Black Holes as Heavy Particles

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DESY-HU theory seminar
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Based on

- Heavy Black Hole Effective Theory, Damgaard, Haddad, AH, 1908.10308
- On-shell heavy particle effective theories, Aoude, Haddad, AH, 2001.09164
- The double copy for heavy particles, Haddad, AH, 2005.13897
- Tidal effects in quantum field theory, Haddad, AH, 2008.04920
- Tidal effects for spinning particles, Aoude, Haddad, AH, 2012.05256

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1. Black Holes and particle scattering
2. Heavy particles
3. Heavy spinor-helicity
4. Tidal effects

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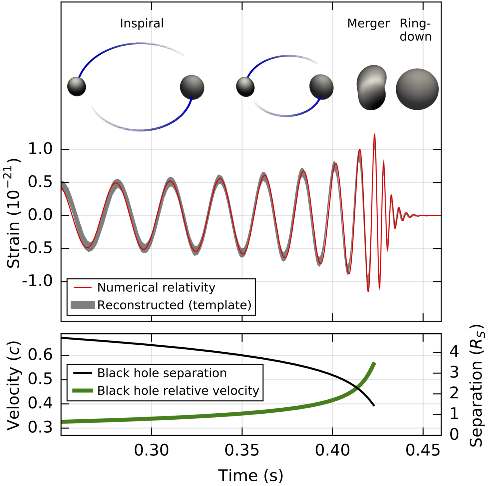
Observation of gravitational waves



LIGO 1602.03839

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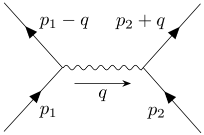
How to use quantum field theory for classical physics?

- Make contact with classical observables

- Take the limit $\hbar \to 0$

Classical potential from scattering amplitudes



Tree-level scattering amplitude in center-of-mass frame

$$\mathcal{M} = \frac{4\pi G m_1 m_2}{\vec{q}^2}$$

Take the Fourier transform

$$V(r) = -\int \frac{d^3 q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \mathcal{M} = -\frac{G m_1 m_2}{r}$$

This is Newton's potential!

Scattering angle from amplitude

Scattering to bound states Kälin, Porto 1910.03008

$$\chi = -\pi + 2b\int_{r_{\min}}^{\infty} \frac{dr}{r\sqrt{r^2(1+\tilde{\mathcal{M}}/p_\infty^2) - b^2}}$$

The scattering angle becomes

$$\chi = \frac{2\omega^2 - 1}{\omega^2 - 1}\,\frac{2E}{m_1 + m_2}\,\frac{G(m_1 + m_2)}{b}$$

Classical observables from amplitudes

Radiated momentum Kosower,Maybee,O'Connell 1811.10950

$$R^\mu = \sum_X \int d\Phi k_X^\mu A(p_1, p_2 \to p_1 + \omega_1, p_2 + \omega_2, k, r_X)$$
$$\times A^*(p_1 + q, p_2 - q \to p_1 + \omega_1, p_2 + \omega_2, k, r_X)$$

Classical vs Quantum: counting $\hbar$

Classical observables from amplitudes Kosower,Maybee,O'Connell 1811.10950

- Coupling constants: $e/\sqrt{\hbar}$, $\kappa/\sqrt{\hbar}$
- Massless momentum: $q = \hbar\bar{q}$
- Massive momentum: p and m fixed

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Classical vs Quantum: counting $\hbar$

Going back to the tree-level scattering amplitude

$$\mathcal{M} = \frac{4\pi G m_1 m_2}{\vec{q}^2} \overset{\hbar \neq 1}{=} \frac{4\pi G m_1 m_2}{\hbar^3 \vec{\bar{q}}^2}$$

$$\mathcal{M} \sim \frac{1}{\hbar^3}$$

$$V(r) = -\int \frac{d^3 q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}/\hbar} \mathcal{M} = -\hbar^3 \int \frac{d^3 \bar{q}}{(2\pi)^3} e^{-i\vec{\bar{q}}\cdot\vec{r}} \mathcal{M}$$

$$V(r) \sim \hbar^0$$

Classical vs Quantum: counting $\hbar$

Loop-level scattering amplitudes contain classical pieces!

$$
\begin{aligned}
\mathcal{M}_{\text{box}} &\sim \frac{1}{\hbar^4} \\
\mathcal{M}_{\text{triangle}} &\sim \frac{1}{\hbar^3} \\
\mathcal{M}_{\text{bubble}} &\sim \frac{1}{\hbar^2}
\end{aligned}
$$

QFT Toolbox

Results using QFT tools

- Generalized unitarity
- Multiloop integration
- Double copy
- Effective field theory

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Other tools

- Heavy limit
- Spinor helicity variables
- Hilbert series for higher-dimensional operators

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Heavy Quark Effective Theory

- Heavy quark (with mass M) interacting with light quarks and gluons.

- Interactions on the scale $k \sim \Lambda_{\mathrm{QCD}} \ll M$

Momentum of the heavy quark

$$p^\mu = Mv^\mu + k^\mu$$

- Expand in $\frac{k}{M}$

Heavy particles and classical physics

But what does this have to do with classical physics?

Heavy particles and classical physics

But what does this have to do with classical physics?

Recall the $\hbar$ counting Kosower,Maybee,O'Connell 1811.10950

- For massless particles, we keep the wavenumber $\bar{q}$ fixed

$$q^\mu = \hbar \bar{q}^\mu$$

- For massive particles, we keep the momentum p^μ and the mass fixed as $\hbar \to 0$

Restoring $\hbar$ in HQET

The photons/gluons are massless and we should keep their wavenumber (and not their momentum) fixed as $\hbar \to 0$.

The interaction scale is $k^\mu = \hbar \bar{k}^\mu$.

$$p^\mu = M v^\mu + \hbar \bar{k}^\mu$$

The HQET expansion is an expansion in $\hbar \bar{k}/M$.

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Black Holes = Heavy Particles?

We must have that

$$\frac{\hbar|\overline{q}|}{M} \ll 1$$

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Also, for a point-particle description to be a good approximation Kosower,Maybee,O'Connell 1811.10950

$$\ell_c \ll \ell_w \ll \sqrt{-b^2}$$

where b^μ is the impact parameter, Compton wavelength $\ell_c = \hbar/M$ and spread in wavepackets ℓ_w.

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$$\ell_c \ll \ell_w \ll \sqrt{-b^2}$$

where b^μ is the impact parameter, Compton wavelength $\ell_c = \hbar/M$ and spread in wavepackets ℓ_w. The scattering length $1/\sqrt{-\overline{q}^2}$ is on the order of the impact parameter.

$$\frac{1}{\sqrt{-\overline{q}^2}} \sim \sqrt{-b^2}$$

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Heavy Quarks vs. Heavy Black Holes

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$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi$$

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$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi$$

Split field into particle and anti-particle, then integrate out the anti-particle

$$\mathcal{L} = \bar{Q}_v \left[iv \cdot D + (i\not{D}_\perp)\frac{1}{2m + iv \cdot D}(i\not{D}_\perp) \right] Q_v$$

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$$S = \int d^4x \sqrt{-g}\,\bar{\psi}(ie^\mu_a \gamma^a \nabla_\mu - m)\psi$$

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$$S = \int d^4x \sqrt{-g}\, \bar{\psi}(ie_a^\mu \gamma^a \nabla_\mu - m)\psi$$

Split field into particle and anti-particle, then integrate out the anti-particle

$$\mathcal{L} = \bar{Q}_v \left[\bar{\nabla} + (\bar{\nabla} P_-) \frac{1}{2m - \bar{\nabla} P_-} (\bar{\nabla}) \right] Q_v$$

$$\bar{\nabla} = ie_a^\mu \gamma^a \nabla_\mu + (e_a^\mu - \delta_a^\mu) m \gamma^a v_\mu$$

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Massive spinor-helicity formalism

Introduce spinors λ_α^I and $\tilde{\lambda}_{\dot{\alpha},I}$ for the momentum p_μ

$$p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = p_{\alpha\dot{\alpha}} = \lambda_\alpha^I \tilde{\lambda}_{\dot{\alpha},I}$$

such that

$$p^2 = m^2 \Rightarrow \det \lambda \times \det \tilde{\lambda} = m^2$$

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Minimal coupling at three-points

The minimal coupling three-point amplitude with one massless and two massive particles is Arkani-Hamed, Huang, Huang 1709.04891

$$\mathcal{A}(1^s, 2^s, 3^+) = x\langle \mathbf{21}\rangle^{2s}$$
$$\mathcal{A}(1^s, 2^s, 3^{+2}) = x^2\langle \mathbf{21}\rangle^{2s}$$

Heavy spinor-helicity formalism

Massive and heavy Dirac spinors are related as

$$u_v^I(p) = \left(\frac{1+\not{v}}{2}\right) u^I(p) = \left(1 - \frac{\not{k}}{2m}\right) u^I(p)$$

We introduce heavy on-shell spinors

$$\begin{pmatrix} |p_v\rangle \\ |p_v] \end{pmatrix} = \left(1 - \frac{\not{k}}{2m}\right) \begin{pmatrix} |\boldsymbol{p}\rangle \\ |\boldsymbol{p}] \end{pmatrix}$$

Spin in on-shell variables

The Pauli-Lubanski pseudovector S^μ

$$S^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}$$

In on-shell variables

$$(S^\mu)_\alpha{}^\beta = \frac{1}{4}\left[(\sigma^\mu)_{\alpha\dot{\alpha}}v^{\dot{\alpha}\beta} - v_{\alpha\dot{\alpha}}(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}\right]$$

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Three-particle amplitude with heavy spinors

$$\mathcal{A}_3^{+|h|,s} = \mathcal{A}_3^{+|h|,0} \sum_{k=0}^{2s} g_{s,k}^{H} \left(\frac{x}{2m} \frac{\langle 2_v q \rangle \langle q 1_v \rangle}{\langle 2_v 1_v \rangle} \right)^k = \mathcal{A}_3^{+|h|,0} \sum_{k=0}^{2s} g_{s,k}^{H} \frac{(2s-k)!}{(2s)!} \left(\frac{\langle q \cdot S \rangle}{m} \right)^k$$

Three-particle amplitude with heavy spinors

$$\mathcal{A}_3^{+|h|,s} = \mathcal{A}_3^{+|h|,0} \sum_{k=0}^{2s} g^H_{s,k} \left(\frac{x}{2m} \frac{\langle 2_v q \rangle \langle q 1_v \rangle}{\langle 2_v 1_v \rangle} \right)^k = \mathcal{A}_3^{+|h|,0} \sum_{k=0}^{2s} g^H_{s,k} \frac{(2s-k)!}{(2s)!} \left(\frac{\langle q \cdot S \rangle}{m} \right)^k$$

For minimal coupling, $g^H_{s,k} = g_0 \binom{2s}{k}$

$$\mathcal{A}_3^{+|h|,s} = \mathcal{A}_3^{+|h|,0} \sum_{k=0}^{2s} g_0 \frac{1}{k!} \left(\frac{\langle q \cdot S \rangle}{m} \right)^k = \mathcal{A}_3^{+|h|,0} e^{\frac{\langle q \cdot S \rangle}{m}}$$

Compton Scattering

For same-helicity amplitude also exponentiates!

$$\mathcal{A}_4^{+|h|,+|h|,s} = \mathcal{A}_4^{+|h|,+|h|,0} e^{\frac{\langle q \cdot S \rangle}{m}}$$

But the opposite-helicity amplitude has a problem

$$\mathcal{A}_4^{-|h|,+|h|,s} = \mathcal{A}_4^{-|h|,+|h|,0} e^{\frac{\langle (\bar{q}+\omega) \cdot S \rangle}{m}}$$

There are spurious poles in $\omega \cdot S$. This can be fixed by looking at spin-5/2 amplitudes

Chiodaroli, Johansson, Pichini 2107.14779

Matching calculation to worldline theory

The worldline theory is

$$S = \int d\sigma \left[-m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{SI}[u^\mu, S_{\mu\nu}, g_{\mu\nu}(y^\mu)] \right]$$

The spin-multipoles are parametrized by Wilson coefficients C_{s^k}. For Kerr black holes they are all $C_{s^k}^{\mathrm{Kerr}} = 1$.

Matching calculation to worldline theory

Matching to heavy spinors gives

$$g^H_{s,k} = \frac{\kappa m}{2} C_{S^k} \sum_{j=0}^{k} \binom{s}{k-j} \binom{s}{j}$$

For minimal coupling, $g^H_{s,k} = \frac{\kappa m}{2}\binom{2s}{k}$,

$$C^{\min}_{S^k} = \binom{2s}{k} \left[\sum_{j=0}^{k} \binom{s}{k-j} \binom{s}{j}\right]^{-1} = 1$$

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Tidal effects from higher-dimensional operators

Tidal deformations are described by higher-dimensional operators Cheung,Solon 2006.06665

$$\Delta S = \int d^4x \sqrt{-g} \frac{1}{4} C_{\mu\alpha\nu\beta} C^{\rho\alpha\sigma\beta} \left(\lambda \phi^2 \delta^\mu_\rho \delta^\nu_\sigma + \frac{\eta}{m^4} \nabla^\mu \nabla^\nu \phi \nabla_\rho \nabla_\sigma \phi \right)$$

In the worldline formalism Bini, Damour, Geralico 2001.00352

$$\Delta S^{\mathrm{WL}} = \int d\tau \left[\frac{\mu_i^{(2)}}{4} (E^i_{\alpha\beta})^2 + \frac{2\sigma_i^{(2)}}{3} (B^i_{\alpha\beta})^2 \right]$$

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Basis of tidal operators

Tidal operators of the form $\phi^2 R^2 D^{2n}$. How to construct systematically?

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Two main redundancies:

- Integration by parts (momentum conservation)
- Field redefinition (on-shell condition)

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Two main redundancies:

- Integration by parts (momentum conservation)
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Two solutions:

- On-shell scattering amplitudes
- Hilbert series (for Lagrangian)

Hilbert series

The Hilbert series is

$$\mathcal{H} = \int d\mu \frac{1}{P} \mathrm{PE}[\chi_\phi]$$

- $\mathrm{PE}[\chi_\phi]$: generates all tensor products of the fields and derivatives
- $1/P$: removes total derivatives
- $\int d\mu$: picks out the invariant operators

Hilbert series for tidal effects

$$\mathcal{H}^{C^2}_{6+2n} = \left\lfloor \frac{n+2}{2} \right\rfloor (C_L^2\phi^2\nabla^{2n} + C_R^2\phi^2\nabla^{2n}) + \left\lfloor \frac{n}{2} \right\rfloor C_L C_R \phi^2 \nabla^{2n}$$

These are all tidal operators. We can systematically enumerate any other class.
Heavy limit is useful to get the 'classically relevant' operators.

Scattering angle for all tidal operators at 2PM

We can calculate the scattering angle for any tidal operator

$$\Delta\chi = \frac{G^2 m_2^3}{E} \sum_{i=0}^{\infty} \frac{n_i}{p_\infty^2 b^{2(i+3)}} g_i(\omega)$$

- n_i: numerical factor
- $g_i(\omega)$: from amplitude, depends on Wilson coefficients

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Match to worldline calculation

The leading term is matched with Bini, Damour, Geralico 2001.00352

$$
\begin{aligned}
c_0^{(1)} &\to -\frac{1}{32} m_1 \sigma^{(3)}, \\
c_1^{(2)} &\to \frac{1}{144 m_1}\left(-\mu^{(3)} - 12\sigma'^{(2)} + \frac{9}{2}\sigma^{(3)}\right), \\
d_0^{(3)} &\to \frac{1}{12 m_1^3}\left(\mu^{(3)} + 3\sigma^{(3)}\right), \\
d_1^{(4)} &\to \frac{1}{36 m_1^5}\left(9\mu'^{(2)} - \mu^{(3)} + 24\sigma'^{(2)} - 3\sigma^{(3)}\right).
\end{aligned}
$$

Outlook

Future directions

- Compton scattering for all spins
- Double copy in heavy limit
- Apply heavy spinors to HQET
- Direct link to worldline QFT Jakobsen, Mogull, Plefka, Steinhoff 2109.04465

Thank you!

2PM spin-spin amplitude

$$\mathcal{A}_{\text{spinless}} = G^2 m_1 m_2 \frac{3(5\omega^2 - 1)}{2}(m_1 + m_2) S \mathcal{U}_1 \mathcal{U}_2$$

$$\mathcal{A}_{\text{spin}_2} = G^2 m_1 m_2 \frac{\omega(5\omega^2 - 3)}{2(\omega^2 - 1)}(3m_1 + 4m_2) S \frac{i\mathcal{U}_1 \mathcal{E}_2}{m_1 m_2^2}$$

$$\mathcal{A}_{\text{spin}_1 - \text{spin}_2} = G^2 \frac{(20\omega^4 - 21\omega^2 + 3)}{2(\omega^2 - 1)}(m_1 + m_2) S \left(q \cdot S_1 q \cdot S_2 - q^2 S_1 \cdot S_2\right)$$

Worldline spin-multipole terms

$$L_{SI} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{S^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \dots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} \dots S^{\mu_{2n}}$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{S^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \dots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} \dots S^{\mu_{2n+1}}$$

For Kerr black holes: $C_{S^k}^{\mathrm{Kerr}} = 1$

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On-shell basis

We construct all non-factorizable on-shell amplitudes with two scalars and two gravitons

$$\mathcal{M}(\phi\phi; g^{2+}(p_1), g^{2+}(p_2)) = [12]^4 c(s_{12}, s_{13}, s_{14})$$
$$\mathcal{M}(\phi\phi; g^{2-}(p_1), g^{2+}(p_2)) = \langle 1|(p_3 - p_4)|2]^4 d(s_{12}, s_{13}, s_{14})$$

$$c(s_{12}, s_{13}, s_{14}) = \sum_{i,j=0}^{\infty} c_{i,j} s_{12}^i (s_{13}s_{14})^j$$
$$d(s_{12}, s_{13}, s_{14}) = \sum_{i,j=0}^{\infty} d_{i,j} s_{12}^i (s_{13}s_{14})^j$$

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Hilbert series

Plethystic exponential

$$\mathrm{PE}_\phi = \exp\left[\sum_{r=0}^{\infty} z^{r+1} \frac{\phi^r}{r\mathcal{D}^{r\Delta_\phi}} \chi_\phi(x_1^r, \ldots, x_k^r)\right]$$

Conformal characters

$$\begin{aligned} \chi_\phi &= \chi_{[1,(0,0)]}(\mathcal{D}; \alpha, \beta) \\ \chi_{C_L} &= \chi_{[3,(2,0)]}(\mathcal{D}; \alpha, \beta) \\ \chi_{C_R} &= \chi_{[3,(0,2)]}(\mathcal{D}; \alpha, \beta) \end{aligned}$$

Tidal operators

$$
\Delta\mathcal{L}_{\rm GR}^{\rm tidal} =
$$

$$
\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor}\Big\{c_k^{(n)}\left[\nabla^{\mu_1\ldots\mu_k}\phi\right]\left[\nabla_{\nu_1\ldots\nu_k}\phi\right]\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{\rho\sigma\alpha\beta}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C^{\rho\sigma\alpha\beta}\right]
$$

$$
+d_k^{(n+2)}\left[\nabla^{\rho\sigma\mu_1\ldots\mu_k}\phi\right]\left[\nabla_{\alpha\beta\nu_1\ldots\nu_k}\phi\right]\left[\nabla_{\mu_1\ldots\mu_k\alpha_1\ldots\alpha_{n-2k}}C_{\lambda\rho\tau\sigma}\right]\left[\nabla^{\nu_1\ldots\nu_k\alpha_1\ldots\alpha_{n-2k}}C^{\lambda\alpha\tau\beta}\right]\Big\}
$$

Scattering angle for tidal corrections

$$n_i = \frac{(-1)^i (2i+4)!(i+3)}{2^{i+2}} \frac{\sqrt{\pi}\Gamma(i+7/2)}{\Gamma(i+4)}$$

$$g_i(\omega) = \sum_{k=0}^{i} \frac{(-1)^k (1/2)_k}{2^k (1)_k} (\omega^2 - 1)^k \Bigg[16 m_1^{2k} c_k^{(i+k)} + \frac{m_1^{2k+4} d_k^{(i+k+2)}}{4(k+2)(k+1)} \left[(2k+5)(2k+7)\omega^4 - 6(2k+5)\omega^2 + (4k^2+12k+11) \right] \Bigg]$$

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