Theory group DESY Zeuthen

Analysing first order factorizable systems of differential equations in one variable

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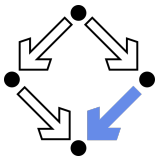
SAGEX
Scattering Amplitudes:
from Geometry to Experiment



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Amplitude computation in Quantum Field Theory (QFT)

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- For that, we use the **integration-by-parts** (IBP) method to reduce the thousands of initial Feynman integrals to a handful of **master integrals** (MIs) $\mathcal{I} = (\mathcal{I}_1(x,\epsilon), \ldots, \mathcal{I}_n(x,\epsilon))$ (with $s = -\frac{(1-x)^2}{x}$ and $\epsilon = D-4$ the dimensional parameter) and a system in the MIs $\mathcal{I}$:

$$\frac{d}{dx}\mathcal{I}(x,\epsilon) = \mathcal{M}(x,\epsilon)\mathcal{I}(x,\epsilon) + \mathcal{R}(x,\epsilon)$$

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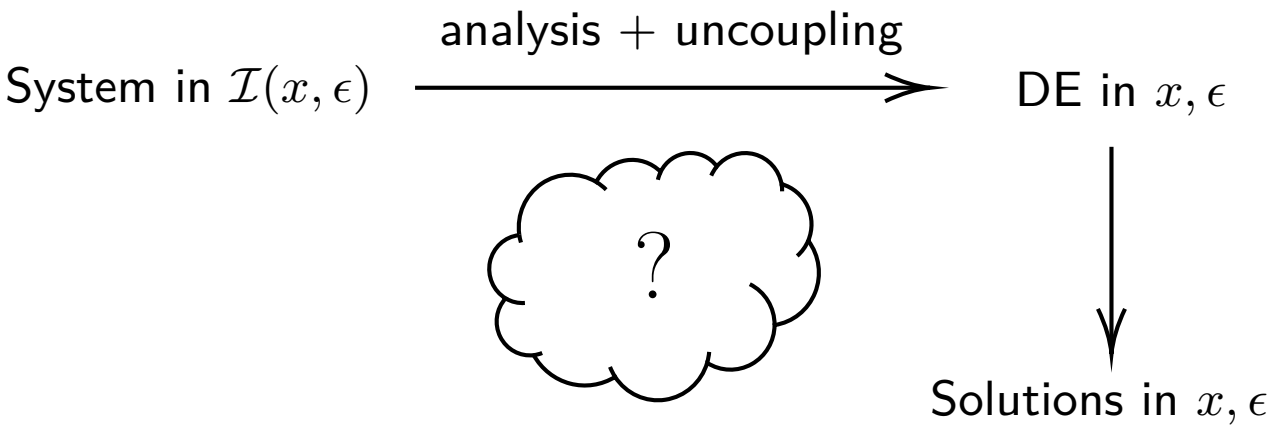
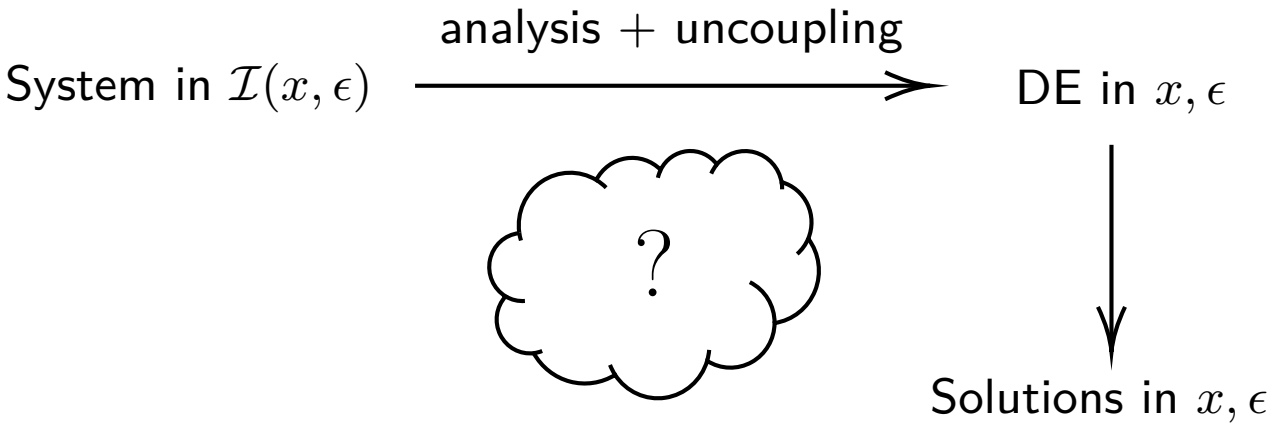
Aim: We need an algorithm to preprocess and solve this system of first order differential equations.


analysis + uncoupling
System in $\mathcal{I}(x, \epsilon)$
DE in x, ϵ
?
Solutions in x, ϵ



- In this presentation, we will concentrate on the preprocessing analysis algorithm, that has been the main focus of our recent work.

1. General setting
   - Assumptions
   - Main ideas for solving the system
   - Why preprocessing the system?
2. Finding the best uncouplings
   - Clustering the system
   - Zürcher scheme
   - Gauß scheme
3. Computing the $\epsilon-$orders for the whole system
   - Local extraction of the $\epsilon-$orders
   - Global correction and trees
   - Building trees in different schemes

- Differentiation/IBP of MIs $\rightarrow$ $\underbrace{\text{elements of } \mathcal{I}}_{\mathcal{MI}} + \underbrace{\text{new integrals (Base Integrals – BIs)}}_{\mathcal{R} \text{ (inhomogeneous part)}}$

$$\frac{d}{dx}\mathcal{I}(x,\epsilon) = \mathcal{M}(x,\epsilon)\mathcal{I}(x,\epsilon) + \mathcal{R}(x,\epsilon) \tag{1}$$

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- Some of them will rely on **differential equations** only, some of them make use of **recurrence/difference equations** (and therefore `Sigma`)
- We concentrate here on the scheme where we expand in ϵ only in the latest stages of the computation and solve at this point recurrence equations

- In order to solve our system, it has to verify several general assumptions, which are the following:

Assumptions

1. $\mathcal{M}(x, \epsilon) \in \mathcal{M}_n(\mathbb{K}(x, \epsilon))$ ($\mathbb{K}$ is a computable subfield of $\mathbb{R}$) is invertible

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2. Each element of $\mathcal{R}(x,\epsilon) = (\mathcal{R}_1(x,\epsilon),\ldots,\mathcal{R}_n(x,\epsilon))$ and $\mathcal{I}(x,\epsilon) = (\mathcal{I}_1(x,\epsilon),\ldots,\mathcal{I}_n(x,\epsilon))$ can be expanded as a **power series** in x:

$$\forall i \in \{1,\ldots,n\} \quad \mathcal{R}_i(x,\epsilon) = \sum_{n=0}^{\infty} R_i(n,\epsilon)x^n, \quad \mathcal{I}_i(x,\epsilon) = \sum_{n=0}^{\infty} I_i(n,\epsilon)x^n$$

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3. The coefficients of the Taylor expansion above can themselves be **Laurent expanded**:

$$R_i(n,\epsilon) = \sum_{k=l}^{\infty} R_{i,k}(n)\epsilon^k, \quad I_i(n,\epsilon) = \sum_{k=l}^{\infty} I_{i,k}(n)\epsilon^k \text{ with } l \in \mathbb{Z}.$$

Where the coefficients $R_i(n,\epsilon)$ and $I_i(n,\epsilon)$ are **indefinite nested sums over hypergeometric products**

The main steps behind the solving algorithm are the following ones:

Solving iteratively the system (1)

1. Uncouple the system using the `OreSys` package. In all generality, we obtain one or several HODE of the form

$$\alpha_{0,i}(x,\epsilon)\mathcal{I}_i(x,\epsilon) + \cdots + \alpha_{m,i}(x,\epsilon)\mathrm{d}_x^{m_i}\mathcal{I}_i(x,\epsilon) = \beta_i(x,\epsilon) \tag{2}$$

where $\alpha_{i,j}(x,\epsilon) \in \mathbb{K}[x,\epsilon]$ and $\beta_i(x,\epsilon) \in \langle\{\mathcal{R}_i(x,\epsilon), \mathrm{d}_x^j\mathcal{R}_i(x,\epsilon)\}\rangle_{\mathbb{K}(x,\epsilon)}$, as well as several linear algebraic relations (LFs) for the other Mis of the form

$$\mathcal{I}_j(x,\epsilon) \in \langle\{\mathrm{d}_x^i\mathcal{I}_i(x,\epsilon), \mathcal{R}_i(x,\epsilon), \mathrm{d}_x^j\mathcal{R}_i(x,\epsilon)\}\rangle_{\mathbb{K}(x,\epsilon)}$$

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2. Taylor expanding $\mathcal{I}_i(x,\epsilon)$, plugging them into the HODEs and comparing coefficients we get recurrences of the form

$$a_0(n,\epsilon)I_i(n,\epsilon) + a_1(n,\epsilon)I_i(n+1,\epsilon) + \cdots + a_d(n,\epsilon)I_i(n+d,\epsilon) = b(n,\epsilon) \tag{3}$$

Where $d \in \mathbb{N}$, $a_i(n,\epsilon) \in \mathbb{K}[n,\epsilon]$ and $b(n,\epsilon)$ can be expanded in Laurent series.

Solve iteratively the system (2)

3. We solve the difference equation (3) order by order in ϵ up to the required order, i.e. we search for a Laurent expansion for $I_1(n,\epsilon)$ of the form

$$I_1(n,\epsilon) = \sum_{k=-l}^{\infty} I_{1,k}(n)\epsilon^k$$

where $l \in \mathbb{N}$ and the $I_{1,k}(n)$ can be given in terms of INSH, and after collecting terms with the same power of ϵ, we solve the $x,\epsilon-$independent difference equations using `Sigma`.

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4. Once we get a solution for the $\tilde{\mathcal{I}}_i(x,\epsilon)$ up to some order ℓ in ϵ, we plug it in the second step's relations in order to get the other MIs up to the required order in ϵ.

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Idea: Write a preprocessing algorithm that analyses the system, finds the best possible uncouplings and computes the required $\epsilon-$orders for the BIs to speed up the actual solving

Remarks:

1. Usually, the system that we initially get is quite huge (of order $n \sim 1000$)
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Preprocessing step 1: Clustering

We "triangularize/cluster" the system in n smaller sub-systems where the ith subsystem of coupled MIs depend effectively only on these MIs and on those from the systems 1 through $i-1$.

Algorithmically, one can proceed in the following way to cluster one big system:

- Write the dependence relations of the LHS on the MIs appearing in the RHS
  Example:

$$\mathrm{d}_x \mathcal{I}_2(x,\epsilon) = 2\mathcal{I}_1(x,\epsilon) + \frac{\mathcal{I}_3(x,\epsilon)}{x} \;\; : \;\; \mathcal{I}_2(x,\epsilon) \to \{\mathcal{I}_1(x,\epsilon), \mathcal{I}_3(x,\epsilon)\}$$

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Explicit example:

$$\begin{cases} \frac{1}{x^2-1} I_5'(x) &= \frac{x(x+2)}{x-2} I_5(x) + x I_3(x) + R_5(x) \\ (x+1) I_4'(x) &= -I_3(x) + \frac{I_1(x)}{x} + R_4(x) \\ \frac{x(x-2)}{3} I_3'(x) &= x I_4(x) + R_3(x) \\ I_2'(x) &= x^3 I_1(x) + R_2(x) \\ 2 I_1'(x) &= x I_1(x) - \frac{8}{x-1} I_2(x) + R_1(x) \end{cases}$$

- We write first all the fundamental rules:

$$I_5 \to \{I_3, I_5\},\ I_4 \to \{I_1, I_3\},\ I_3 \to \{I_4\},\ I_2 \to \{I_1\},\ I_1 \to \{I_1, I_2\}$$

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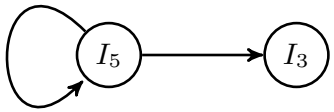

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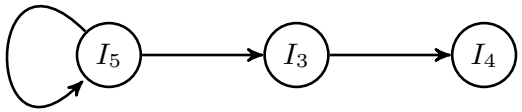

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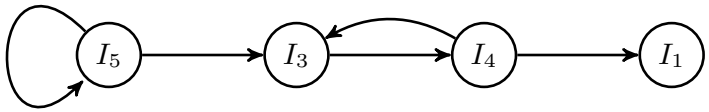

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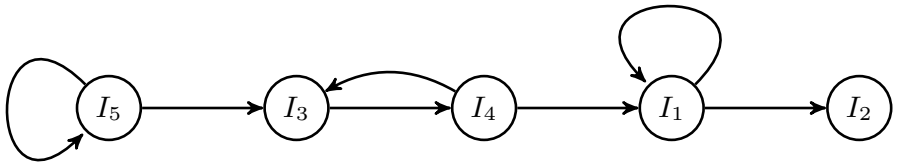

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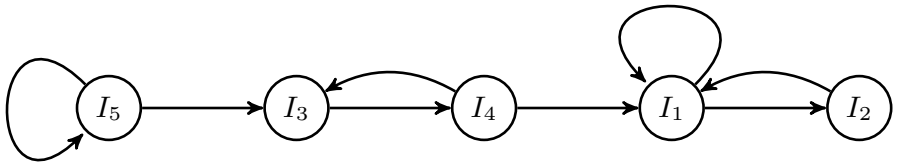

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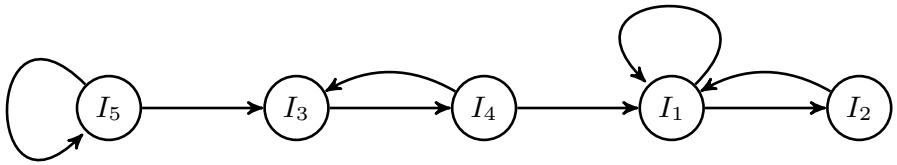

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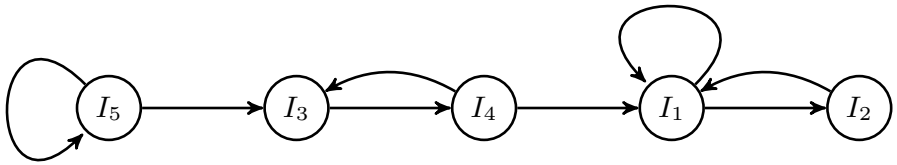

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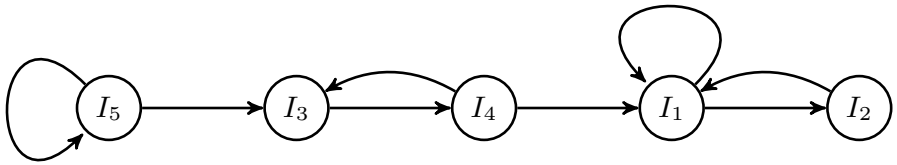

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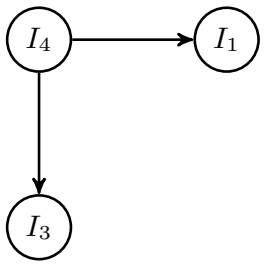

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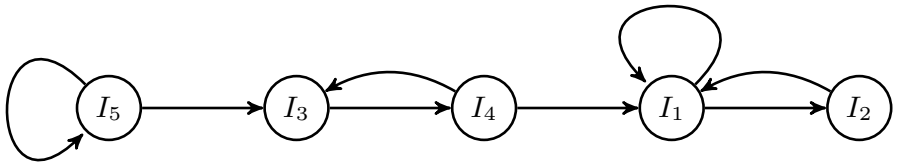

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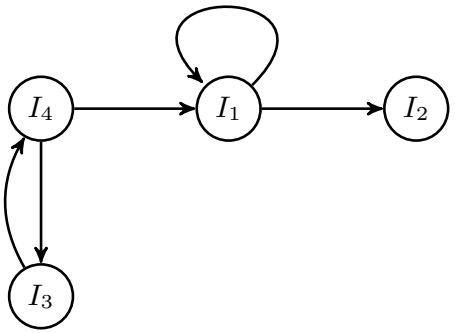
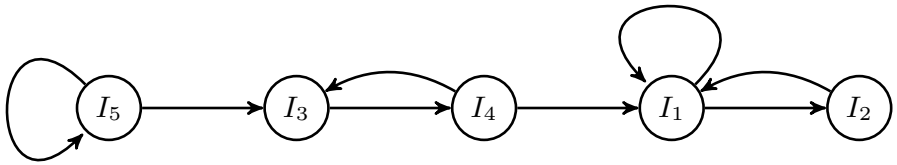


- We write first all the fundamental rules:

$$I_5 \to \{I_3, I_5\},\ I_4 \to \{I_1, I_3\},\ I_3 \to \{I_4\},\ I_2 \to \{I_1\},\ I_1 \to \{I_1, I_2\}$$



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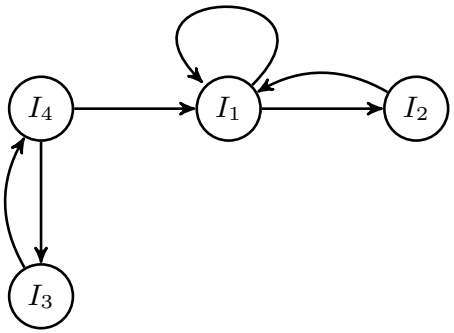
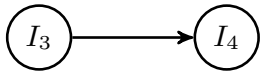
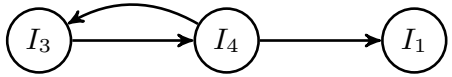
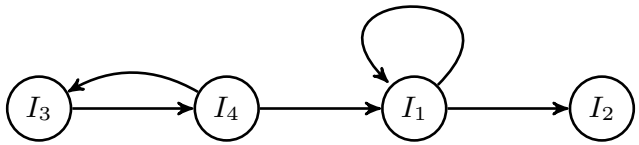


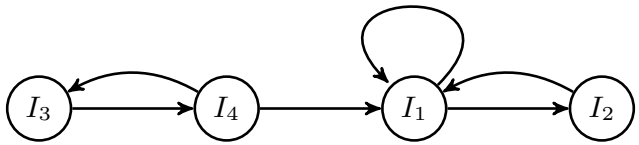
$$I_4 \to \{I_1, I_2, I_3, I_4\}$$


I_3


I_3
I_4


I_3
I_4
I_1


I_3
I_4
I_1
I_2



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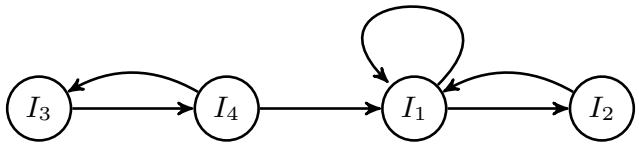

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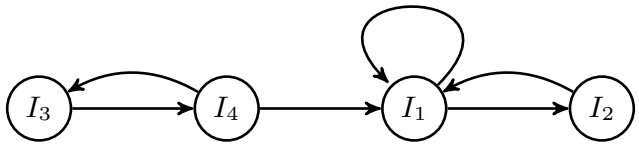

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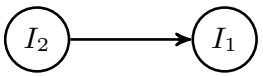

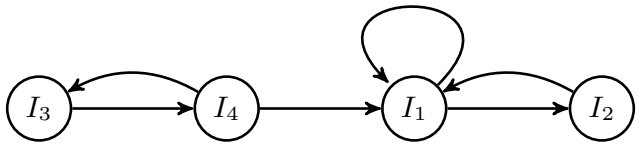

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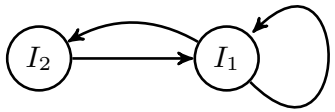

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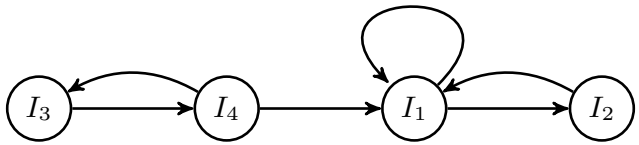

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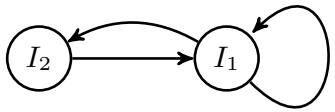

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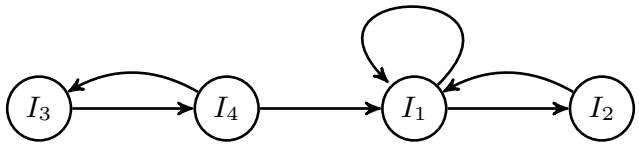

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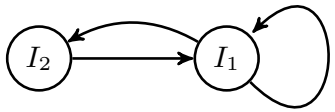

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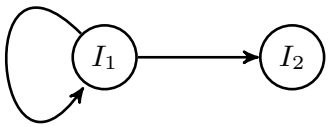

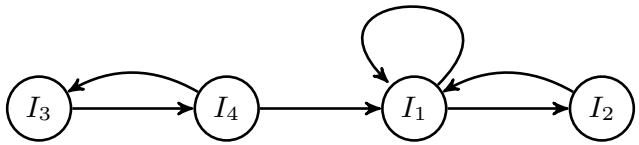

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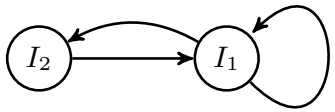

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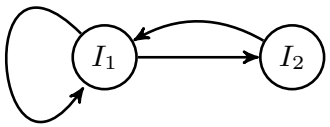

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The full rules are therefore the following:

$$\begin{cases} I_5 & \rightarrow & \{I_1, I_2, I_3, I_4, I_5\} \\ I_4 & \rightarrow & \{I_1, I_2, I_3, I_4\} \\ I_3 & \rightarrow & \{I_1, I_2, I_3, I_4\} \\ I_2 & \rightarrow & \{I_1, I_2\} \\ I_1 & \rightarrow & \{I_1, I_2\} \end{cases}$$

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  then $\mathcal{I}_i(x, \epsilon)$ is actually a trivial MI that can be moved to the base case integrals (that by assumption can be computed easily)
- A MI that appears as a BI in a subsystem is called an **induced BI** (IBI), as opposed to the real BI, that we call **fundamental BI** (FBI)

From now on, we consider a subsystem of the original system, and to distinguish them we will denote all the quantities with a tilde:

$$\frac{\mathrm{d}}{\mathrm{d}x}\tilde{\mathcal{I}}(x,\epsilon) = \tilde{\mathcal{M}}(x,\epsilon)\tilde{\mathcal{I}}(x,\epsilon) + \tilde{\mathcal{R}}(x,\epsilon) \quad \text{(of order } n\text{)}$$

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- All of these equations will depend **explicitely** on the $\tilde{\mathcal{R}}_i(x,\epsilon)$ variables and their derivatives (that we keep generic to speed up the uncoupling)

- More precisely, the `OreSys` Zürcher inital uncoupling actually makes use of dummy variables (here denoted with W_i) and we have to get rid of them:

Zürcher uncoupling (OreSys output)

- $\mathrm{HODE}\left(\underline{W_1, \{\mathrm{d}_x^j W_1\}}; \{\tilde{\mathcal{R}}_i, \mathrm{d}_x^j \tilde{\mathcal{R}}_i\}\right)$
- $$\begin{cases} W_2 &= F_2(W_1, \{\mathrm{d}_x^j W_1\}; \{\tilde{\mathcal{R}}_i, \mathrm{d}_x^j \tilde{\mathcal{R}}_i\}) \\ W_3 &= F_3(W_1, \{\mathrm{d}_x^j W_1\}; W_2, \{\mathrm{d}_x^j W_2\}; \{\tilde{\mathcal{R}}_i, \mathrm{d}_x^j \tilde{\mathcal{R}}_i\}) \\ \vdots & \qquad\qquad \vdots \\ W_n &= F_n(W_1, \{\mathrm{d}_x^j W_1\}; \cdots; W_{n-1}, \{\mathrm{d}_x^j W_{n-1}\}; \{\tilde{\mathcal{R}}_i, \mathrm{d}_x^j \tilde{\mathcal{R}}_i\}) \end{cases}$$
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Notes:

- Eliminating the W_i dummy variables and replacing the generic $\tilde{\mathcal{R}}_i$ by their actual values is sometimes as costly as the uncoupling itself
- Non-trivial case when $G_1(\{W_i, \mathrm{d}_x^j W_i\}) \neq W_1$

The very first step is to convert the `OreSys` output into the actual uncoupling with the full dependence on the BIs, that we note from now on $\tilde{\mathcal{J}}_1(x,\epsilon),\ldots,\tilde{\mathcal{J}}_m(x,\epsilon)$:

Zürcher uncoupling (with full dependence)

- $$\sum_{k=0}^{m} p_k(x,\epsilon)\mathrm{d}_x^k\tilde{\mathcal{I}}_1(x,\epsilon) = r(x,\epsilon)$$ HODE in $\tilde{\mathcal{I}}_1(x)$, $p_i(x,\epsilon)\in\mathbb{K}(x,\epsilon)$
- $$r(x,\epsilon) = \sum_{i=0}^{p}\sum_{j=1}^{m} r_{i,j}(x,\epsilon)\mathrm{d}_x^i\tilde{\mathcal{J}}_j(x,\epsilon)$$ $p\in\mathbb{N}$, $r_{i,j}(x,\epsilon)\in\mathbb{K}(x,\epsilon)$
- $$\tilde{\mathcal{I}}_j(x,\epsilon) = \sum_{i=0}^{m-1} a_{j,i}(x,\epsilon)\mathrm{d}_x^i\tilde{\mathcal{I}}_1(x,\epsilon) + \rho_k(x,\epsilon)$$ $j\in\{2,\ldots,m\}$, $a_{k,i}(x,\epsilon)\in\mathbb{K}(x,\epsilon)$, $\rho_j(x)$ like $r(x,\epsilon)$

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$$\tilde{\mathcal{I}}_2(x,\epsilon) = F(\{\tilde{\mathcal{I}}_1, \mathrm{d}_x^i \tilde{\mathcal{I}}_1\}) + \frac{1}{\epsilon^2}\tilde{\rho}_{2,1}(x,\epsilon)\tilde{\mathcal{J}}(x,\epsilon) + \epsilon\tilde{\rho}_{2,2}(x,\epsilon)\mathrm{d}_x\tilde{\mathcal{J}}(x,\epsilon)$$

Where $\mathbb{K}(x,\epsilon) \ni \tilde{\rho}_{2,i\in\{1,2\}}(x,\epsilon) \underset{\epsilon\to 0}{\longrightarrow} \tilde{\rho}_{2,i\in\{1,2\}}(x) \in \mathbb{K}(x)^\star$

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- This means that if we want for instance to solve the system up to the 1^{st} order in ϵ, we need to give an expansion of $\tilde{\mathcal{J}}(x,\epsilon) = \sum_{n=l}^{\infty} \tilde{\mathcal{J}}_n(x)\epsilon^n$ up to the 3^{rd} order in ϵ:

$$\begin{aligned}\tilde{\mathcal{I}}_2(x,\epsilon) \sim \tfrac{1}{\epsilon^2}\tilde{\mathcal{J}}(x,\epsilon) &= \tfrac{1}{\epsilon^2}\left(\cdots + \tilde{\mathcal{J}}_0(x) + \epsilon\tilde{\mathcal{J}}_1(x) + \epsilon^2\tilde{\mathcal{J}}_2(x) + \epsilon^3\tilde{\mathcal{J}}_3(x) + \cdots\right)\\ &= \cdots + \tfrac{1}{\epsilon^2}\tilde{\mathcal{J}}_0(x) + \tfrac{1}{\epsilon}\tilde{\mathcal{J}}_1(x) + \tilde{\mathcal{J}}_2(x) + \boxed{\epsilon\tilde{\mathcal{J}}_3(x)} + \cdots\end{aligned}$$

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Answer: To compare two uncouplings, we compare the $\epsilon-$orders for the BIs appearing in the inhomogeneous part of the HODE and in the LFs: **the less the expansion orders are, the better is the uncoupling**

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- Simple version: permute cyclically the MIs vector $\tilde{\mathcal{I}}(x,\epsilon) = \left(\tilde{\mathcal{I}}_1(x,\epsilon),\ldots,\tilde{\mathcal{I}}_n(x,\epsilon)\right)$ with the permutation $\sigma = (1\cdots n)$, so that we uncouple first in $\tilde{\mathcal{I}}_1$, then in $\tilde{\mathcal{I}}_2$, etc. until $\tilde{\mathcal{I}}_n$.
  - We compare the n resulting uncouplings through $\epsilon-$orders comparison.
  - Minimizes the number of HODEs (only one per uncoupling), but not optimal

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  - We compare the n resulting uncouplings through $\epsilon-$orders comparison.
  - Minimizes the number of HODEs (only one per uncoupling), but not optimal
- Advanced version: uncouple cyclically as above, but choose in a recursive way the best equations (HODEs or LFs) at each step for each MI among the n^2 available equations
  - Increases potentially the number of **initial conditions** (ICs) required, but improves the $\epsilon-$orders

Example:

- We have a system of size $n = 3$, with MIs $\tilde{\mathcal{I}}_{i=1,2,3}(x, \epsilon)$ and 4 BIs $\tilde{\mathcal{J}}_{j=1,\ldots,4}(x, \epsilon)$
- From now on, a couple $(\tilde{\mathcal{J}}_i, m_i)$ means that in the HODE/one of the LFs we encounter a factor of the form

$$\epsilon^{m_i} \frac{\alpha_i(x, \epsilon)}{\beta_i(x, \epsilon)} \tilde{\mathcal{J}}_i(x, \epsilon) \text{ with } \alpha_i(x, \epsilon), \beta_i(x, \epsilon) \in \mathbb{K}[x, \epsilon] \text{ and } \epsilon \nmid \alpha_i(x, \epsilon), \beta_i(x, \epsilon)$$

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MIs	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\text{HODE}(\tilde{\mathcal{I}}_1) = \tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$\tilde{\mathcal{I}}_2 = f_2(\tilde{\mathcal{I}}_1) = \tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_3 = f_3(\tilde{\mathcal{I}}_1) = \tilde{\mathcal{I}}_3^{(1)}$	-1	0	-1	-1
$\text{HODE}(\tilde{\mathcal{I}}_2) = \tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$\tilde{\mathcal{I}}_1 = f_1(\tilde{\mathcal{I}}_2) = \tilde{\mathcal{I}}_1^{(2)}$	-2	-1	0	-2
$\tilde{\mathcal{I}}_3 = f_3(\tilde{\mathcal{I}}_2) = \tilde{\mathcal{I}}_3^{(2)}$	0	0	0	0
$\text{HODE}(\tilde{\mathcal{I}}_3) = \tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$\tilde{\mathcal{I}}_1 = f_1(\tilde{\mathcal{I}}_3) = \tilde{\mathcal{I}}_1^{(3)}$	0	0	0	0
$\tilde{\mathcal{I}}_2 = f_2(\tilde{\mathcal{I}}_3) = \tilde{\mathcal{I}}_2^{(3)}$	0	1	0	0

- First we select the best HODE between the three equations $\tilde{\mathcal{I}}_1^{(H)}$, $\tilde{\mathcal{I}}_2^{(H)}$ and $\tilde{\mathcal{I}}_3^{(H)}$:

	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_1^{(H)}$				
$n_2^{(H)}$				
$n_3^{(H)}$				

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	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_1^{(H)}$	1			
$n_2^{(H)}$	0			
$n_3^{(H)}$	1			

- First we select the best HODE between the three equations $\tilde{\mathcal{I}}_1^{(H)}$, $\tilde{\mathcal{I}}_2^{(H)}$ and $\tilde{\mathcal{I}}_3^{(H)}$:

	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_1^{(H)}$	1	2		
$n_2^{(H)}$	0	0		
$n_3^{(H)}$	1	2		

- First we select the best HODE between the three equations $\tilde{\mathcal{I}}_1^{(H)}$, $\tilde{\mathcal{I}}_2^{(H)}$ and $\tilde{\mathcal{I}}_3^{(H)}$:

	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_1^{(H)}$	1	2	2	
$n_2^{(H)}$	0	0	1	
$n_3^{(H)}$	1	2	2	

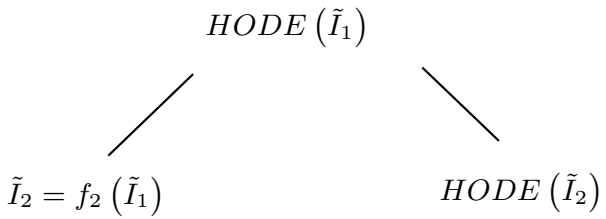
- First we select the best HODE between the three equations $\tilde{\mathcal{I}}_1^{(H)}$, $\tilde{\mathcal{I}}_2^{(H)}$ and $\tilde{\mathcal{I}}_3^{(H)}$:

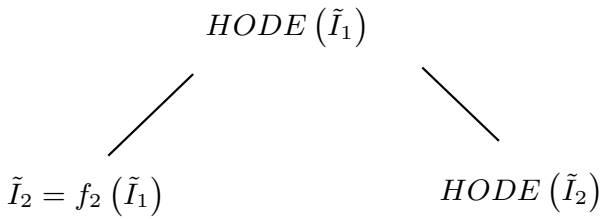
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_1^{(H)}$	1	2	2	(3)
$n_2^{(H)}$	0	0	1	1
$n_3^{(H)}$	1	2	2	(3)

- First we select the best HODE between the three equations $\tilde{\mathcal{I}}_1^{(H)}$, $\tilde{\mathcal{I}}_2^{(H)}$ and $\tilde{\mathcal{I}}_3^{(H)}$:

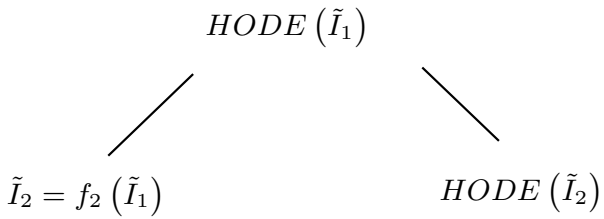
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_1^{(H)}$	1	2	2	(3)
$n_2^{(H)}$	0	0	1	1
$n_3^{(H)}$	1	2	2	(3)

This means that there are 2 possibilities: $\tilde{\mathcal{I}}_1^{(H)}$ and $\tilde{\mathcal{I}}_3^{(H)}$. We'll have to examine both of them.

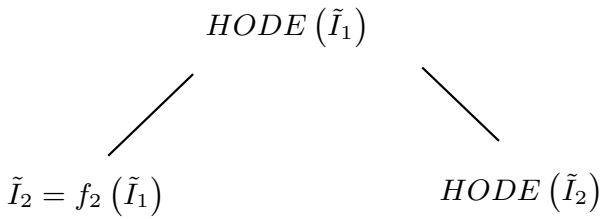

$HODE\left(\tilde{I}_1\right)$
$\tilde{I}_2 = f_2\left(\tilde{I}_1\right)$
$HODE\left(\tilde{I}_2\right)$



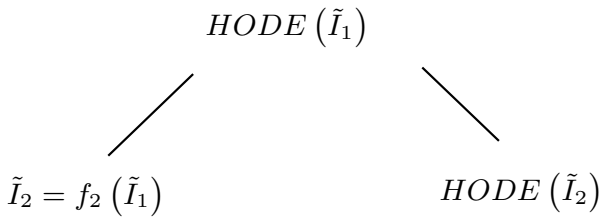
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$				
$n_2^{(H)}$				



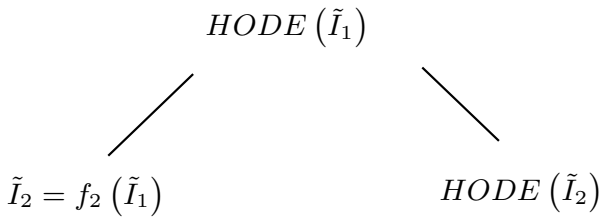
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1			
$n_2^{(H)}$	0			



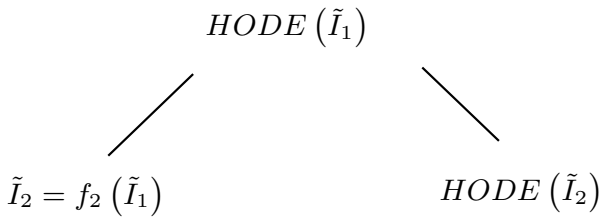
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2		
$n_2^{(H)}$	0	1		



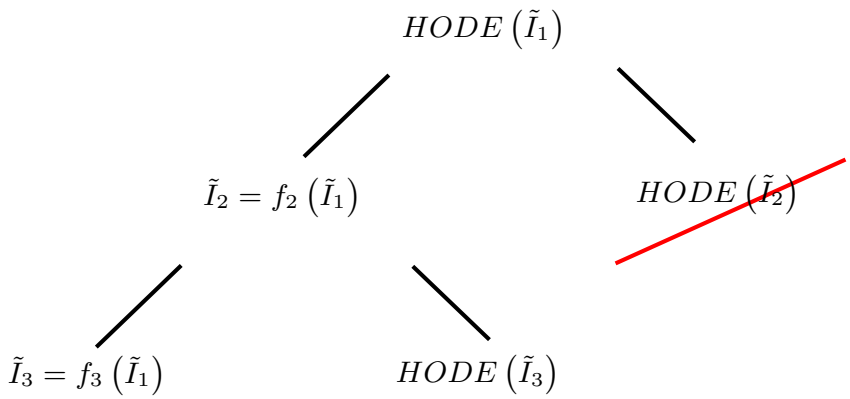
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2	2	
$n_2^{(H)}$	0	1	2	

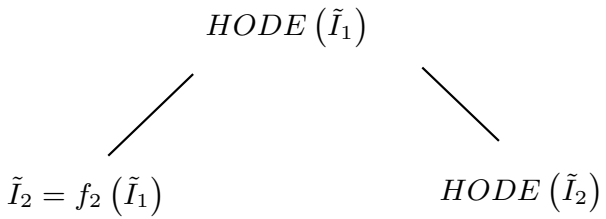


	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2	2	(3)
$n_2^{(H)}$	0	1	2	2

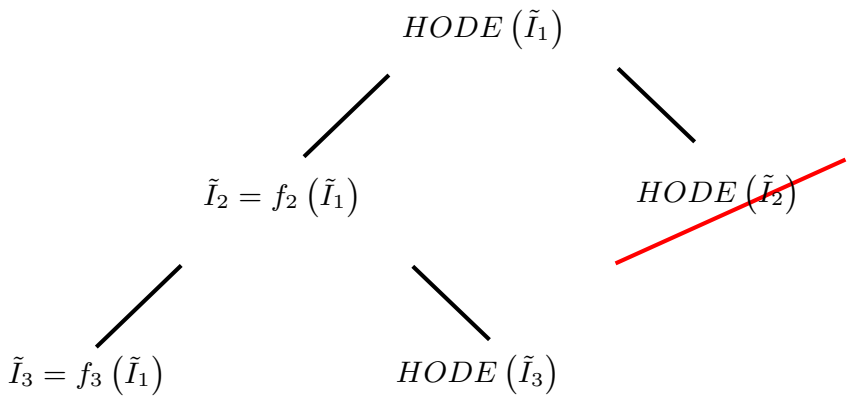


	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2	2	(3)
$n_2^{(H)}$	0	1	2	2

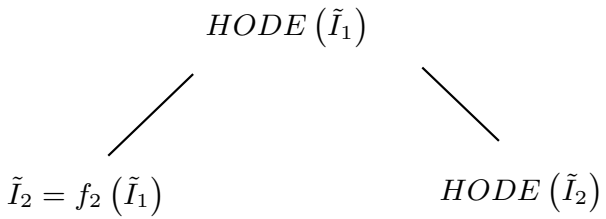




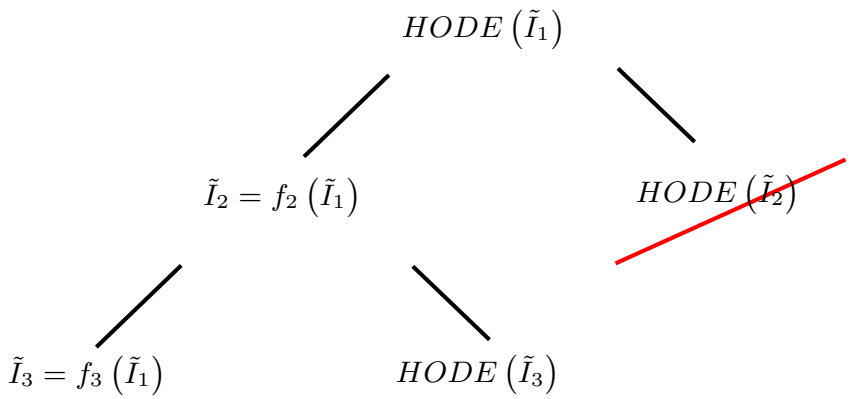
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2	2	(3)
$n_2^{(H)}$	0	1	2	2



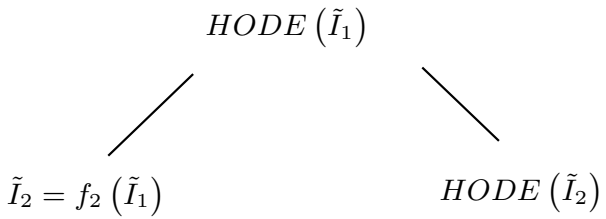
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_3^{(1)}$	-1	0	-1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_3^{(1)}$				
$n_3^{(H)}$				



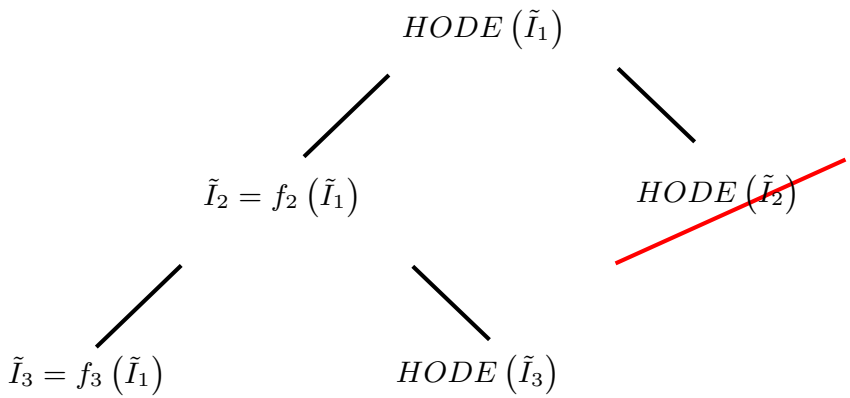
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2	2	(3)
$n_2^{(H)}$	0	1	2	2



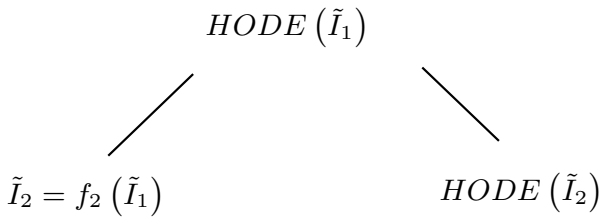
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_3^{(1)}$	-1	0	-1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_3^{(1)}$	0			
$n_3^{(H)}$	1			



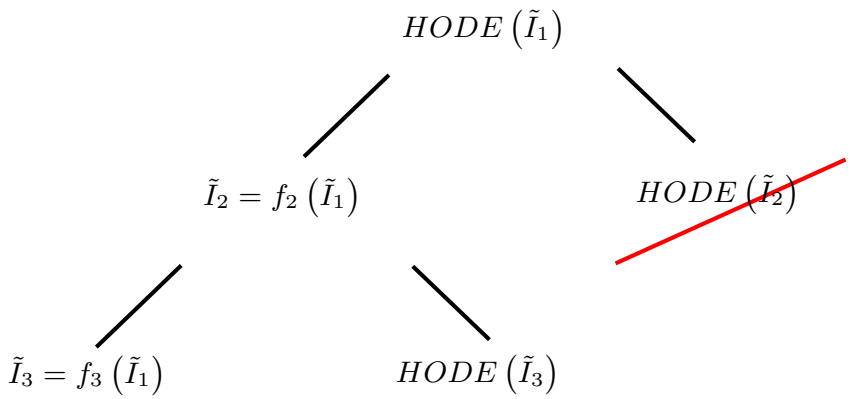
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2	2	(3)
$n_2^{(H)}$	0	1	2	2



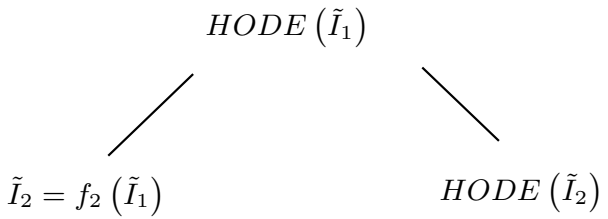
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_3^{(1)}$	-1	0	-1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_3^{(1)}$	0	0		
$n_3^{(H)}$	1	2		



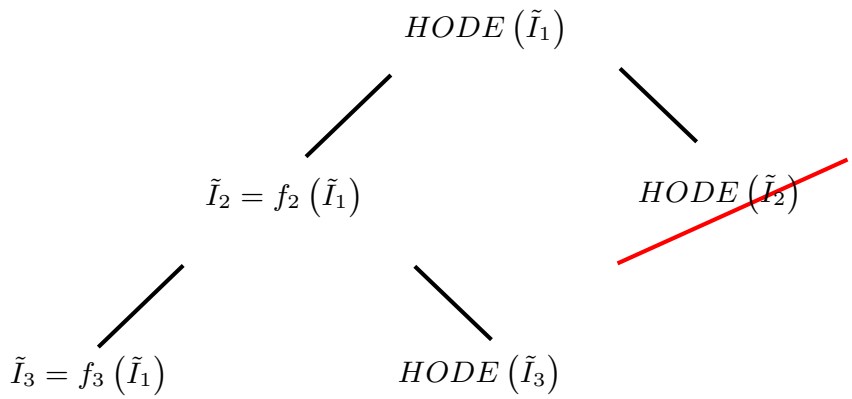
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2	2	(3)
$n_2^{(H)}$	0	1	2	2



	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_3^{(1)}$	-1	0	-1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_3^{(1)}$	0	0	0	
$n_3^{(H)}$	1	2	3	

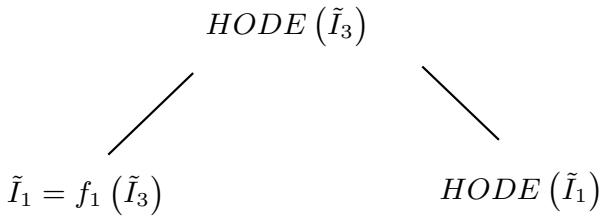


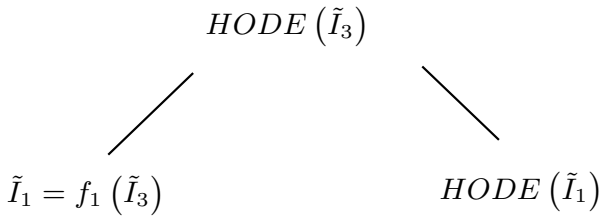
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_2^{(1)}$	0	0	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$	1	2	2	(3)
$n_2^{(H)}$	0	1	2	2



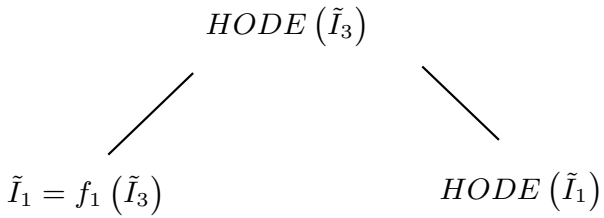
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_3^{(1)}$	-1	0	-1	-1
$\tilde{\mathcal{I}}_3^{(H)}$	0	1	0	0
$n_3^{(1)}$	0	0	0	0
$n_3^{(H)}$	1	2	3	(4)

First solution: $\mathrm{HODE}(\tilde{\mathcal{I}}_1),\ \tilde{\mathcal{I}}_2 = f_2(\tilde{\mathcal{I}}_1),\ \mathrm{HODE}(\tilde{\mathcal{I}}_3)$

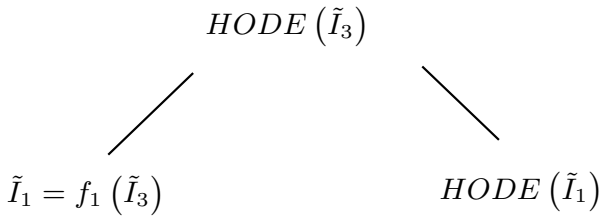

$HODE\left(\tilde{I}_3\right)$
$\tilde{I}_1 = f_1\left(\tilde{I}_3\right)$
$HODE\left(\tilde{I}_1\right)$



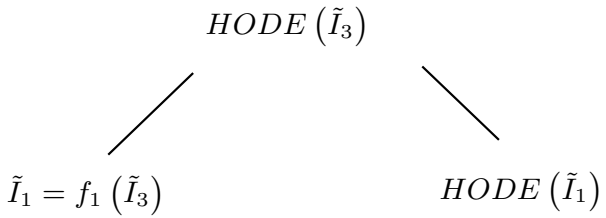
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(3)}$	0	0	0	0
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$n_1^{(3)}$				
$n_1^{(H)}$				



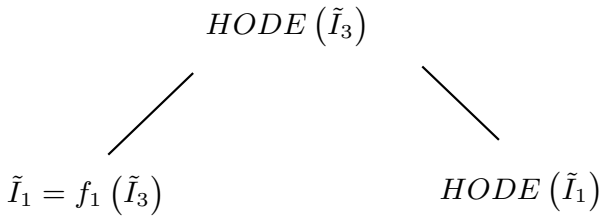
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(3)}$	0	0	0	0
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$n_1^{(3)}$	1			
$n_1^{(H)}$	1			



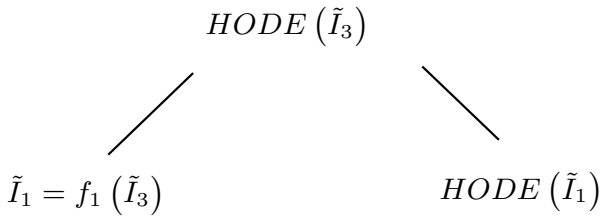
	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(3)}$	0	0	0	0
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$n_1^{(3)}$	1	1		
$n_1^{(H)}$	1	2		



	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(3)}$	0	0	0	0
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$n_1^{(3)}$	1	1	2	
$n_1^{(H)}$	1	2	3	



	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
$\tilde{\mathcal{I}}_1^{(3)}$	0	0	0	0
$\tilde{\mathcal{I}}_1^{(H)}$	0	1	0	0
$n_1^{(3)}$	1	1	2	3
$n_1^{(H)}$	1	2	3	(4)



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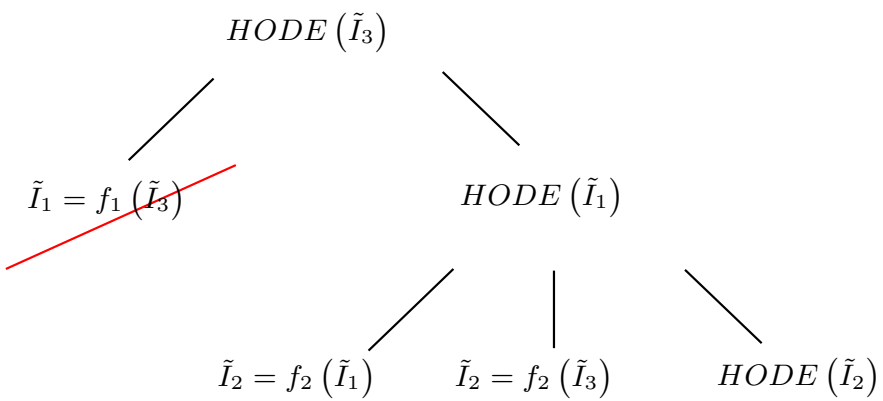

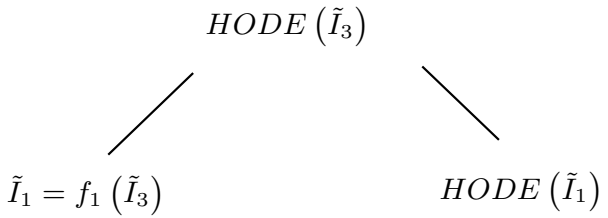

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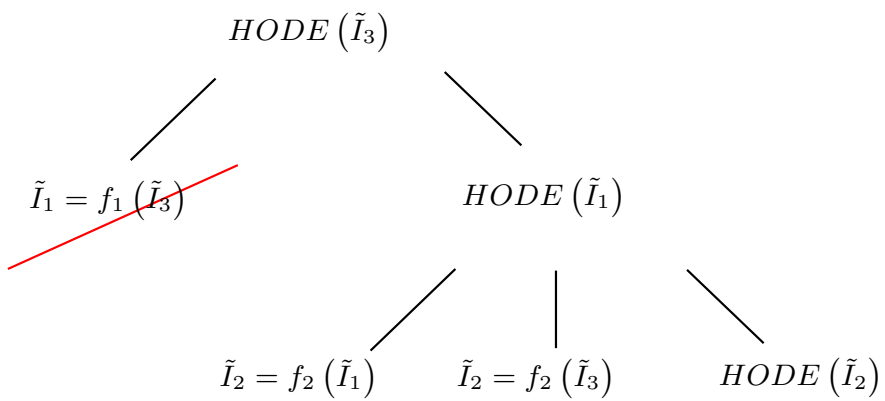

	$\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_2$	$\tilde{\mathcal{J}}_3$	$\tilde{\mathcal{J}}_4$
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$\tilde{\mathcal{I}}_2^{(3)}$	0	1	0	0
$\tilde{\mathcal{I}}_2^{(H)}$	-1	0	1	-1
$n_2^{(1)}$				
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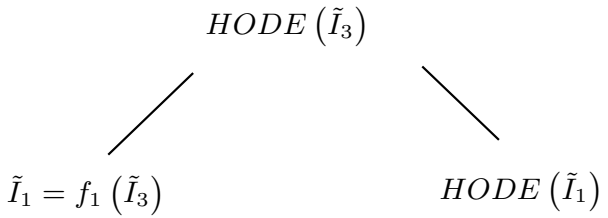

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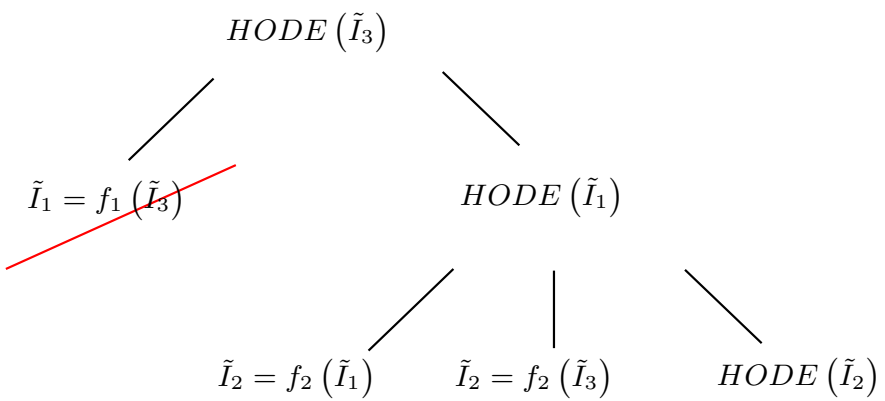

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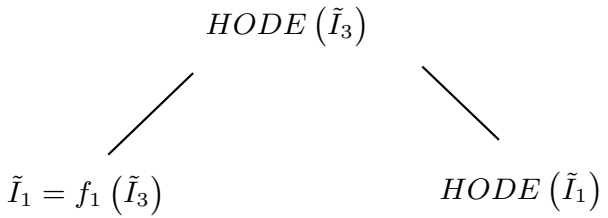

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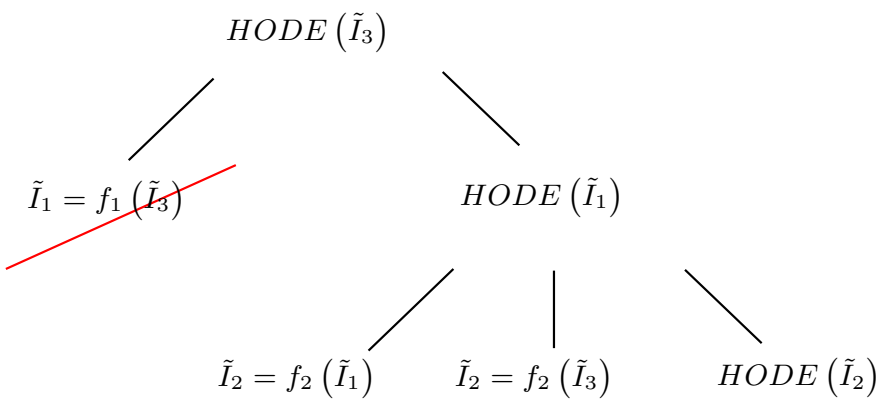

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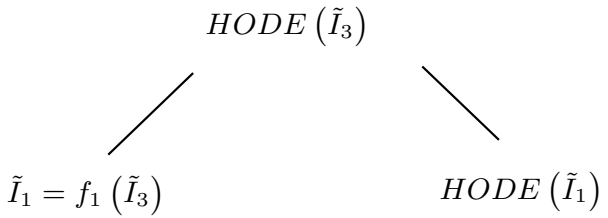

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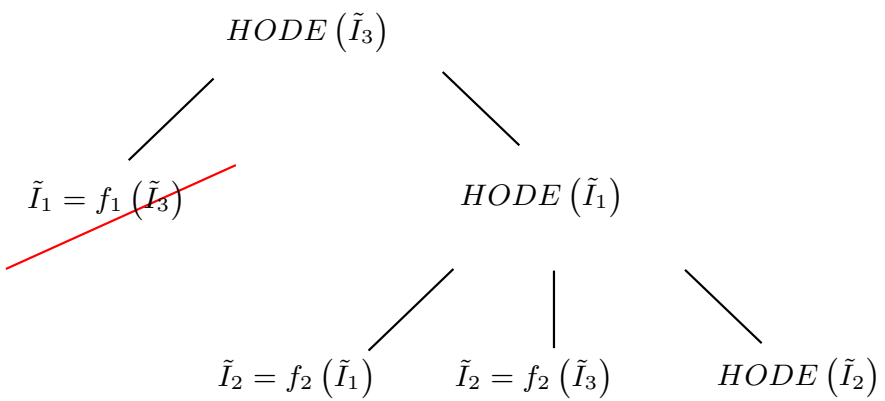

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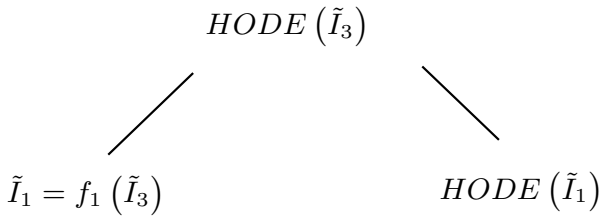

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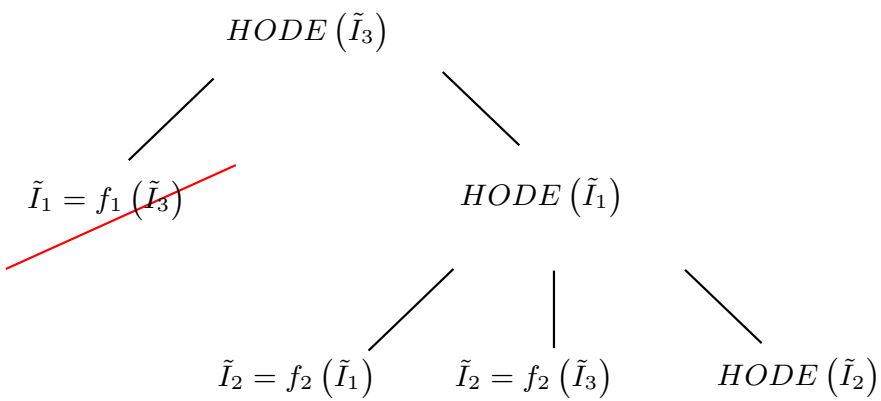

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Second solution: $\mathrm{HODE}(\tilde{\mathcal{I}}_1),\ \tilde{\mathcal{I}}_2 = f_2(\tilde{\mathcal{I}}_3),\ \mathrm{HODE}(\tilde{\mathcal{I}}_3)$

Note: Doing this procedure means selecting the equations based only on the ϵ-orders. This might not be the actual optimal solution $\rightarrow$ **HODE order**, **recurrence order** play also an important role

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We have included four weight functions:

1. Equal weight (default): $f_1(\{-n_\epsilon, n_d, n_r\}) := -n_\epsilon + n_d + n_r$
2. Only ϵ-order: $f_2(\{-n_\epsilon, n_d, n_r\}) := -n_\epsilon$
3. Only HODE order: $f_3(\{-n_\epsilon, n_d, n_r\}) := n_d$
4. Only recurence order: $f_4(\{-n_\epsilon, n_d, n_r\}) := n_r$

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- The user can also define himself the weight function f and give it as an optional argument to the main function

The procedure described above can actually be improved even further: in this recursive choice of equations, we prioritize the $\epsilon-$orders at the expense of the number of IC (several HODEs each of order n possible) $\rightarrow$ can be improved

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- The next equation we need to find is for $\tilde{\mathcal{I}}_2$. This time, instead of the previous choice, we uncouple the subsystem $\tilde{S}_1(\tilde{\mathcal{I}}_2, \ldots, \tilde{\mathcal{I}}_n) \subsetneq S(\tilde{\mathcal{I}}_1, \ldots, \tilde{\mathcal{I}}_n)$ **where $\tilde{\mathcal{I}}_1$ is seen as a BI**

$$\begin{array}{cccc} & S & & \tilde{S}_1 \\ \text{MIs:} & \tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2, \ldots, \tilde{\mathcal{I}}_n & \longrightarrow & \tilde{\mathcal{I}}_2, \ldots, \tilde{\mathcal{I}}_n \\ \text{BIs:} & \tilde{\mathcal{J}}_1, \ldots, \tilde{\mathcal{J}}_m & \longrightarrow & \tilde{\mathcal{I}}_1, \tilde{\mathcal{J}}_1 \ldots, \tilde{\mathcal{J}}_m \end{array}$$

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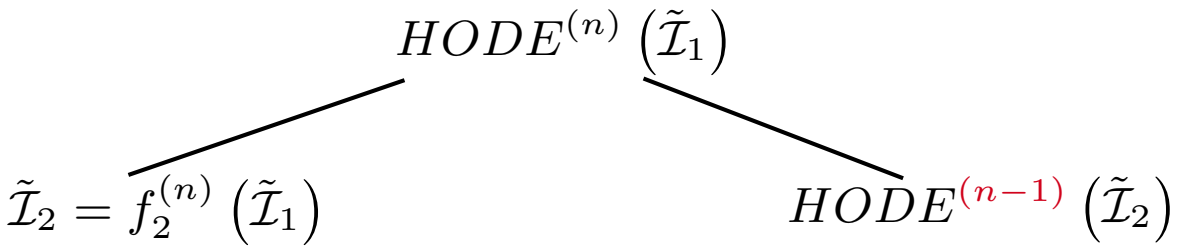
- Use the new HODE for $\tilde{\mathcal{I}}_2$, of order $n-1$, instead of that obtained during the original uncoupling
- Reiterate for $\tilde{\mathcal{I}}_3$, etc.

- This time, we have the following choice tree:



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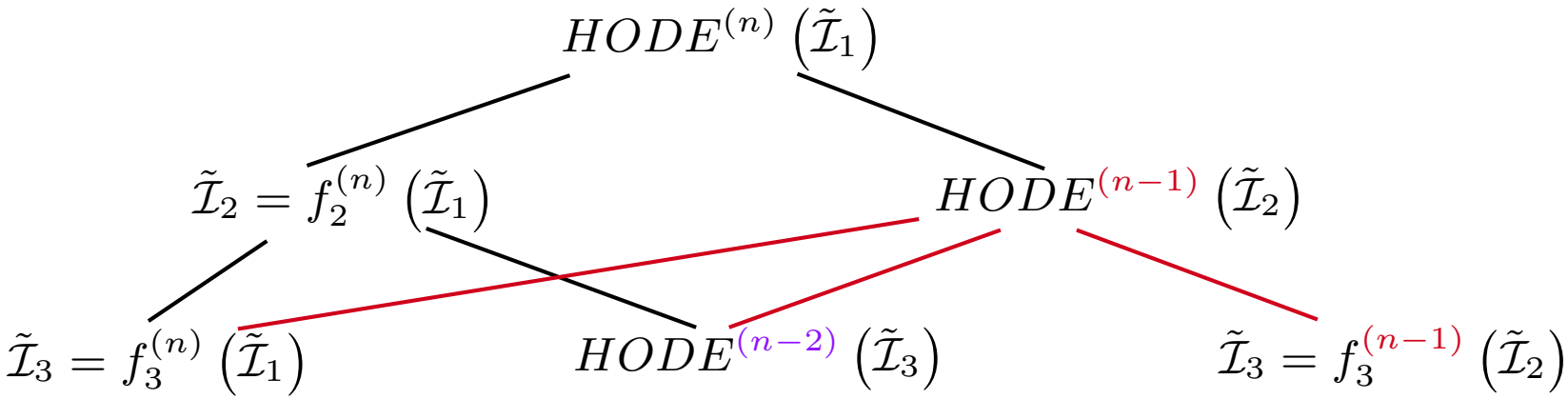

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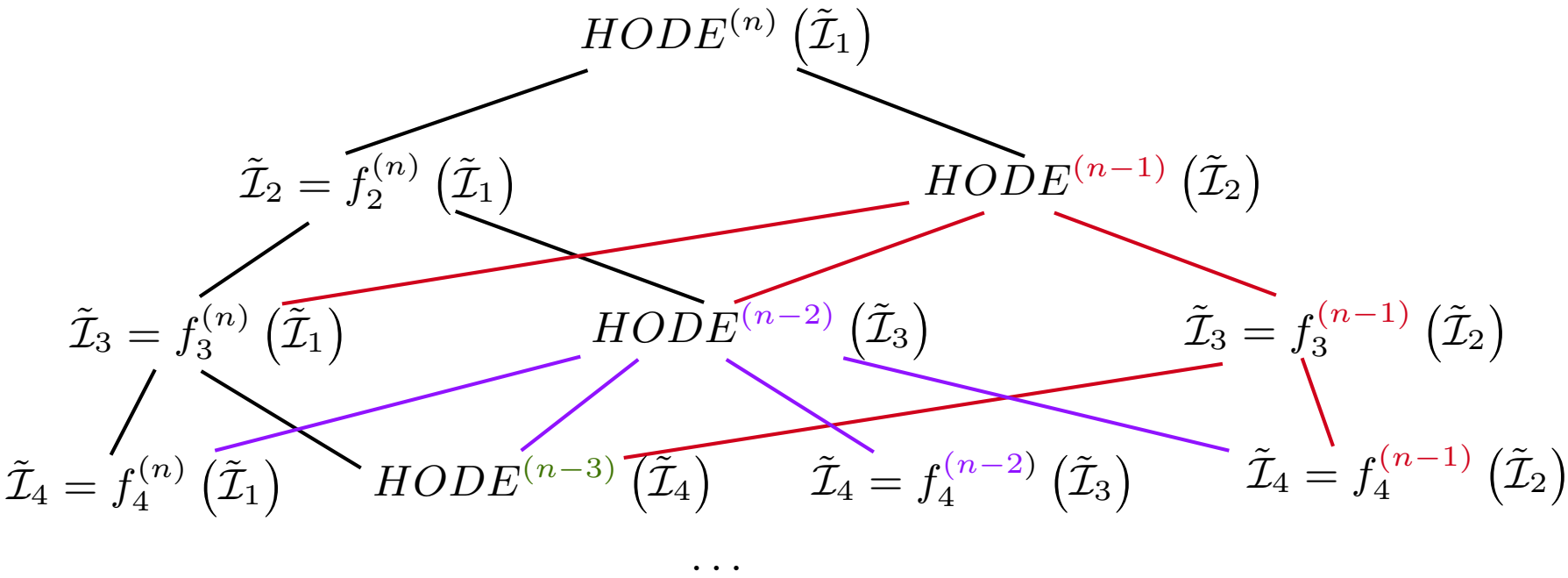

- In short: uncouple at step i in $\tilde{\mathcal{I}}_i$ by considering that all $\tilde{\mathcal{I}}_1, \ldots, \tilde{\mathcal{I}}_{i-1}$ are BIs, so that we reduce the orders of the HODEs

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- For each "good" uncoupling, let's us denote with $m_{i,j}$ the $\epsilon-$order for the BI $\tilde{\mathcal{J}}_j(x,\epsilon)$, $1 \leq j \leq m$ appearing in the expression for the MI $\tilde{\mathcal{I}}_i(x,\epsilon)$, $1 \leq i \leq n$, and write the orders matrix:

$$M = (m_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$$

Example:

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} \tilde{\mathcal{J}}_1^{(H)} \\ \tilde{\mathcal{J}}_2^{(1)} \\ \tilde{\mathcal{J}}_3^{(H)} \end{matrix}, \quad M_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{matrix} \tilde{\mathcal{J}}_1^{(H)} \\ \tilde{\mathcal{J}}_2^{(3)} \\ \tilde{\mathcal{J}}_3^{(H)} \end{matrix}$$

- Define the functions

$$\sigma : M \mapsto \sum_{i,j} m_{i,j}$$

$$\|\bullet\| : M \mapsto \sqrt{\mathrm{Tr}(M^\top M)}$$

And finally combine them in a function

$$\boxed{\mathcal{N} : \begin{cases} \mathcal{M}_{n,m}(\mathbb{Z}) & \longrightarrow & \mathbb{Z} \times \mathbb{R}_- \\ M & \longmapsto & (\sigma(M), -\|M\|) \end{cases}}$$

- We obtain the "best" solution by finding the "minimal" $\mathcal{N}(M)$, i.e. the element that maximizes the sum of orders and minimizes their variance (in this order)

- One last improvement we can make to the Zürcher uncoupling is choosing among *all* the possible remaining MIs at each step and not only one of them

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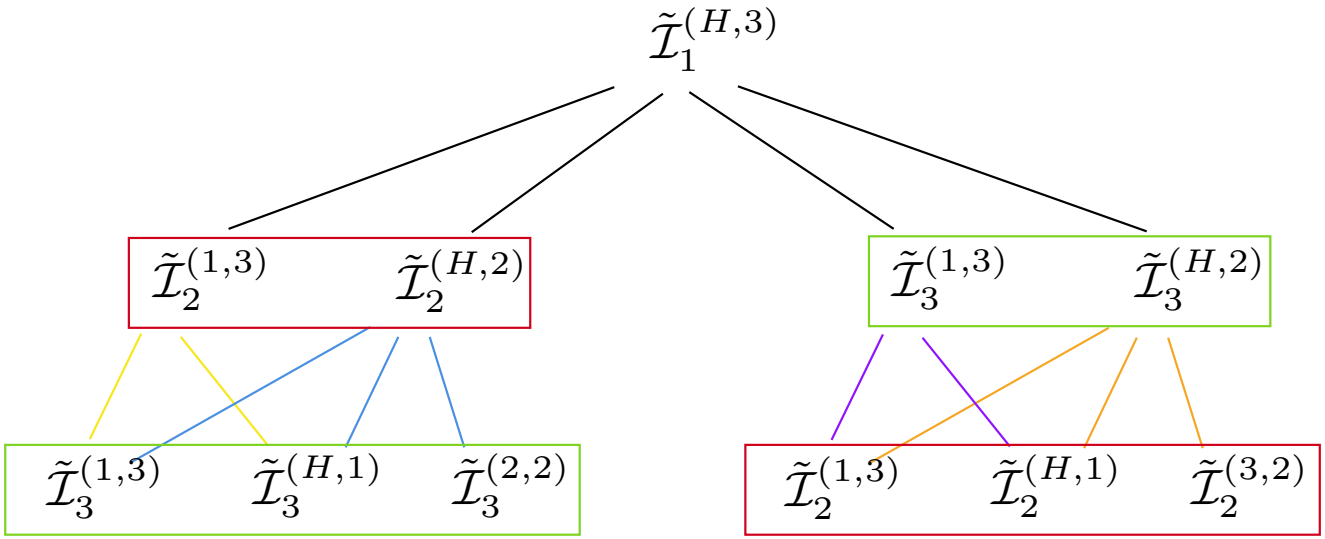
Example: Defining a shorter notation $\tilde{\mathcal{I}}_k^{(i,j)} := f_k^{(j)}(\tilde{\mathcal{I}}_i)$, $\tilde{\mathcal{I}}_k^{(H,i)} := \mathrm{HODE}^{(i)}(\tilde{\mathcal{I}}_k)$, for a system of order 3, here's the choice tree we would get:



- The second possible uncoupling scheme is the Gauß scheme. The result returned by `OreSys` in whole generality takes the following form:

Gauß uncoupling (OreSys output)

$$\begin{cases} \text{HODE}(\underline{\tilde{\mathcal{I}}_{i_1}, \{\mathrm{d}_x^j \tilde{\mathcal{I}}_{i_1}\}}; \{\tilde{\mathcal{R}}_i, \mathrm{d}_x^j \tilde{\mathcal{R}}_i\}) \\ \text{HODE}(\underline{\tilde{\mathcal{I}}_{i_2}, \{\mathrm{d}_x^j \tilde{\mathcal{I}}_{i_2}\}}; \tilde{\mathcal{I}}_{i_1}, \{\mathrm{d}_x^j \tilde{\mathcal{I}}_{i_1}\}; \{\tilde{\mathcal{R}}_i, \mathrm{d}_x^j \tilde{\mathcal{R}}_i\}) \\ \qquad\qquad \vdots \\ \text{HODE}(\underline{\tilde{\mathcal{I}}_{i_n}, \{\mathrm{d}_x^j \tilde{\mathcal{I}}_{i_n}\}}; \cdots; \tilde{\mathcal{I}}_{i_1}, \{\mathrm{d}_x^j \tilde{\mathcal{I}}_{i_1}\}; \{\tilde{\mathcal{R}}_i, \mathrm{d}_x^j \tilde{\mathcal{R}}_i\}) \end{cases}$$

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- Compared to the Zürcher uncoupling, Gauß is more sensible to the order of the equations for the uncoupling, so we simple uncouple in all the possible ways, i.e. we get $n!$ uncouplings
- Finally, we can combine **both** Zürcher and Gauß uncouplings, i.e. at each step choose among **all** available equations (HODE and LF from Zürcher and Gauß) for remaining MIs: this is actually the scheme that is ultimately used, because it allows for the most freedom

Because we get to do a lot of different uncouplings ($2n-1$ for Zürcher and $n!$ for Gauß), we have implemented different "speed" schemes:

- **Slowest (greedy)**: The algorithm computes everything. Gives the best result, but finishes in reasonable time only for rather small systems (up to order ~ 5, and not too much subsystems, around ~ 10)

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- **Fastest (lenient)**: The algorithm computes in parallel the Zürcher and the Gauß schemes and only gets the one that finishes first. Mostly for test purposes, or for exceptionally big systems for which the other schemes take too much time

Mathematica: All of this is implemented in our (beta) package `SystemAnalysis` into an algorithm called `bestUncpl`, that returns the best uncouplings (i.e. the collection of best HODEs/LFs) for Zürcher and Gauß schemes in a list

Recap:

- We clustered the system into smaller irreducible subsystems
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What remains to do?

- Compute for each subsystem in a symbolic way the order up to which we need to give the BIs in order to solve the system up to the user-required order in the MIs
- Using these results, compute the real $\epsilon-$order correction for each BI in a recursive way for the whole system

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- We uncoupled them in Zürcher and Gauß scheme and, using the method described above, obtained a set of "best" equations $\epsilon-$order-wise

What remains to do?

- Compute for each subsystem in a symbolic way the order up to which we need to give the BIs in order to solve the system up to the user-required order in the MIs
- Using these results, compute the real $\epsilon-$order correction for each BI in a recursive way for the whole system

Why?

- A non-irreducible system clusters non-trivially into several subsystems
- This gives rise to **induced BIs** (IBIs) in the subsystems, i.e. MIs that become locally BIs, and therefore the dependence of **fundamental BIs** (FBIs) might depend not only on the MIs where they appear, but also on other subsystems' MIs which have in the RHS the BIs' MIs as IBIs

- First, we extract the different $\epsilon-$order corrections in a completely general and symbolic way and make them into Mathematica rules.

Toy example 1: System in two variables $\tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2$, with one BI $\mathcal{J}$, with the following uncoupling (for simplicity's sake, the dependence in x is suppressed):

$$\text{HODE}(\tilde{\mathcal{I}}_1) = -2\epsilon^8(1-2\epsilon^2)\mathcal{J}(x) + \epsilon^{27}\mathcal{J}'(x)$$

$$\tilde{\mathcal{I}}_2 = -\frac{\epsilon^{14}}{2}\tilde{\mathcal{I}}_1(x) + \frac{\epsilon^{19}}{2}\tilde{\mathcal{I}}_1' - 3\epsilon^{19}\mathcal{J}(x)$$

Initialisation:

$$(\tilde{\mathcal{I}}_1, a) \quad (\tilde{\mathcal{I}}_2, b)$$

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Problem: This only works if we choose equations from the simple n Zürcher, but if we choose from Gauß and sub-Zürcher uncouplings, this adds a recursive dependence of the BIs ϵ-orders on the chosen equations

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Toy example 2: Let's take as system of order $n = 4$ with 4 MIs $\tilde{\mathcal{I}}_{i=1,\dots,4}$ and 2 BIs $\tilde{\mathcal{J}}_{i=1,2}$, with the following chosen equations:

$$\begin{cases} \mathrm{HODE}(\tilde{\mathcal{I}}_1) &= \epsilon^{\alpha_1}\tilde{\mathcal{J}}_1 + \epsilon^{\beta_1}\tilde{\mathcal{J}}_2 \\ \mathrm{HODE}(\tilde{\mathcal{I}}_4) &= \epsilon^{\alpha_2}\tilde{\mathcal{J}}_1 + \epsilon^{\beta_2}\tilde{\mathcal{J}}_2 + \epsilon^{\gamma_2}\tilde{\mathcal{I}}_1 \\ \tilde{\mathcal{I}}_3 &= \epsilon^{\alpha_3}\tilde{\mathcal{J}}_1 + \epsilon^{\beta_3}\tilde{\mathcal{J}}_2 + \epsilon^{\gamma_3}\tilde{\mathcal{I}}_1 + \epsilon^{\delta_3}\tilde{\mathcal{I}}_4 \\ \tilde{\mathcal{I}}_2 &= \epsilon^{\alpha_4}\tilde{\mathcal{J}}_1 + \epsilon^{\beta_4}\tilde{\mathcal{J}}_2 + \epsilon^{\gamma_4}\tilde{\mathcal{I}}_1 + \epsilon^{\delta_4}\tilde{\mathcal{I}}_4 + \epsilon^{\eta_4}\tilde{\mathcal{I}}_3 \end{cases}$$

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Initialization: $(\tilde{\mathcal{I}}_1, a)$, $(\tilde{\mathcal{I}}_2, b)$, $(\tilde{\mathcal{I}}_3, c)$, $(\tilde{\mathcal{I}}_4, d)$

- We have to update recursively the ϵ-orders, from the last to the first equation, let us show how it works explicitely for $\tilde{\mathcal{J}}_1$

1. Updating with 4^{th} equation $\left(\tilde{\mathcal{I}}_2 = \epsilon^{\alpha_4}\tilde{\mathcal{J}}_1 + \epsilon^{\beta_4}\tilde{\mathcal{J}}_2 + \epsilon^{\gamma_4}\tilde{\mathcal{I}}_1 + \epsilon^{\delta_4}\tilde{\mathcal{I}}_4 + \epsilon^{\eta_4}\tilde{\mathcal{I}}_3\right)$

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The correction for $\tilde{\mathcal{J}}_1$ is:

$$\max(a_4, b_1, c_2, d_3)$$

- This generalizes very easily if there are derivatives of the MIs/BIs, e.g. if the second equation of the example was

$$\mathrm{HODE}(\tilde{\mathcal{I}}_4) = \epsilon^{\alpha_2}\tilde{\mathcal{J}}_1 + \epsilon^{\beta_2}\tilde{\mathcal{J}}_2 + \epsilon^{\gamma_2}\tilde{\mathcal{I}}_1 + \epsilon^{\gamma_2^{(1)}}\mathrm{d}_x\tilde{\mathcal{I}}_1$$

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$$\mathrm{HODE}(\tilde{\mathcal{I}}_4) = \epsilon^{\alpha_2}\tilde{\mathcal{J}}_1 + \epsilon^{\beta_2}\tilde{\mathcal{J}}_2 + \epsilon^{\gamma_2}\tilde{\mathcal{I}}_1 + \epsilon^{\gamma_2^{(1)}}\mathrm{d}_x\tilde{\mathcal{I}}_1$$

Then at step 3 the correction would be

$$(\tilde{\mathcal{I}}_1, a_3 = \max(a_2, d_2 - \gamma_2, d_2 - \gamma_2^{(1)})$$

- Let us use this recursive algorithm with the toy example 1 as a sanity check:

$$\text{HODE}(\tilde{\mathcal{I}}_1) = -2\epsilon^8(1-2\epsilon^2)\mathcal{J} + \epsilon^{27}\mathcal{J}'$$

$$\tilde{\mathcal{I}}_2 = -\frac{\epsilon^{14}}{2}\tilde{\mathcal{I}}_1 + \frac{\epsilon^{19}}{2}\tilde{\mathcal{I}}_1' - 3\epsilon^{19}\mathcal{J}$$

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Mathematica: A subfunction in our package does this computation and returns a symbolic "function-rule" that we need for the next step. In our example, the rule would look like:

$$\{\{\tilde{\mathcal{I}}_1, a\_\}, \{\tilde{\mathcal{I}}_2, b\_\}\} \longrightarrow \{\{\{\tilde{\mathcal{I}}_1, \max(a, b-14)\}, \{\tilde{\mathcal{I}}_2, b\}\}, \{\{\{\mathcal{J}, \max[\max(a, b-14) - 8, b-19]\}\}\}$$

Now that we have the local correction rules, we are almost done:

1. We identify what we call the **root MIs** (RMIs), i.e. the MIs that do not appear as IBIs in any equation, since these won't get any retro-correction and define in a sense the "root" of the correction trees

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Note: It is clear that this procedure terminates as each step will either:

- Add a new variable (MI or BI) $\rightarrow$ there is only a finite number of them
- Nothing happens, but that means the algorithm is finished

Example:

- We have an initial system of order $n = 5$, i.e. the MIs are $\tilde{\mathcal{I}}_i(x, \epsilon)$, $i \in \{1, \ldots, 5\}$, and there is one FBI $\mathcal{J}$
- Suppose the system splits into three irreducible subsystems:

MIs	(Full) Dependence
$\tilde{\mathcal{I}}_1$	$\tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2, \tilde{\mathcal{I}}_3, \tilde{\mathcal{I}}_4, \tilde{\mathcal{I}}_5; \mathcal{J}$
$\tilde{\mathcal{I}}_2, \tilde{\mathcal{I}}_3$	$\tilde{\mathcal{I}}_2, \tilde{\mathcal{I}}_3, \tilde{\mathcal{I}}_4, \tilde{\mathcal{I}}_5; \mathcal{J}$
$\tilde{\mathcal{I}}_4, \tilde{\mathcal{I}}_5$	$\tilde{\mathcal{I}}_4, \tilde{\mathcal{I}}_5; \mathcal{J}$

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$\tilde{\mathcal{I}}_4, \tilde{\mathcal{I}}_5$	$\tilde{\mathcal{I}}_4, \tilde{\mathcal{I}}_5; \mathcal{J}$

- We have only one RMI in this case: $\tilde{\mathcal{I}}_1$

- We suppose that after the computation of the local $\epsilon-$order corrections, we obtain the following rules:

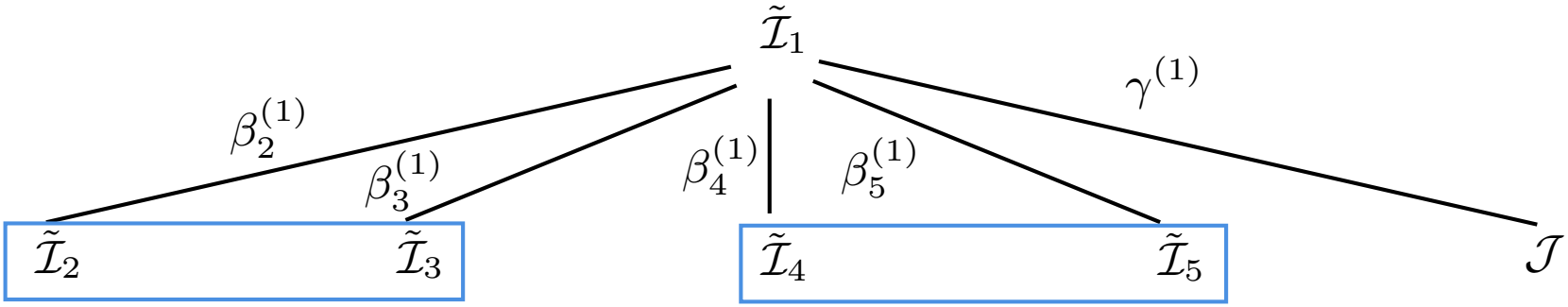
$$\{\{\tilde{\mathcal{I}}_1, a_1\}\} \to \{\{\{\tilde{\mathcal{I}}_1, \alpha_1^{(1)}\}\}, \{\{\{\tilde{\mathcal{I}}_2, \beta_2^{(1)}\}, \{\tilde{\mathcal{I}}_3, \beta_3^{(1)}\}, \{\tilde{\mathcal{I}}_4, \beta_4^{(1)}\}, \{\tilde{\mathcal{I}}_5, \beta_5^{(1)}\}, \{\mathcal{J}, \gamma^{(1)}\}\}\}\}$$

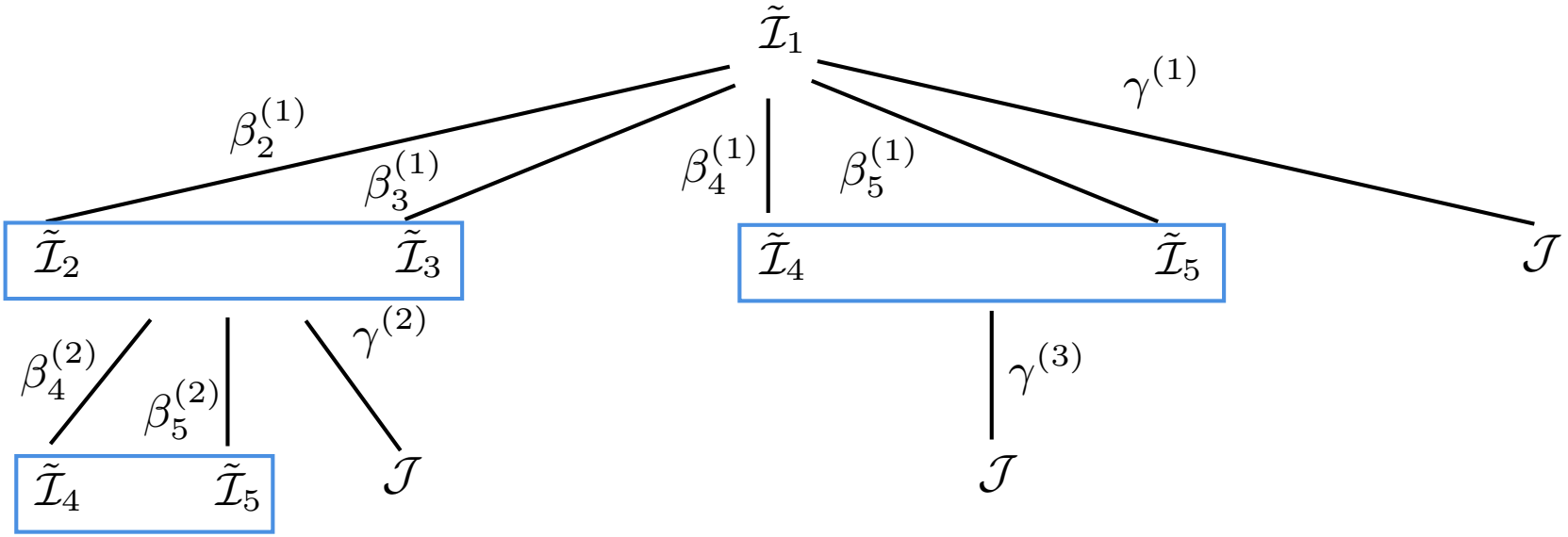
$$\{\{\tilde{\mathcal{I}}_2, a_2\}, \{\tilde{\mathcal{I}}_3, a_3\}\} \to \{\{\{\tilde{\mathcal{I}}_2, \alpha_2^{(2)}\}, \{\tilde{\mathcal{I}}_3, \alpha_3^{(2)}\}\}, \{\{\{\tilde{\mathcal{I}}_4, \beta_4^{(2)}\}, \{\tilde{\mathcal{I}}_5, \beta_5^{(2)}\}, \{\mathcal{J}, \gamma^{(2)}\}\}\}$$

$$\{\{\tilde{\mathcal{I}}_4, a_4\}, \{\tilde{\mathcal{I}}_5, a_5\}\} \to \{\{\{\tilde{\mathcal{I}}_4, \alpha_4^{(3)}\}, \{\tilde{\mathcal{I}}_5, \alpha_5^{(3)}\}\}, \{\{\{\mathcal{J}, \gamma^{(3)}\}\}\}$$

Where the $\alpha_i^{(j)}, \beta_i^{(j)}, \gamma^{(j)}$ are functions of the a_i

$$\tilde{\mathcal{I}}_1$$


$\tilde{\mathcal{I}}_1$
$\beta_2^{(1)}$
$\beta_3^{(1)}$
$\beta_4^{(1)}$
$\beta_5^{(1)}$
$\gamma^{(1)}$
$\tilde{\mathcal{I}}_2$
$\tilde{\mathcal{I}}_3$
$\tilde{\mathcal{I}}_4$
$\tilde{\mathcal{I}}_5$
$\mathcal{J}$


$\tilde{\mathcal{I}}_1$
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$\tilde{\mathcal{I}}_2$
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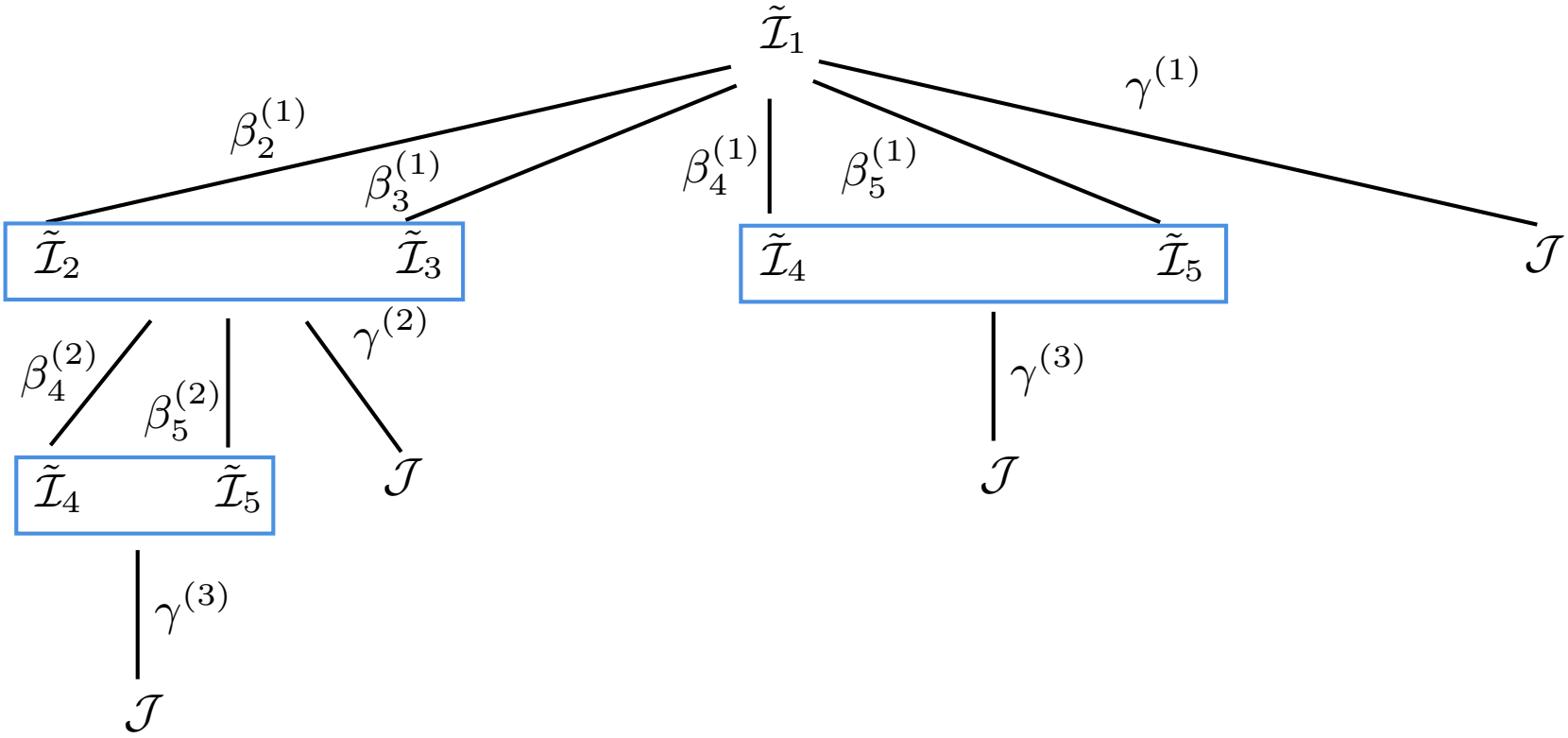
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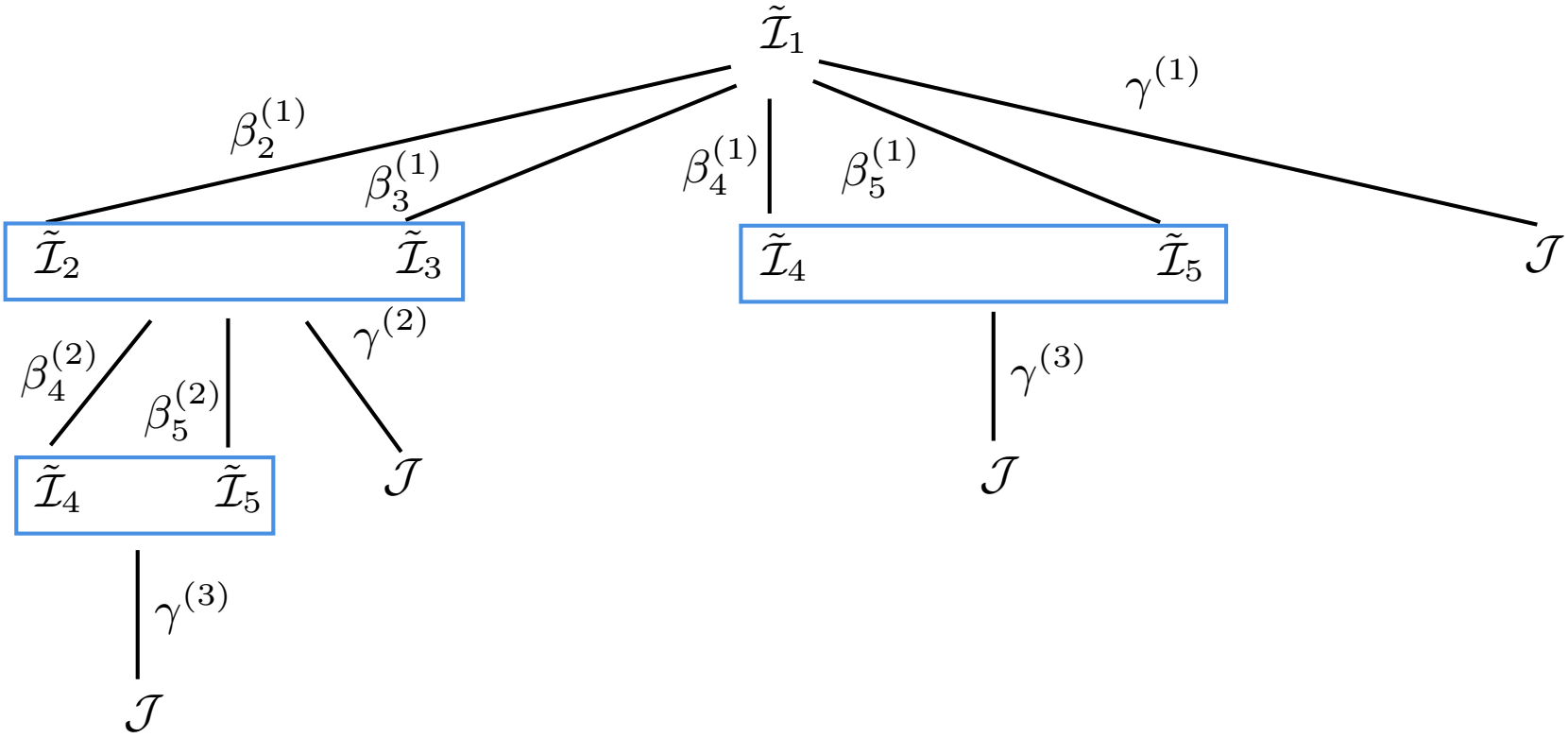
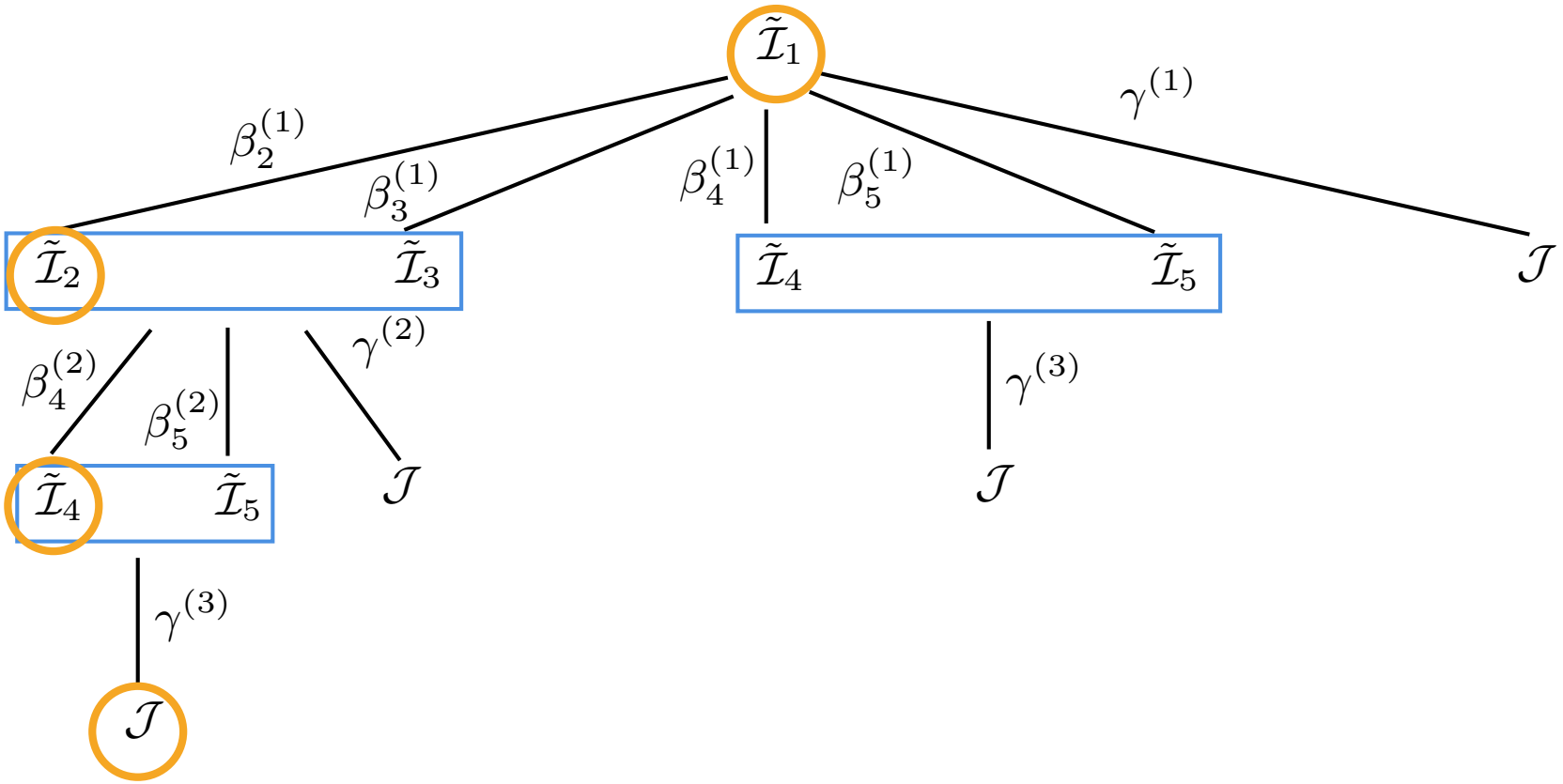


Note: The corrections are applied in a recursive way, i.e. following a branch we need to compose the $\alpha_i^{(j)}/\beta_i^{(j)}/\gamma^{(j)}$ functions from the outermost node all the way down to the root.



Note: The corrections are applied in a recursive way, i.e. following a branch we need to compose the $\alpha_i^{(j)}/\beta_i^{(j)}/\gamma^{(j)}$ functions from the outermost node all the way down to the root. Example:

$$\boxed{\gamma^{(3)} \circ \beta_4^{(2)} \circ \beta_2^{(1)}(a_1)}$$

- We collect in this way all the corrections at the end of the branches, and we get the following list:

$$L_{\text{corr}} : \left\{ \begin{array}{ccccc} \gamma^{(3)} \circ \beta_2^{(4)} \circ \beta_2^{(1)}, & \gamma^{(3)} \circ \beta_2^{(1)} \circ \beta_3^{(1)}, & \gamma^{(3)} \circ \beta_4^{(1)}, & \gamma^{(3)} \circ \beta_5^{(1)}, & \gamma^{(1)} \\ \gamma^{(3)} \circ \beta_5^{(2)} \circ \beta_2^{(1)}, & \gamma^{(3)} \circ \beta_5^{(2)} \circ \beta_3^{(1)}, & & & \\ \gamma^{(2)} \circ \beta_1^{(2)}, & \gamma^{(2)} \circ \beta_1^{(3)}, & & & \end{array} \right\}$$

- Since we have to satisfy all of these corrections, i.e. $\mathcal{J}$ will effectively get at some step of the computations ϵ coefficients to the power of each element of this list, we simply return $\max(L_{\text{corr}})$ as the $\epsilon-$order correction for $\mathcal{J}$

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- It suffices to replace now all the symbolic orders by the actual orders given as input by the user in order to get the corrections for the BIs

Notes:

- In the case of several sets of RMIs, i.e. several trees, we would have to return the maximum maximorum of all the lists $L_{i,\text{corr}}$, i.e. $\max(\max(L_{i,\text{corr}}))$

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- In the case of several sets of RMIs, i.e. several trees, we would have to return the maximum maximorum of all the lists $L_{i,\text{corr}}$, i.e. $\max(\max(L_{i,\text{corr}}))$
- Actually, it might happen that some subsystem contains an incomplete set of IBIs, i.e. only some of the MIs that make up the main variables of another subsystem. In this case, we simply compute "reduced" rules by replacing the missing variables symbolic a_i' by their real values, given as input by the user.

Example: Supposing that $\tilde{\mathcal{I}}_5$ is missing for the IBI, we compute the following reduced rule:

$$\{\{\tilde{\mathcal{I}}_4, a_4\}, \{\tilde{\mathcal{I}}_5, a_5\}\} \to \{\{\{\tilde{\mathcal{I}}_4, \alpha_4^{(3)}\}, \{\tilde{\mathcal{I}}_5, \alpha_5^{(3)}\}\}, \{\{\{\mathcal{J}, \gamma^{(3)}\}\}\}\}$$

$$\Downarrow$$

$$\{\{\tilde{\mathcal{I}}_4, a_4\}\} \to \{\{\{\tilde{\mathcal{I}}_4, \alpha_4^{(3)}\}, \{\tilde{\mathcal{I}}_5, \alpha_5^{(3)}\}\}, \{\{\{\mathcal{J}, \gamma^{(3)}\}\}\}\Big|_{a_5 \to a_5^{\text{input}}}$$

Remark: Since we computed the local symbolic rule in the Gauß and Zürcher schemes, there are actually different trees that we can build:

- Purely Gauß: We apply recursively the local Gauß rules to the RMIs to build the tree
- Purely Zürcher: We apply recursively the local Zürcher rules to the RMIs to build the tree
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- Combined Zürcher and Gauß: We apply recursively the local combined rules to the RMIs to build the tree
- Mixed scheme: Given the list of Zürcher rules $L_Z = (r_1^Z, r_2^Z, \cdots, r_n^Z)$ and that of Gauß $L_G - (r_1^G, r_2^G, \cdots, r_n^G)$, we intertwine them in all the possible ways, i.e.:

$$(r_1^Z, r_2^Z, \cdots, r_n^Z), (r_1^G, r_2^Z, \cdots, r_n^Z), (r_1^Z, r_2^G, \cdots, r_n^Z), \cdots, (r_1^G, r_2^G, \cdots, r_n^G)$$

Then we construct a tree for each of these intertwined lists, replace the symbolic orders by the real ones and select the best tree (the lowest average correction)

Examples of system computations results:

(n; subsystems,BI)	BI orders (Carsten)	BI orders (NF)
$(3; 1; 4)$	$(1, 0, 0, 1)$	$(1, 0, 0, 1)$
$(48; 19; 7)$	$(5, 2, 4, 1, 1, 4, 0)$	$(3, 2, 3, 1, 1, 3, 0)$

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- Main application: faster solving and speed up for the large momentum method

To do:

- Intensive testing of the algorithm to debug the problems and make sure that it's really giving the optimal solution!
- Speed up some (resilient) parts of the algorithm: uncoupling, converting `OreSys` output into a usable form,...
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Thank you!

Inspired by and built on I


J. Ablinger et al. *A toolbox to solve coupled systems of differential and difference equations*. 2016. arXiv: 1601.01856 [cs.SC].


J. Ablinger et al. *Algorithms to solve coupled systems of differential equations in terms of power series*. 2016. arXiv: 1608.05376 [cs.SC].


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J. Blümlein, A. De Freitas, and C. Schneider. *Recent Symbolic Summation Methods to Solve Coupled Systems of Differential and Difference Equations*. 2014. arXiv: 1407.2537 [cs.SC].

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J. Blümlein, A. De Freitas, and C. Schneider. "Recent Symbolic Summation Methods to Solve Coupled Systems of Differential and Difference Equations". In: *PoS* LL2014 (2014), p. 017. DOI: 10.22323/1.211.0017.

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