$\mathcal{O}(\alpha_s^3)$ calculations for the inclusive determination of $|V_{cb}|$

Zeuthen 2021

Kay Schönwald | November 4, 2021

TTP KARLSRUHE

[based on: Fael, Schönwald, Steinhauser PRL 125 (2020); JHEP 10 (2020); PRD 103 (2021); PRD 104 (2021)]

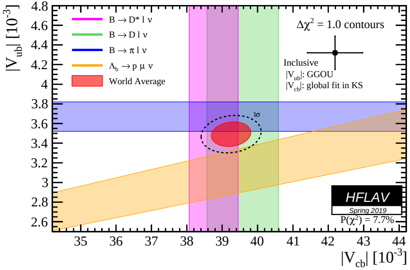
Institute for Theoretical Particle Physics

TRR 257 - Particle Physics Phenomenology after the Higgs Discovery

Outline

Motivation

- $b \to c\ell\nu$ is an important ingredient in the inclusive determination of $|V_{cb}|$:
  - Currently there is a tension between inclusive and exclusive determination of $|V_{cb}|$.
  - Errors are mostly theory dominated.
  - Precise measurements of the CKM matrix elements $|V_{ib}|$ are among main goals of Belle II and LHCb.
  - The semi-leptonic decay rate is an important ingredient in the global fit for the inclusive determination.
  - The global fits are performed in the kinetic scheme.
- $\mu \to e\nu\nu$ is the most precise way to determine G_F.

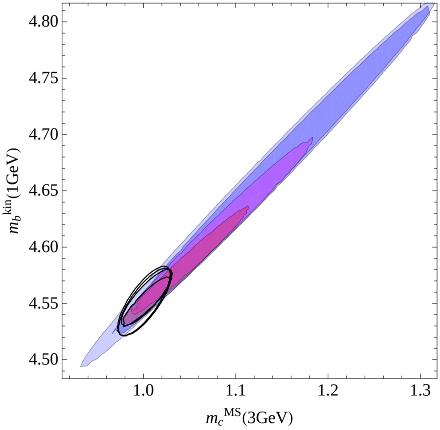


Motivation

The Heavy Quark Expansion (HQE):

$$\mathrm{d}\Gamma = \mathrm{d}\Gamma_0 + \mathrm{d}\Gamma_{\mu_\pi}\frac{\mu_\pi^2}{m_b^2} + \mathrm{d}\Gamma_{\mu_G}\frac{\mu_G^2}{m_b^2} + \mathrm{d}\Gamma_{\rho_D}\frac{\rho_D^3}{m_b^3} + \mathrm{d}\Gamma_{\rho_{LS}}\frac{\rho_{LS}^3}{m_b^3} + \dots$$

- $\mathrm{d}\Gamma_i$ are computed in perturbative QCD
- dependece on non-perturbative HQE parameters:
  $\mu_\pi, \mu_G, \rho_D, \rho_{LS}, \dots$
- Perturbative corrections to $b \to c\ell\nu$ exhibit a bad convergence in the on-shell and the $\overline{\mathrm{MS}}$ scheme for the heavy quark masses.
- Better knowledge of scheme conversion can further constrain the global fit.



	tree	α_s	α_s^2	α_s^3
1	✓	✓	✓	this talk
$1/m_b^2$	✓	✓	!	
$1/m_b^3$	✓	✓		
$1/m_b^{4,5}$	✓			
$\overline{m}_b - m_b^{\mathrm{kin}}$		✓	✓	this talk

[Jezabek, Kühn, NPB 314 (1989); Gambino et al., NPB 719 (2005)]
[Melnikov, PLB 666 (2008); Pak, Czarnecki, PRD (2008)]
[Fael, KS, Steinhauser, PRD 104 (2021)]

[Alberti, Gambino, Nandi, JHEP 1401 (2014)]
[Mannel, Pivovarov, Rosenthal, PRD 92 (2015)]
[Becher, Boos, Lunghi, JHEP 0712 (2007)]

[Mannel, Pivovarov, PRD 100 (2019)]

[Dassinger, Mannel, Turczyk, JHEP 0703 (2007); JHEP 1011 (2010)]
[Fael, Mannel, Vos, JHEP 02 (2019); JHEP 12 (2019)]

[Bigi et al., PRD 56 (1997); Czarnecki, Melnikov, Uraltsev, PRL 80 (1998)]
[Fael, Steinhauser, KS, PRL 125 (2020); PRD 103 (2021)]

The Kinetic Heavy Quark Mass

The kinetic heavy quark mass

- The kinetic mass scheme is tailored for $b \to c$ transitions.
- It is based on the heavy quark – hadron mass relation in HQET:

Definition

$$M_B = m_b - \overline{\Lambda}(\mu) - \frac{\mu_\pi^2(\mu)}{2m_b} + \dots$$

[Bigi et al., PRD 56 (1997); Czarnecki, Melnikov, Uraltsev, PRL 80 (1998)]

- Key mass scale is m_b, not M_B:

$$\Gamma_{\mathrm{sl}} \simeq \frac{G_F^2 |V_{cb}|^2}{192\pi^3} (M_B - \overline{\Lambda})^5$$

The kinetic heavy quark mass

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Definition

$$m^{\text{kin}} = m^{\text{OS}} - \overline{\Lambda}(\mu)\Big|_{\text{pert}} - \frac{\mu_\pi^2(\mu)\Big|_{\text{pert}}}{2m^{\text{kin}}} + \ldots$$

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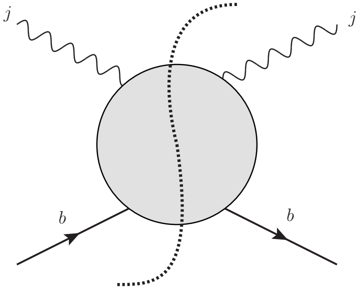
$$\Gamma_{\text{sl}} \simeq \frac{G_F^2 |V_{cb}|^2}{192\pi^3}(M_B - \overline{\Lambda})^5$$

The Small Velocity Sum Rules

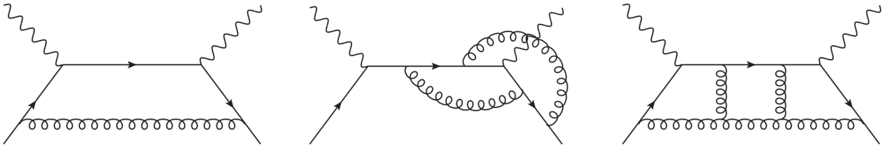


- $W(\omega, \vec{v}) = 2\,\mathrm{Im}\left[\frac{i}{2m}\int \mathrm{d}^4x\, e^{-iqx}\,\langle Q|\, TJ(x)J(0)\,|Q\rangle\right]$
  (J is an arbitrary current)
- $\vec{v}$: velocity of the heavy quark
- ω: excitation energy of the heavy quark

$$\overline{\Lambda}(\mu)\Big|_{\text{pert}} = \lim_{\vec{v}\to 0}\lim_{m\to\infty}\frac{2}{\vec{v}^2}\frac{\int\limits_0^\mu \mathrm{d}\omega\,\omega\, W(\omega,\vec{v})}{\int\limits_0^\mu \mathrm{d}\omega\, W(\omega,\vec{v})}, \qquad \mu_\pi^2(\mu)\Big|_{\text{pert}} = \lim_{\vec{v}\to 0}\lim_{m\to\infty}\frac{3}{\vec{v}^2}\frac{\int\limits_0^\mu \mathrm{d}\omega\,\omega^2\, W(\omega,\vec{v})}{\int\limits_0^\mu \mathrm{d}\omega\, W(\omega,\vec{v})}$$

Calculation Strategy

- We have to calculate the imaginary part of the forward scattering amplitudes:



$$W(\omega, \vec{v}) = W_{\mathrm{el}}(\vec{v})\delta(\omega) + \frac{\vec{v}^2}{\omega} W_{\mathrm{real}}(\omega)\theta(\omega) + \mathcal{O}(v^4, \frac{\omega}{m_b})$$

- Virtual corrections: only contribute to the denominator
  - given by the static limit of the massive form factor
    [Ravidran, Neerven, PLB 445 (1998),. . . , Ablinger et al. PRD 97 (2018), . . .]
- real corrections: only contribute to the numerator
  - needs to be computed in the appropriate limit

Calculation Strategy

- We can translate to a kovariant expansion:

$$
\begin{aligned}
y = m_b^2 - s &= m_b\omega(2+v^2) + \mathcal{O}(\omega^2, v^4) \\
q^2 &= -m_b v^2 (m_b - \omega) + \mathcal{O}(\omega^2, v^4) \\
2p\cdot q &= y - q^2
\end{aligned}
$$

- We realize the threshold expansion via expansion by regions: [Beneke, Smirnov (Nucl. Phys. B (1998))]
  - the loop momenta can scale: hard (h) $k_i \sim m_b$ or ultrasoft (u) $k_i \sim y/m_b$
  - not all k_i can be hard, otherwise no imaginary part is produced.
  - we checked that we reproduce all regions with these scalings with `Asy.m` [Pak, Smirnov (Eur. Phys. J. C (2011))]
- expansion in $\vec{v}$ reduces to a Taylor expansion in q.

Calculation Strategy

Different kinds of master integrals appear in hard or ultra-soft regions:

- hard regions: up to two-loop on-shell master integrals.
  [Melnikov, van Ritbergen (Nucl. Phys. B (2000) ; Lee, Smirnov (JHEP (2011))]
- soft regions: three-loop ultra-soft master integrals with (massive) eikonal propagators

hard region ($k_1 \sim m_b$):

$$
\begin{aligned}
&\int \frac{d^d k_1}{(2\pi)^d} \frac{1}{[k_1^2][(p+q+k_1)^2 - m_b^2][(p+k_1)^2 - m_b^2]} \\
=&\int \frac{d^d k_1}{(2\pi)^d} \frac{1}{[k_1^2][k_1^2 + 2(p+q)\cdot k_1 - y][k_1^2 + 2p\cdot k_1]} \\
\to&\int \frac{d^d k_1}{(2\pi)^d} \frac{1}{[k_1^2][k_1^2 + 2p\cdot k_1]^2} + \dots
\end{aligned}
$$

Calculation Strategy

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ultrasoft region ($k_1 \sim y/m_b$):

$$\int \frac{d^d k_1}{(2\pi)^d} \frac{1}{[k_1^2][(p+q+k_1)^2 - m_b^2][(p+k_1)^2 - m_b^2]}$$

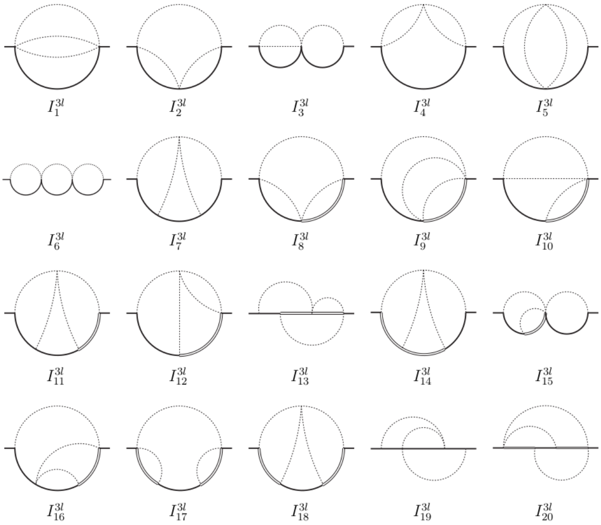
$$= \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{[k_1^2][k_1^2 + 2(p+q)\cdot k_1 - y][k_1^2 + 2p \cdot k_1]}$$

$$\to \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{[k_1^2][2p \cdot k_1 - y][2p \cdot k_1]} + \dots$$

Calculation of the Master Integrals – ultrasoft regions

Three types of propagators left:

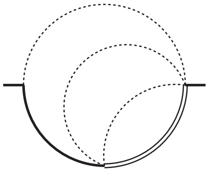
- massless propagators: k_i^2 (dotted lines)
- eikonal propagators: $2k_i.p$ (double lines)
- 'massive' eikonal propagators: $2k_i.p - y$ (solid lines)



Calculation - Complexity Class 1

Strategy:

- direct integration of Feynman parameters in terms of Γ-functions



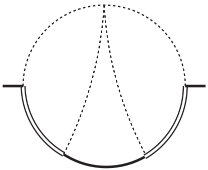
I_9^{3l}

$$= \prod_{i=1}^{3} \int \frac{\mathrm{d}k_i}{(2\pi)^d} \frac{1}{[-k_1^2][-(k_1+k_3)^2][-(k_2+k_3)^2][-2p.k_2][-2p.k_1+y]}$$
$$= y^{3d-8} m^{6-3d} \frac{\Gamma^3(d/2-1)\Gamma(5-2d)\Gamma(3d-7)\Gamma(8-3d)}{\Gamma(d-2)}$$
$$= y^{4-6\epsilon} m^{6\epsilon-6} S_\epsilon^3 \left\{ -\frac{1}{144\epsilon^2} - \frac{7}{108\epsilon} - \frac{127}{324} - \frac{29\pi^2}{576} - \epsilon\left(\frac{478}{243} + \frac{203}{432}\pi^2 - \frac{23}{144}\zeta_3\right) + \ldots \right\}$$

Calculation - Complexity Class 2

Strategy:

- We can find an up to 2-fold Mellin-Barnes representation.
- Two approaches for the solution:
  1. High precision numerical integration together with `PSLQ`.
  2. Analytic summation of residue sums with the help of `Sigma`, `EvaluateMultiSums` and `HarmonicSums` [Schneider (2007-); Ablinger et al (2011-)] .



I_{18}^{3l}

$$= \prod_{i=1}^{3} \int \frac{\mathrm{d}k_i}{(2\pi)^d} \frac{1}{[-k_1^2][-k_3^2][-(k_1-k_2)^2][-(k_2-k_3)^2][-2p.k_1][-2p.k_3][-2p.k_2+y]}$$

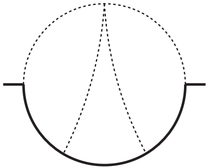
$$= y^{3d-11} m^{8-3d} \frac{\Gamma(11-3d)\Gamma(3d-10)}{\Gamma^2(d-3)} \left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} dw_1 \int_{-i\infty}^{+i\infty} dw_2 \ldots$$

$$= y^{1-6\epsilon} m^{6\epsilon-4} S_\epsilon^3 \left\{ -\frac{\zeta_3}{\epsilon} - \frac{3}{5}\zeta_2^2 - 8\zeta_3 + \epsilon\left(\frac{9}{2}\zeta_2\zeta_3 - \frac{24}{5}\zeta_2^2 - 52\zeta_3 - 57\zeta_5\right) + \mathcal{O}(\epsilon^2) \right\}$$

Calculation - Complexity Class 3

Strategy:

- We find a 3- or 4-fold Mellin-Barnes representation.
- Approache for the solution:
  1. Introduce a new scale $x \neq y$ into some of the massive eikonal propagators.
  2. Derive differential equations and solve with the help of `Sigma`, `OreSys` and `HarmonicSums` [Schneider (2007-); Ablinger et al (2011-)] .
  3. Fix boundary for $x \to 0$.



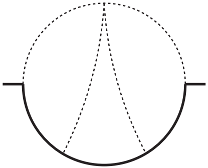
I_7^{3l}

$$= \prod_{i=1}^{3} \int \frac{\mathrm{d}k_i}{(2\pi)^d} \frac{1}{[-k_1^2][-k_3^2][-(k_1-k_2)^2][-(k_2-k_3)^2][-2p.k_1+y][-2p.k_3+y][-2p.k_2+y]}$$

Calculation - Complexity Class 3

Strategy:

- We find a 3- or 4-fold Mellin-Barnes representation.



I_7^{3l}

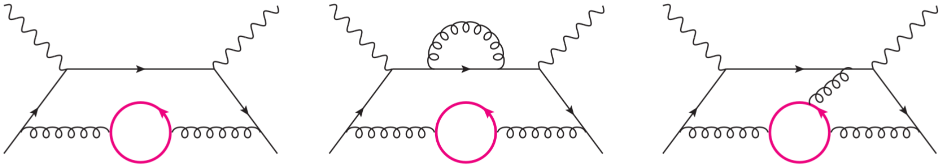
$$= \prod_{i=1}^{3} \int \frac{\mathrm{d}k_i}{(2\pi)^d} \frac{1}{[-k_1^2][-k_3^2][-(k_1-k_2)^2][-(k_2-k_3)^2][-2p.k_1+y][-2p.k_3+y][-2p.k_2+y]}$$

$$= y^{1-6\epsilon} m^{6\epsilon-2} \left(\frac{1}{2\pi i}\right)^4 \int_{-i\infty}^{+i\infty} dw_1 \int_{-i\infty}^{+i\infty} dw_2 \int_{-i\infty}^{+i\infty} dw_3 \int_{-i\infty}^{+i\infty} dw_4 \ldots$$

$$= y^{1-6\epsilon} m^{6\epsilon-2} \Big\{ -\frac{\zeta_2}{3\epsilon^2} + \frac{1}{\epsilon}\left(\frac{\zeta_3}{3} - \frac{8\zeta_2}{3}\right) - \frac{52\zeta_2}{3} + \frac{8\zeta_3}{3} - \frac{49\zeta_2^2}{30} + \epsilon\Big(-\frac{320}{3}\zeta_2 +$$

$$\frac{52}{3}\zeta_3 - \frac{196}{15}\zeta_2^2 - \frac{11}{6}\zeta_2\zeta_3 - \frac{83}{3}\zeta_5\Big) + \mathcal{O}(\epsilon^2)\Big\}$$

Charm mass dependence



- Previously no charm mass effects for the kinetic mass had been known.
- We assume $|y| \ll m_c^2, m_b^2$, i.e. no cuts through a charm loop.
- The bare result has a non-trivial dependence on m_c.
- After renormalization only decoupling effects are left.

$\Rightarrow$ The kinetic mass conversion has no explicit m_c dependence if parametrized in terms of $\alpha_s^{(3)}$.

Results

$$
\begin{aligned}
\frac{m^{\text{kin}}}{m^{\text{OS}}} = 1 &- \frac{\alpha_s^{(n_l)}}{\pi} C_F \left(\frac{4}{3} \frac{\mu}{m^{\text{OS}}} + \frac{1}{2} \frac{\mu^2}{(m^{\text{OS}})^2} \right) + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^2 C_F \left\{ \frac{\mu}{m^{\text{OS}}} \left[C_A \left(-\frac{215}{27} + \frac{2\pi^2}{9} + \frac{22}{9} l_\mu \right) + n_l T_F \left(\frac{64}{27} - \frac{8}{9} l_\mu \right) \right] \right. \\
&\left. + \frac{\mu^2}{(m^{\text{OS}})^2} \left[C_A \left(-\frac{91}{36} + \frac{\pi^2}{12} + \frac{11}{12} l_\mu \right) + n_l T_F \left(\frac{13}{18} - \frac{1}{3} l_\mu \right) \right] \right\} + \left(\frac{\alpha_s^{(n_l)}}{\pi} \right)^3 C_F \left\{ \frac{\mu}{m^{\text{OS}}} \left[C_A^2 \left(-\frac{130867}{1944} \right. \right. \right. \\
&+ \frac{511\pi^2}{162} + \frac{19\zeta_3}{2} - \frac{\pi^4}{18} + \left(\frac{2518}{81} - \frac{22\pi^2}{27} \right) l_\mu - \left. \frac{121}{27} l_\mu^2 \right) + C_A n_l T_F \left(\frac{19453}{486} - \frac{104\pi^2}{81} - 2\zeta_3 \right. \\
&+ \left. \left(-\frac{1654}{81} + \frac{8\pi^2}{27} \right) l_\mu + \frac{88}{27} l_\mu^2 \right) + C_F n_l T_F \left(\frac{11}{4} - \frac{4\zeta_3}{3} - \frac{2}{3} l_\mu \right) + \left. n_l^2 T_F^2 \left(-\frac{1292}{243} + \frac{8\pi^2}{81} + \frac{256}{81} l_\mu - \frac{16}{27} l_\mu^2 \right) \right] \\
&+ \frac{\mu^2}{(m^{\text{OS}})^2} \left[C_A^2 \left(-\frac{96295}{5184} + \frac{445\pi^2}{432} + \frac{57\zeta_3}{16} - \frac{\pi^4}{48} + \left(\frac{2155}{216} - \frac{11\pi^2}{36} \right) l_\mu - \frac{121}{72} l_\mu^2 \right) + C_A n_l T_F \left(\frac{13699}{1296} - \frac{23\pi^2}{54} \right. \right. \\
&- \left. \frac{3\zeta_3}{4} + \left(-\frac{695}{108} + \frac{\pi^2}{9} \right) l_\mu + \frac{11}{9} l_\mu^2 \right) + C_F n_l T_F \left(\frac{29}{32} - \frac{\zeta_3}{2} - \frac{1}{4} l_\mu \right) + \left. \left. n_l^2 T_F^2 \left(-\frac{209}{162} + \frac{\pi^2}{27} + \frac{26}{27} l_\mu - \frac{2}{9} l_\mu^2 \right) \right] \right\}, \quad (4)
\end{aligned}
$$

Results

Using as inputs $\overline{m}_b(\overline{m}_b) = 4163\,\text{MeV}$, $\alpha_s^{(5)}(M_Z) = 0.1179$:

$m_c = 0$:

$n_l = 3$: $\quad m_b^{\text{kin}}(1\,\text{GeV}) = (4163 + 248 + 81 + 30)\,\text{MeV} = 4521(15)\,\text{MeV}$

$n_l = 4$: $\quad m_b^{\text{kin}}(1\,\text{GeV}) = (4163 + 259 + 77 + 25)\,\text{MeV} = 4523(12)\,\text{MeV}$

To be compared with:

- scheme conversion uncertainty at two loops: $\delta m_b^{\text{kin}} = 30\,\text{MeV}$ [Gambino, JHEP 09 (2011)]
- m_b from $b \to c\ell\nu$ global fit: $m_b^{\text{kin}}(1\,\text{GeV}) = 4554 \pm 18\,\text{MeV}$ [HFLAV, EPJC 81 (2021)]

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$n_l = 4$: $\quad m_b^{\text{kin}}(1\,\text{GeV}) = (4163 + 259 + 77 + 25)\,\text{MeV} = 4523(12)\,\text{MeV}$

$m_c \neq 0$:

$n_l = 3$: $\quad m_b^{\text{kin}}(1\,\text{GeV}) = (4163 + 248 + 80 + 30)\,\text{MeV} = 4520(15)\,\text{MeV}$

$n_l = 4$: $\quad m_b^{\text{kin}}(1\,\text{GeV}) = (4163 + 259 + 78 + 26)\,\text{MeV} = 4526(12)\,\text{MeV}$

To be compared with:

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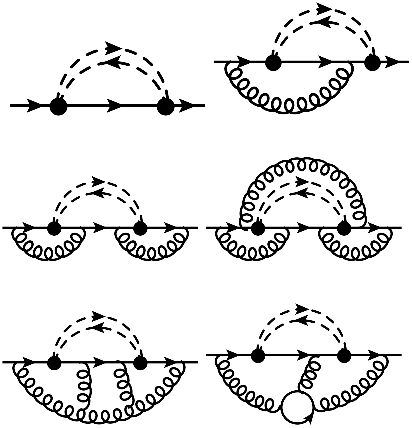
  [Gambino, JHEP 09 (2011)]
- m_b from $b \to c\ell\nu$ global fit: $m_b^{\text{kin}}(1\,\text{GeV}) = 4554 \pm 18\,\text{MeV}$
  [HFLAV, EPJC 81 (2021)]

The Inclusive Semileptonic Decay Rate

Method of Calculation

- We calculate the inclusive decay rate to third order via the optical theorem, i.e. we consider the imaginary part of 5-loop forward scattering diagrams.
- We consider massless leptons, i.e. we have two dimensionful scales, the bottom mass m_b and the charm mass m_c.
- Analytical dependence on charm and bottom mass seems out of reach:
  $\Rightarrow$ consider expansion in mass difference

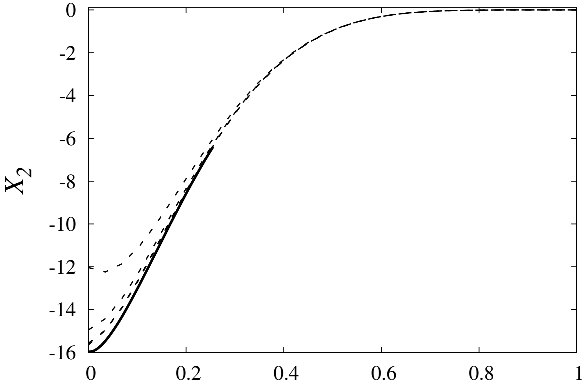


The Heavy-Daughter Expansion

- Perform the expansion in the limit $m_c \sim m_b$: $\delta = 1 - \rho = 1 - \frac{m_c}{m_b} \ll 1$
- Limit has been shown to converge well down to $m_c/m_b \to 0$ at 2-loop order.
  [Czarnecki, Dowling, Piclum (Phys. Rev. D 78 (2008))]

$$\Gamma(b \to c\ell\nu) = \Gamma_0 \left[X_0 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n \right]$$

with $\Gamma_0 = G_F^2 m_b^2 |V_{cb}|^2 / 192\pi^3$



Asymptotic Expansion

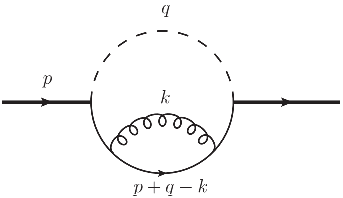
The expansion in $\delta = 1 - m_c/m_b$ is very similar to the threshold expansion for the kinetic mass:

- We use the method of regions to perform the expansion. [Beneke, Smirnov (Nucl. Phys. B (1998))]
- Loop momenta can either scale hard $k_i \sim m_b$ or ultra-soft $k_i \sim \delta m_b$ (regions have been cross-checked with `Asy`). [Pak, Smirnov (Eur. Phys. J. C (2011))]
- The momentum of the electron-neutrino loop can be integrated trivially.
- The properties of the asymptotic expansion allow to factorize the leptonic system completely.

$\Rightarrow$ We can reduce our 5-loop to 3-loop integrals with integer powers without any integration-by-parts.

Asymptotic Expansion – Example

Look at the 1-loop integral (we already integrated out the electron-neutrino loop):



$$\sim \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[(p+q)^2-m_c^2]^2[k^2][(p+q+k)^2-m_c^2]}$$

- case 1: q has to be ultra-soft, k is hard ($q \sim \delta m_b$, $k \sim m_b$);

$$\to \int \frac{\mathrm{d}q}{[q^2]^\alpha[2p\cdot q+2m_b^2\delta]^2} \times \int \frac{\mathrm{d}k}{[k^2][(k+p)^2-m_b^2]}$$

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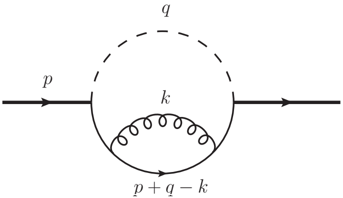

$$\sim \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[(p+q)^2 - m_c^2]^2[k^2][(p+q+k)^2 - m_c^2]}$$

- case 2: q and k are ultra-soft ($q, k \sim \delta m_b$);

$$\to \int \frac{\mathrm{d}q\mathrm{d}k}{[q^2]^\alpha[2p\cdot q + 2m_b^2\delta]^2[k^2][2p\cdot k + \underbrace{2p\cdot q + 2m_b^2\delta}_{\text{fixed combination}}]}$$

$$= \int \frac{\mathrm{d}q}{[q^2]^\alpha[2p\cdot q + 2m_b^2\delta]^\beta} \times \int \frac{\mathrm{d}k}{[k^2][2p\cdot k + 1]}$$

Details on the Calculation

- We can always perform the integrations over the electron-neutrino loop and lepton momenta analytically via 1-loop tensor reduction. The remaining loop integration have the following scalings:

	scaling	n. regions
$\mathcal{O}(\alpha_s)$	h, u	2
$\mathcal{O}(\alpha_s^2)$	hh, hu, uu	4
$\mathcal{O}(\alpha_s^3)$	hhh, huu, hhu, uuu	8

- In case a single region with either hard or ultra-soft scaling remains we can also integrate it out analytically.
- The remaining two- or three-loop integrals have integer powers of the propagators and can be reduced to master integrals via IBP reduction.

Details on the calculation

- We have to consider 1450 five-loop diagrams.
- Several subtleties with `FORM`:
  - Major obstacle is to keep the size of `FORM` expressions as low as possible.
  - Efficient expansion of denominators and memory management.
  - Automated partial fractioning and sector/families mapping.
    [LIMIT, F. Herren, PhD Thesis, KIT, 2020]
- Intermediate `FORM` expressions of $O(100)$ GB.
- About 25M three-loop integrals with positive and negative indices up to 12 had to be reduce. We used (a private version of) `FIRE` together with `LiteRed` (also standalone).

Different kinds of master integrals appear in hard or ultra-soft regions:

- hard regions: up to three loop on-shell master integrals.
  [Melnikov, van Ritbergen (Nucl. Phys. B (2000) ; Lee, Smirnov (JHEP (2011))]
- soft regions: three-loop ultra-soft master integrals with (massive) eikonal propagators

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Renormalization

- For the renormalization of the decay width we need
  - the wave function renormalization constant Z_2
  - the mass renormalization constant Z_m

  with two massive quarks in the expansion $m_c \sim m_b$ up to $O(\alpha_s^3)$.
- Previously they were only known in the expansion $x = m_c/m_b \sim 0$ and numerically for larger values of m_c [Bekavac, Grozin, Seidel, Steinhauser, JHEP 10 (2007)] .

$\Rightarrow$ new calculation necessary

	one mass-scale	two mass-scales
$\mathcal{O}(\alpha_s)$	✓	–
$\mathcal{O}(\alpha_s^2)$	✓ [Tarrach, NPB 183 (1981)]	✓ [Broadhurst et al, ZPC 48 (1990); 52 (1991)]
$\mathcal{O}(\alpha_s^3)$	✓ [Chetyrkin et al., PRL 83 (1999); . . .]	✓ [Bekavac et al. JHEP(2007)]
$\mathcal{O}(\alpha_s^4)$	✓ [Marquard et al., PRL 114 (2015); . . .]	✗

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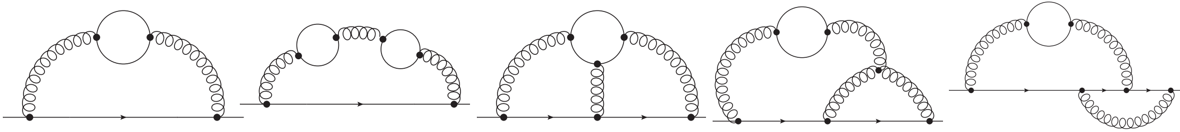
Definition

- The masses and the wave function can be renormalized multiplicatively:

$$m^0 = Z_m^{\mathrm{OS}} m_{\mathrm{OS}}, \qquad\qquad \psi^0 = \sqrt{Z_2^{\mathrm{OS}}}\psi^{\mathrm{OS}}.$$

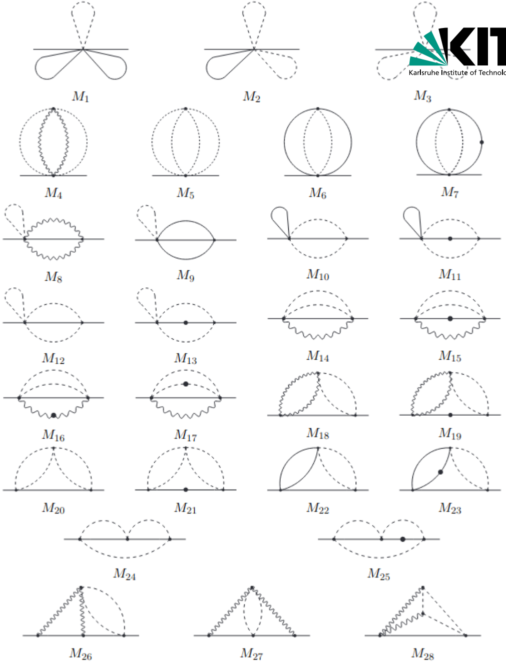
- The renormalization constants can be calculated from the quark self energy:

$$\Sigma(q^2, m) = m\Sigma_1(q^2, m) + (\not{q} - m)\Sigma_2(q^2, m).$$



Computation Strategy

- Generate graphs with `qgraf`.
- Generate amplitudes with `q2e/exp`.
- Reduce to master integrals with `Crusher` and `LiteRed`.
- Derive a closed system of differential equations in the variable $x = m_c/m_b$ for the master integrals and solve.



Computation Strategy

To solve the coupled system of differential equations we use the algorithm presented in [Ablinger, Blümlein, Marquard, Rana, Schneider, NPB 939 (2019)] :

- Uncouple coupled systems using `OreSys` to a single differential equation of higher order.
- Solve the higher order differential equations by factorizing the differential operator (as implemented in `HarmonicSums`).
- Fix the boundary constants from the known values at $x = 0$ and/or $x = 1$.

- We find 4 master integrals which cannot be expressed in terms of harmonic polylogarithms.

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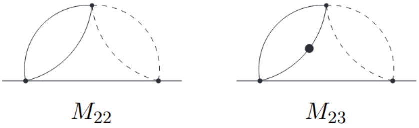
Computation Strategy

For example we find the homogenous differential equation:

$$M_{22}'' = \frac{1-4x^2}{x(1-x^2)} M_{22}' + \frac{4}{1-x^2} M_{22} \,.$$



With solutions:

$$M_{22}^{(1)} = x^2(4-x^2)\,,$$

$$M_{22}^{(2)} = (2-3x^2+x^4)\sqrt{1-x^2} + x^2(4-x^2)\left[1 + G\left(\left\{\frac{\sqrt{1-\tau^2}}{\tau}\right\}, x\right)\right].$$

$$G\left(\{f(\tau), \vec{g}(\tau)\}, x\right) = \int_0^x d\tau_1 f(\tau_1) G\left(\{\vec{g}(\tau)\}, \tau_1\right)$$

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Results - Analytic Structure

- For the full solution we need the letters:

$$\frac{1}{1-\tau}, \quad \frac{1}{\tau}, \quad \frac{1}{1+\tau}, \quad \sqrt{1-\tau^2}, \quad \frac{\sqrt{1-\tau^2}}{\tau}$$

- The last two letters can be rationalized with:

$$x = \frac{2y}{1+y^2},$$

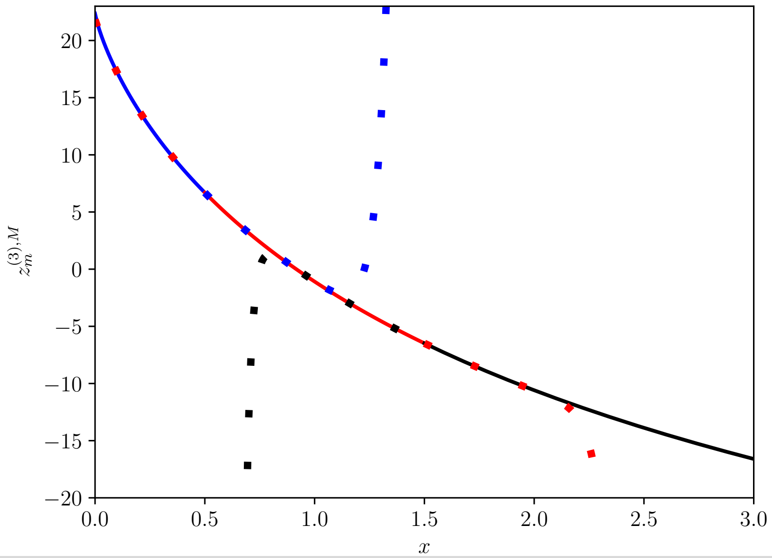
then we have cyclotomic harmonic polylogarithms [Ablinger, Blümlein, Schneider, JMP 52 (2011)]
:

$$\frac{1}{1-\tau}, \quad \frac{1}{\tau}, \quad \frac{1}{1+\tau}, \quad \frac{1}{1+\tau^2}, \quad \frac{\tau}{1+\tau^2}$$

- The constants at $x = y = 1$ are known (at least up to weight 5).

Results - Series Expansions

$$\frac{Z_m^{\mathrm{OS}}}{Z_m^{\overline{\mathrm{MS}}}} = 1 + \frac{\alpha_s}{\pi} z_m^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \left(z_m^{(2)} + n_m z_m^{(2),M}\right) + \left(\frac{\alpha_s}{\pi}\right)^3 \left(z_m^{(3)} + n_m z_m^{(3),M}\right)$$



Results - Decay Width

$$\Gamma(b \to c\ell\nu) = \Gamma_0 \left[X_0 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n X_n \right]$$

$$X_3 = \delta^5 \Bigg(\frac{266929}{810} - \frac{5248 a_4}{27} + \frac{2186\pi^2\zeta_3}{45} - \frac{4094\zeta_3}{45} - \frac{1544\zeta_5}{9} - \frac{656 l_2^4}{81} + \frac{1336}{405}\pi^2 l_2^2$$
$$+ \frac{44888\pi^2 l_2}{135} - \frac{9944\pi^4}{2025} - \frac{608201\pi^2}{2430} \Bigg) + \delta^6 \Bigg(-\frac{284701}{540} + \frac{2624 a_4}{9} - \frac{1093\pi^2\zeta_3}{15}$$
$$+ \frac{391\zeta_3}{3} + \frac{772\zeta_5}{3} + \frac{328 l_2^4}{27} - \frac{668}{135}\pi^2 l_2^2 - \frac{1484\pi^2 l_2}{3} + \frac{4972\pi^4}{675} + \frac{591641\pi^2}{1620} \Bigg) + \mathcal{O}(\delta^7 \ln^2(\delta)) ,$$

with $l_2 = \ln(2)$, $a_4 = \mathrm{Li}_4(1/2)$, $\zeta_i = \sum_{j=1}^{\infty} j^{-i}$ and $\mu_s = m_b$.

- We have calculated the expansion up to δ^{12} (for general color factors).
- A subset of color factors has been independently been computed up to δ^9.

[Czakon, Czarnecki, Dowling (2021)]

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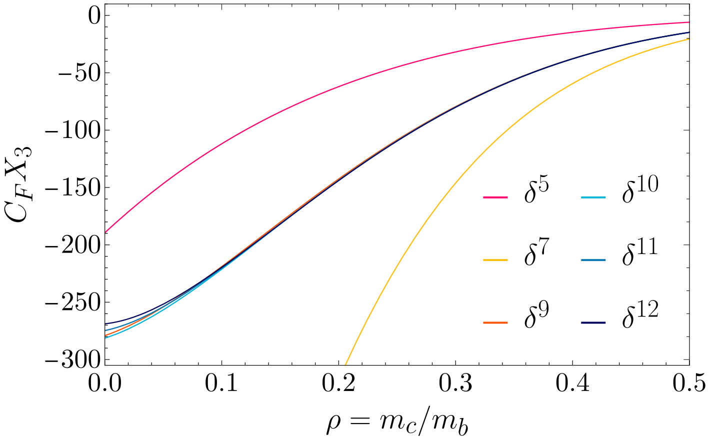
Convergence – Quark Decays

- We see a good convergence at the physical point of $\rho = m_c/m_b \approx 0.28$.
- We find:

$$X_3(\rho = 0.28) = -68.4 \pm 0.3$$

- We use the difference of the last two expansion terms to estimate the uncertainty.
- For $\rho \to 0$ we can extract values for $b \to u\ell\nu$:

$$X_3^u = -202 \pm 20$$



Different Renormalization Schemes

- The total decay rate of quarks expressed in terms of on-shell masses converges poorly:

$$\Gamma_{\rm sl} \sim 1 - 1.72\frac{\alpha_s(m_b)}{\pi} - 13.1\left(\frac{\alpha_s(m_b)}{\pi}\right)^2 - 163\left(\frac{\alpha_s(m_b)}{\pi}\right)^3$$

- Also the $\overline{\rm MS}$ scheme usually behaves poorly, since the scale has to be chosen rather low.
- Different threshold masses like the $\rm PS$ [Beneke (1998)] , $\rm 1S$ [Hoang, Ligeti, Manohar (1998)] or kinetic mass [Bigi, Shifman, Uraltsev, Vainshtein (1996)] have been proposed to improve the convergence.
- We see a much better behavior in the convergence for the schemes used for the global fits of inclusive quantities.
- E.g. for the kinetic mass:

$$m_b^{\rm kin}, m_c^{\rm kin}: \qquad \Gamma(b\to c\ell\nu)/\Gamma_0 = 0.633\,(1-0.066-0.018-0.007) \quad \approx 0.575$$

$$m_b^{\rm kin}, \overline{m}_c(3\text{ GeV}): \qquad \Gamma(b\to c\ell\nu)/\Gamma_0 = 0.700\,(1-0.116-0.035-0.010) \quad \approx 0.587$$

Different Renormalization Schemes

BLM and non-BLM part

$$\Gamma(b \to c\ell\nu) = \frac{G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} X_0 \left[1 + C_F \sum_{n\geq 1} \left(\frac{\alpha_s}{\pi}\right)^n Y_n \right]$$

$$Y_2 = \beta_0 Y_2^{\beta_0} + Y_2^{\mathrm{rem}}$$
$$Y_3 = \beta_0^2 Y_3^{\beta_0} + Y_3^{\mathrm{rem}}$$

	Y_1	Y_2^{rem}	$\beta_0 Y_2^{\beta_0}$	Y_3^{rem}	$\beta_0^2 Y_3^{\beta_0^2}$
$m_b^{\mathrm{OS}}, m_c^{\mathrm{OS}}$	-1.72	3.08	-16.17	48.8	-212.1
$m_b^{\mathrm{kin}}, m_c^{\mathrm{kin}}$	-0.94	0.33	-4.08	-5.4	-15.4
$m_b^{\mathrm{kin}}, \overline{m}_c(3\ \mathrm{GeV})$	-1.67	-3.39	-3.85	-97.7	69.1
$m_b^{\mathrm{kin}}, \overline{m}_c(2\ \mathrm{GeV})$	-1.25	-1.21	-2.43	-68.8	67.9
$\overline{m}_b(\overline{m}_b), \overline{m}_c(3\ \mathrm{GeV})$	3.07	-21.81	35.17	-56.7	119.4
$m_b^{\mathrm{PS}}, \overline{m}_c(2\ \mathrm{GeV})$	-0.47	-6.10	-2.31	-93.1	-7.19
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m_b^{1S}, m_c via HQET	-1.38	0.73	-7.05	5.04	-38.09

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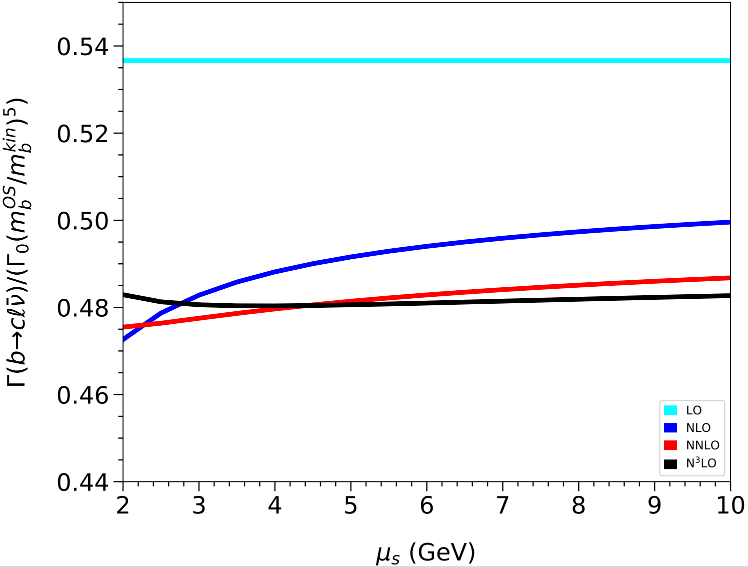
Different Renormalization Schemes – kinetic mass

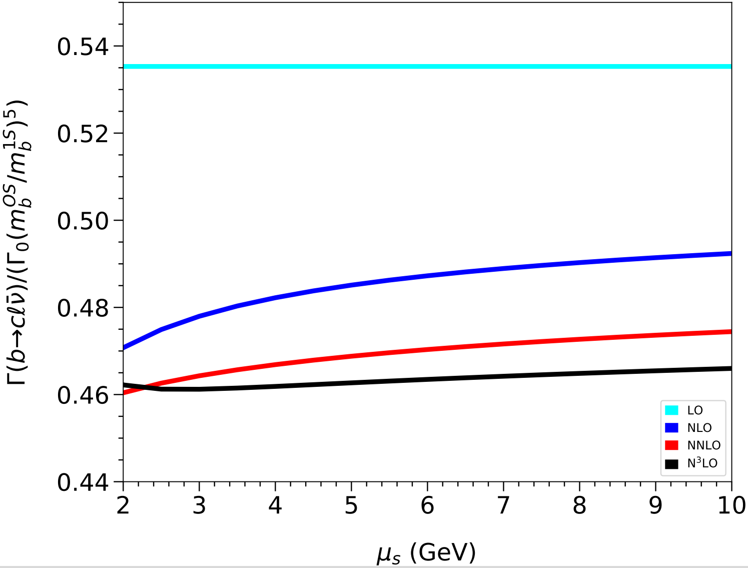
- m_b is expressed in the kinetic scheme.
- m_c is expressed in the $\overline{\mathrm{MS}}$ scheme at 2 GeV.



Different Renormalization Schemes – $1\mathrm{S}$ mass

- m_b is expressed in the $1\mathrm{S}$ scheme.
- m_c is expressed through meson masses via the HQET relation:

$$m_b^{\mathrm{OS}} - m_c^{\mathrm{OS}} = \overline{m}_B - \overline{m}_D + \ldots$$



The Updated Fit

- experimental moments from 2014 [Belle,Babar,CDF,CLEO,DELPHI]
- $\mathcal{O}(\alpha_s^3)$ corrections to $\Gamma(B \to X_c \ell \bar{\nu})$ [Fael,Schönwald,Steinhauser (2020)]
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- $\overline{m}_c(3\text{ GeV}) = 0.988(7)$ GeV , $\overline{m}_b(\overline{m}_b) = 4.198(12)$ GeV $\longrightarrow m_b^{\mathrm{kin}} = 4.565(19)$ GeV [FLAG (2019)]

$$|V_{cb}| = 42.16(30)_{\mathrm{th}}(32)_{\mathrm{exp}}(25)_{\Gamma} \times 10^{-3}$$

- $\Gamma(B \to X_c \ell \bar{\nu})_{\mathcal{O}(\alpha_s^3)}$:
  - shift: $|V_{cb}|$ by $+0.6\%$
  - reduce uncertainty: $(50)_\Gamma$ ➪ $(25)_\Gamma$
- 1.2% uncertainty
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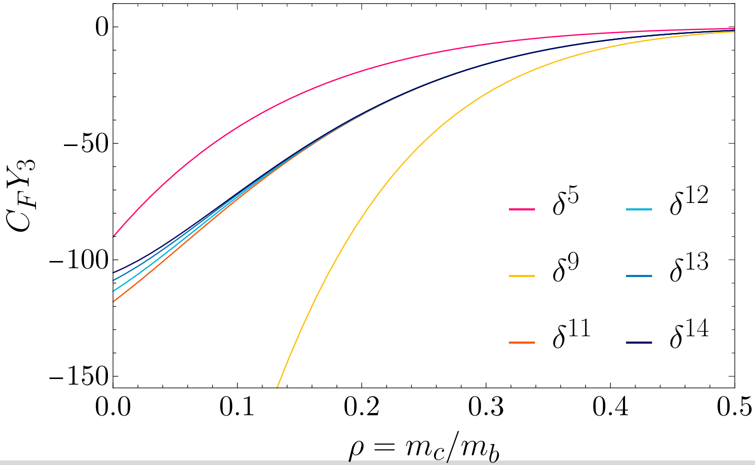
Moments of Differential Distributions

- The method can be used to calculate inclusive moments of differential distributions.
- For example we can calculate-q^2 moments:

$$Q_n = \frac{1}{\Gamma_0} \int dq^2 \left(\frac{q^2}{m_b^2} \right)^n \frac{\mathrm{d}\Gamma}{\mathrm{d}q^2}$$



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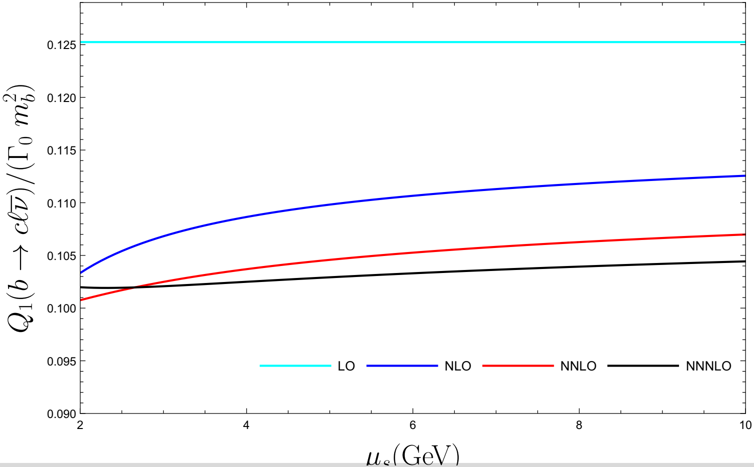


Convergence – Muon Decays

- Specifying the color factor to QED and setting $\rho = m_e/m_\mu \approx 0$ we get the 3-loop contributions to the muon decay.
- We find:

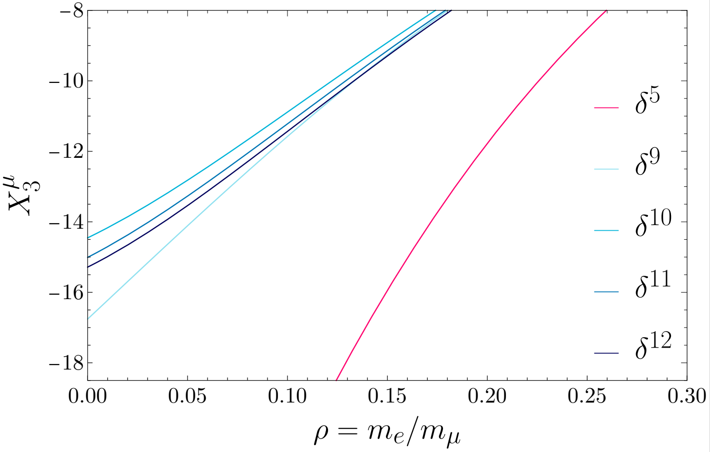
$$X_3^\mu = -15.3 \pm 2.3$$

- This leads to the shift:

$$\Delta\tau_\mu = (-9 \pm 1) \cdot 10^{-8}\,\mu s$$

- The current experimental value reads:

$$\tau_\mu = (2.1969811 \pm 0.0000022)\,\mu s$$



Conclusions and Outlook

Conclusions

- We have calculated the relation between the kinetic and on-shell mass up to $\mathcal{O}(\alpha_s^3)$.
- We have computed the α_s^3 corrections to the width of $b \to c\ell\nu$.
- We performed an expansion in the limit $1 - m_c/m_b \ll 1$ and demonstrated its good convergence.
- The result is one of the few third order corrections involving two mass scales.
- The results have been used to improve the inclusive determination of $|V_{cb}$.
- The results are also relevant for $b \to u\ell\nu$ and the muon decay.

Outlook

- The method of calculation can be applied for the calculation of moments of the differential distributions.