Using improved auxiliary mass flow method to compute master integrals

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Hu Berlin/DESY Zeuthen theory seminar
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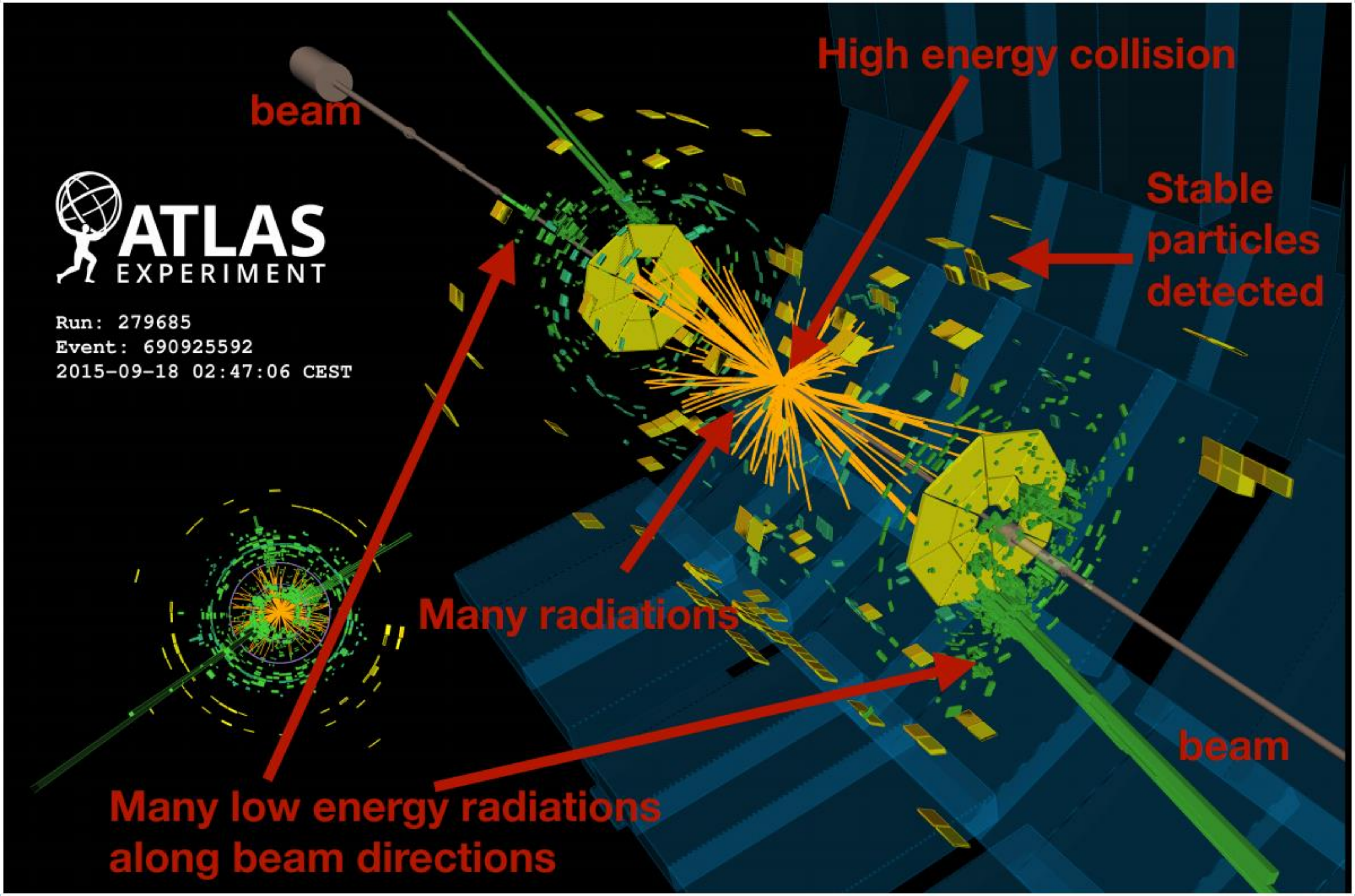
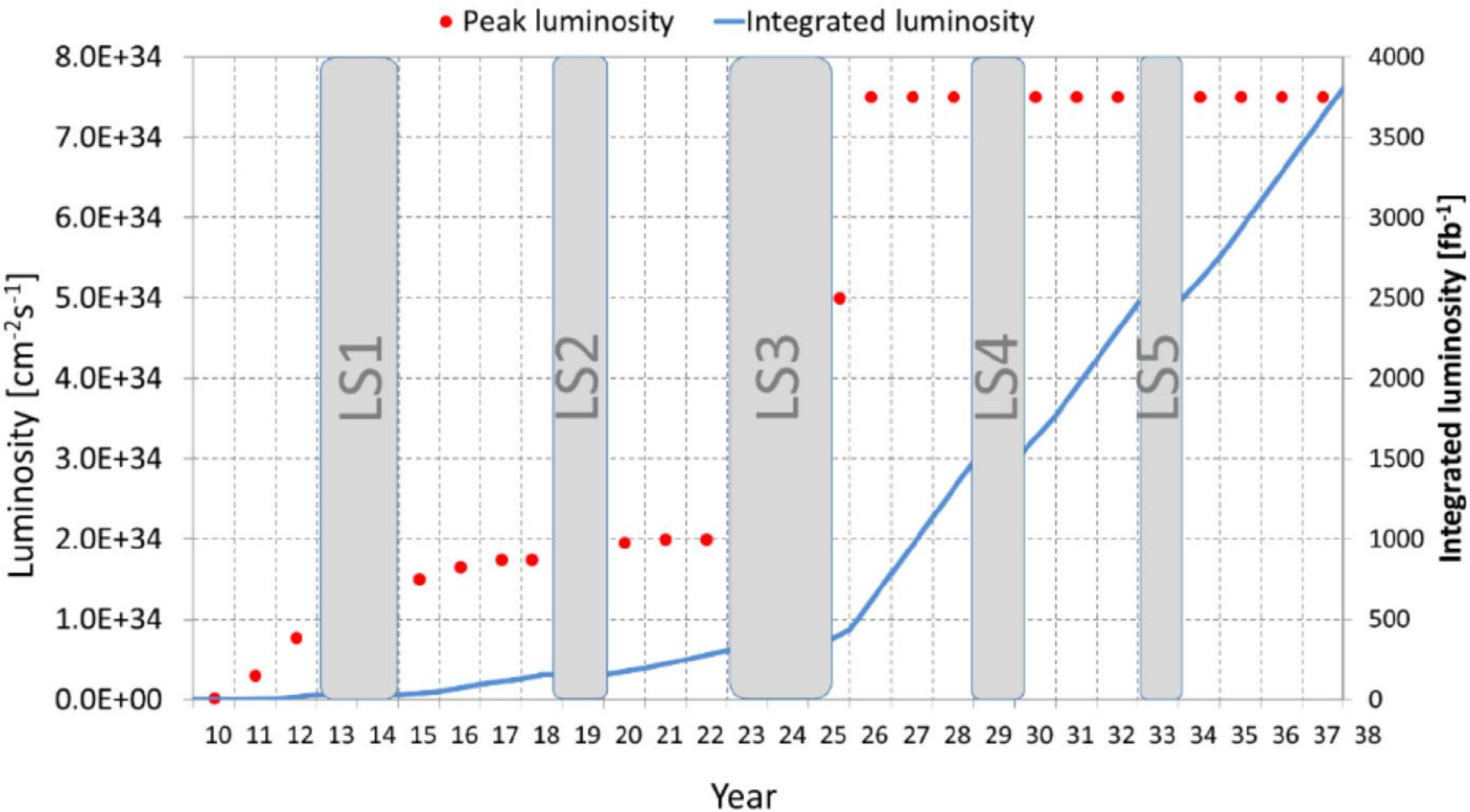
Outline

High precision physics



High precision physics



LHC luminosity. Figure from [Apollinari et al. 2015]

Theoretical predictions

➢ Perturbative QCD

- NLO revolution [Ossola, Papadopoulos and Pittau. *Nucl. Phys. B*, 2007] [Berger, Bern, Dixon et al. *Phys. Rev. D*, 2008] [Ellis, Giele and Kunszt. *JHEP*, 2008] [Giele Kunszt and Melnikov. *JHEP*, 2008]
- NNLO techniques
  - $pp \to 2j$ [Currie Ridder, Gehrmann et al. *Phys. Rev. Lett.*, 2017]
  - $pp \to Z + j$ [Ridder, Gehrmann, Glover et al. *Phys. Rev. Lett.*, 2016]
  - $pp \to W + j$ [Boughezal, Focke, Liu et al. *Phys. Rev. Lett.*, 2015]
  - $pp \to \gamma + j$ [Campbell, Ellis and Williams. *Phys. Rev. Lett.*, 2017]
  - $pp \to \gamma\gamma$ [Cieri, Coradeschi and Florian. *JHEP*, 2015]
  - $pp \to t\bar{t}$ [Czakon Heymes and Mitov. *Phys. Rev. Lett.*, 2016]
  - …
  - first 2 to 3: $pp \to \gamma\gamma\gamma$ [Chawdhry, Czakon, Mitov et al. *JHEP*, 2020]
  - $pp \to \gamma\gamma + j$ [Chawdhry, Czakon, Mitov et al. 2105.06940]

Multi-loop scattering amplitudes

- master integrals calculation
  - differential equations [Kotikov. *Phys. Lett. B*, 1991] [Henn. *Phys. Rev. Lett.*, 2013] [Czakon. *Phys. Lett. B*, 2008]
  - sector decomposition [Binoth and Heinrich, *Nucl. Phys. B*, 2000]
  - Mellin-Barnes representation [Smirnov. *Phys. Lett. B*, 1999]
  - dimensional recurrence relations [Tarasov. *Phys. Rev. D*, 1996] [Lee. *Nucl. Phys. B*, 2010]
- "basis" of special functions not fully known
  - elliptic sectors in $H + j$ production in full QCD [Bonciani, Duca, Frellesvig et al. *JHEP*, 2016] [Bonciani, Duca, Frellesvig et al. *JHEP*, 2020] [Frellesvig, Hidding, Maestri et al. *JHEP*, 2020]
- auxiliary mass flow method [Liu, Ma and Wang. Phys. Lett. B, 2018] [paper in preparation]

Outline

I. Introduction

II. Auxiliary mass flow

III. The improved version

IV. Cutting-edge examples

V. Summary and outlook

Introduction of auxiliary mass

➢ Integral family with auxiliary mass [Liu, Ma and Wang, *Phys. Lett. B*, 2018]

$$I[\vec{\nu}](D;\vec{s};\eta) \equiv \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \eta)^{\nu_1} \cdots (\mathcal{D}_K - \eta)^{\nu_K}}$$

- η: the auxiliary mass parameter

$$I[\vec{\nu}](D;\vec{s}) = \lim_{\eta \to \mathrm{i}0^-} I[\vec{\nu}](D;\vec{s};\eta)$$

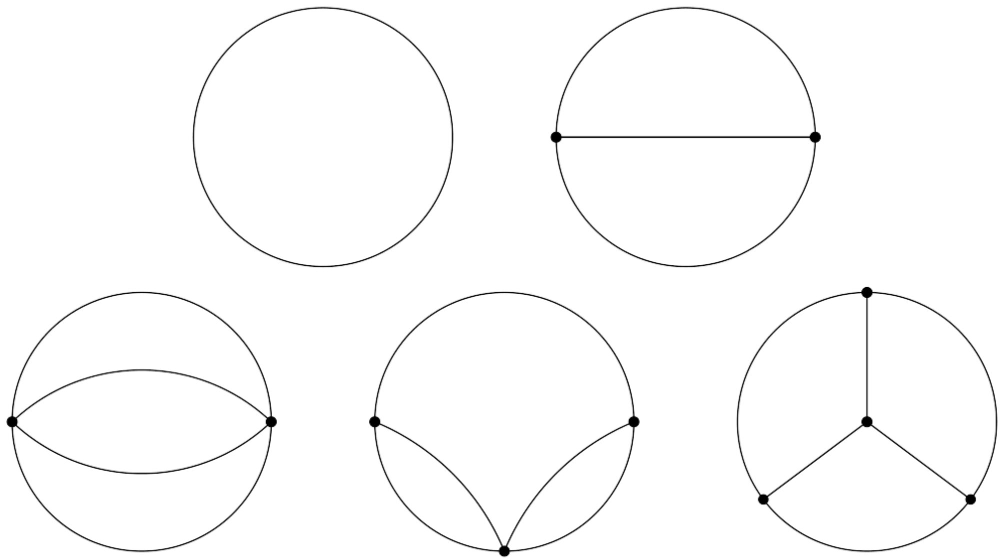
- near $\eta = \infty$, only one integration region:

$$\ell_i^\mu \sim \sqrt{\eta}$$

$$\frac{1}{((\ell+p)^2 - m^2 - \eta)^\nu} = \frac{1}{(\ell^2 - \eta)^\nu} \sum_{i=0}^{\infty} \frac{(\nu)_i}{i!} \left(-\frac{2\ell \cdot p + p^2 - m^2}{\ell^2 - \eta} \right)^i$$

Introduction of auxiliary mass

- equal-mass vacuum integrals:



[Davydychev and Tausk. *Nucl. Phys. B*, 1993]
[Broadhurst. *Eur. Phys. J. C*, 1999]
[Schroder and Vuorinen. *JHEP*, 2005]
[Kniehl, Pikelner and Veretin. *JHEP*, 2017]
[Luthe. phdthesis, 2015]
[Luthe, Maier, Marquard et al. *JHEP*, 2017]

- differential equations:

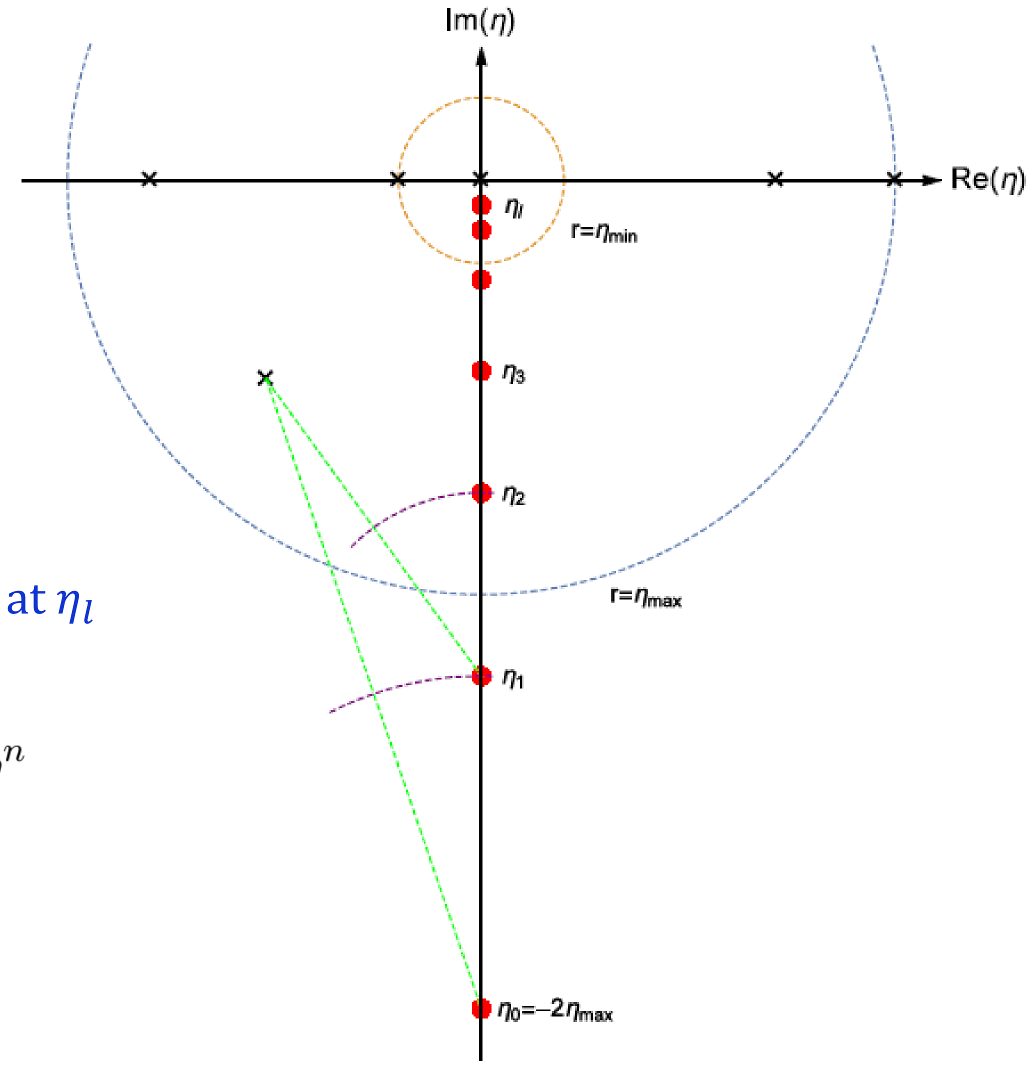
$$\frac{\partial}{\partial \eta}\vec{\mathcal{I}}(\eta) = A(\eta)\vec{\mathcal{I}}(\eta)$$

➢ Numerical evaluation

- list of regular points: $\{\eta_0, \eta_1, \ldots, \eta_l\}$
  - η_0: outside the large circle
  - η_l: inside the small circle
- expand at $\eta = \infty$ to estimate $\vec{I}(\eta_0)$
- expand at $\eta = \eta_i$ to estimate $\vec{I}(\eta_{i+1})$
- expand formally at $\eta = 0$ and match at η_l

$$\vec{I}(\eta) = \sum_{\mu \in S} \eta^{\mu} \sum_{k=0}^{k_\mu} \log^k(\eta) \sum_{n=0}^{\infty} \vec{I}_{\mu,k,n} \eta^n$$

- take the limit $\eta \to 0$
- auxiliary mass flow



Auxiliary mass flow

➢ Advantages

- systematic
  - process independent
  - general algorithm
  - physical region ~ Euclidean region
- efficient
  - $e \sim \left|\frac{\eta_{i+1}-\eta_i}{r}\right|^N, N \sim t$
  - $p \sim -\log(e) \sim N \sim t$

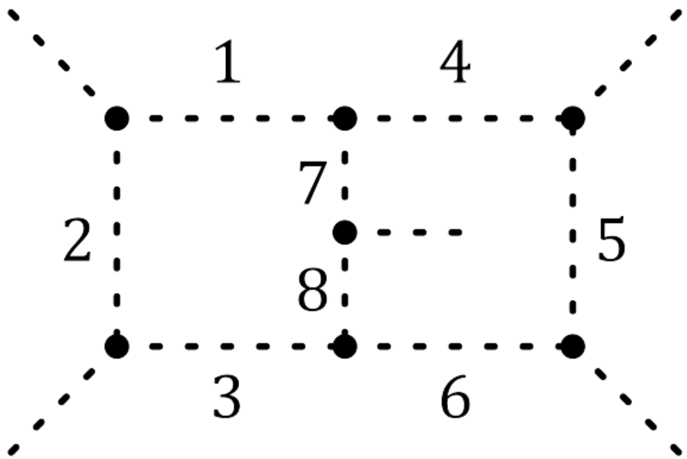
➢ Problems

- effect of η: enlargement of the number of master integrals

Outline

Improved auxiliary mass flow

➢ A simple observation



108 master integrals

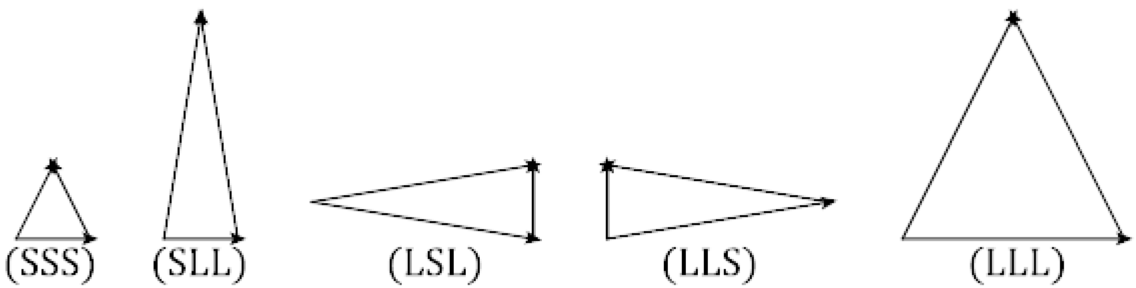
Mode	Propagators	#MIs
all	{1,2,3,4,5,6,7,8}	476
loop	{4,5,6,7,8}	305
	{1,2,3,4,5,6}	319
branch	{4,5,6}	233
	{7,8}	234
propagator	{4}	178
	{5}	176
	{7}	220

Integration regions

➢ Integration regions [Beneke and Smirnov. *Nucl. Phys. B,* 1998]

- principles: loop momentum of each branch can be either of $O(1)$ or $O(\sqrt{\eta})$
- regions for one-loop:

- regions for two-loop:



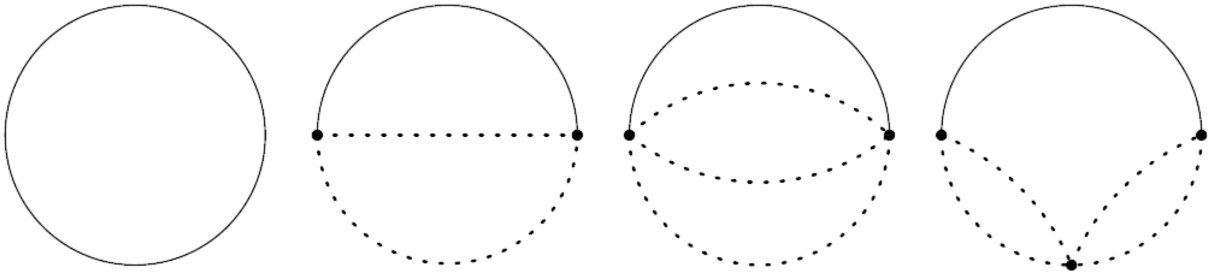
  - (LSS), (SLS), (SSL) excluded by momentum conservation
- $R_1 = 2, R_2 = 5, R_3 = 15, R_4 = 47, \ldots$

Integration regions

➢ Expansion in each region

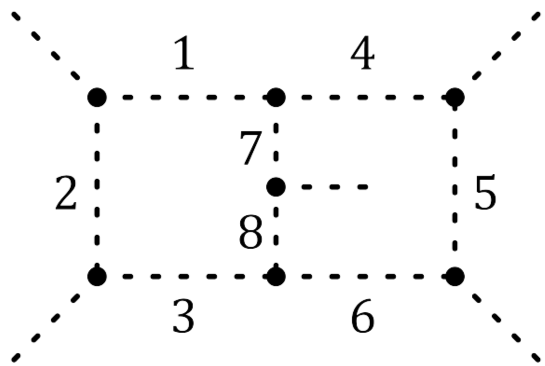
- (L … L): $\frac{1}{((\ell+p)^2-m^2-k\eta)^\nu} \sim \frac{1}{(\ell^2-k\eta)^\nu}$ ⇒ vacuum
- (L … S): $\frac{1}{((\ell_\mathrm{L}+\ell_\mathrm{S}+p)^2-m^2-k\eta)^\nu} \sim \frac{1}{(\ell_\mathrm{L}^2-k\eta)^\nu}$ ⇒ factorized
- (S … S): $\frac{1}{((\ell+p)^2-m^2-\eta)^\nu} \sim \frac{1}{(-\eta)^\nu}$ ⇒ sub-family

Integrals can be solved by iteration!



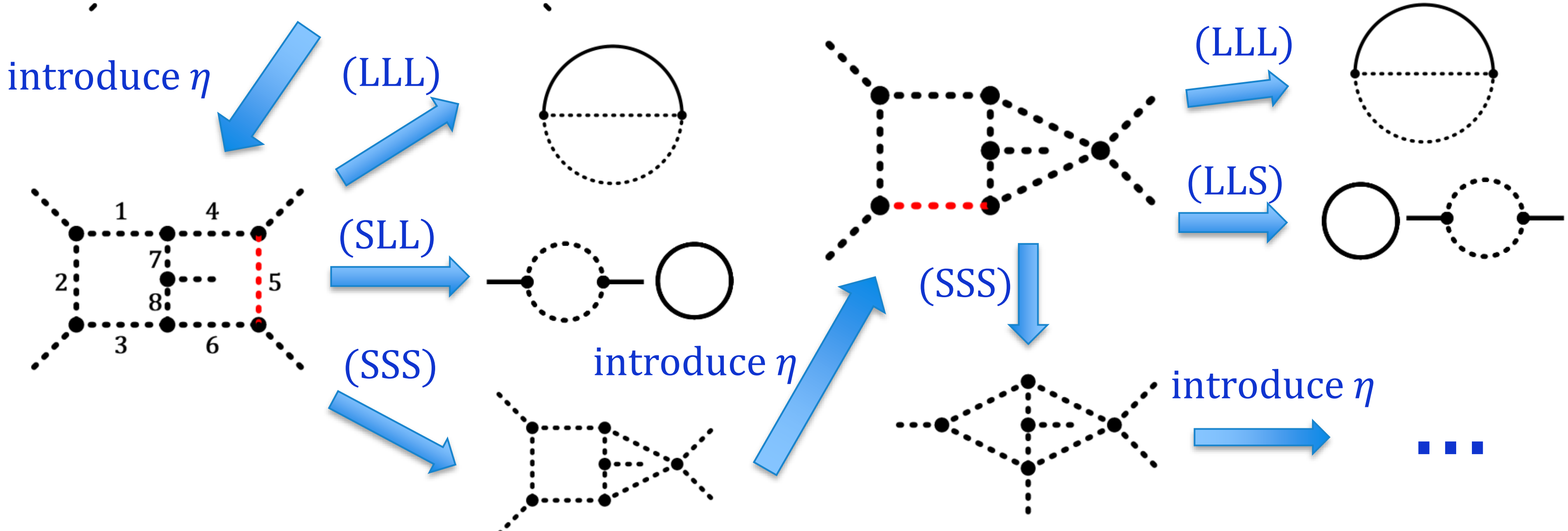
Examples

➢ Two-loop five-point massless double-pentagon



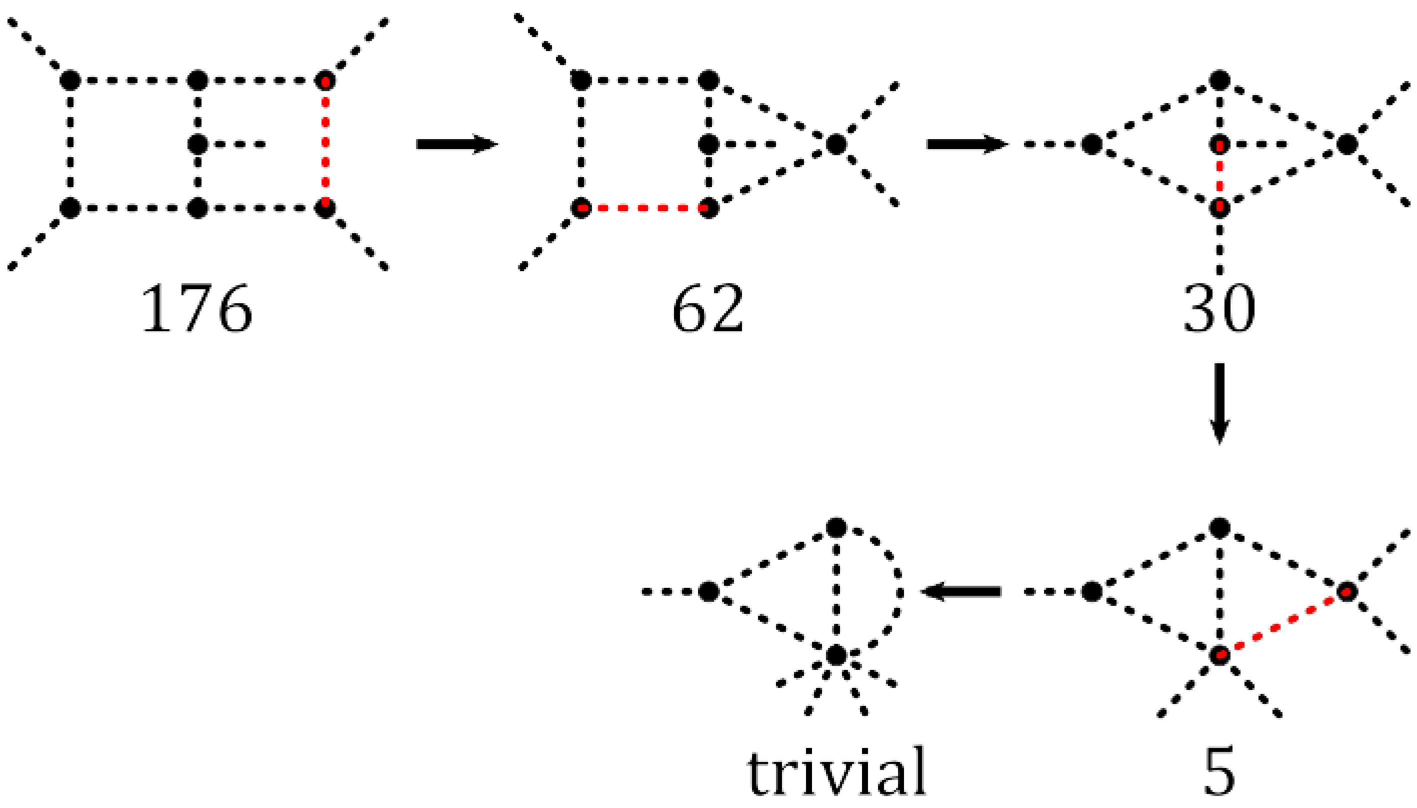
$$\mathcal{D}_1 = \ell_1^2,\ \mathcal{D}_2 = (\ell_1 - p_1)^2,\ \mathcal{D}_3 = (\ell_1 - p_1 - p_2)^2,$$
$$\mathcal{D}_4 = \ell_2^2,\ \mathcal{D}_5 = (\ell_2 + p_5)^2,\ \mathcal{D}_6 = (\ell_2 + p_4 + p_5)^2,$$
$$\mathcal{D}_7 = (\ell_1 - \ell_2)^2,\ \mathcal{D}_8 = (\ell_1 - \ell_2 + p_3)^2,\ \mathcal{D}_9 = (\ell_1 + p_5)^2,$$
$$\mathcal{D}_{10} = (\ell_2 - p_1)^2,\ \mathcal{D}_{11} = (\ell_2 - p_1 - p_2)^2,$$

$$\vec{s} \equiv \{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$$



Examples

- all-small region iteration



Examples

- $\vec{s}_0 = \{4, -\frac{113}{47}, \frac{281}{149}, \frac{349}{257}, -\frac{863}{541}\}$
- setup (block-triangular systems) + solve: 6 CPU hour + 7 CPU hour
- first step in the iteration: $P_1 = 95\%$

$$
\begin{aligned}
I_{\text{phy}}[1,1,1,1,1,1,1,1,0,0,0] = \\
&- 0.06943562517263776\epsilon^{-4} \\
&+ (1.162256636711287 + 1.416359853446717i)\epsilon^{-3} \\
&+ (37.82474332116938 + 15.91912443581739i)\epsilon^{-2} \\
&+ (86.2861798369034 + 166.8971535711277i)\epsilon^{-1} \\
&- (4.1435965578662 - 333.0996040071305i) \\
&- (531.834114822928 - 1583.724672502141i)\epsilon \\
&- (2482.240253232612 - 2567.398291724192i)\epsilon^2 \\
&- (8999.90369367113 - 19313.42643829926i)\epsilon^3 \\
&- (28906.9558269676 2 - 17366.82954322838i)\epsilon^4
\end{aligned}
$$

checked with analytic solutions

[Chicherin, Gehrmann, Henn et al. *Phys. Rev. Lett.*, 2019]

[Chicherin and Sotnikov. *JHEP*, 2020]

new, self-consistence check

$$\frac{\partial}{\partial r}\vec{\mathcal{I}}_{\text{phy}}(r) = A(r)\vec{\mathcal{I}}_{\text{phy}}(r)$$

$$\vec{s} = \vec{s}_0 + r(\vec{s}_1 - \vec{s}_0)$$

Examples

- block-triangular systems [Liu, Ma and Wang. *Phys. Lett. B*, 2018] [Guan, Liu and Ma. *Chin. Phys. C*, 2020]
  - each block can be solved independently
  - much smaller size and much better structure than IBP systems
  - 30~100 times faster in general for finite field computations

- block-triangular system V.S. IBP system (LiteRed + FiniteFlow) [Lee. *J. Phys. Conf. Ser.*, 2014] [Peraro. *JHEP*, 2019]

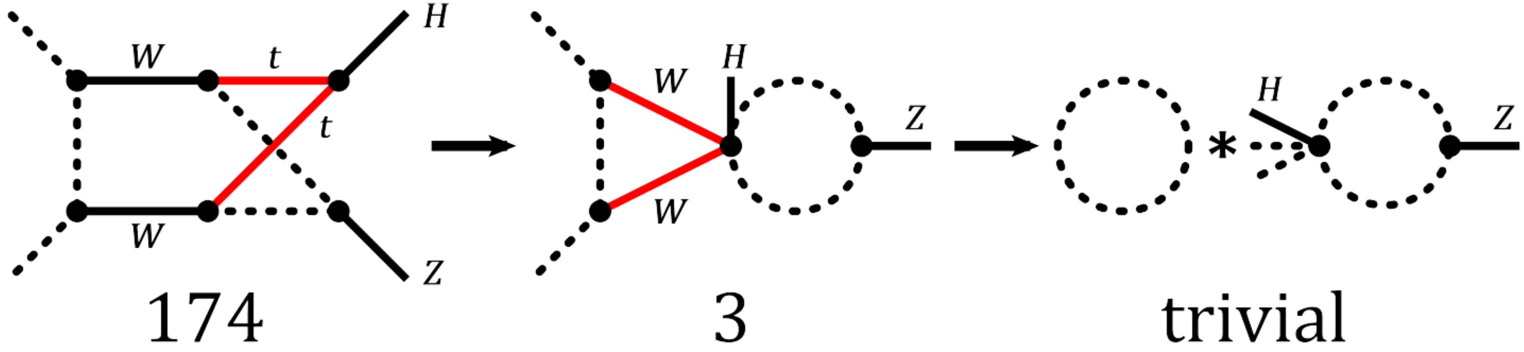
system	block-triangular	IBP
# relations	869	212847
t_{FF}	0.029s	3.93s

Much more efficient!

Outline

Examples

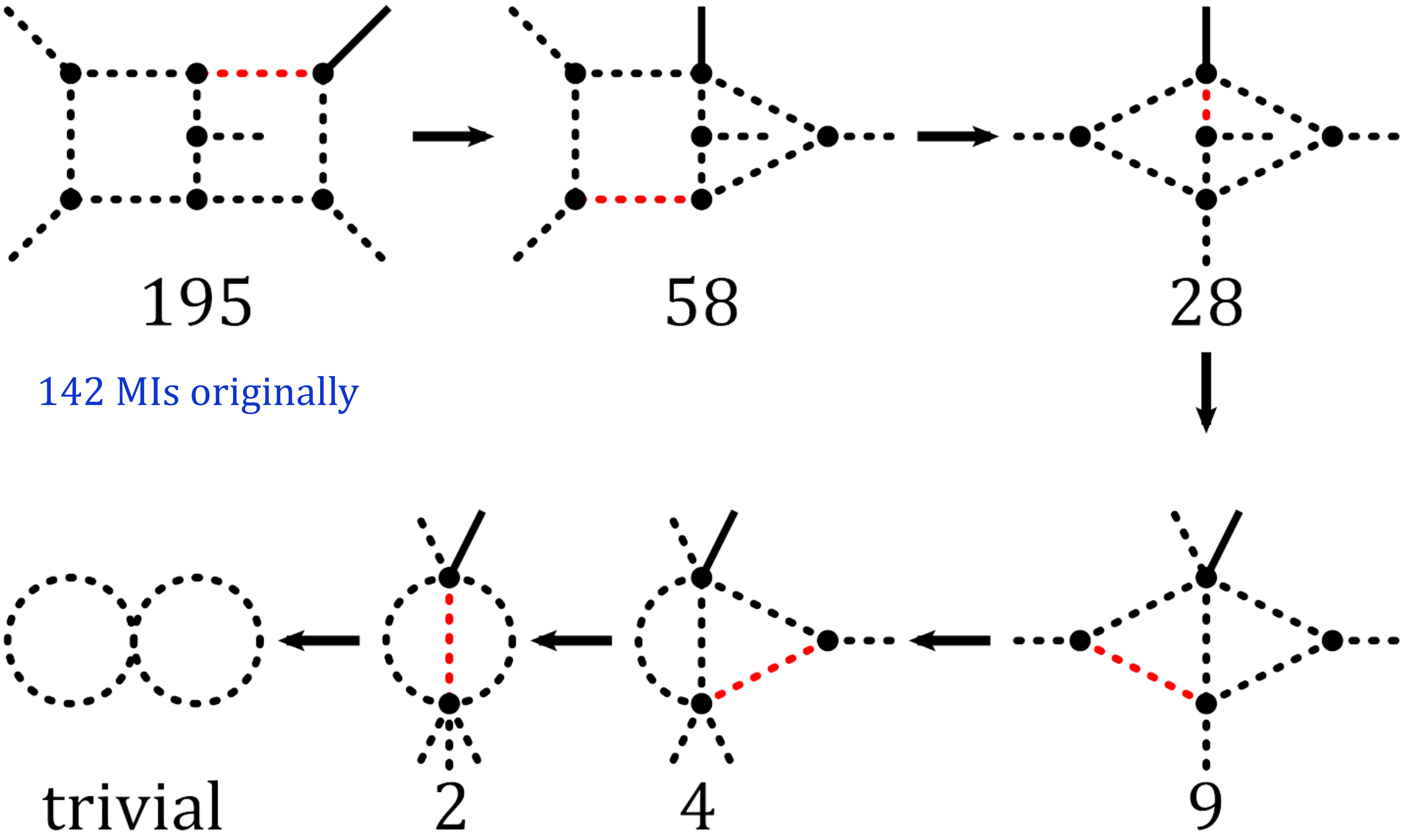
- two-loop electroweak correction to $e^+e^- \to H + Z$ [Song and Freitas. *JHEP*, 2021]



- mass mode [Hansen and Wang. *JHEP*, 2021] [Hansen and Wang. *JHEP*, 2021]
- $m_t^2 \to m_t^2 + \eta$
- The number of master integrals does not change.

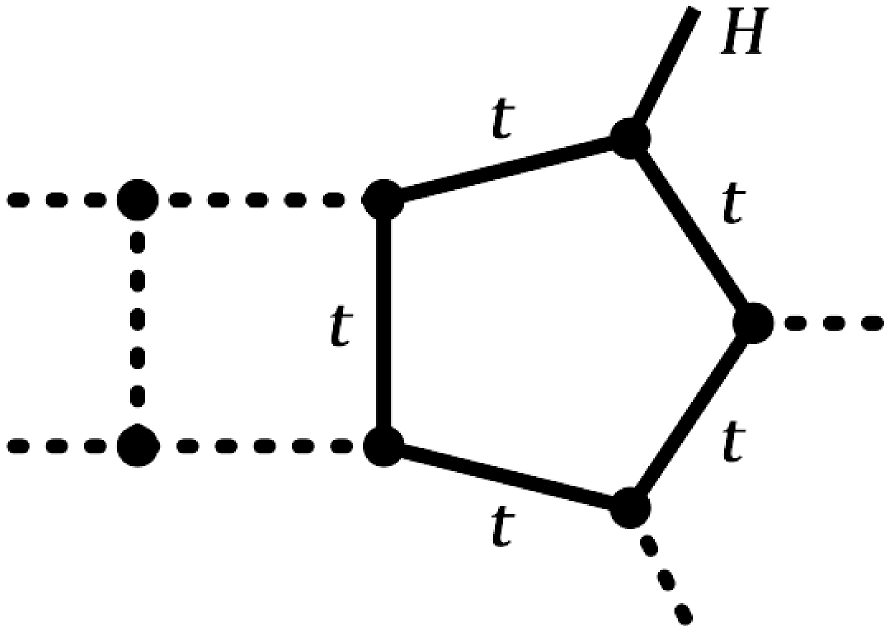
Examples

- $W/Z/H + 2j$ production at two loop



Examples

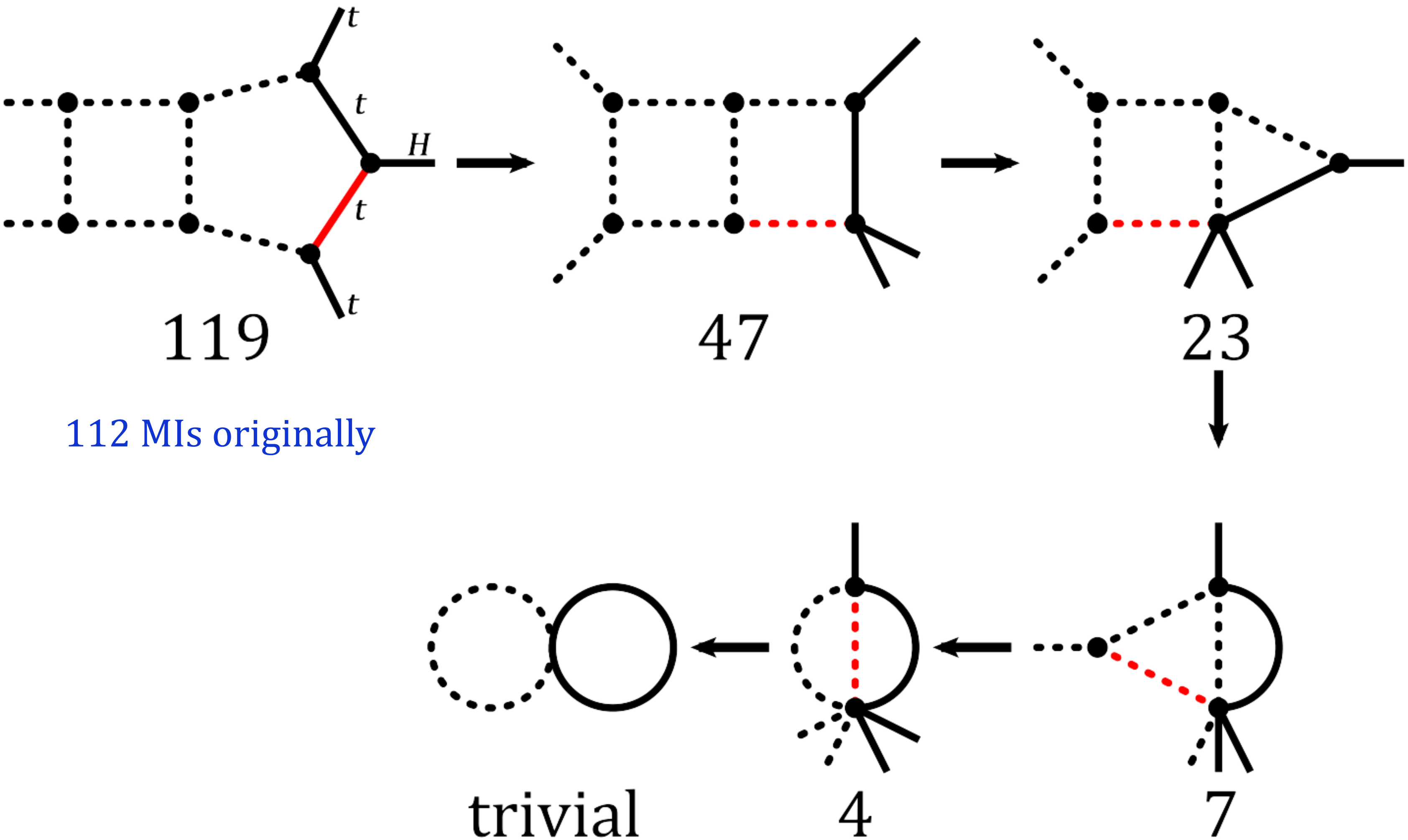
- $H + 2j$ production at two loop in full QCD



- mass mode: $m_t^2 \to m_t^2 + \eta$
- 173 MIs

Examples

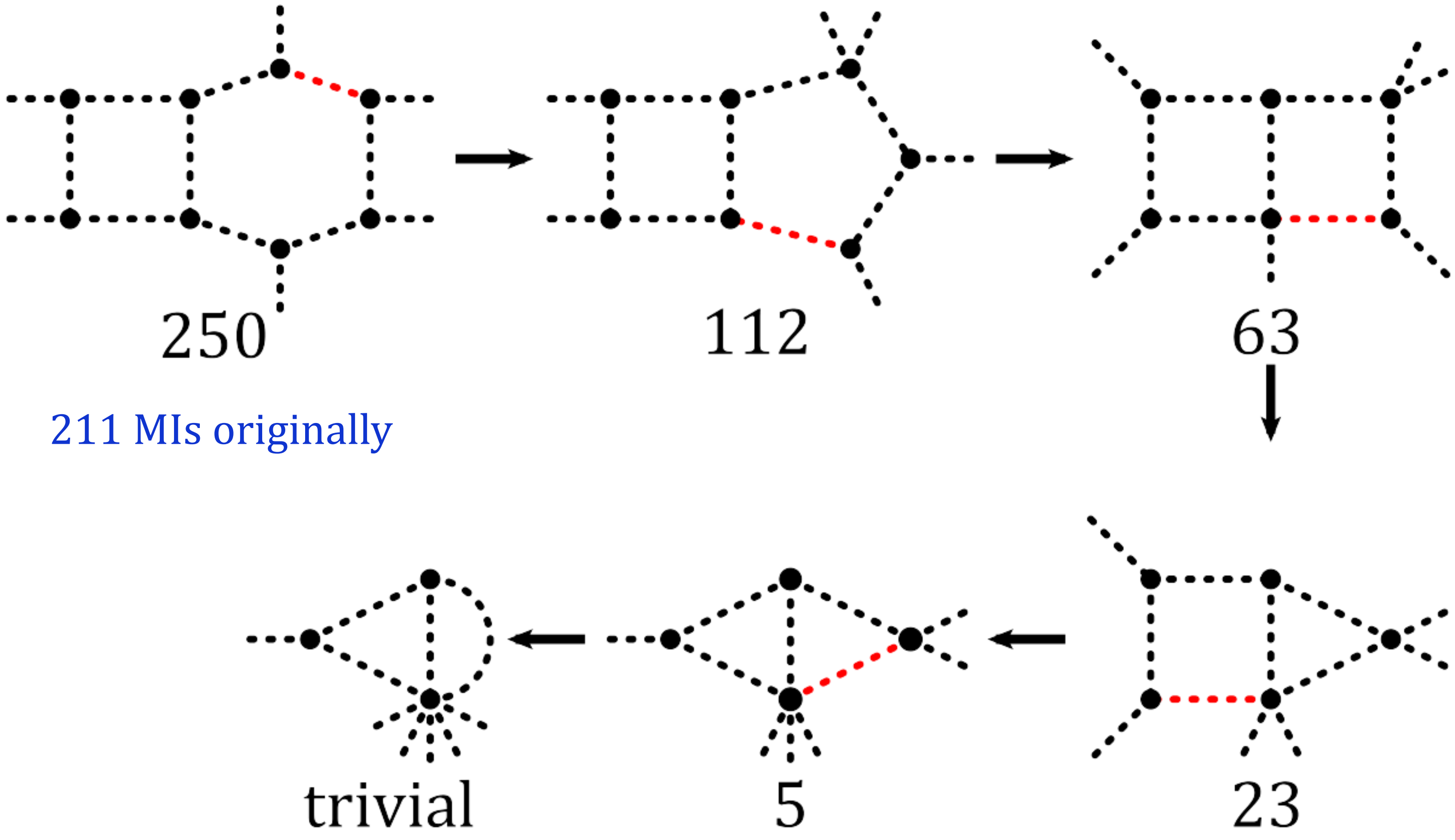
- $t\bar{t}H$ production at two loop



112 MIs originally

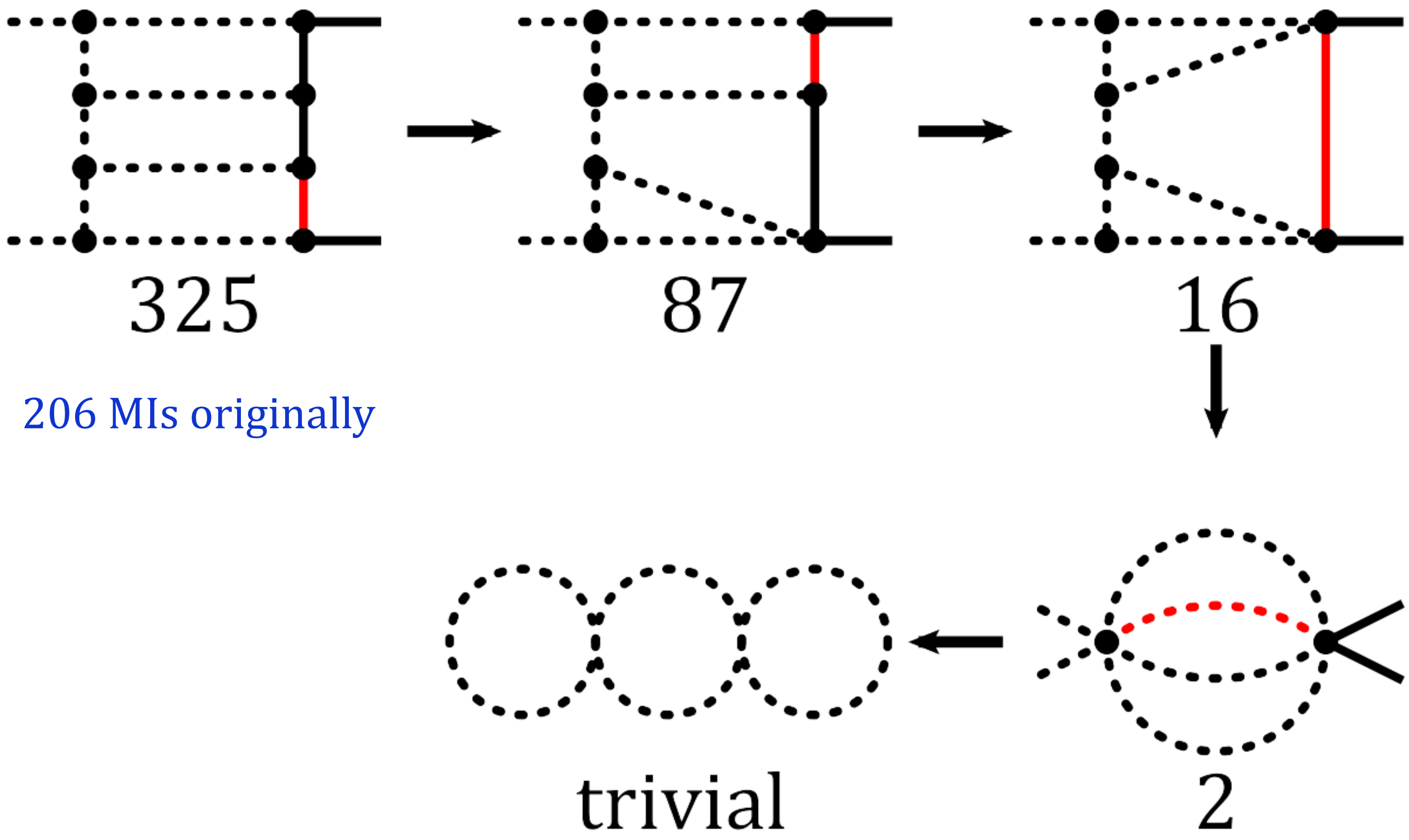
Examples

- $4j$ production at two loop

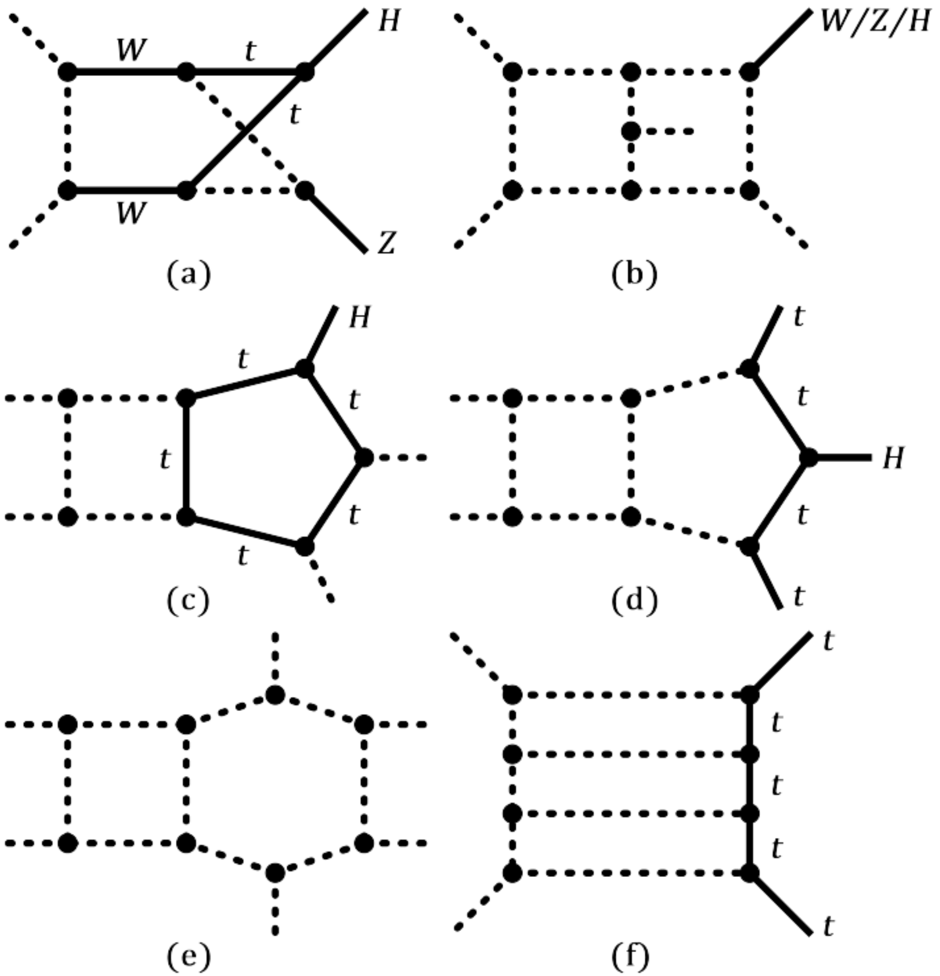


Examples

- $t\bar{t}$ production at three loop



➢ Other cutting-edge examples



Family	(a)	(b)	(c)	(d)	(e)	(f)
T_{setup}	20	18	8	1	25	30
T_{solve}	11	15	6	3	15	42
P_1	99%	96%	99%	98%	94%	93%

- $O(10)\sim O(100)$ CPU hours
- $P_1 > 90\%$, first step dominant

Outline

Summary and outlook

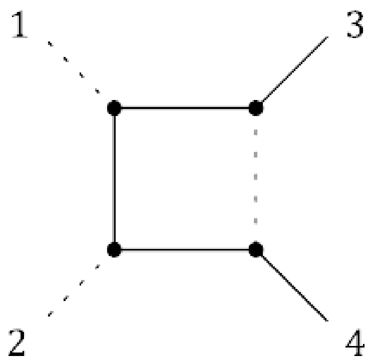
- We improve the original auxiliary mass flow method by introducing η in an iterative way.

- We compute master integrals involved in cutting-edge problems which may be challenging for all other methods.

- The method could be used directly or combined with traditional differential equations for future phenomenological studies.

Thank you!

➢ Example: one-loop four-point integral



$s = (p_1 + p_2)^2 = 10, t = (p_1 + p_3)^2 = -3, m^2 = 1$

$$I[1,1,1,1] = \frac{0.0665971 - 0.101394i}{\epsilon} + (-0.0133705 + 0.287857i)$$

- η-reg: $I[1,1,1,1](\eta) \sim (0.0665971 - 0.101394i)\log(\eta) + 0.0250704 + 0.22933i$
- ϵ-reg: $I[1,1,1,1](\eta) \sim \eta^{-\epsilon} f_1 + f_2 + \eta^{1/2-\epsilon} f_3$

$$f_1 = \frac{-0.0665971 + 0.101394i}{\epsilon} + (0.0384409 - 0.0585265i),$$
$$f_2 = \frac{0.0665971 - 0.101394i}{\epsilon} + (-0.0133705 + 0.287857i),$$
$$f_3 = 0.1309.$$

- take $\eta \to 0$, only f_2 survives