A pedagogical penguin integral

Finding master integrals of uniform weight

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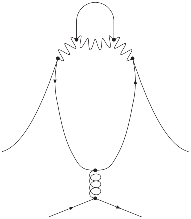


Motivation

- The method of differential equations is a popular method to compute Feynman integrals.

  (Kotikov '90, Remiddi '97, Gehrmann and Remiddi '99)

- We may **systematically** derive a differential equation for any Feynman integral.

  (`Reduze`, `Fire`, `Kira`)

- The essential step is to transform the differential equation into a particular nice form (ε-form).

  (Henn '13)

  In fact, the problem of computing Feynman integrals is reduced to finding an appropriate transformation for the differential equation.

Feynman integrals

Notation:

$$I_{\nu_1 \nu_2 \dots \nu_n}(\varepsilon, x) = e^{l \varepsilon \gamma_E} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \int \prod_{r=1}^{l} \frac{d^D k_r}{i \pi^{\frac{D}{2}}} \prod_{j=1}^{n_{\mathrm{int}}} \frac{1}{\left(-q_j^2 + m_j^2\right)^{\nu_j}}$$

Kinematic variables $x_1, \dots, x_{N_B+1}$:

$$\frac{-p_i \cdot p_j}{\mu^2}, \qquad \frac{m_i^2}{\mu^2}.$$

As μ^2 is arbitrary, we may set one kinematic variable to one.

Integration-by-parts allows us to express any $I_{\nu_1 \nu_2 \dots \nu_n}$ as a linear combination of master integrals.

Notation

N_F $= N_{\text{Fibre}}$: Number of master integrals,
master integrals denoted by $I = (I_1, ..., I_{N_F})$.

N_B $= N_{\text{Base}}$: Number of kinematic variables,
kinematic variables denoted by $x = (x_1, ..., x_{N_B})$.

The method of differential equations

We want to calculate

$$I = (I_1, \ldots, I_{N_F})$$

1. *Find a differential equation with respect to the kinematic variables for the Feynman integrals.*

$$\left[d + A(\varepsilon, x) \right] I = 0.$$

2. *Transform the differential equation into a simple form.*

$$\left[d + \varepsilon A(x) \right] I = 0.$$

3. *Solve the latter differential equation with appropriate boundary conditions.*

Transformations

- **Change the basis of the master integrals**

$$I' = UI,$$

where $U(\varepsilon, x)$ is a $N_F \times N_F$-matrix. The new connection matrix is

$$A' = UAU^{-1} + UdU^{-1}.$$

- **Perform a coordinate transformation on the base manifold**:

$$x_i' = f_i(x), \qquad 1 \leq i \leq N_B.$$

The connection transforms as

$$A = \sum_{i=1}^{N_B} A_i dx_i \qquad \Rightarrow \qquad A' = \sum_{i,j=1}^{N_B} A_i \frac{\partial x_i}{\partial x_j'} dx_j'.$$

Heuristic method

- We consider a change of the basis of master integrals: $I' = UI$.
- Suppose we have an **educated guess** for $U(\varepsilon, x)$. It is easy to check, if this transformation factors out ε: Simply compute

$$A' = UAU^{-1} + UdU^{-1}.$$

- Compare to the following situation: Suppose N is the product of two prime numbers. It is simple to check if p is a factor of N, this requires only one division.
- No mathematical rigour required for our educated guess.
- Well-known: **Constant leading singularities** and **maximal cuts** are helpful for finding U.

Content

A heuristic method, which gives an educated guess for $U(\varepsilon, x)$. The method is simple and involves even in non-trivial situations not more than back-of-the-envelope calculations.

Outline

1. The Baikov representation
2. Fibre transformations
3. Maximal cuts and constant leading singularities
4. A two-loop penguin integral
5. Discussion and outlook

Section 1

The Baikov representation

The Baikov representation

Let $p_1, p_2, ..., p_{n_\text{ext}}$ denote the external momenta and denote by

$$e \quad = \quad \dim \langle p_1, p_2, ..., p_{n_\text{ext}} \rangle$$

the dimension of the span of the external momenta.
For generic external momenta and $D \geq n_\text{ext} - 1$ we have $e = n_\text{ext} - 1$.
Lorentz invariants involving the loop momenta are of the form

$$\begin{array}{rl} -k_i^2, & 1 \leq i \leq l, \\ -k_i \cdot k_j, & 1 \leq i < j \leq l, \\ -k_i \cdot p_j, & 1 \leq i \leq l, \quad 1 \leq j \leq e. \end{array}$$

In total we have

$$N_V \quad = \quad \frac{1}{2} l \left(l + 1 \right) + e l$$

linear independent scalar products involving the loop momenta. Set

$$\sigma \quad = \quad \left(\sigma_1, ..., \sigma_{N_V} \right) = \left(-k_1 \cdot k_1, -k_1 \cdot k_2, ..., -k_l \cdot k_l, -k_1 \cdot p_1, ..., -k_l \cdot p_e \right).$$

The Baikov representation

A Feynman graph G has a Baikov representation if

1. $$N_V = n_{\text{int}}$$

2. Any internal inverse propagator can be expressed as a linear combination of the linear independent scalar products involving the loop momenta and terms independent of the loop momenta.

The second condition says, that there is an invertible $N_V \times N_V$-dimensional matrix C and a loop-momentum independent N_V-dimensional vector f such that

$$-q_s^2 + m_s^2 = C_{st}\sigma_t + f_s$$

for all $1 \leq s \leq n_{\text{int}}$.

Gram determinants

Define

$$\det G(q_1, ..., q_n) = \det(-q_i \cdot q_j),$$

e.g.

$$\det G(q_1, q_2) = \begin{vmatrix} -q_1^2 & -q_1 \cdot q_2 \\ -q_1 \cdot q_2 & -q_2^2 \end{vmatrix} = q_1^2 q_2^2 - (q_1 \cdot q_2)^2.$$

We change the integration variables to the **Baikov variables** z_j:

$$z_j = -q_j^2 + m_j^2.$$

The determinant $\det G(k_1, ..., k_l, p_1, ..., p_e)$ expressed in the variables z_j's is called the **Baikov polynomial**:

$$\mathcal{B}(z_1, ..., z_{N_V}) = \det G(k_1, ..., k_l, p_1, ..., p_e).$$

The Baikov representation

Baikov representation:

$$I = \frac{e^{l\varepsilon\gamma_E} \left(\mu^2\right)^{\nu - \frac{lD}{2}} \left[\det G(p_1, ..., p_e)\right]^{\frac{-D+e+1}{2}}}{\pi^{\frac{1}{2}(N_V - l)} (\det C) \prod\limits_{j=1}^{l} \Gamma\left(\frac{D-e+1-j}{2}\right)} \int\limits_{\mathcal{C}} d^{N_V} z \, \left[\mathcal{B}(z)\right]^{\frac{D-l-e-1}{2}} \prod_{s=1}^{N_V} z_s^{-\nu_s}.$$

The domain of integration $\mathcal{C}$ is given by

$$\mathcal{C} = \mathcal{C}_1 \cap \mathcal{C}_2 \cap \cdots \cap \mathcal{C}_l$$

with

$$\mathcal{C}_j = \left\{ \frac{\det G(k_j, k_{j+1}, ..., k_l, p_1, ..., p_e)}{\det G(k_{j+1}, ..., k_l, p_1, ..., p_e)} \geq 0 \right\}.$$

Comments

- Sketch of the proof:
  - Decompose loop momenta $k = k_{||} + k_{\perp}$ into a parallel space $k_{||} \in \langle p_1, p_2, ..., p_{n_{\mathrm{ext}}} \rangle$ and a component living in the complement, called the orthogonal space.
  - Integrate out the orthogonal space.
  - Change variables in the parallel space to the Baikov variables.
  - The Gram determinants enter through the Jacobian.
- The Baikov representation is very convenient for computing **cuts** of Feynman integrals. This corresponds to taking residues in the Baikov representation.
- We may use the **loop-by-loop approach**, this often leads to a one-dimensional integral representation for the maximal cut.

Baikov representation: Executive summary

- The **integrand** of the Baikov representation is **easily computable**.
- The integration domain in the Baikov representation is in general rather complicated.

Section 2

Fibre transformations

Overview

We seek a transformation $\vec{I}' = U\vec{I}$ such that $A' = UAU^{-1} + UdU^{-1}$ is simpler.

- **Block decomposition**
- Reduction to an univariate problem
- Picard-Fuchs operators
- Exploitung a master integral known to be of uniform weight
- Magnus expansion
- Moser's algorithm
- Leinartas decomposition
- **Maximal cuts and constant leading singularities**

Block decomposition

Order the set of master integrals $\vec{I} = (I_{\boldsymbol{\nu}_1}, \dots, I_{\boldsymbol{\nu}_{N_{\mathrm{master}}}})^T$ such that $I_{\boldsymbol{\nu}_1}$ is the simplest integral and $I_{\boldsymbol{\nu}_{N_{\mathrm{master}}}}$ the most complicated integral.

The matrix A has a lower block-triangular structure:

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ A_3 & \multicolumn{2}{c}{A_2} & \begin{matrix} 0 \\ 0 \end{matrix} \\ A_6 & \multicolumn{2}{c}{A_5} & A_4 \end{pmatrix}$$

Diagonal blocks:	A_1, A_2, A_4
Non-diagonal blocks:	A_3, A_5, A_6

Diagonal blocks

Let's consider block A_2. We consider a transformation of the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_2^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformed A' is given by

$$A' = \begin{pmatrix} A_1 & 0 & 0 \\ U_2 A_3 & \mathbf{U_2 A_2 U_2^{-1} + U_2 d U_2^{-1}} & 0 \\ A_6 & A_5 U_2^{-1} & A_4 \end{pmatrix}.$$

Non-diagonal blocks

Let us now consider block A_3. At this stage we would like to preserve the blocks A_1 and A_2. We consider a transformation of the form

$$U = \begin{pmatrix} 1 & 0 & 0 \\ U_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -U_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformed A' is given by

$$A' = \begin{pmatrix} A_1 & 0 & 0 \\ \mathbf{A_3 - A_2 U_3 + U_3 A_1 - dU_3} & A_2 & 0 \\ A_6 - A_5 U_3 & A_5 & A_4 \end{pmatrix}.$$

Section 3

Maximal cuts and constant leading singularities

Heuristic methods

- Suppose somebody gives us a transformation matrix U

$$\vec{I}' = U\vec{I}.$$

- It is easy to check if this fibre transformation transforms the differential equation to an ε-form. We simply calculate

$$A' = UAU^{-1} + UdU^{-1}$$

  and check if A' is in ε-form.

- This is a situation where a **heuristic method** may work well: Guessing a suitable U may outperform any systematic algorithm to construct the matrix U.

Feynman integrals with cuts

Recall: Baikov representation

$$I_{\nu_1 \dots \nu_n}\left(D, x_1, \dots, x_{N_B}\right) = C \int\limits_{\mathcal{C}} d^{N_V} z \, \left[\mathcal{B}(z)\right]^{\frac{D-l-e-1}{2}} \prod_{s=1}^{N_V} z_s^{-\nu_s}$$

with integration contour $\mathcal{C}$.

Consider a **modified integration contour** $\mathcal{C}'$ such that

1. Integration-by-parts identities still hold.
2. The variation of the integral with respect to the kinematic variables comes entirely from the integrand.
3. The symmetries among the integrals are respected.

The maximal cut

Definition (Feynman integral with the internal edge e_j cut)

Baikov integral with a modified integration domain $\mathcal{C}'$:

- a small anti-clockwise circle around $z_j = 0$ in the complex z_j-plane,
- in all other variables the intersection of the original integration domain $\mathcal{C}$ with the hyperplane $z_j = 0$.

We may iterate the procedure and take multiple cuts. Of particular importance is the maximal cut:

Definition (Maximal cut)

Take for a Feynman integral $I_{\nu_1 \dots \nu_{n_{\mathrm{int}}}}$ the cut for all edges e_j for which $\nu_j > 0$.

Constant leading singularities

- Denote the **integrands** of the master integrals by $\varphi_1, \dots, \varphi_{N_{\mathrm{master}}}$.
- Choose N_{master} **independent integration domains** $\mathcal{C}_1, \dots, \mathcal{C}_{N_{\mathrm{master}}}$.
  The integration domains are independent, if the $N_{\mathrm{master}} \times N_{\mathrm{master}}$-matrix with entries

$$\langle \varphi_i | \mathcal{C}_j \rangle = \int\limits_{\mathcal{C}_j} \varphi_i$$

  has full rank.
- We are interested in choosing the integration domains $\mathcal{C}_j$ as simple as possible. Particular simple integration domains are products of circles around singular points. These correspond to residue calculations.

Constant leading singularities

- Let φ be the integrand of a Feynman integral I.
- Define d_{min} by

$$d_{\mathrm{min}} = \min_j \left(\mathrm{ldegree} \left(\langle \varphi | \mathcal{C}_j \rangle, \varepsilon \right) \right),$$

- We say that the Feynman integral I has **constant leading singularities**, if for all j

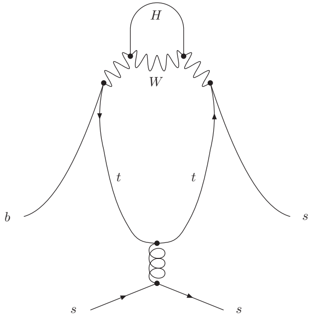
$$\mathrm{coeff} \left(\langle \varphi | \mathcal{C}_j \rangle, \varepsilon^{d_{\mathrm{min}}} \right) = \text{constant of weight zero},$$

- Integrals with constant leading singularities are a guess for a basis of master integrals, which puts the differential equation into an ε-form.

Section 4

A two-loop penguin integral

A two-loop penguin integral

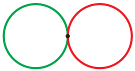


Three kinematic variables:

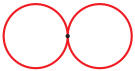
$$x_1 = \frac{-s}{m_t^2}, \quad x_2 = \frac{m_W^2}{m_t^2}, \quad x_3 = \frac{m_H^2}{m_t^2}.$$

Master integrals

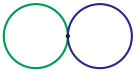
J_1



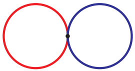
J_2



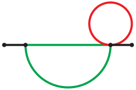
J_3



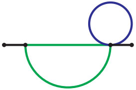
J_4



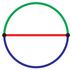
J_5



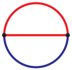
J_6



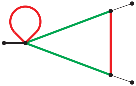
J_7



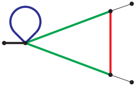
J_8



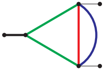
J_9



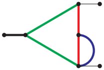
J_{10}



$J_{11} - J_{14}$



J_{15}



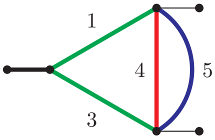
Topologies with one master integral

Candidate for a master integral of uniform weight: Divide by the maximal cut and replace $\pi \to \varepsilon^{-1}$.

Topologies with several master integrals

Four master integrals:

$$I_{1011100}, I_{1011200}, I_{1012100}, I_{1021100}$$



Maximal cuts

Involves only determinants and series expansion:

$$\begin{aligned}
\mathrm{MaxCut}\; I_{1012100}(4) &= 4\pi^2 i \int\limits_{\mathcal{C}_{\mathrm{MaxCut}}} \frac{m_t^2\left(m_H^2 - z_2\right) dz_2}{s\left(z_2 - m_W^2\right)\sqrt{4 m_W^2 m_H^2 - \left(z_2 + m_H^2\right)^2}}, \\
\mathrm{MaxCut}\; I_{1011200}(4) &= 4\pi^2 i \int\limits_{\mathcal{C}_{\mathrm{MaxCut}}} \frac{m_t^2\left(2m_W^2 - m_H^2 - z_2\right) dz_2}{s\left(z_2 - m_W^2\right)\sqrt{4 m_W^2 m_H^2 - \left(z_2 + m_H^2\right)^2}}, \\
\frac{1}{\varepsilon}\mathrm{MaxCut}\; I_{1011100}(2) &= -8\pi^2 i \int\limits_{\mathcal{C}_{\mathrm{MaxCut}}} \frac{m_t^4 dz_2}{P_2\sqrt{4 m_W^2 m_H^2 - \left(z_2 + m_H^2\right)^2}},
\end{aligned}$$

with the polynomial P_2 quadratic in z_2:

$$P_2 = \left(z_2 + m_t^2 - m_W^2\right)^2 - s\left(z_2 - m_W^2\right)$$

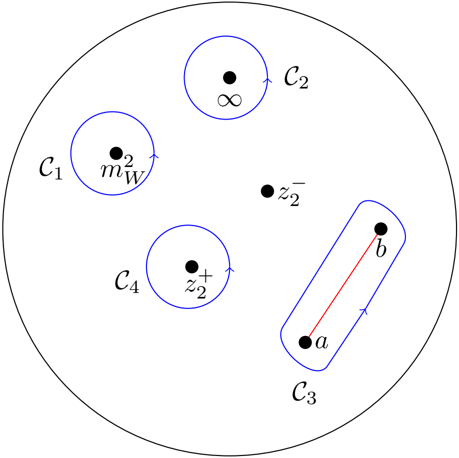
Singularities

From the denominator:

$$
\begin{aligned}
P_1 &= z_2 - m_W^2, \\
P_2 &= \left(z_2 + m_t^2 - m_W^2\right)^2 - s\left(z_2 - m_W^2\right) \\
&= \left(z_2 - z_2^+\right)\left(z_2 - z_2^-\right), \\
R &= \sqrt{4 m_W^2 m_H^2 - \left(z_2 + m_H^2\right)^2} \\
&= \sqrt{-\left(z_2 - a\right)\left(z_2 - b\right)}.
\end{aligned}
$$

P_1 is a linear polynomial in z_2, P_2 is a quadratic polynomial in z_2 and R is the square root of a quadratic polynomial in z_2.

Master contours



Integrals with constant leading singularities

1. $$\frac{1}{2\pi i} \int\limits_{\mathcal{C}} \frac{dz}{\sqrt{(\mathbf{z}-\mathbf{a})(\mathbf{z}-\mathbf{b})}} = \pm 1,$$

   where $\mathcal{C}$ is a cycle around the slit $[a,b]$.

2. $$\frac{\sqrt{(c-a)(c-b)}}{2\pi i} \int\limits_{\mathcal{C}'} \frac{dz}{(\mathbf{z}-\mathbf{c})\sqrt{(\mathbf{z}-\mathbf{a})(\mathbf{z}-\mathbf{b})}} = \pm 1,$$

   where $\mathcal{C}'$ is either a cycle around the slit $[a,b]$ or a small circle around $z=c$.

Master integrands

Master integrands, which give constants of weight 0 when integrated along $\mathcal{C}_1$-$\mathcal{C}_4$

$$
\begin{aligned}
\varphi_1 &= i\pi^2\varepsilon^3 \frac{\left(m_H^2 - z_2\right)}{P_1 R} dz_2, \\
\varphi_2 &= i\pi^2\varepsilon^3 \frac{\left(2m_W^2 - m_H^2 - z_2\right)}{P_1 R} dz_2, \\
\varphi_3 &= \pi^2\varepsilon^3 \frac{m_t^2 r_4 \left(z_2 - z_2^-\right)}{P_2 R} dz_2, \\
\varphi_4 &= \pi^2\varepsilon^3 \frac{m_t^2 r_5 \left(z_2 - z_2^+\right)}{P_2 R} dz_2,
\end{aligned}
$$

where

$$
r_4 = \frac{R\left(z_2^+\right)}{m_t^2}, \qquad r_5 = \frac{R\left(z_2^-\right)}{m_t^2}.
$$

Period matrix

We compute the period matrix

$$\langle \varphi_i | \mathcal{C}_j \rangle = 2i\pi^3\varepsilon^3 \begin{pmatrix} 1 & 1 & -2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The period matrix has full rank, hence φ_1-φ_4 form a basis for the integrands.

Translating back

$$\varphi_1 = i\pi^2\varepsilon^3 \frac{\left(\mathbf{m_H^2 - z_2}\right)}{\mathbf{P_1 R}} dz_2$$

$$\Rightarrow \quad \varepsilon^3 \frac{s}{m_t^2} I_{1012100}$$

is a candidate for an integral of uniform weight.

$$\text{MaxCut } I_{1012100}(4) = 4\pi^2 i \int\limits_{\mathcal{C}_{\text{MaxCut}}} \frac{m_t^2 \left(\mathbf{m_H^2 - z_2}\right) dz_2}{s\, \mathbf{P_1 R}},$$

$$\text{MaxCut } I_{1011200}(4) = 4\pi^2 i \int\limits_{\mathcal{C}_{\text{MaxCut}}} \frac{m_t^2 \left(2m_W^2 - m_H^2 - z_2\right) dz_2}{s P_1 R},$$

$$\frac{1}{\varepsilon} \text{MaxCut } I_{1011100}(2) = -8\pi^2 i \int\limits_{\mathcal{C}_{\text{MaxCut}}} \frac{m_t^4\, dz_2}{P_2 R},$$

Translating back

$$\varphi_3 = \pi^2 \varepsilon^3 \frac{m_t^2 r_4 \left(z_2 - z_2^-\right)}{\mathbf{P_2 R}} dz_2$$

It follows that

$$\frac{\varepsilon^2 r_4}{m_t^2} \left[m_t^2 \mathbf{D}^- I_{1(-1)11100} - z_2^- \mathbf{D}^- I_{1011100} \right]$$

is a candidate for an integral of uniform weight.

$$\text{MaxCut } I_{1012100}(4) = 4\pi^2 i \int\limits_{\mathcal{C}_{\text{MaxCut}}} \frac{m_t^2 \left(m_H^2 - z_2\right) dz_2}{s P_1 R},$$

$$\text{MaxCut } I_{1011200}(4) = 4\pi^2 i \int\limits_{\mathcal{C}_{\text{MaxCut}}} \frac{m_t^2 \left(2m_W^2 - m_H^2 - z_2\right) dz_2}{s P_1 R},$$

$$\frac{1}{\varepsilon} \text{MaxCut } I_{1011100}(2) = -8\pi^2 i \int\limits_{\mathcal{C}_{\text{MaxCut}}} \frac{m_t^4 dz_2}{\mathbf{P_2 R}},$$

Translating back

In summary we obtain from φ_1 *-* φ_4 *(with* $r_1 = \sqrt{x_1(4+x_1)}$*)*

$$
\begin{aligned}
J_{11} &= \varepsilon^3 x_1 I_{1012100}, \\
J_{12} &= \varepsilon^3 x_1 I_{1011200}, \\
J_{13} &= \varepsilon^2 r_4 \left[\mathbf{D}^- I_{1(-1)11100} + \left(1 + \frac{1}{2}x_1 - x_2 + \frac{1}{2}r_1\right) \mathbf{D}^- I_{1011100} \right], \\
J_{14} &= \varepsilon^2 r_5 \left[\mathbf{D}^- I_{1(-1)11100} + \left(1 + \frac{1}{2}x_1 - x_2 - \frac{1}{2}r_1\right) \mathbf{D}^- I_{1011100} \right].
\end{aligned}
$$

By computing

$$A' = UAU^{-1} + UdU^{-1}$$

we may check that this brings the differential equation into ε-form.

Computational cost

- This only involves:
  - computation of Gram determinants,
  - computation of residues,
  - computation of series expansions,
  - matching with simple integrals of constant leading singularities.
- No computational expensive calculations are involved!

Section 5

Discussion and outlook

A necessary condition?

Question:

Are constant leading singularities a necessary condition for having master integrals of uniform weight?

Answer:

No.

In the elliptic case: For the sunrise integral we may choose master integrals of uniform weight with leading singularities

$$P^{\text{leading}} = 2i \begin{pmatrix} (2\pi i \varepsilon)^2 & (2\pi i \varepsilon)^2 \tau \\ 0 & -(2\pi i \varepsilon) \end{pmatrix}.$$

A sufficient condition?

Question:

Are constant leading singularities a sufficient condition for having master integrals of uniform weight?

Answer:

No.

Suppose that I has constant leading singularities and is of uniform weight. Then $I' = (1+\varepsilon) I$ has also constant leading singularities but is not of uniform weight.

Summary

- The computation of Feynman integrals can be reduced to finding an appropriate transformation for the differential equation.
- We are interested in a transformation, which factors out ε.
- As we may easily check, if a given transformation achieves this, we may use heuristic methods.
- Method based on master contours and master integrands, such that the period matrix has full rank and only involves constant entries of weight zero.
- Very efficient heuristic method.
- Warning: The condition above is neither necessary nor sufficient.