

# Renormalisation of twist-two operators in covariant gauges

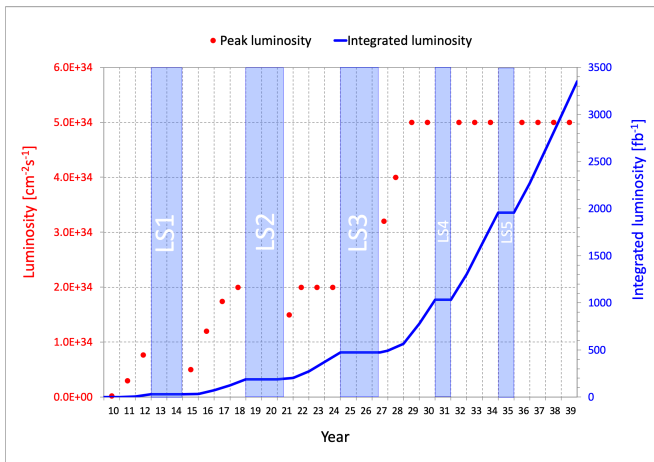
DESY-HU Theorie-Seminar  
Giulio Falcioni

Based on **JHEP 05 (2022) 177**; arxiv:**2203.11181**  
in collaboration with Franz Herzog



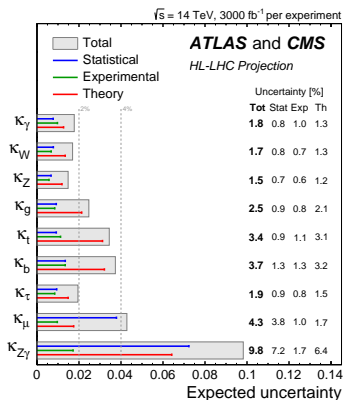
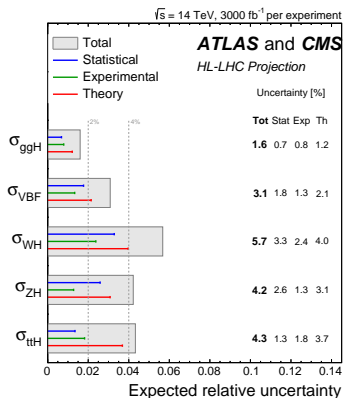
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# The LHC is back!



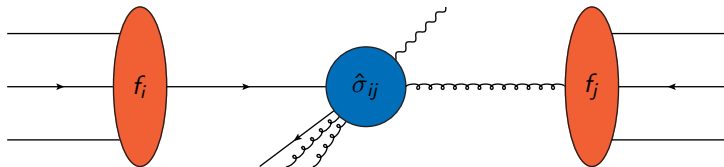
# 1% uncertainties at the LHC

Projections for HL-LHC: experimental uncertainties  $\simeq 1\%$   
 Error budget dominated by theory.



Vol. 7/2019 of CERN Yellow Reports: Monograph.

# Ingredients for theoretical predictions



$$\sigma_{PP \rightarrow X}(s) = \sum_{i,j} \int dx_1 dx_2 f_i(x_1, \mu^2) f_j(x_2, \mu^2) \hat{\sigma}_{ij \rightarrow X}(s, x_1 x_2, \mu^2)$$

- $\hat{\sigma}$  partonic cross section, computed perturbatively.
- $f_i(x, \mu^2)$  probability parton  $i$  carries momentum fraction  $x$ .
- $\mu$  separates short-distance from long-distance dynamics.

# LHC phenomenology to N<sup>3</sup>LO in QCD

N<sup>3</sup>LO corrections are *crucial* to reach 1% accuracy.

	Q [GeV]	$\delta\sigma^{\text{N}^3\text{LO}}$	$\delta(\text{scale})$	$\delta(\text{PDF-TH})$
$gg \rightarrow \text{Higgs}$	$m_H$	3.5%	+0.21% -2.37%	$\pm 1.2\%$
$b\bar{b} \rightarrow \text{Higgs}$	$m_H$	-2.3%	+3.0% -4.8%	$\pm 2.5\%$
NCDY	30	-4.8%	+1.53% -2.54%	$\pm 2.8\%$
	100	-2.1%	+0.66% -0.79%	$\pm 2.5\%$
CCDY( $W^+$ )	30	-4.7%	+2.5% -1.7%	$\pm 3.2\%$
	150	-2.0%	+0.5% -0.5%	$\pm 2.1\%$
CCDY( $W^-$ )	30	-5.0%	+2.6% -1.6%	$\pm 3.2\%$
	150	-2.1%	+0.6% -0.5%	$\pm 2.13\%$

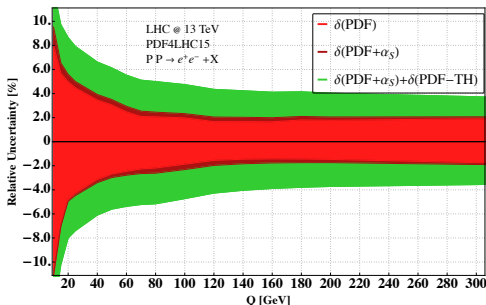
Numbers from J. Baglio, C. Duhr, B. Mistlberger, R. Szafron 2209.06138

$$\delta\sigma^{\text{N}^3\text{LO}} = \frac{\sigma^{\text{N}^3\text{LO}} - \sigma^{\text{NNLO}}}{\sigma^{\text{NNLO}}}$$

# Missing N<sup>3</sup>LO PDFs

Additional errors on N<sup>3</sup>LO predictions from missing N<sup>3</sup>LO PDFs

$$\delta(\text{PDF} - \text{TH}) = \frac{1}{2} \left| \frac{\sigma^{\text{NNLO, NNLO-PDFs}}(Q^2) - \sigma^{\text{NNLO, NLO-PDFs}}(Q^2)}{\sigma^{\text{NNLO, NNLO-PDFs}}(Q^2)} \right| \simeq O(2\% - 3\%)$$



Neutral Current DY at N<sup>3</sup>LO (Duhr, Mistlberger JHEP 03 (2022) 116).

# Beyond LHC

- Deep Inelastic Scattering (DIS) is key to extract  $f_i(x, \mu^2)$
- There is substantial information on N<sup>3</sup>LO DIS  
(Blümlein, Marquard, Schneider and Schönwald 2208.14325 and refs.)
- Forthcoming DIS experiment

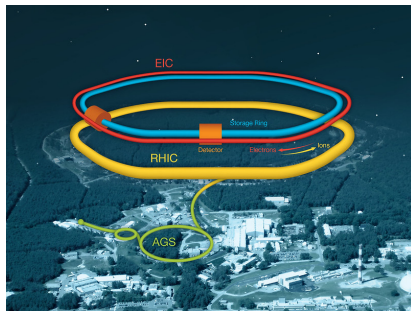
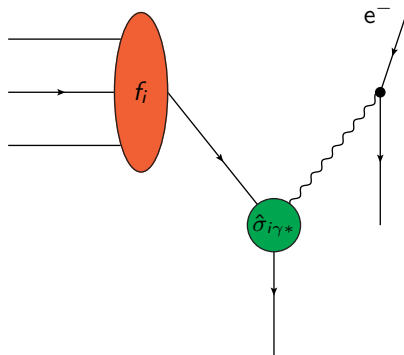


Image: Brookhaven National Laboratory



# Scale evolution of the PDFs

## N<sup>3</sup>LO PDFs → 4-loop evolution kernels

$$\mu^2 \frac{d}{d\mu^2} f_i(x, \mu^2) = \int_x^1 \frac{dy}{y} P_{ij}(\alpha_s, y) f_j\left(\frac{x}{y}, \mu^2\right), \quad i = g, q, \bar{q}$$

$$P_{ij}(\alpha_s, x) = \underbrace{a P_{ij}^{(0)}}_{\text{LO}} + \underbrace{a^2 P_{ij}^{(1)}}_{\text{NLO}} + \underbrace{a^3 P_{ij}^{(2)}}_{\text{NNLO}} + \underbrace{a^4 P_{ij}^{(3)}}_{\text{N}^3\text{LO}}, \quad a = \frac{\alpha_s}{4\pi}$$

## Formulation in Mellin space

$$\gamma_{ij}^{(N)} = - \int_0^1 x^{N-1} P_{ij}(\alpha_s, x) dx$$



## Strategies to compute $N^3$ LO splitting functions

- Expansion of the DIS Compton amplitude  
(Gorishnii, Larin, Tkachov 1983)
  - Calculation of NNLO evolution (Moch, Vermaseren, Vogt 2004)
  - Provided a subset  $N = 2 \dots 8$  at  $N^3$ LO  
(Moch, Ruijl, Ueda, Vermaseren, Vogt 2021)
  - Going beyond may be too expensive with this method.
- **Operator Product Expansion**
  - **Complete**  $N^3$ LO evolution of Non Singlet (NS) PDFs  
(Moch, Ruijl, Ueda, Vermaseren, Vogt 2017)
  - NS and polarised singlet at NNLO  
(Blümlein, Marquard, Schneider, Schönwald 2021)<sup>2</sup>

### Warning!

Conceptual problem for gluon and quark singlet PDFs!

## Pitfalls in the singlet sector

- Discrepancies in the evolution of the gluon PDF at **2 loops**
  - **Feynman gauge** calculation by different groups (1977-1980) in agreement with each other
  - Another approach in **axial gauge** gives a **different** result (Curci, Furmanski, Petronzio 1980)

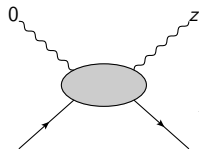
### Inconsistency

The evolution of PDFs **cannot** depend on the gauge choice.

- **Axial** gauge result correctly fulfils supersymmetry Ward id.
- Conflict was resolved **only** at 2 loops (Hamberg, van Neerven 1993).

# The OPE approach

Lightcone OPE of DIS (Gross, Wilczek 1973; Georgi, Politzer 1973)



$$\simeq_{z^2 \rightarrow 0} \sum_i C_i(z^2) \times z^{\mu_1} \dots z^{\mu_N} \underbrace{\langle P | \mathcal{O}_{\mu_1 \dots \mu_N}^{(N),i}(0) | P \rangle}_{\text{Operator of min. Twist = dim. - spin = 2}}$$

- Separate short-distance ( $C_i$ ) from long-distance physics  $\langle P | \dots | P \rangle$ .
- Scale evolution of the operators  $\rightarrow$  scale evolution of the PDFs

$$\mu^2 \frac{d}{d\mu^2} \mathcal{O}_{\mu_1 \dots \mu_N}^{(N),i} = -\gamma_{ij}^{(N)} \mathcal{O}_{\mu_1 \dots \mu_N}^{(N),j}$$

Moments of the splitting functions from *renormalisation* of twist 2.

# Operator renormalisation

- $\gamma_{ij}^{(N)}$  are computed in terms of *renormalisation constants*

$$\gamma_{ij}^{(N)} = - \left( \frac{d}{d\mu^2} Z_{ik}^{(N)} \right) Z_{kj}^{(N)}. \quad (1)$$

- $Z_{ij}^{(N)}$  are defined to render *all* correlators of  $\mathcal{O}^{(N),i}$  *finite*.

$$\mathcal{O}^{(N),i} = Z_{ij}^{(N)} \mathcal{O}_{\text{bare}}^{(N),j}.$$

- In Minimal Subtraction, eq. (1) becomes

$$\gamma_{ij}^{(N)} = a \left. \frac{\partial}{\partial a} Z_{ij}^{(N)} \right|_{\frac{1}{\epsilon}}$$

# Twist-2 operators

## Twist-2 operators of QCD fields

$$\mathcal{O}_{g;\mu_1\dots\mu_N}^{(N)} = \frac{1}{2} \mathcal{S}_T \left\{ F_{\rho\mu_1}^{a_1} D_{\mu_2}^{a_1 a_2} \dots D_{\mu_{N-1}}^{a_{N-2} a_{N-1}} F_{\mu_N}^{a_N; \rho} \right\},$$

$$\mathcal{O}_{q;\mu_1\dots\mu_N}^{(N)} = \mathcal{S}_T \left\{ \bar{\psi}_{i_1} \gamma_{\mu_1} D_{\mu_2}^{i_1 i_2} \dots D_{\mu_N}^{i_{N-1} i_N} \psi_{i_N} \right\},$$

$$\mathcal{O}_{ns;\mu_1\dots\mu_N}^{(N),\rho} = \mathcal{S}_T \left\{ \bar{\psi}_{i_1} (\lambda^\rho) \gamma_{\mu_1} D_{\mu_2}^{i_1 i_2} \dots D_{\mu_N}^{i_{N-1} i_N} \psi_{i_N} \right\},$$

$\lambda^\rho \rightarrow$  generator of  $SU(n_f)$ .

$\mathcal{S}_T \rightarrow$  symmetrise over  $\mu_1 \dots \mu_N$  and remove trace terms.

- Construct projectors with a **lightlike** vector  $\Delta^\mu$

$$F_\nu^a = F_{\nu\mu}^a \Delta^\mu, \quad D = D_\mu \Delta^\mu, \quad \partial = \partial_\mu \Delta^\mu, \quad A^a = A_\mu^a \Delta^\mu,$$

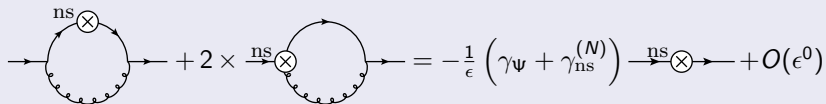
- In this talk  $\rightarrow$  focus on **gluonic** operator

$$\mathcal{O}_1^{(N)} = \mathcal{O}_{g;\mu_1\dots\mu_N}^{(N)} \Delta^{\mu_1} \dots \Delta^{\mu_N} = \frac{1}{2} \text{Tr} \left[ F_\rho D^{N-1} F^\rho \right]$$

# Renormalisation: Non-singlet vs Singlet operators

## Non-singlet operators • Diagonal RGE

$$\mathcal{O}_{\text{ns}}^{(N),R}(\mu^2) = Z_{\text{ns}}^{(N)}(\mu^2) \mathcal{O}_{\text{ns}}^{(N),\text{bare}}$$

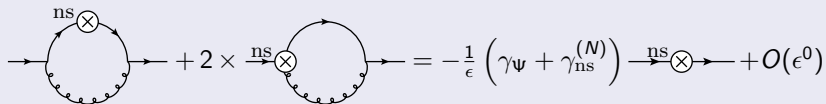


$$+ 2 \times \text{ns} \text{ loop with self-energy} = -\frac{1}{\epsilon} \left( \gamma_{\psi} + \gamma_{\text{ns}}^{(N)} \right) \text{ns vertex} + O(\epsilon^0)$$

# Renormalisation: Non-singlet vs Singlet operators

## Non-singlet operators • Diagonal RGE

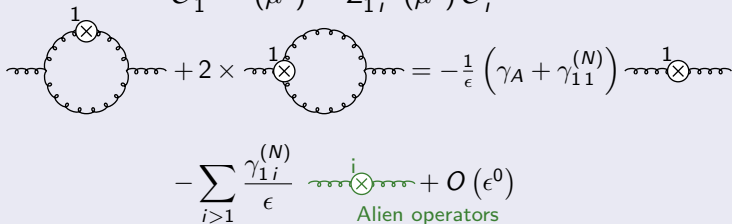
$$\mathcal{O}_{\text{ns}}^{(N),R}(\mu^2) = Z_{\text{ns}}^{(N)}(\mu^2) \mathcal{O}_{\text{ns}}^{(N),\text{bare}}$$



$$+ 2 \times \text{[Diagram]} = -\frac{1}{\epsilon} \left( \gamma_{\Psi} + \gamma_{\text{ns}}^{(N)} \right) \text{[Cross]} + \mathcal{O}(\epsilon^0)$$

## Singlet operators • Mixing

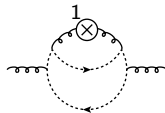
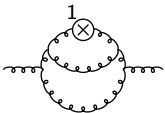
$$\mathcal{O}_1^{(N),R}(\mu^2) = Z_{1i}^{(N)}(\mu^2) \mathcal{O}_i^{(N),\text{bare}}$$



$$+ 2 \times \text{[Diagram]} = -\frac{1}{\epsilon} \left( \gamma_A + \gamma_{11}^{(N)} \right) \text{[Cross]} - \sum_{i>1} \frac{\gamma_{1i}^{(N)}}{\epsilon} \text{[Cross]} + \mathcal{O}(\epsilon^0)$$

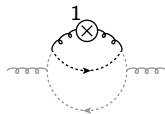
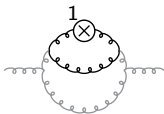
Alien operators

# Multiloop renormalisation

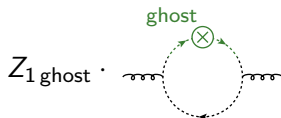
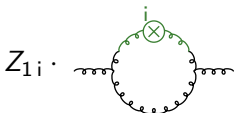




# Multiloop renormalisation



Alien operators, including **ghost** operators enter in subdivergences



To renormalise the gauge invariant operator we need

- The whole **set** of **aliens**, including *ghost* operators.
- The mixing counterterms  $Z_{1i}$ .

# What do we know about aliens

- Basis of all **Alien operators** entering at **2 loops** (Dixon, Taylor 1974)

$$\bar{O}_\lambda = \bar{F}^{aa} \bar{D}_\alpha^a \partial^{m-2} \bar{A}^b - \bar{g} f^{abc} \bar{F}_\alpha^a \sum_{i=1}^{m-2} \frac{\kappa_i}{\eta} \partial^\alpha [(\partial^{i-1} \bar{A}^b)(\partial^{m-2-i} \bar{A}^c)] + \mathcal{O}(\bar{g}^2) \quad (2.9)$$

$$\bar{O}_\omega = -\xi^a \partial^m \bar{\omega}^a - \bar{g} f^{abc} \xi^a \sum_{i=1}^{m-2} \frac{\eta_i}{\eta} \partial [(\partial^{m-2-i} \bar{A}^b)(\partial^i \bar{\omega}^c)] + \mathcal{O}(\bar{g}^2) \quad (2.10)$$

Solve puzzle in the gluon evolution (Hamberg, van Neerven 1993).

- General theorem: characterisation of all aliens (Joglekar, Lee 1975)
  - BRST-exact operators
  - Operators proportional to the Equation of Motion

## This talk

- General construction of aliens and **explicit** basis to **4 loops**
- Renormalisation mixing constants  $Z_{1i}^{(N)}$  and physical ones

# Setup $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{G.F.}+\text{G}}$

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu}^a F^{a;\mu\nu}$$

$$\mathcal{L}_{\text{GF}+\text{G}} = s \left[ \bar{c}^a \left( \partial^\mu A_\mu^a - \frac{\xi L}{2} b^a \right) \right] = \bar{s} \left[ c^a \left( \frac{\xi L}{2} b^a - \partial^\mu A_\mu^a \right) \right]$$

- Gauge invariance of  $\mathcal{L}_0$  under

$$\delta_\omega A_\mu^a = D_\mu^{ab} \omega^b = \partial_\mu \omega^a + g f^{abc} A_\mu^b \omega^c$$

- BRST( $s$ )/anti-BRST( $\bar{s}$ ) invariance, s.t.  $s^2 = \bar{s}^2 = \{s, \bar{s}\} = 0$

$$s(A_\mu^a) = D_\mu^{ab} c^b$$

$$s(c^a) = -\frac{g}{2} f^{abc} c^b c^c$$

$$s(\bar{c}^a) = -b^a$$

$$s(b^a) = 0$$

$$\bar{s}(A_\mu^a) = D_\mu^{ab} \bar{c}^b$$

$$\bar{s}(\bar{c}^a) = -\frac{g}{2} f^{abc} \bar{c}^b \bar{c}^c$$

$$\bar{s}(c^a) = -\bar{b}^a$$

$$\bar{s}(\bar{b}^a) = 0$$

# Strategy

Follow the approach of Dixon and Taylor, by

- 1 Extend the lagrangian including physical and **EOM** operators
- 2 These maintain a **generalised gauge invariance** under
$$A_{\mu}^a \rightarrow A_{\mu}^a + \delta A_{\mu}^a$$
- 3 The **ghost** operators are chosen to maintain **generalised BRST invariance**  $A_{\mu}^a \rightarrow A_{\mu}^a + \theta s'(A_{\mu}^a)$

Imposing also **generalised anti-BRST invariance**

$A_{\mu}^a \rightarrow A_{\mu}^a + \bar{\theta} \bar{s}'(A_{\mu}^a)$  provides non-trivial relations among operators.

## Equation of Motion Operators

- Variations of  $S_0 = \int \mathcal{L}_0 d^d x$  under  $A_\mu^a \rightarrow A_\mu^a + \mathcal{G}_\mu^a$

$$\mathcal{O}_{\text{EOM}}^{(N)} = \frac{\delta S_0}{\delta A_\mu^a} \mathcal{G}_\mu^a(A_{\mu_1}^{a_1}, \partial_{\mu_1} A_{\mu_2}^{a_1}, \dots) = \underbrace{(D^\nu F_\nu)^a}_{\text{Lead Twist}} \mathcal{G}^a(A^{a_1}, \partial A^{a_1} \dots)$$

# Equation of Motion Operators

- Variations of  $S_0 = \int \mathcal{L}_0 d^d x$  under  $A_\mu^a \rightarrow A_\mu^a + \mathcal{G}_\mu^a$

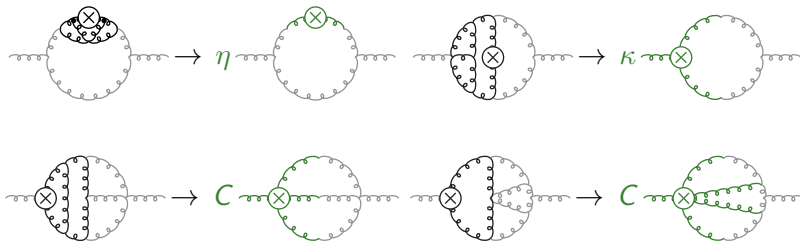
$$\mathcal{O}_{\text{EOM}}^{(N)} = \frac{\delta S_0}{\delta A_\mu^a} \mathcal{G}_\mu^a(A_{\mu_1}^{a_1}, \partial_{\mu_1} A_{\mu_2}^{a_2}, \dots) = \underbrace{(D^\nu F_\nu)^a}_{\text{Lead Twist}} \mathcal{G}^a(A^{a_1}, \partial A^{a_1} \dots)$$

- Concrete ansatz with general couplings

$$\begin{aligned} \mathcal{O}_{\text{EOM}}^{(N)} = & (D^\mu F_\mu)^a \left[ \eta \partial^{N-2} A^a + g f^{aa_1 a_2} \sum_{i_1+i_2=N-3} \kappa_{i_1 i_2} (\partial^{i_1} A^{a_1}) (\partial^{i_2} A^{a_2}) \right. \\ & + g^2 \sum_{\substack{i_1+i_2+i_3 \\ N-4}} \left( \underbrace{\kappa_{i_1 i_2 i_3}^{(1)} f^{aa_1 z} f^{a_2 a_3 z} + \kappa_{i_1 i_2 i_3}^{(2)} d_4^{aa_1 a_2 a_3} + \kappa_{i_1 i_2 i_3}^{(3)} d_{4ff}^{aa_1 a_2 a_3}}_{C_{i_1 i_2 i_3}^{aa_1 a_2 a_3}} \right) (\partial^{i_1} A^{a_1}) \dots (\partial^{i_3} A^{a_3}) \\ & \left. + g^3 \sum_{\substack{i_1+\dots+i_4 \\ N-5}} \left( \underbrace{\kappa_{i_1 \dots i_4}^{(1)} (f f f)^{aa_1 a_2 a_3 a_4} + \kappa_{i_1 \dots i_4}^{(2)} d_{4f}^{aa_1 a_2 a_3 a_4}}_{C_{i_1 \dots i_4}^{aa_1 \dots a_4}} \right) (\partial^{i_1} A^{a_1}) \dots (\partial^{i_4} A^{a_4}) + O(g^4) \right] \end{aligned}$$

# Diagrammatic interpretation

- $\mathcal{O}_{\text{EOM}}^{(N)}$  cancel subdivergences in **gluonic** diagrams at 4 loops



Coefficients (e.g.  $C_{i_1 \dots i_3}^{a \dots a_3}$ ) have colour structure of  $n$ -point graphs.

# Generalised Gauge Invariance

Define  $\mathcal{L}_{\text{EGI}}$  to include all purely *gluonic* terms

$$S_{\text{EGI}} = \int \mathcal{L}_{\text{EGI}} d^d x = \int \underbrace{\mathcal{L}_0 + \mathcal{C}_1 \mathcal{O}_1^{(N)}}_{\text{Gauge invariant}} + \underbrace{\mathcal{O}_{\text{EOM}}^{(N)}}_{\sim \delta S_0} d^d x$$

$\mathcal{C}_1 \rightarrow$  coupling of  $\mathcal{O}_1^{(N)}$ , analogous to  $\eta, \kappa_{ij}, \kappa_{ijk}^{(1)}, \dots$

## Generalise gauge transformations

$$\delta A_\mu^a = \delta_\omega A_\mu^a + \delta_\omega^\Delta A_\mu^a \iff \delta S_{\text{EGI}} = 0.$$

We find the *general solution* for every class of EOM fixed by  $\mathcal{G}_\mu^a$ .

$$\delta_\omega^\Delta A_\mu^a = -\delta_\omega \mathcal{G}_\mu^a + g f^{abc} \mathcal{G}_\mu^b \omega^c$$



# Generalised BRST transformations

Generalised gauge transf.  $\rightarrow$  Generalised BRST transf.

$$\delta A_\mu^a = \delta_\omega A_\mu^a + \delta_\omega^\Delta A_\mu^a \xrightarrow{\omega^a \rightarrow c^a} s'(A_\mu^a) = s(A_\mu^a) + s_\Delta(A_\mu^a)$$

From the expression of  $\delta_\omega^\Delta A_\mu^a$  we get

$$s_\Delta A_\mu^a = -s(G_\mu^a) + g f^{abc} G_\mu^b c^c \quad (2)$$

Rest of BRST transf. **unchanged** i.e.  $s_\Delta(c^a) = s_\Delta(\bar{c}^a) = s_\Delta(b^a) = 0$ .  
 Automatically nilpotent by eq.(2)

$$s'^2(A_\mu^a) = \{s, s_\Delta\} A_\mu^a = 0$$

# Ghost operators

Define complete Lagrangian invariant under generalised BRST

$$\tilde{\mathcal{L}} = \underbrace{\mathcal{L}_0 + c_1 \mathcal{O}_1^{(N)} + \mathcal{O}_{\text{EOM}}^{(N)}}_{\mathcal{L}_{\text{EGI}}} + s' \underbrace{\left[ \bar{c}^a \left( \partial^\mu A_\mu^a - \frac{\xi L}{2} b^a \right) \right]}_{\text{Original ancestor for gauge fix.}}$$

Ghost operators given by  $s_\Delta$ , once  $\mathcal{G}_\mu^a$  is known

$$\mathcal{O}_G^{(N)} = -\bar{c}^a \partial^\mu (s_\Delta A_\mu^a) = \bar{c}^a \partial^\mu \left( s(\mathcal{G}_\mu^a) - g f^{abc} \mathcal{G}_\mu^b c^c \right)$$

EOM and ghost operators are related to each other.

# Diagrammatic interpretation

Spin  $N = 2$

$$\mathcal{G}^a = \eta A^a$$

$$\mathcal{O}_{\text{EOM}}^{(2)} = \eta (D^\nu F_\nu)^a A^a, \quad \mathcal{O}_G^{(2)} = \eta \bar{c}^a \partial^2 c^a$$

There is a **single** alien operator

$$\mathcal{O}_2^{(2)} = \eta \left[ (D^\nu F_\nu)^a A^a + \bar{c}^a \partial^2 c^a \right].$$

**Same** mixing counterterm into **gluon** and **ghost** operators



## Increasing spin

The ansatz for  $\mathcal{G}^a$  is larger

$$\mathcal{G}^a = \eta \partial^2 A^a + 2 g \kappa_{01} f^{aa_1 a_2} A^{a_1} \partial A^{a_2} + g^2 \kappa_{000}^{(2)} d^{aa_1 a_2 a_3} A^{a_1} A^{a_2} A^{a_3}$$

which leads to

Spin  $N = 4$

$$\mathcal{O}_{\text{EOM}}^{(4)} = (D.F)^a \left[ \eta \partial^2 A^a + 2 g \kappa_{01} f^{aa_1 a_2} A^{a_1} \partial A^{a_2} + g^2 \kappa_{000}^{(2)} d^{aa_1 a_2 a_3} A^{a_1} A^{a_2} A^{a_3} \right]$$

$$\begin{aligned} \mathcal{O}_G^{(4)} = & - \eta \partial \bar{c}^a \left[ \partial^3 c^a + g f^{abc} \left( 2 \partial A^b \partial c^c + A^b \partial^2 c^c \right) \right] \\ & - 2 g \kappa_{01} \partial \bar{c}^a \left[ f^{abc} \left( A^b \partial^2 c^c - \partial A^b \partial c^c \right) + g f^{abz} f^{cdz} A^b A^c \partial c^d \right] \\ & - 3 g^2 \kappa_{000}^{(2)} d_4^{abcd} \partial \bar{c}^a A^b A^c \partial c^d \end{aligned}$$

## Generalised anti-BRST invariance

$$\tilde{\mathcal{L}} = \mathcal{L}_0 + c_1 \mathcal{O}_1^{(N)} + \mathcal{O}_{\text{EOM}}^{(N)} + \bar{s}' \left[ c^a \left( \frac{\xi_L}{2} b^a - \partial^\mu A_\mu^a \right) \right]$$

where

$$\bar{s}' A_\mu^a = \bar{s} A_\mu^a + \bar{s}_\Delta A_\mu^a, \quad \bar{s}_\Delta A_\mu^a = -\bar{s} (\mathcal{G}_\mu^a) + g f^{abc} \mathcal{G}_\mu^b \bar{c}^c$$

New expression for **ghost operators**  $\rightarrow$  new non-trivial relations

$$\mathcal{O}_G^{(N)} = \underbrace{-\bar{c}^a \partial^\mu (s_\Delta A_\mu^a)}_{\text{from BRST}} = \underbrace{c^a \partial^\mu (\bar{s}_\Delta A_\mu^a)}_{\text{from anti-BRST}}$$

For instance, at  $N = 4$ , this implies  $2\kappa_{01} = \eta$ .

# Summary

- Complete set of gluonic gauge invariant + Aliens

$$\tilde{\mathcal{L}} = \mathcal{L}_0 - \frac{(\partial^\mu A_\mu^a)^2}{2\xi_L} - \bar{c}^a \partial^\mu D_\mu^{ab} c^b + \underbrace{\mathcal{C}_1 \mathcal{O}_1^{(N)}}_{\text{Gauge inv.}} + \underbrace{\sum_{i>1} \mathcal{C}_i \mathcal{O}_i^{(N)}}_{\mathcal{O}_{\text{EOM}}^{(N)} + \mathcal{O}_G^{(N)}}$$

- $\mathcal{O}_{\text{EOM}}^{(N)} = (D^\mu F_\mu)^a \mathcal{G}^a$ , where  $\mathcal{G}^a$  has known ansatz.
- $\mathcal{O}_G^{(N)} = -(\partial \bar{c}^a) (s(\mathcal{G}^a) - g f^{abc} \mathcal{G}^b c^c)$ , **automatically**.
- Consistency with anti-BRST  $\rightarrow \{\mathcal{C}_{i>2}\}$  restricted to minimal set.  
 E.g. for  $N = 4$ ,

$$\{\mathcal{C}_i\} = \{\eta, \kappa_{000}^{(2)}\} \text{ or } \{\mathcal{C}_i\} = \{\kappa_{01}, \kappa_{000}^{(2)}\}.$$

## Renormalising operators with $N = 2$

$$\mathcal{O}_1^{(2)} = \frac{1}{2} F_\mu^a F^{a;\mu}, \quad \mathcal{O}_2^{(2)} = (D^\nu F_\nu)^a A^a + \bar{c}^a \partial^2 c^a$$

- Renormalisation constants: mixing

$$\delta Z_{12}^{(2)} = -a \frac{C_A}{2\epsilon} + a^2 C_A^2 \left[ \frac{19}{24\epsilon^2} + \frac{5}{48} \frac{\xi}{\epsilon} - \frac{35}{48\epsilon} \right] + a^3 C_A^3 \left[ -\frac{779}{432\epsilon^3} + \frac{1}{\epsilon^2} \left( \frac{2807}{864} - \frac{35\xi}{216} + \frac{5\xi^2}{288} \right) + \frac{1}{\epsilon} \left( -\frac{16759}{7776} - \frac{11\zeta_3}{72} + \frac{377\xi}{1728} + \frac{5\zeta_3 \xi}{72} - \frac{65\xi^2}{1728} \right) \right] + O(a^4)$$

- Renormalisation constants: gauge invariant operator

$$Z_{11}^{(2)} = 1 + O(a^5)$$

Agreement with general theorem (Freedman, Muzinich, Weinberg 1974) and explicit calculations (Moch, Ruijl, Ueda, Vermaseren, Vogt 2021).

# Renormalising operators with $N = 4$

- Operator basis and mixing

$$O_1^{(4)} = \frac{1}{2} \text{Tr}[F_\nu D^2 F^\nu],$$

$$O_2^{(4)} = (D.F)^a \left[ \partial^2 A^a + g f^{abc} A^b \partial A^c \right] - \partial \bar{c}^a \partial^3 c^a \\
 - g f^{abc} \partial \bar{c}^a \left[ 2A^b \partial^2 c^c + \partial A^b \partial c^c \right] \\
 - g^2 f^{abe} f^{cde} \partial \bar{c}^a A^b A^c \partial c^d,$$

$$O_3^{(4)} = d^{abcd} \left[ (D.F)^a A^b A^c A^d - 3 \partial \bar{c}^a A^b A^c \partial c^d \right]$$

$$\delta Z_{12}^{(4)} = -\frac{a C_A}{12\epsilon} - a^2 C_A^2 \left[ \frac{97}{1440\epsilon^2} - \frac{\xi}{320\epsilon} + \frac{8641}{86400\epsilon} \right] \\
 + a^3 C_A^3 \left[ \frac{9437}{86400\epsilon^3} - \frac{1}{\epsilon^2} \left( \frac{1520341}{15552000} - \frac{853\xi}{86400} \right) \right. \\
 \left. - \frac{1}{\epsilon} \left( \frac{166178237}{466560000} + \frac{\zeta_3}{2400} - \frac{37199\xi}{648000} - \frac{37\zeta_3 \xi}{9600} \right) \right]$$

$$\delta Z_{13}^{(4)} = \frac{a C_A}{24\epsilon}$$

- Gauge invariant operator renormalisation agrees with (Moch, Ruijl, Ueda, Vermaseren, Vogt 2021)

$$\delta Z_{11}^{(4)} = a \frac{21 C_A}{5\epsilon} + a^2 C_A^2 \left( \frac{28}{25\epsilon^2} + \frac{7121}{1000\epsilon} \right) - a^3 C_A^3 \left( \frac{1316}{1125\epsilon^3} + \frac{151441}{45000\epsilon^2} - \frac{103309639}{4050000\epsilon} \right) \\
 + a^4 \left\{ C_A^4 \left[ \frac{11186}{5625\epsilon^4} + \frac{1512989}{450000\epsilon^3} - \frac{5437269017}{162000000\epsilon^2} + \frac{1}{\epsilon} \left( \frac{1502628149}{13500000} + \frac{1146397\zeta_3}{45000} \right. \right. \right. \\
 \left. \left. \left. - \frac{126\zeta_5}{5} \right) \right] + \frac{d_{AA}}{N_A} \left( \frac{21623}{600\epsilon} + \frac{3899\zeta_3}{15\epsilon} - \frac{1512\zeta_5}{5\epsilon} \right) \right\}$$



# Operator basis for $N = 6$

Alien operators  $\mathcal{O}_2^{(6)} \dots \mathcal{O}_6^{(6)}$

$$\begin{aligned} \mathcal{O}_2^{(6)} = & (D.F)^a \partial^4 A^a - \partial \bar{c}^a \partial^5 c^a + g f^{abc} \left[ \frac{5}{3} (D.F)^a A^b \partial^3 A^c - \partial \bar{c}^a \left( \frac{8}{3} A^b \partial^4 c^c \right. \right. \\ & \left. \left. + 4 \partial A^b \partial^3 c^c + 6 \partial^2 A^b \partial^2 c^c + \frac{7}{3} \partial^3 A^b \partial c^c \right) \right] + g^2 (ff)^{abcd} \left[ \frac{5}{3} (D.F)^a \partial A^b \partial A^c A^d \right. \\ & - \frac{5}{3} \partial \bar{c}^a \left( A^b A^c \partial^3 c^d + 4 A^b \partial A^c \partial^2 c^d + 3 A^b \partial^2 A^c \partial c^d + \partial A^b \partial A^c \partial c^d \right. \\ & \left. \left. - 2 \partial A^b A^c \partial^2 c^d \right) \right] - g^3 (fff)^{abcde} \left[ \frac{2}{3} (D.F)^a A^b A^c A^d \partial A^e \right. \\ & \left. + \frac{1}{3} \partial \bar{c}^a \left( 2 A^b A^c A^d \partial^2 c^e + 6 A^b A^c \partial A^d \partial c^e - A^b \partial A^c A^d \partial c^e \right. \right. \\ & \left. \left. - 8 \partial A^b A^c A^d \partial c^e \right) \right] \end{aligned}$$

Simplest vertex  $\rightarrow$  **2-point**. Higher-point interactions have the same coupling.

$$\begin{aligned} \mathcal{O}_3^{(6)} = & g f^{abc} \left[ (D.F)^a \left( 2 \partial A^b \partial^2 A^c + \frac{4}{3} A^b \partial^3 A^c \right) - \frac{4}{3} \partial \bar{c}^a \left( A^b \partial^4 c^c + \frac{3}{2} \partial A^b \partial^2 c^c \right. \right. \\ & \left. \left. - \frac{3}{2} \partial^2 A^b \partial^2 c^c - \partial^3 A^b \partial c^c \right) \right] + g^2 (ff)^{abcd} \left[ - \frac{10}{3} (D.F)^a \partial A^b A^c \partial A^d \dots \right. \end{aligned}$$

$\mathcal{O}_3^{(6)}$  has a leading **3-point** vertex.  $\mathcal{O}_4^{(6)} \dots \mathcal{O}_6^{(6)}$  start with a 4-point.

# Renormalisation constants for $N = 6$

- Mixing with alien operators

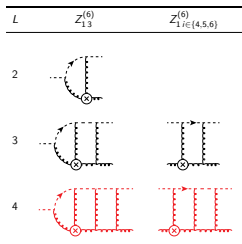
$$\delta Z_{12}^{(6)} = -a \frac{C_A}{30\epsilon} - a^2 C_A^2 \left[ \frac{653}{10080\epsilon^2} + \frac{19\xi}{20160\epsilon} + \frac{185093}{4233600\epsilon} \right] + O(a^3)$$

$$\delta Z_{13}^{(6)} = -a \frac{C_A}{48\epsilon} - a^2 C_A^2 \left[ \frac{2021}{40320\epsilon^2} + \frac{235813}{8467200\epsilon} + O(\xi) \right] + O(a^3),$$

$$\delta Z_{14}^{(6)} = -a \frac{C_A}{32\epsilon} + O(a^2),$$

$$\delta Z_{15}^{(6)} = a \frac{C_A}{24\epsilon} + O(a^2),$$

$$\delta Z_{16}^{(6)} = O(a^2).$$



- Gauge invariant operator renormalisation agrees with  
 (Larin, Nogueira, van Ritbergen, Vermaseren 1996)

$$\delta Z_{11}^{(6)} = a \frac{83 C_A}{14\epsilon} + a^2 C_A^2 \left( \frac{7885}{1176\epsilon^2} + \frac{1506899}{148176\epsilon} \right) + a^3 C_A^3 \left( -\frac{465215}{148176\epsilon^3} + \frac{243375989}{18670176\epsilon^2} + \frac{96390174479}{2613824640\epsilon} \right) + O(a^4),$$

## Conclusion

- Obj: OPE approach to get *moments* of  $N^3$ LO splitting functions.
- Conceptual obstacle: **alien** operators mixing in renormalisation.

### This work

- General method to construct all the aliens.
- Explicit bases of aliens up to 4 loops.
- Results of moments  $N = 2, 4$  to 4 loops and  $N = 6$  to 3 loops.

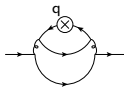
### In progress

- Extension to fermionic operators.
- Calculation of higher moments to 4 loops.

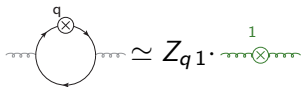
# Thank you for the attention!

# Quark operators mixing

Consider quark pure singlet e.g.

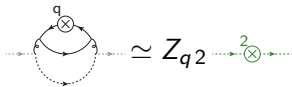
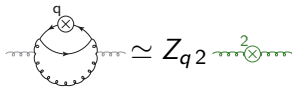


Mixing of quark singlet operators into the aliens starts at 3 loops



Only the physical operator  $\mathcal{O}_1^{(N)}$  enters **subdiagrams** at **1 loop**

By BRST invariance, aliens involve both gluon and ghost.



Ghost diagrams start at 2 loops  $\rightarrow$  subdivergence at 3 loops.

# Comparison with Joglekar and Lee

$\mathcal{O}_{\text{EOM}}$  generated by *classical* EOM.  $\mathcal{O}_G = s_\Delta (\bar{c} \partial^\mu A_\mu^a)$  *generalised-BRST-exact*. But

$$\mathcal{O}_{\text{EOM}}^{(N)} = \frac{\delta S_0}{\delta A_\mu^a} \mathcal{G}_\mu^a, \quad \mathcal{O}_G^{(N)} = -s (\bar{c}^a \partial^\mu \mathcal{G}_\mu^a) + \underbrace{\left[ \frac{1}{\xi_L} \partial^\mu (\partial^\nu A_\nu^a) + g f^{abc} (\partial^\mu \bar{c}^b) c^c \right]}_{\text{Gauge fix. and ghost terms in EOM of } A_\mu^a} \mathcal{G}_\mu^b$$

$$\mathcal{O}_{\text{EOM}} + \mathcal{O}_G = s ((\partial^\mu \bar{c}^a) \mathcal{G}_\mu^a) + \frac{\delta S}{\delta A_\mu^a} \mathcal{G}_\mu^a$$

Now we compare with Joglekar and Lee operators of class I and II

$$\mathcal{O}_I = s (F(A, c, \bar{c})) + \frac{\delta S}{\delta A_\mu^a} \frac{\partial F(A, c, \bar{c})}{\partial (\partial_\mu \bar{c}^a)} \Rightarrow F = (\partial^\mu \bar{c}^a) \mathcal{G}_\mu^a$$

$$\mathcal{O}_{II} = \frac{\delta S}{\delta c^a} X^a(A, c, \bar{c}) \Rightarrow X^a = 0 \quad \frac{\delta S}{\delta c^a} \text{ is twist-3!}$$

# Space of alien operators

Spin $N$	2	4	6	8	10	12	14	16
w aBRST	1	2	5	12	25	50	87	140
w/o aBRST	1	3	11	30	66	126	215	339

# Generalised Gauge transformation (I)

Expression of EOM operators

$$\mathcal{G}^a = \sum_{k=1}^{\infty} g^{k-1} \sum_{\substack{i_1+\dots+i_k \\ =N-k-1}} C_{i_1\dots i_k}^{a;a_1\dots a_k} \left( \partial^{i_1} A^{a_1} \right) \dots \left( \partial^{i_k} A^{a_k} \right)$$

$C_{i_1\dots i_k}^{a;a_1\dots a_k}$  given up to 4 loops. Then  $\delta_{\omega}^{\Delta} A_{\mu}^a = -\Delta_{\mu} (\delta_{\omega} \mathcal{G}^a - g f^{abc} \mathcal{G}^b \omega^c)$  reads

$$\begin{aligned} \delta_{\omega}^{\Delta} A_{\mu}^a &= -\Delta_{\mu} \sum_{k=1}^{\infty} \sum_{\substack{i_1+\dots+i_k \\ N-k-1}} \left( \partial^{i_1} A^{a_1} \right) \dots \left( \partial^{i_k+1} \omega^{a_k} \right) \sum_{\sigma \in Z_k} C_{i_{\sigma(1)} \dots i_{\sigma(k)}}^{a;a_{\sigma(1)} \dots a_{\sigma(k)}} \\ &+ g \Delta_{\mu} \sum_{k=1}^{\infty} \sum_{\substack{i_1+\dots+i_{k+1} \\ N-k-2}} \left( \partial^{i_1} A^{a_1} \right) \dots \left( \partial^{i_k} A^{a_k} \right) \left( \partial^{i_{k+1}+1} \omega^{a_{k+1}} \right) \sum_{m=1}^k \binom{i_m + i_{k+1} + 1}{i_m} \\ &\quad \times C_{i_1 \dots i_m + i_{k+1} + 1 \dots i_k}^{a;a_1 \dots a_{m-1} b a_{m+1} \dots a_k} f^{b a_m a_{k+1}} \\ &= -\Delta_{\mu} \sum_{k=1}^{\infty} g^{k-1} \sum_{\substack{i_1+\dots+i_k \\ =N-k-1}} \tilde{C}_{i_1 \dots i_k}^{a;a_1 \dots a_k} \left( \partial^{i_1} A^{a_1} \right) \dots \left( \partial^{i_{k-1}} A^{a_{k-1}} \right) \left( \partial^{i_k+1} \omega^{a_k} \right) \end{aligned}$$



# Generalised Gauge transformation (II)

Explicit coefficients up to 4 loops

$$\begin{aligned}\tilde{C}_{i_1 i_2}^{a; a_1 a_2} &= \eta_{i_1 i_2}^{(1)} f^{a; a_1 a_2}, \\ \tilde{C}_{i_1 i_2 i_3}^{a; a_1 a_2 a_3} &= \eta_{i_1 i_2 i_3}^{(1)} (ff)^{aa_1 a_2 a_3} + \eta_{i_1 i_2 i_3}^{(2)} d^{aa_1 a_2 a_3} + \eta_{i_1 i_2 i_3}^{(3)} d_{4ff}^{aa_1 a_2 a_3}, \\ \tilde{C}_{i_1 i_2 i_3 i_4}^{a; a_1 a_2 a_3 a_4} &= \eta_{i_1 i_2 i_3 i_4}^{(1)} (fff)^{aa_1 a_2 a_3 a_4} + \eta_{i_1 i_2 i_3 i_4}^{(2a)} d_{4f}^{aa_1 a_2 a_3 a_4} + \eta_{i_1 i_2 i_3 i_4}^{(2b)} d_{4f}^{aa_4 a_1 a_2 a_3}.\end{aligned}$$

The  $\eta$  coefficients are given by

$$\begin{aligned}\eta_{i_1 i_2 i_3}^{(1)} &= 2\kappa_{i_1(i_2+i_3+1)} \binom{i_2+i_3+1}{i_2} + 2 \left[ \kappa_{i_1 i_2 i_3}^{(1)} + \kappa_{i_3 i_2 i_1}^{(1)} \right], & \eta_{i_1 i_2}^{(1)} &= 2\kappa_{i_1 i_2} + \eta \left( \binom{i_1+i_2+1}{i_1} \right), \\ \eta_{i_1 i_2 i_3 i_4}^{(1)} &= 2 \left[ \kappa_{i_1 i_2(i_3+i_4+1)}^{(1)} + \kappa_{(i_3+i_4+1) i_2 i_1}^{(1)} \right] \binom{i_3+i_4+1}{i_3} \\ &+ 2 \left[ \kappa_{i_1 i_2 i_3 i_4}^{(1)} + \kappa_{i_1 i_4 i_3 i_2}^{(1)} + \kappa_{i_4 i_1 i_3 i_2}^{(1)} + \kappa_{i_4 i_3 i_1 i_2}^{(1)} \right], & \eta_{i_1 i_2 i_3}^{(2)} &= 3\kappa_{i_1 i_2 i_3}^{(2)}, \\ \eta_{i_1 i_2 i_3 i_4}^{(2a)} &= 3\kappa_{i_1 i_2(i_3+i_4+1)}^{(2)} \binom{i_3+i_4+1}{i_3} + 2\kappa_{i_1 i_2 i_3 i_4}^{(2)}, & \eta_{i_1 i_2 i_3}^{(3)} &= 2 \left[ \kappa_{i_1 i_2 i_3}^{(3)} - \kappa_{i_3 i_2 i_1}^{(3)} \right], \\ & & \eta_{i_1 i_2 i_3 i_4}^{(2b)} &= 2\kappa_{i_4 i_1 i_2 i_3}^{(2)}.\end{aligned}$$

# Anti-BRST identities

Consistency with anti-BRST symmetry requires

$$-\bar{c}^a \partial^\mu (s_\Delta A_\mu^a) = c^a \partial^\mu (\bar{s}_\Delta A_\mu^a)$$

which implies constraints on the  $\eta$  coefficients in  $s_\Delta A_\mu^a$  and  $\bar{s}_\Delta A_\mu^a$

$$\eta_{i_1 i_2}^{(1)} = - \sum_{s_1=0}^{i_1} (-1)^{s_1+i_2} \binom{s_1+i_2}{s_1} \eta_{(i_1-s_1)(i_2+s_1)}^{(1)},$$

$$\eta_{i_1 i_2 i_3}^{(1)} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1+s_2+i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta_{(i_2-s_2)(i_1-s_1)(i_3+s_1+s_2)}^{(1)},$$

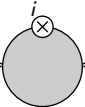
$$\eta_{i_1 i_2 i_3}^{(2)} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1+s_2+i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta_{(i_1-s_1)(i_2-s_2)(i_3+s_1+s_2)}^{(2)},$$

$$\eta_{i_1 i_2 i_3}^{(3)} = \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \frac{(s_1+s_2+i_3)!}{s_1! s_2! i_3!} (-1)^{s_1+s_2+i_3} \eta_{(i_2-s_2)(i_1-s_1)(i_3+s_1+s_2)}^{(3)}$$

# Anti-BRST identities (continued)

$$\begin{aligned}
 \eta_{i_1 i_2 i_3 i_4}^{(1)} &= - \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \sum_{s_3=0}^{i_3} \frac{(s_1 + s_2 + s_3 + i_4)!}{s_1! s_2! s_3! i_4!} \\
 &\quad \times (-1)^{s_1 + s_2 + s_3 + i_4} \eta_{(i_3 - s_3)(i_2 - s_2)(i_1 - s_1)(i_4 + s_1 + s_2 + s_3)}^{(3)}, \\
 \eta_{i_1 i_2 i_3 i_4}^{(2a)} &= - \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \sum_{s_3=0}^{i_3} \frac{(s_1 + s_2 + s_3 + i_4)!}{s_1! s_2! s_3! i_4!} \\
 &\quad \times (-1)^{s_1 + s_2 + s_3 + i_4} \eta_{(i_1 - s_1)(i_2 - s_2)(i_3 - s_3)(i_4 + s_1 + s_2 + s_3)}^{(2a)}, \\
 \eta_{i_1 i_2 i_3 i_4}^{(2b)} &= \eta_{i_1 i_3 i_2 i_4}^{(2a)} - \eta_{i_1 i_2 i_3 i_4}^{(2a)} + \sum_{s_1=0}^{i_1} \sum_{s_2=0}^{i_2} \sum_{s_3=0}^{i_3} \frac{(s_1 + s_2 + s_3 + i_4)!}{s_1! s_2! s_3! i_4!} \\
 &\quad \times (-1)^{s_1 + s_2 + s_3 + i_4} \eta_{(i_1 - s_1)(i_2 - s_2)(i_3 - s_3)(i_4 + s_1 + s_2 + s_4)}^{(2b)}.
 \end{aligned}$$

# Calculation in the background field method

$$\Gamma_{i;BB}^{(N)}(g, \xi) = \text{Diagram} = \begin{cases} \Gamma_{i;BB}^{(N),0} + \delta\Gamma_{i;BB}^{(N)}(g, \xi) & i = 1 \\ \delta\Gamma_{i;BB}^{(N)}(g, \xi) & i \neq 1 \end{cases}$$


Extraction of the physical renormalisation constant  $Z_{11}^{(N)} = 1 + \delta Z_{11}^{(N)}$

$$\delta Z_{11}^{(N)} = -\frac{1}{Z_B \Gamma_{1;BB}^{(N),0}} K_\epsilon \left[ Z_B \sum_{i \geq 1} Z_{1i}^{(N)} \delta\Gamma_{i;BB}^{(N)}(g_B, \xi_B) \right]$$

- $Z_B = Z_a^{-1}$  B-field renormalisation  
 $K_\epsilon$  extracts poles in  $\epsilon$
- $Z_{1i}^{(N)}$  cancel subdivergences  
 → 2-, 3-pt subgraphs up to 3 loops  
 → 4-pt subgraphs up to 2 loops  
 → 5-pt subgraphs up to 1 loop

$L$	$Z_{12}^{(N \geq 2)}$	$Z_{13}^{(4)}$	$Z_{13}^{(6)}$	$Z_{1i}^{(6)}(4,5,6)$
2				
3				
4				