Multiloop Feynman diagrams in a quantum computer
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Based on: S. Ramírez Uribe, A.E. Rentería Olivo, GR, G.F.R. Sborlini, L. Vale Silva, arXiv 2105.08703
factorization into short distance (hard scattering) and long distance (initial and final state)

QFT = quantum mechanics + special relativity + creation-annihilation of particles
RECENT QUANTUM APPLICATIONS

- **Parton densities:**
  Pérez-Salinas, Cruz-Martínez, Alhajri, Carrazza, PRD 103, 034027 (2021)

- **Parton showers:**
  Bauer, de Jong, Nachman, Provasoli, PRL 126, 062001 (2021)
  Bauer, Freytsis, Nachman, arXiv:2102.05044
  Bepari, Malik, Spannowsky, Williams, PRD103, 076020 (2021)

- **Quantum machine learning:**

- **Tree-level helicity amplitudes:**
  Bepari, Malik, Spannowsky, Williams, PRD103, 076020 (2021)

- **Multiloop scattering amplitudes:**
  generally accepted to be beyond the reach of quantum computers, since it would require a prohibitive number of qubits

- **Jet clustering:**
  Wei, Naik, Harrow, Thaler, PRD 101, 094015 (2020)
  Pires, Bargassa, Seixas, Omar, arXiv:2101.05618
  Pires, Omar, Seixas, arXiv:2012.14514
Multi-loop scattering amplitude: $n$ sets of internal momenta, each set depends on a specific linear combination of the $L$ loop momenta

$$A^{(L)}_N = \mathcal{N}(\{\ell_i\}_L, \{p_j\}_N) G_F(1,\ldots,n) = \prod_{i \in 1 \cup \ldots \cup n} (G_F(q_i))^{a_i}$$

Starting from the integrand in the Feynman representation, take the residues of propagators in the first set (in each loop, integrates out one component of the loop momentum)

$$A_D(1; 2,\ldots,n) = -2\pi i \sum_{i_1 \in 1} \text{Res} \left( A_F(1,\ldots,n), \text{Im}(\eta q_{i_1}) < 0 \right)$$

Cauchy contour always from below the real axis

valid for arbitrary powers and Lorentz invariant

reverse momenta, if necessary, to keep a coherent momentum flow: $s \to \bar{s}$ ($q_i \to -q_i$)

The nested residue involving several sets

$$A_D(1,\ldots,r; r+1,\ldots) = -2\pi i \sum_{i_r \in r} \text{Res} \left( A_D(1,\ldots,r-1; r,\ldots), \text{Im}(\eta q_{i_r}) < 0 \right)$$

If $\eta = (1,0)$ integrate out the energy component and the integration domain is Euclidean
What if we reorder the loop lines (order of residues)? Do we get a different representation?

In fact no (math vs physics), and if we sum up all the nested residues, e.g. scalar integral

\[ \mathcal{A}^{(L)}_{\text{MLT}} = \int \prod \frac{1}{2q_{i,0}^{(+)} + \frac{1}{\lambda^{+}_{1,n}} + \frac{1}{\lambda^{-}_{1,n}}} q_{i,0}^{(+)} = \sqrt{q_{i}^{2} + m_{i}^{2} - \imath 0} \]

- Independent of the initial momentum flow assignments
- **Causal conjecture:** the LTD representation if manifestly free of non-causal singularities for all topologies and internal configurations
Causal Propagators

\[ \frac{1}{\lambda^+_1,n} = \frac{1}{\sum_{i=1}^{n} q_{i,0}^{(+)} + k_{(1,n),0}} \]

- **Incoming** external momenta, \( k_{(1,n),0} < 0 \), a causal threshold / soft / collinear singularity arises when all the internal momenta are on shell and aligned in the same direction.

\[ \frac{1}{\lambda^-_1,n} = \frac{1}{\sum_{i=1}^{n} q_{i,0}^{(+)} - k_{(1,n),0}} \]

- **Outgoing** external momenta, \( k_{(1,n),0} > 0 \), a causal threshold / soft / collinear singularity arises when all the internal momenta are on shell and aligned in the opposite direction.
CAUSAL LTD

CAUSAL REPRESENTATION

\[
\mathcal{A}^{(L)} = \int \frac{1}{\ell_1 \cdots \ell_L} \prod 2q_{i,0}^{(+)} \sum_{\sigma \in \Sigma} \frac{\mathcal{N}_{\sigma(i_1, \ldots, i_{n-L})}}{\lambda_{\sigma(i_1)}^{h_1} \cdots \lambda_{\sigma(i_{n-L})}^{h_{n-L}}} + (\lambda_\uparrow \leftrightarrow \lambda_\downarrow)
\]

- Each combination of compatible causal propagators in Σ fixes the momentum flows of all the internal momenta
- Conversely, if we fix the causal momentum flows we can bootstrap the causal LTD representation

○ Quantum algorithm
NON-CAUSAL SINGULARITIES ARE NONPHYSICAL

CAUSALITY IN LOOP-TREE DUALITY

- Integrands in the Feynman representation have singularities that are nonphysical \( \equiv \) not related to the optical theorem
- LTD leads to manifestly causal representations (free of non-causal singularities): more stable numerically

- Integrand numerical instabilities across a noncausal threshold
- manifestly causal LTD representation

- geometric interpretation: Sborlini PRD 104 (2021) 036014
- Wolfram Mathematica package: Torres Bobadilla JHEP 2104, 183 and EPJC 81 (2021) 514
FEYNMAN DIAGRAMS AT TREE LEVEL

**CAUSALITY**

*Cause-effect* relationships are unbreakable. (You cannot be born before your grandfather.)

[Symmetry Magazine]

- $e^{-}$ and $\mu^{-}$ get closer, and the **electron** emits a photon which is absorbed by the muon
- $e^{-}$ and $\mu^{-}$ get closer, and the **muon** emits a photon which is absorbed by the electron

\[ G_F(q_i) = \frac{1}{q_i^2 - m_i^2 + i0} = \frac{1}{(q_{i,0} - q_{i,0}^{(+)})(q_{i,0} + q_{i,0}^{(+))})} \]

- Positive frequencies are propagated forwards in time, and negative are propagated backwards
- **Two on-shell states**, one with positive and another with negative energy:
  \[ q_{i,0} \to \pm q_{i,0}^{(+)} = \pm \sqrt{q_i^2 + m_i^2 - i0} \]
\[ \mathcal{A}^{(1)} = \int_{\ell} \mathcal{N}(\ell, p_i) G_F(q_1, q_2) \]

- the gluon fluctuates into a quark-antiquark pair, which merges back into a gluon
- a **causal discontinuity** (imaginary part) arises when the quark and antiquark are produced as real (on-shell)
  \[ \equiv \text{Cutkosky rule / optical theorem} \]
- the gluon emits a quark, which travels backwards in time to the point of emission
- each configuration generates a **non-causal singularity** of the integrand: they are nonphysical and must somehow be cancelled out
- there are \(2^n\) configurations for \(n\) propagators
SELECTING CAUSAL CONFIGURATIONS

LOOPS IN A QUANTUM COMPUTER

- Each Feynman propagator has **two on-shell states**, one with positive energy $|1\rangle$, and another with negative energy $|0\rangle \equiv$ momentum flow in one direction or the opposite.

- A classical algorithm requires to test causal conditions (≡ directed acyclic graph) for $O(2^n)$ states.

- A **quantum algorithm** exploits superposition and entanglement to test causality at once, in $O(1)$ iterations (still a number of measurements $O(\text{causal states})$)

\[
q_1 = \ell_1
\]

\[
q_2 = \ell_2
\]

\[
q_{12} = -\ell_1 - \ell_2 + p
\]

\[
\mathcal{A}^{(2)} = \int_{\ell_1,\ell_2} G_F(q_1, q_2, q_{12})
\]

\[
\mathcal{A}_{\text{LTD}}^{(2)} = -\int_{\ell_1,\ell_2} \frac{1}{x_{12}} \left( \frac{1}{q_{1,0}^{(+)}} + \frac{1}{q_{2,0}^{(+)}} + \frac{1}{q_{12,0}^{(+)}} + p_0 + \frac{1}{q_{1,0}^{(+)}} + \frac{1}{q_{2,0}^{(+)}} + \frac{1}{q_{12,0}^{(+)}} - p_0 \right)
\]

\[
x_{12} = \prod_{i=1,2,12} 2q_{i,0}^{(+)}
\]
Seven loops $L = 7$

Equivalent to 3 eLoops (Mercedes) regarding causal bootstrapping
GROVER’S ALGORITHM

- A uniform superposition of $N = 2^n$ states
  
  $$|q\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$$

- which is a superposition of the winning state $|w\rangle$ encoding the $r$ causal states, and the orthogonal state $|q_\perp\rangle$ collecting the noncausal states

  $$|q\rangle = \sin \theta |w\rangle + \cos \theta |q_\perp\rangle$$

  $$|w\rangle = \frac{1}{\sqrt{r}} \sum_{x \in w} |x\rangle \quad |q_\perp\rangle = \frac{1}{\sqrt{N-r}} \sum_{x \not\in w} |x\rangle$$

- The mixing angle is

  $$\theta = \arcsin \sqrt{r/N}$$
The **oracle operator**

\[ U_w = I - 2 |w\rangle\langle w| \]

- **flips** the state \( |x\rangle \) if \( x \in w \): \( U_w |x\rangle = - |x\rangle \)
- leaves it unchanged otherwise: \( U_w |x\rangle = |x\rangle \) if \( x \not\in w \)
**GROVER’S ALGORITHM**

- The *oracle operator*

\[ U_w = I - 2 |w\rangle\langle w| \]

- *flips* the state \(|x\rangle\) if \(x \in w: U_w |x\rangle = - |x\rangle\)

- leaves it unchanged otherwise: \(U_w |x\rangle = |x\rangle\) if \(x \notin w\)

- The *diffusion operator*

\[ U_q = 2 |q\rangle\langle q| - I \]

- performs a *reflection* around the initial state.

The iterative application \(t\) times leads to

\[ (U_q U_w)^t |q\rangle = \sin \theta_t |w\rangle + \cos \theta_t |q_\perp\rangle \quad \theta_t = (2t + 1) \theta \]
Grover’s Algorithm

- If \( r \ll N \), requires \( \Theta(\sqrt{N/r}) \) iterations instead of \( \Theta(N) \) from a classical computation.

- If \( r = N/4 \) or \( \theta = 30^\circ \), then one single iteration brings to \( \theta_t = 90^\circ \).

- If \( r = 3N/4 \) or \( \theta = 60^\circ \), then one single iteration brings to \( \theta_t = 180^\circ \) which suppresses the probability of the winning state.

- If \( r = N/2 \) or \( \theta = 45^\circ \), it does not matter how many iterations are applied, probabilities remain unchanged.

- Increasing the total number of states, e.g. maximum of two ancillary qubits [see also Nielsen-Chuang 2000].

- Given a causal state, the mirror state with all the momentum flows reversed is also causal.
Loop Quantum Algorithm

- The $|q\rangle$ register encodes the states of the edges/internal propagators: the qubit $q_i$ is in the state $|1\rangle$ if the momentum flow of the corresponding edge is oriented in the direction of the original assignment, and $|0\rangle$ if it is in the opposite direction.

- The $|c\rangle$ register stores the binary clauses that probe if two qubits representing two adjacent edges are in the same state (oriented in the same direction):

  $$c_{ij} \equiv (q_i = q_j) \quad \bar{c}_{ij} \equiv (q_i \neq q_j)$$

- The $|a\rangle$ register stores the loop clauses that probe if all the qubits (edges) in each subloop form a cyclic circuit.

- The Grover’s marker initialized to the Bell state $|out_0\rangle = |\rangle - \rangle = (|0\rangle - |1\rangle)/\sqrt{2}$.

- The oracle

  $$U_w |q\rangle |c\rangle |a\rangle |out_0\rangle = |q\rangle |c\rangle |a\rangle |out_0 \otimes f(a, q)\rangle$$

  $$|out_0 \otimes 0\rangle = |out_0\rangle$$

  $$|out_0 \otimes 1\rangle = - |out_0\rangle$$

- The diffuser $U_q$ from IBM Qiskit.
### NUMBER OF QUBITS

#### LOOP QUANTUM ALGORITHM

| eloops (edges per set) | $|q\rangle$ | $|c\rangle$ | $|a\rangle$ | Total |
|------------------------|-------------|-------------|--------------|-------|
| one ($n$)              | $n + 1$     | $n - 1$     | 1            | $2n + 2$ |
| two ($n_0, n_1, n_2$)  | $n$         | $n$         | 3            | $2n + 4$ |
| three ($n_0, \ldots, n_5$) | $n$ | $n + (2$ to 3$)$ | 4 to 7 | $2n + (7$ to 11$)$ |
| four ($n_0, \ldots, n_7$) | $n$ | $n + (3$ to 6$)$ | 5 to 13 | $2n + (9$ to 20$)$ |
| four ($n_0, \ldots, n_8(t,s)$) | $n$ | $n + (4$ to 7$)$ | 5 to 13 | $2n + (10$ to 21$)$ |
| four ($n_0, \ldots, n_8(u)$) | $n$ | $n + (5$ to 8$)$ | 9 to 13 | $2n + (15$ to 22$)$ |

- 19 qubits if $n_i = 1$,
- 25 qubits if $n_i = 1$,
- 28 qubits if $n_i = 1$,
- 33 qubits if $n_i = 1$,
\[
a_0 = \neg (c_{01} \land c_{12}) \\
f^{(1)}(a, q) = a_0 \land q_0 \land q_3
\]

- \( |q\rangle = H^\otimes n |0\rangle \)
- The \( |c\rangle \) and \( |a\rangle \) registers initialized to \( |0\rangle \)

ANCILLARY QUBIT
QUANTUM SIMULATION

TWO ELOOPS

Qiskit

\[ a_0 = \neg (c_{01} \land c_{13} \land c_{34}) \]
\[ a_1 = \neg (c_{01} \land \bar{c}_{12}) \]
\[ a_2 = \neg (c_{23} \land c_{34}) \]
\[ f^{(2)}(a, q) = (a_0 \land a_1 \land a_2) \land q_2 \]
 QUANTUM SIMULATION

THREE ELOOPS

\[ a_0 = \neg(c_{01} \land c_{12}) \quad a_1 = \neg(\bar{c}_{04} \land \bar{c}_{34}) \]

\[ a_2 = \neg(\bar{c}_{15} \land \bar{c}_{45}) \quad a_3 = \neg(\bar{c}_{23} \land \bar{c}_{35}) \]

\[ f^{(3)}(a, q) = (a_0 \land \ldots \land a_3) \land q_0 \]
First nonplanar graph starting at four loops

- 115/512 causal states
- requires 33 qubits > IBM Qiskit capacity
First **nonplanar** graph starting at four loops
- 115/512 causal states
- **QUTE simulator**, up to 38 logical qubits

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CONCLUSIONS

- **Causal configurations** of multiloop Feynman integrals successfully identified with an efficient modification of Grover’s quantum algorithm for querying over unstructured databases.

- Beyond particle physics, finds application for **directed acyclic graphs**.

- Still limited by quantum volume in real devices, and number of qubits in quantum simulators.

- Given the rapid progress in the field, flagship application for future developments.