a y_b + \mathbf{y}^{(1)}_4 y_b y^*_b y_a + \mathbf{y}^{(1)}_5 y_b \mathrm{Tr}[\tilde{y}_b y_a]$$



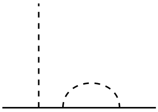
$\mathbf{y}^{(1)}_1$



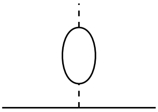
$\mathbf{y}^{(1)}_2$



$\mathbf{y}^{(1)}_3$



$\mathbf{y}^{(1)}_4$

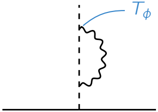


$\mathbf{y}^{(1)}_5$

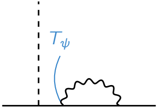
Weyl consistency conditions: 3-loop example

Parametrization of 1-loop Yukawa β-functions:

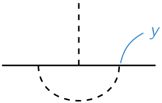
$$\beta_{aij}^{(1)} = \mathbf{y}_1^{(1)} y_b [T_\phi^A a_{AB} T_\phi^B]_{ba} + \mathbf{y}_2^{(1)} y_a [T_\psi^A a_{AB} T_\psi^B] + \mathbf{y}_3^{(1)} y_b y_a^* y_b + \mathbf{y}_4^{(1)} y_b y_b^* y_a + \mathbf{y}_5^{(1)} y_b \mathrm{Tr}[\tilde{y}_b y_a]$$



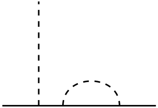
$\mathbf{y}_1^{(1)}$



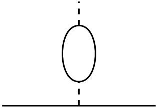
$\mathbf{y}_2^{(1)}$



$\mathbf{y}_3^{(1)}$

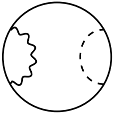
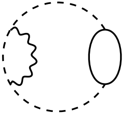


$\mathbf{y}_4^{(1)}$

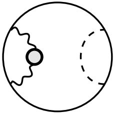
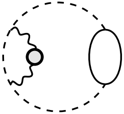
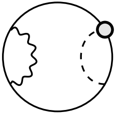
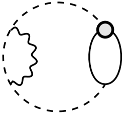


$\mathbf{y}_5^{(1)}$

The 3-loop A-function contains mixed gauge-Yukawa terms

$\hat{A} \supset \mathbf{A}_{10}^{(3)}$  $+ \mathbf{A}_{11}^{(3)}$ 

Weyl consistency conditions: 3-loop example

$\partial^I \hat{A} \supset \mathbf{A}^{(3)}_{10}$  $+\, \mathbf{A}^{(3)}_{11}$  $+\, 2\mathbf{A}^{(3)}_{10}$  $+\, 2\mathbf{A}^{(3)}_{11}$ 

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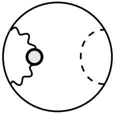
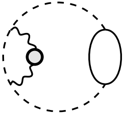
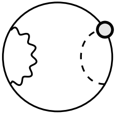
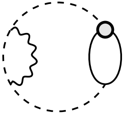
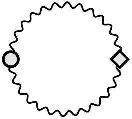
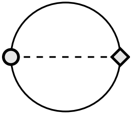
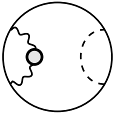
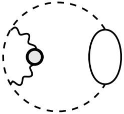
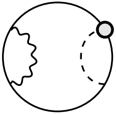
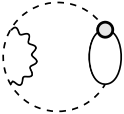
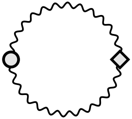
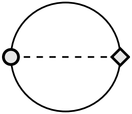
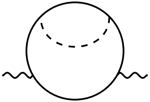
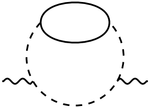
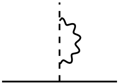
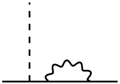
$\partial^I \hat{A} \supset \mathbf{A}^{(3)}_{10}$  $+\, \mathbf{A}^{(3)}_{11}$  $+\, 2\mathbf{A}^{(3)}_{10}$  $+\, 2\mathbf{A}^{(3)}_{11}$ 

$\hat{T}^{IJ} \supset$ $\mathbf{T}^{(1)}_{gg}$  $+$ $\mathbf{T}^{(2)}_{yy}$ 

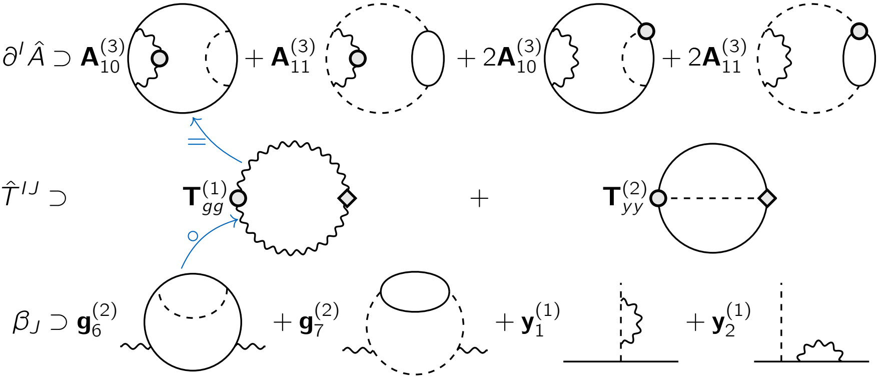
Weyl consistency conditions: 3-loop example

$\partial^I \hat{A} \supset \mathbf{A}_{10}^{(3)}$  $+\, \mathbf{A}_{11}^{(3)}$  $+\, 2\mathbf{A}_{10}^{(3)}$  $+\, 2\mathbf{A}_{11}^{(3)}$ 

$\hat{T}^{IJ} \supset \mathbf{T}_{gg}^{(1)}$  $+\, \mathbf{T}_{yy}^{(2)}$ 

$\beta_J \supset \mathbf{g}_6^{(2)}$  $+\, \mathbf{g}_7^{(2)}$  $+\, \mathbf{y}_1^{(1)}$  $+\, \mathbf{y}_2^{(1)}$ 

Weyl consistency conditions: 3-loop example

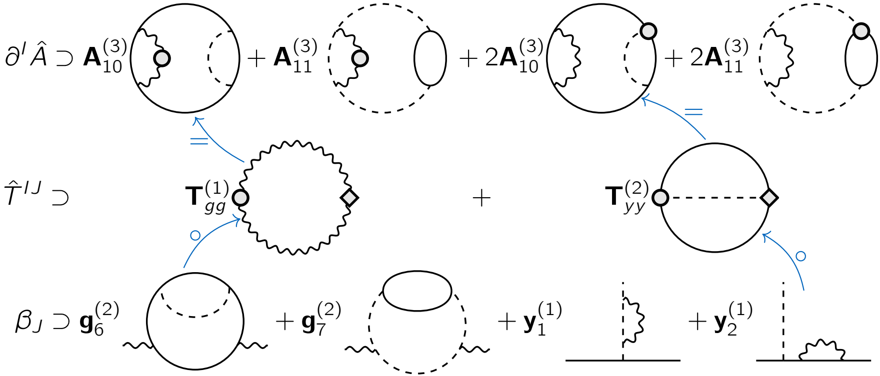


$\partial^I \hat{A} \supset \mathbf{A}^{(3)}_{10} + \mathbf{A}^{(3)}_{11} + 2\mathbf{A}^{(3)}_{10} + 2\mathbf{A}^{(3)}_{11}$

$\hat{T}^{IJ} \supset \mathbf{T}^{(1)}_{gg} + \mathbf{T}^{(2)}_{yy}$

$\beta_J \supset \mathbf{g}^{(2)}_{6} + \mathbf{g}^{(2)}_{7} + \mathbf{y}^{(1)}_{1} + \mathbf{y}^{(1)}_{2}$

Weyl consistency conditions: 3-loop example



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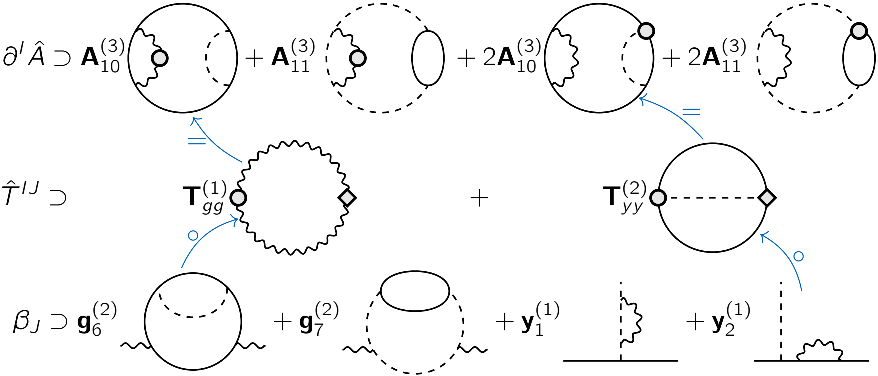
Weyl consistency conditions: 3-loop example



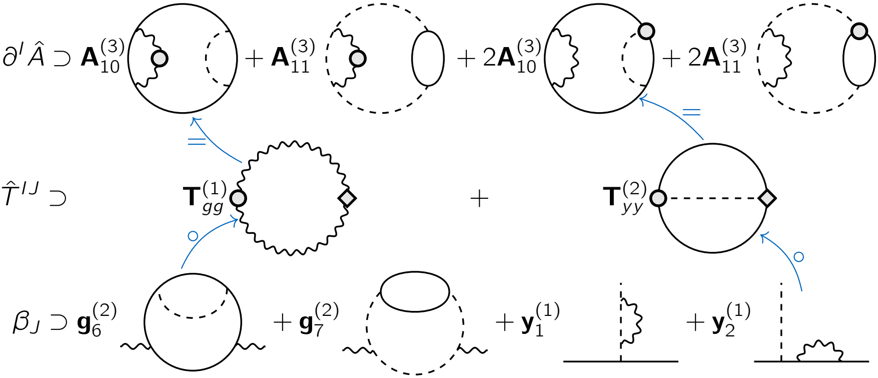
$\partial^I \hat{A} \supset \mathbf{A}_{10}^{(3)} \quad + \mathbf{A}_{11}^{(3)} \quad + 2\mathbf{A}_{10}^{(3)} \quad + 2\mathbf{A}_{11}^{(3)}$

$\hat{T}^{IJ} \supset \quad \mathbf{T}_{gg}^{(1)} \quad + \quad \mathbf{T}_{yy}^{(2)}$

$\beta_J \supset \mathbf{g}_6^{(2)} \quad + \mathbf{g}_7^{(2)} \quad + \mathbf{y}_1^{(1)} \quad + \mathbf{y}_2^{(1)}$

$$\partial^I \hat{A} = \hat{T}^{IJ} \beta^J \qquad \Longrightarrow \qquad \begin{cases} \mathbf{A}_{10}^{(3)} = \mathbf{T}_{gg}^{(1)} \mathbf{g}_6^{(2)} & \mathbf{A}_{11}^{(3)} = \mathbf{T}_{gg}^{(1)} \mathbf{g}_7^{(2)} \\ 2\mathbf{A}_{10}^{(3)} = \mathbf{T}_{yy}^{(2)} \mathbf{y}_2^{(1)} & 2\mathbf{A}_{11}^{(3)} = \mathbf{T}_{yy}^{(2)} \mathbf{y}_1^{(1)} \end{cases}$$

Weyl consistency conditions: 3-loop example



$$\partial^I \hat{A} = \hat{T}^{IJ} \beta^J \quad \Longrightarrow \quad \begin{cases} \mathbf{A}_{10}^{(3)} = \mathbf{T}_{gg}^{(1)} \mathbf{g}_6^{(2)} & \mathbf{A}_{11}^{(3)} = \mathbf{T}_{gg}^{(1)} \mathbf{g}_7^{(2)} \\ 2\mathbf{A}_{10}^{(3)} = \mathbf{T}_{yy}^{(2)} \mathbf{y}_2^{(1)} & 2\mathbf{A}_{11}^{(3)} = \mathbf{T}_{yy}^{(2)} \mathbf{y}_1^{(1)} \end{cases}$$

$$\Longrightarrow \mathbf{g}_7^{(2)} \mathbf{y}_2^{(1)} = \mathbf{g}_6^{(2)} \mathbf{y}_1^{(1)} \qquad 0 \cdot (-6) = (-1) \cdot 0$$

Weyl Consistency Conditions to 4-3-2 loop order

	No. of coefficients							
ℓ	$\hat{A}^{(\ell+1)}$	$\hat{T}_{IJ}^{(\ell)}$	$\upsilon^{(\ell-1)}$	$\beta_{AB}^{(\ell)}$	$\beta_{aij}^{(\ell-1)}$	$\beta_{abcd}^{(\ell-2)}$	TS basis	CCs
1	4	1		3			4	
2	14	4		7	5		16	1
3	49	27		33	33	5	91	26
4	257	260	9	198	303	33	703	265
4 (γ_5)	4			4	5		9	5

🟩: Previously known Macachek, Vaughn '83, '84; Jack, Osborn, '84; Pickering, Gracey, Jones [hep-ph/0104247]

🟨: Now determined Poole, AET [1901.02749]; Bednyakov, Pikelner [2105.09918]; Davies, Herren, AET [2110.05496]

Challenges: Poole, AET [1906.04625]

- Generate all unique graphs
- Use gauge identities to reduce to a basis
- Perform the relevant contractions

Flavored Trouble

How I learned to stop worrying
and tolerate divergence

Flavor in the RG

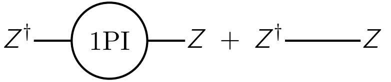
Callan–Symanzik equation for renormalized *n*-point functions:

$$\left(\frac{\partial}{\partial t} + \beta_g \frac{\partial}{\partial g} + n\gamma \right) G^{(n)}(\{p\}) = 0, \qquad \beta_I(g_I) = \frac{\mathrm{d}}{\mathrm{d}t} g_I \equiv \frac{\mathrm{d}}{\mathrm{d}\ln\mu} g_I$$

- The SM (and its extensions) has non-trivial flavor structure (matrix couplings)
- Improved precision necessitates their inclusion in the RG function
- Their inclusion causes new conceptual problems starting at 3-loop order:
  - The RG flow can generate spurious limit cycles
  - The $\overline{\text{MS}}$ counterterms are no longer uniquely defined
  - *RG functions can seemingly be divergent!*

γ-pole at 3-loop order

Renormalization condition for 2-point functions: ($\overline{\text{MS}}$, $d = 4 - 2\epsilon$)



finite,

$$Z = \mathbb{1} + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

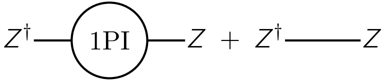
The field anomalous dimension

$$\gamma = Z^{-1} \frac{\mathrm{d}}{\mathrm{d}t} Z = \sum_{n=0}^{\infty} \frac{\gamma^{(n)}}{\epsilon^n} \quad \Longrightarrow \quad \gamma^{(0)} = -\zeta z^{(1)}, \qquad \zeta = k_I g_I \partial^I$$

loop-counting operator

γ-pole at 3-loop order

Renormalization condition for 2-point functions: ($\overline{\text{MS}}$, $d = 4 - 2\epsilon$)



$$= \text{finite}, \qquad Z = \mathbb{1} + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^{n}}$$

The field anomalous dimension

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loop-counting operator

In the SM $\gamma^{(1)} \neq 0$ at 3-loop order for $Z^{\dagger} = Z$:

Bednyakov, Pikelner, Velizhanin [1406.7171] Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6\gamma_q^{(1)} = \frac{g_1^2}{96}\big[y_u y_u^{\dagger},\, y_d y_d^{\dagger}\big] + \frac{1}{32}\big[y_u y_u^{\dagger} y_u y_u^{\dagger},\, y_d y_d^{\dagger}\big] + \frac{1}{32}\big[y_d y_d^{\dagger} y_d y_d^{\dagger},\, y_u y_u^{\dagger}\big]$$

$$(4\pi)^6\gamma_u^{(1)} = \frac{1}{16} y_u^{\dagger}\big[y_d y_d^{\dagger},\, y_u y_u^{\dagger}\big] y_u$$

γ-pole at 3-loop order

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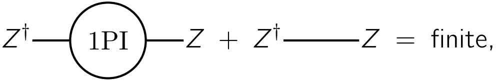

$$Z^\dagger \text{—}\boxed{\text{1PI}}\text{—} Z \;+\; Z^\dagger \text{———} Z \;=\; \text{finite}, \qquad Z = \mathbb{1} + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

The field anomalous dimension

loop-counting operator

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$$\gamma^{(1)} - \gamma^{(1)\dagger} = \left[z^{(1)},\, \zeta z^{(1)} \right] \qquad \text{for} \quad Z^\dagger = Z$$

$\gamma^{(1,2)}$ can be made to vanish with $Z' = UZ$ for some divergent rotation U.

The RG with divergent RG functions

The evolution of *renormalized* amplitudes is governed by the CS Eq.:

$$0 = \Delta^{\mathrm{RG}}\mathcal{W} = \left(\frac{\partial}{\partial t} + \beta_I \partial^I + \int \mathrm{d}^d x \, \mathcal{J}_\beta \gamma^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right) \mathcal{W} \qquad \text{(FSCC)}$$

The RG with divergent RG functions

The evolution of *renormalized* amplitudes is governed by the CS Eq.:

$$\left(\frac{\partial}{\partial t} + \left(\epsilon\beta_I^{(-1)} + \beta_I^{(0)}\right)\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(0)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

$$= -\sum_{n=1}^{\infty} \frac{1}{\epsilon^n}\left(\beta_I^{(n)}\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{(n)\beta}{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}$$

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The Ward identity for the flavor symmetry group G_F (FSCC):

$$0 = \Delta_\omega^F \mathcal{W} = \left((\omega\, g)_I \partial^I - \int \mathrm{d}^d x\, \mathcal{J}_\beta \omega^\beta{}_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha}\right)\mathcal{W}, \qquad \omega \in \mathfrak{g}_F$$

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The RG flow is finite due to

RG Finiteness *(theorem)*

$$\gamma^{(n)} \in \mathfrak{g}_F \quad \text{and} \quad \beta_I^{(n)} = -\left(\gamma^{(n)}\, g\right)_I, \qquad n \geq 1$$

RG finiteness in the SM

3-loop RG divergences in the SM:

using counterterms from Herren, Mihaila, Steinhauser [1712.06614]

$$(4\pi)^6\gamma_q^{(1)} = \frac{g_1^2}{96}\big[y_u y_u^\dagger,\, y_d y_d^\dagger\big] + \frac{1}{32}\big[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\big] + \frac{1}{32}\big[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\big]$$

$$(4\pi)^6\gamma_u^{(1)} = \frac{1}{16} y_u^\dagger\big[y_d y_d^\dagger,\, y_u y_u^\dagger\big] y_u$$

$$\begin{aligned}(4\pi)^6\beta_{y_u}^{(1)} = &-\frac{g_1^2}{96}\big[y_u y_u^\dagger,\, y_d y_d^\dagger\big] y_u - \frac{1}{32}\big[y_u y_u^\dagger y_u y_u^\dagger,\, y_d y_d^\dagger\big] y_u \\ &- \frac{1}{32}\big[y_d y_d^\dagger y_d y_d^\dagger,\, y_u y_u^\dagger\big] y_u + \frac{1}{16} y_u y_u^\dagger\big[y_d y_d^\dagger,\, y_u y_u^\dagger\big] y_u\end{aligned}$$

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$$(\omega\, y_u)^i{}_j = \omega_q{}^i{}_k y_u{}^k{}_j - y_u{}^i{}_k \omega_u{}^k{}_j + \omega_h y_u{}^i{}_j$$

RG finiteness in the SM

3-loop RG divergences in the SM:

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$\beta_{y_u}^{(1)} = -(\gamma^{(1)}\, y_u)$, $\beta_{y_u}^{(2)} = -(\gamma^{(2)}\, y_u)$, *etc.* in the SM

SM RG functions are RG finite at 3-loop order

Renormalization ambiguity

$\mathcal{W}$ is invariant under flavor rotations $R \in G_F$: e.g., $y_u \longrightarrow R_q y_u R_u^\dagger$ in the SM

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}[\gamma,\, Rg,\, R\mathcal{J},\, a^R] =$$

$$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Rg_0,\, R\mathcal{J}_0,\, a_0^R], \quad (R\, g_0)_I = g_{0,I}(R\, g)$$

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Take instead a divergent rotation of $\mathcal{W}_0$:

$$U = \exp\Big[-\sum_{n=1}^{\infty} \frac{1}{\epsilon^n} u^{(n)}(g) \Big], \qquad u^{(n)} \in \mathfrak{g}_F$$

$$\mathcal{W}[\gamma,\, g,\, \mathcal{J},\, a] = \mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, Ug_0,\, U\mathcal{J}_0,\, a_0^U]$$

It results in a change of counterterms, e.g., Ambiguity in taking $\sqrt{Z^\dagger Z}$

$$(U\mathcal{J}_0)_\alpha = \mathcal{J}_{0,\beta} U^{\dagger\beta}{}_\alpha = \mathcal{J}_\beta (Z^{-1}U^\dagger)^\beta{}_\alpha \implies Z^{U\alpha}{}_\beta = U^\alpha{}_\gamma Z^\gamma{}_\beta$$

Ambiguity in RG the functions

$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, U g_0,\, U\mathcal{J}_0,\, a_0^U]$ *but produce different RG functions!*

Herren, AET [2104.07037]

$$\Delta\gamma \equiv \gamma^U - \gamma = -\beta_I U \partial^I U^\dagger \in \mathfrak{g}_F$$
$$\Delta\beta_I \equiv \beta_I^U - \beta_I = -(\Delta\gamma\, g)_I$$
$$\Delta\upsilon \equiv \upsilon^U - \upsilon = -\Delta\gamma$$

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i) By choosing U, one can engineer any $\Delta\gamma = \alpha(g) \in \mathfrak{g}_F$
  - The RG ambiguity reproduces an ambiguity in defining the Weyl symmetry:

$$\Delta^W_\sigma \to \Delta^W_\sigma + \Delta^F_{\sigma\alpha}$$

Ambiguity in RG the functions

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$$\Delta\upsilon \equiv \upsilon^U - \upsilon = -\Delta\gamma$$

i) By choosing U, one can engineer any $\Delta\gamma = \alpha(g) \in \mathfrak{g}_F$

ii) RG-finiteness is conserved $\qquad \beta_I^{(n)} = -\big(\gamma^{(n)}\, g\big)_I, \quad \gamma^{(n)} \in \mathfrak{g}_F, \quad \forall n \geq 1$

  - Either *all or none* of the RG functions are RG finite

Ambiguity in RG the functions

$\mathcal{W}_0[\gamma,\, g_0,\, \mathcal{J}_0,\, a_0] = \mathcal{W}_0[\gamma,\, U g_0,\, U\mathcal{J}_0,\, a_0^U]$ *but produce different RG functions!*

Herren, AET [2104.07037]

$$\begin{aligned}\Delta\gamma &\equiv \gamma^U - \gamma = -\beta_I U \partial^I U^\dagger \in \mathfrak{g}_F \\ \Delta\beta_I &\equiv \beta_I^U - \beta_I = -(\Delta\gamma\, g)_I \\ \Delta\upsilon &\equiv \upsilon^U - \upsilon = -\Delta\gamma\end{aligned}$$

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iii) If $(\beta_I,\, \gamma)$ are RG finite, U can be chosen to make them finite

iv) One can choose counterterms such that the RG functions coincide with the flavor-improved $(B_I,\, \Gamma) = (\beta_I - (\upsilon\, g)_I,\, \gamma + \upsilon)$
   - Choosing $\Delta\gamma = \upsilon$, $(\beta_I^U,\, \gamma^U) = (B_I,\, \Gamma)$
   - $(B_I,\, \Gamma)$ are invariant under the renormalization ambiguity

Proof of RG-finiteness

The flavor-improved Weyl symmetry, $\widehat{\Delta}^W_\sigma \mathcal{W}$, yields the operator identity

$$[T^\mu{}_\mu] = B_I [\mathcal{O}^I] + \mathcal{J}_\alpha \Big((\Delta_\alpha - d) \delta^\alpha{}_\beta + \Gamma^\alpha{}_\beta \Big) [\Phi^\beta] \qquad \text{(FSCC)}$$

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Flavor-improved RG functions

The RG functions $\big(B_I,\, \Gamma\big)$ are unambiguous and finite

We confirmed this explicitly at 3-loop order in the Yukawa sector of the SM

Summary

i) The occurrence of certain ϵ poles in the RG functions is consistent with the Callan–Symanzik equation and not a sign of the theory or renormalization scheme breaking down

ii) The flavor symmetry causes an ambiguity in the choice of renormalization constants

iii) Using the ambiguity, it is always possible to recover finite RG functions (β_I, γ)

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Summary

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No need to panic if you encounter an RG pole!

Backup

General β-functions in 4D QFTs

- General formulas for β-functions are long known to order 3–2–2 ($\overline{\text{MS}}$)

  Macachek, Vaughn '83,'84; Jack, Osborn '84; Pickering, Gracey, Jones [hep-ph/0104247]

- Computer packages with implementation of the general formulas: `SARAH 4`, `PyR@TE 3`, `ARGES`, `RGBeta`

- Weyl Consistency Conditions from Osborn's Eq. establishes self-consistency constraints on β-functions $\partial_I A = T_{IJ}\beta^J$

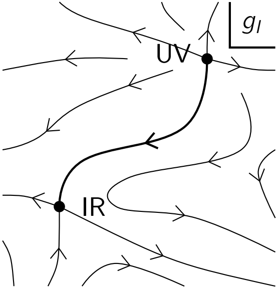
  Antipin *et al.* [1306.3234]; Jack, Poole [1505.05400]; Poole, AET [1906.04625]

- Recent results for the general 4–3–2 β-functions

  Bednyakov, Pikelner [2105.09918]; Davies, Herren, AET [2110.05496]

How to recognize a CFT

Fixed Points



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$$[T^{\mu}{}_{\mu}] = \beta_I [\mathcal{O}^I] = 0$$

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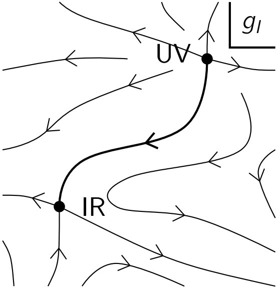

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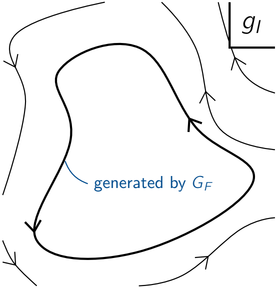
Limit Cycles



Limit cycles can be (are?) CFTs

Fortin, Grinstein, Stergiou [1206.2921, 1208.3674]

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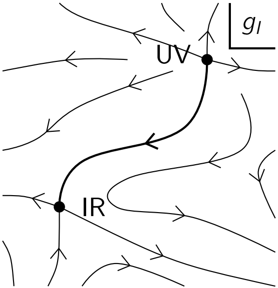
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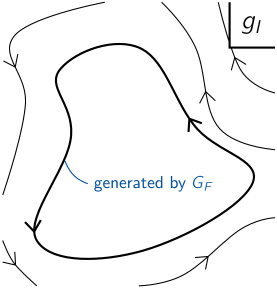
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$$[T^\mu{}_\mu] = B_I[\mathcal{O}^I] = 0$$

B_I is a more physical β-function

Flavor symmetry in the SM

The quark sector of the SM,

$$\mathcal{L} = i\bar{q}\not{D}q + i\bar{u}\not{D}u + i\bar{d}\not{D}d + |D_\mu H|^2 - \big(\bar{q}\, y_u\, u\tilde{H} + \bar{q}\, y_d\, dH \ + \ \text{h.c.}\ \big),$$

has flavor symmetry (maximal symmetry of the kinetic terms)

$$G_F = \mathrm{SU}(3)_q \times \mathrm{SU}(3)_u \times \mathrm{SU}(3)_d \times \mathrm{U}(1)^3 \supset \mathrm{U}(1)_B$$

Physics is invariant under transformations

$$\left.\begin{aligned} y_u &\longrightarrow U_q\, y_u\, U_u^\dagger \\ y_d &\longrightarrow U_q\, y_d\, U_d^\dagger \end{aligned}\right\} \quad \text{e.g.,} \quad (y_u\,, y_d) \longrightarrow (V_{\text{CKM}}^\dagger \hat{y}_u,\, \hat{y}_d)$$

If flavor transformations are unphysical, one can perform arbitrary flavor rotations along the RG flow...

How to compute RG functions

In $\overline{\text{MS}}$ ($d = 4 - 2\epsilon$) the counterterms are arranged by poles

$$g_{0,I} = \mu^{k_I \epsilon}(g + \delta g), \qquad \delta g_I = \sum_{n=1}^{\infty} \frac{\delta g_I^{(n)}}{\epsilon^n}, \qquad Z = 1 + \sum_{n=1}^{\infty} \frac{z^{(n)}}{\epsilon^n}$$

The RG functions are determined recursively from the poles

$$\beta_I^{(-1)} = -k_I g_I, \qquad \beta_I^{(n)} = (\zeta - k_I)\delta g_I^{(n+1)} - \sum_{k=0}^{n-1} \beta_J^{(k)} \partial^J \delta g_I^{(n-k)}, \qquad n \geq 0$$

$$\gamma^{(n)} = -\zeta z^{(n+1)} + \sum_{k=0}^{n-1} \left[\beta_I^{(k)} \partial^I z^{(n-k)} - z^{(n-k)} \gamma^{(k)}\right], \qquad n \geq 0$$

Contrary to common perception, we can have

$$\beta_I^{(n)} = \gamma^{(n)} \neq 0, \qquad n \geq 1$$

RG symmetry

Accounting identity for mass dimension:

$$\Delta^{\mu}\mathcal{W} = 0, \qquad \Delta^{\mu} = \mu\frac{\partial}{\partial\mu} + \int \mathrm{d}^d x \left(2\gamma^{\mu\nu}\frac{\delta}{\delta\gamma^{\mu\nu}} + (d-\Delta_\alpha)\mathcal{J}_\alpha\frac{\delta}{\delta\mathcal{J}_\alpha}\right)$$

The generator of the RG is $\Delta^{\mathrm{RG}} = \Delta^{\mu} - \Delta^{W}_{\sigma=1}$, from which we recover the *CS equation*

$$0 = \Delta^{\mathrm{RG}}\mathcal{W} = \left(\frac{\partial}{\partial t} + \beta_I\partial^I + \int \mathrm{d}^d x\, \mathcal{J}_\beta \gamma^{\beta}{}_{\alpha}\frac{\delta}{\delta\mathcal{J}_\alpha}\right)\mathcal{W} \qquad \text{(FSCC)}$$

Exactly what we would get from $\frac{\mathrm{d}\mathcal{W}}{\mathrm{d}t} = 0$:

$$\left(\frac{\partial}{\partial t} + \beta_g\frac{\partial}{\partial g} + n\gamma\right) G^{(n)}(\{p\}) = 0$$

Flavor symmetry

G_F is a symmetry of S with generator

$$\Delta_\omega^F = \int \mathrm{d}^d x \left(D_\mu \omega \cdot \frac{\delta}{\delta a_\mu} - (\omega\, g)_I \frac{\delta}{\delta g_I} - (\omega\, \mathcal{J})_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right), \qquad \omega \in \mathfrak{g}_F,$$

but it is typically anomalous: Keren-Zur [1406.0869]

$$\Delta_\omega^F \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}_\omega^F(\gamma,\, g,\, a)$$

The Weyl generator can be combined with a flavor rotation to generate a class of Weyl symmetries:

$$\Delta_\sigma^{W'} = \Delta_\sigma^W + \Delta_{\sigma\alpha}^F, \qquad \alpha(g) \in \mathfrak{g}_F,$$

$$\left[\Delta_\omega^F,\, \Delta_\sigma^{W'}\right] = \left[\Delta_\sigma^{W'},\, \Delta_{\sigma'}^{W'}\right] = 0, \qquad \Delta_\sigma^{W'} \mathcal{W} = \int \mathrm{d}^d x\, \mathcal{A}_\sigma^{W'}$$

Ambiguity in the RG

Ambiguity in RG functions defined by the Weyl transformation:

$$\beta'_I = \beta_I + (\alpha\, g)_I, \qquad \upsilon' = \upsilon + \alpha, \qquad \rho'^I = \rho^I - \partial^I \alpha, \qquad \gamma'^\alpha{}_\beta = \gamma^\alpha{}_\beta - \alpha^\alpha{}_\beta$$

The RG flow has a flavor rotation ambiguity

$$\Delta^W_\sigma = \int \mathrm{d}^d x \bigg(2\sigma\gamma^{\mu\nu} \frac{\delta}{\delta\gamma^{\mu\nu}} - \sigma\beta_I \frac{\delta}{\delta g_I} + \sigma \mathcal{J}_\beta \big[(d - \Delta_\alpha)\delta^\beta{}_\alpha - \gamma^\beta{}_\alpha \big] \frac{\delta}{\delta \mathcal{J}_\alpha} \\ + \Big[\partial_\mu \sigma\, \upsilon - \sigma\, D_\mu g_I\, \rho^I \Big] \cdot \frac{\delta}{\delta a_\mu} \bigg)$$

$$\Delta^F_\alpha = \int \mathrm{d}^d x \left(D_\mu \alpha \cdot \frac{\delta}{\delta a_\mu} - (\alpha\, g)_I \frac{\delta}{\delta g_I} - (\alpha\, \mathcal{J})_\alpha \frac{\delta}{\delta \mathcal{J}_\alpha} \right)$$

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The RG flow has a flavor rotation ambiguity

Flavor-improved RG functions are invariant:

$$B_I = \beta_I - (\upsilon\, g)_I, \qquad \Gamma^{\alpha}{}_{\beta} = \gamma^{\alpha}{}_{\beta} + \upsilon^{\alpha}{}_{\beta}, \qquad P^I = \rho^I + \partial^I \upsilon$$

We can choose a ‘gauge’ where $\upsilon = 0$:

$$\begin{aligned}\widehat{\Delta}^W_\sigma = \Delta^W_\sigma + \Delta^F_{-\sigma\,\upsilon} = \int \mathrm{d}^d x \bigg(& 2\sigma\gamma^{\mu\nu} \frac{\delta}{\delta\gamma_{\mu\nu}} - \sigma B_I \frac{\delta}{\delta g_I} \\ & + \sigma \mathcal{J}_\beta \Big[(d - \Delta_\alpha)\delta^{\beta}{}_{\alpha} - \Gamma^{\beta}{}_{\alpha} \Big] \frac{\delta}{\delta \mathcal{J}_\alpha} - \sigma\, D_\mu g_I\, P^I \cdot \frac{\delta}{\delta a_\mu} \bigg)\end{aligned}$$

But generally $B_I \neq \frac{\mathrm{d} g_I}{\mathrm{d} t}$ ...