RENORMALIZATION OF THE FLAVOR-SINGLET AXIAL-VECTOR CURRENT

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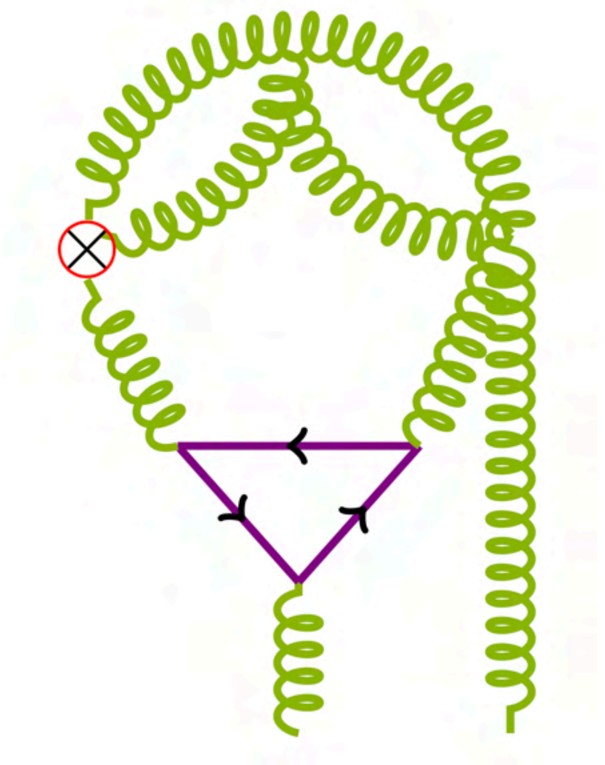
Seminar at DESY



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with Long Chen, Michal Czakon

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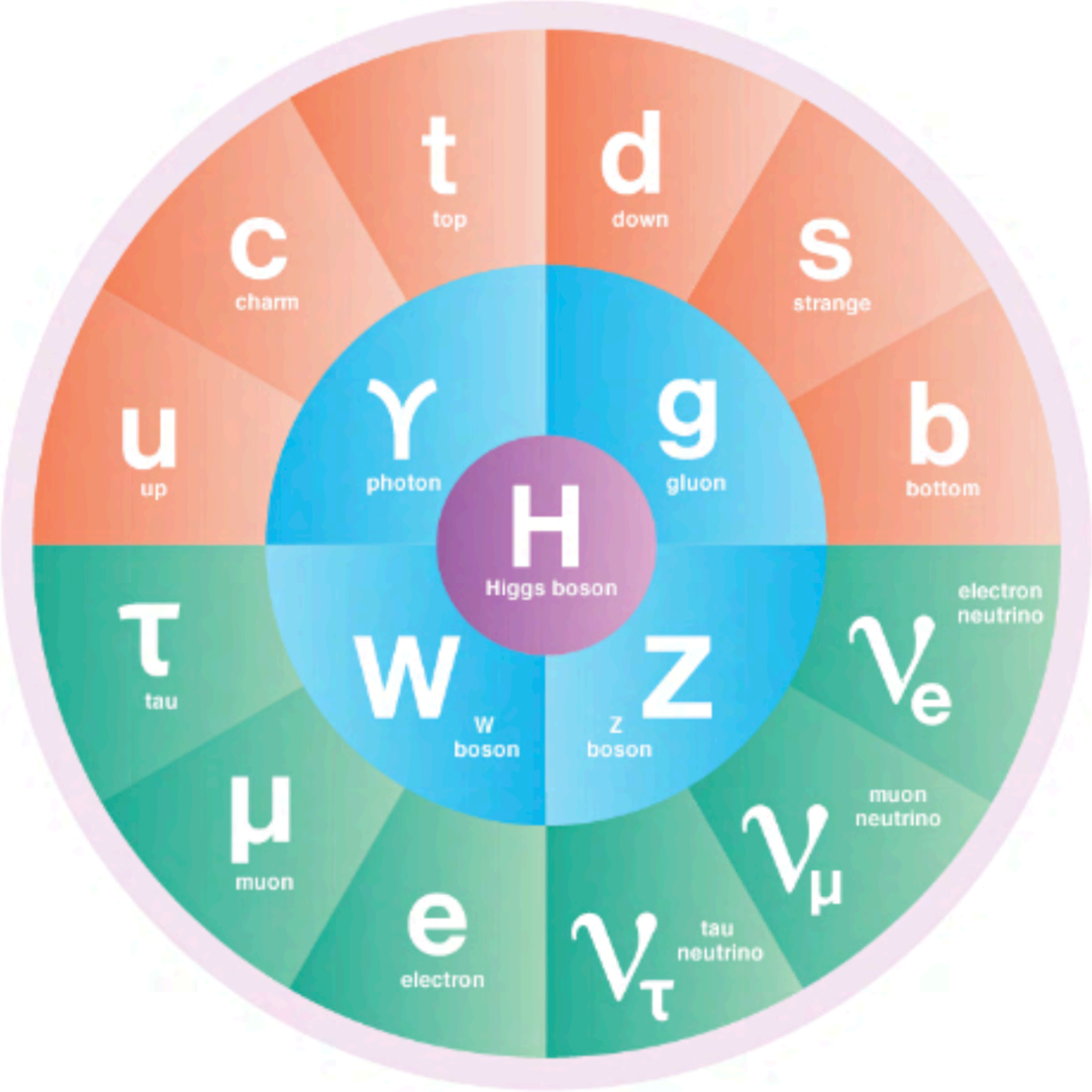
UNIVERSITÀ DEGLI STUDI DI TORINO

PREAMBLE

The larger question - What are the fundamental building blocks of Nature?

Periodic Table
for
Particle Physicists



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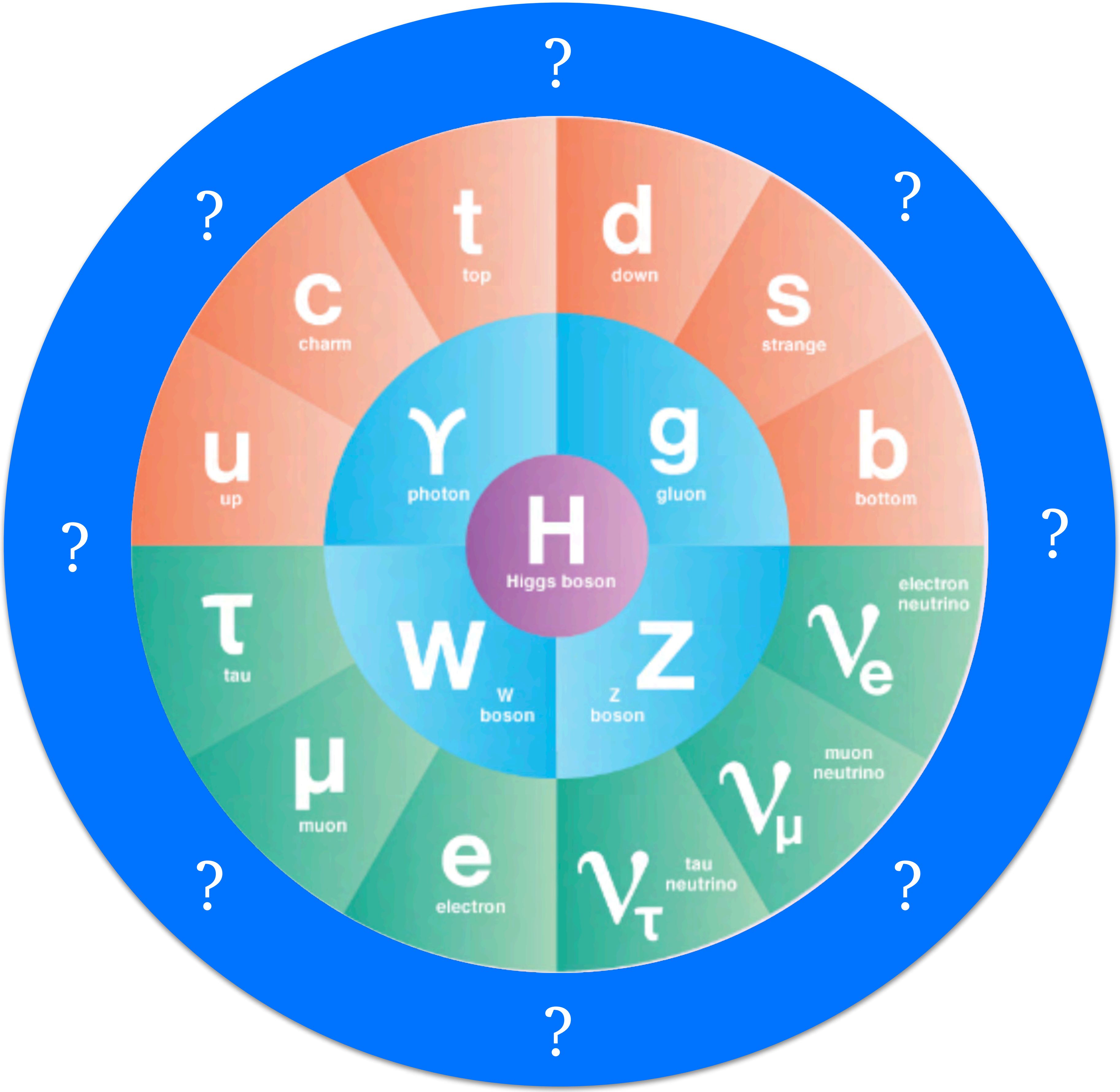
MODEL

PREAMBLE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muon
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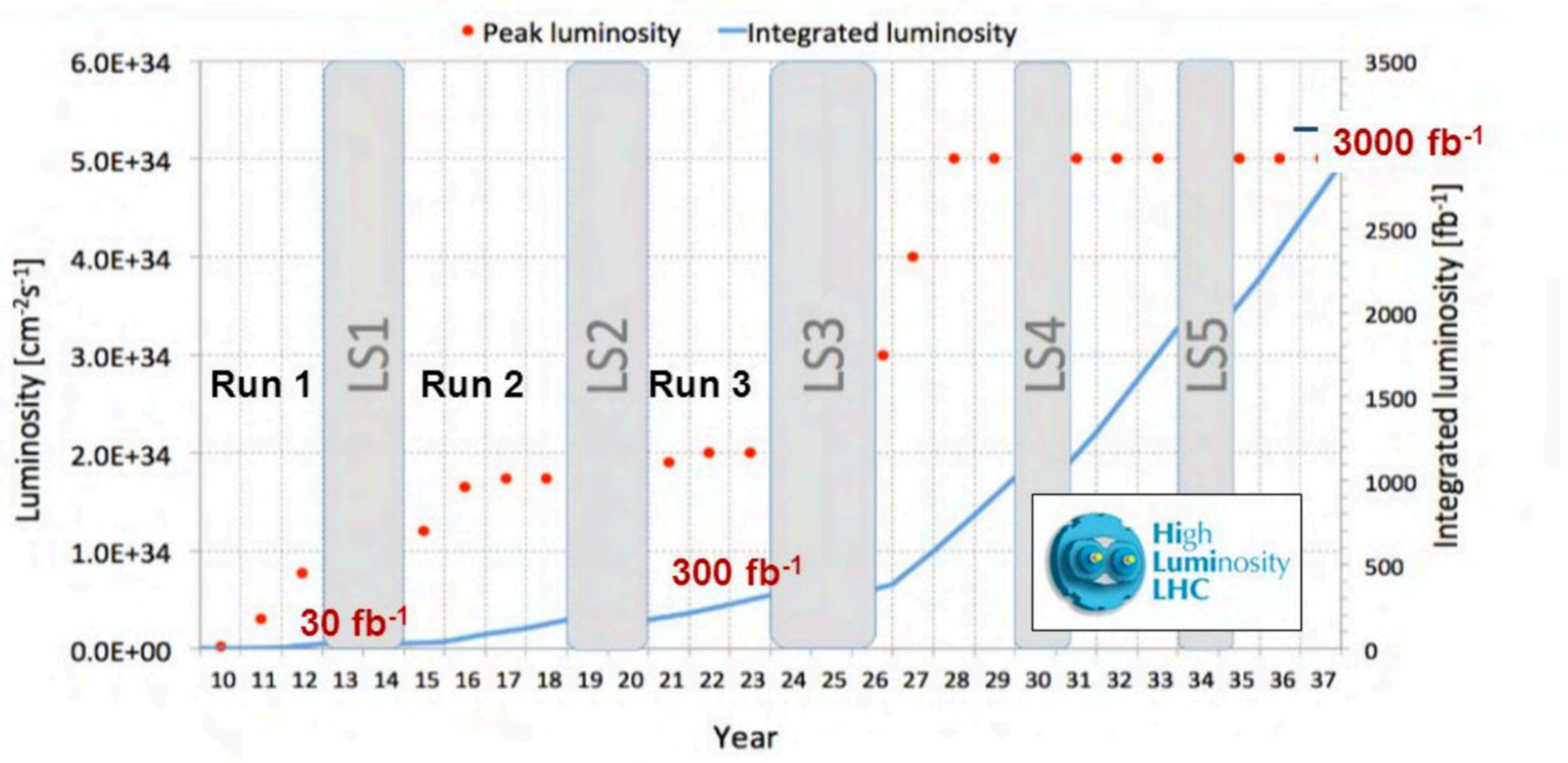
ν_τ
tau
neutrino

WHY PRECISION PHYSICS?

We are still in the early stages of LHC physics: $139\ \mathrm{fb}^{-1}$

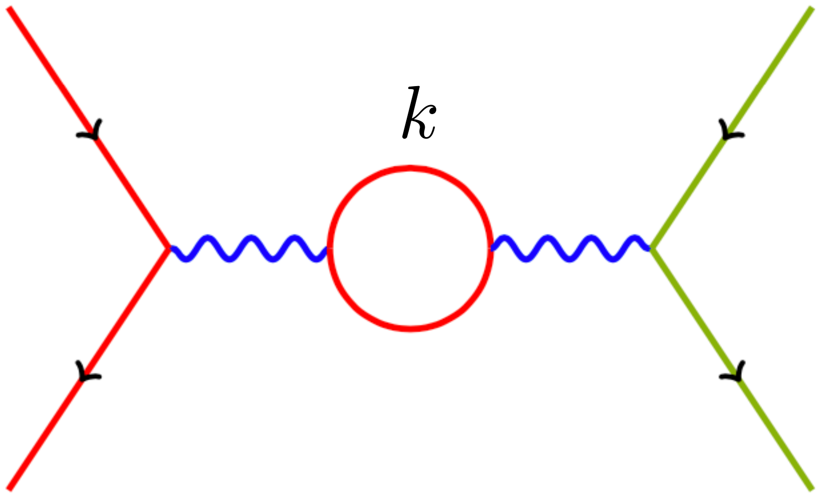


- No striking evidence for new physics so far at the LHC (modulo LHCb recent finding)
- Best hope: try to reveal new physics through deviations of the SM
- To exploit this possibility, abundant amount of data will be collected at LHC
- The exploitation of these data requires highly accurate theoretical predictions

→ Precision Physics

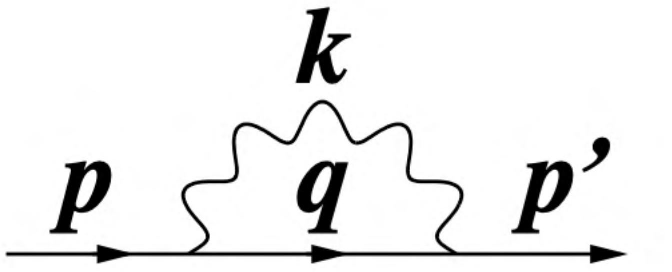
DIVERGENCES

Feynman graphs with closed loops is associated to unconstrained momenta



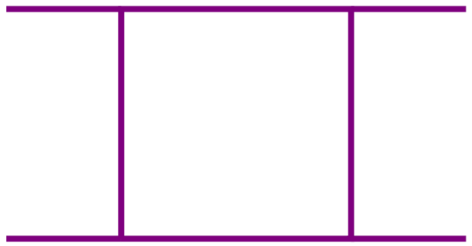
$$\int \frac{d^4k}{(2\pi)^4} \cdots$$

e.g. Electron self-energy in QED



$$\bar{u}(p) \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{p} - \not{k} + m)\gamma_\mu}{k^2[(p-k)^2 - m^2]} u(p)$$

$$\int^\infty d^4k \, \frac{k}{k^2(p-k)^2}$$ UV divergent



$$= \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2(k-p_4)^2}$$

IR (soft & collinear) divergent

REGULARISATION

$$\int^{\Lambda} d^4k \cdots \sim \log\left(\frac{\Lambda}{m}\right)$$ Regularisation of integrals

→ A procedure to identify the nature of divergences

"Pauli - Villars method"

- Equivalent to introducing a fictitious particle
- Not suitable for multi-loop: many fictitious particles needed
- Not suitable for non-abelian

Ideal candidate: dimensional regularisation

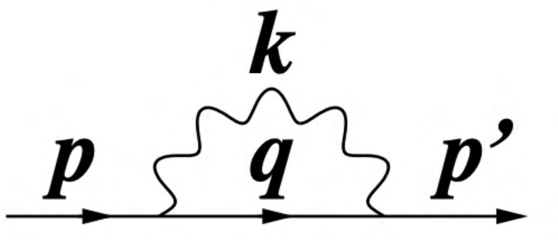
① $$\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^dk}{(2\pi)^d}$$

② $$d = 4 \to (4 - 2\epsilon)$$

- analytic continuation with small +ve parameter such that we live infinitesimally close to the original one
- respect gauge invariance

DIMENSIONAL REGULARISATION

- Divergences appears as poles in ϵ



$$\sim \log\left(\frac{\Lambda}{m}\right) \sim \frac{1}{\epsilon}$$

- L-loop UV div integral: $\dfrac{1}{\epsilon^L}$
- L-loop IR div integral: $\dfrac{1}{\epsilon^{2L}}$

- Loop integrals can be expressed in terms of analytic functions of space-time dimension d

- Action is dimensionless in QFT

$$\int d^dx\,\mathcal{L} \longrightarrow [\mathcal{L}] = d$$

$$\Big\downarrow$$

$$[\psi_0] = \frac{d-1}{2} = \frac{3}{2} - \epsilon\,, \quad [A_0^\mu] = \frac{d-2}{2} = 1 - \epsilon\,, \quad [m_0] = 1\,, \quad [e_0] = \frac{4-d}{2} = \epsilon\,.$$

To keep dimensionless gauge coupling (essential for renormalisability) $e_0 \equiv \mu^\epsilon \tilde{e}_0(\mu)$

auxiliary mass scale/renormalisation scale

dimensionless

Physical quantities should be independent of this scale

→ Renormalization group equation

TODAY'S FOCUS AND GOAL

Way to get rid of UV singularities: UV Renormalisation

Flavour-singlet axial quantity

Massive vector boson

Pseudo-scalar

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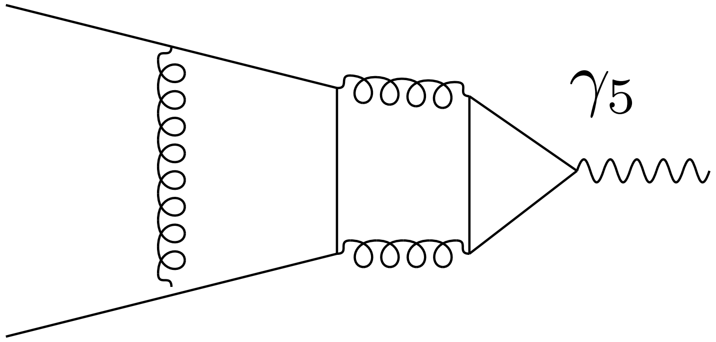


- Axial anomaly
- Operator renormalisation

Our goal

ISSUE OF CHIRAL QUANTITY

DR preserves Lorentz & gauge invariance but not chiral invariance

$$P_\pm = \frac{1}{2}\left(1 \pm \gamma_5\right)$$

inherently 4-dimensional

Algebraically, impossible to define d-dimensional γ algebra with [Hooft, Veltman '72]

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,, \qquad \mu = 0, 1, \cdots d-1$$

satisfying both

4D rules

1. $\{\gamma^\mu, \gamma_5\} = 0$

   Kreimer, Körner and Schilcher (KKS) scheme

2. $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = -4i\varepsilon^{\mu\nu\rho\sigma}$

   't Hooft–Veltman prescription [Hooft, Veltman '72]

   Breitenlohner-Maison [Breitenlohner-Maison '77]

’tHV / BM PRESCRIPTION

’t Hooft–Veltman prescription: $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] = -4i\varepsilon^{\mu\nu\rho\sigma}$ is preserved

$$\gamma^\mu = \hat{\gamma}^\mu + \tilde{\gamma}^\mu$$

d 4 (d-4)

$$\{\gamma^\mu, \gamma_5\} \neq 0$$

$$\{\hat{\gamma}^\mu, \gamma_5\} = 0, \quad [\tilde{\gamma}^\mu, \gamma_5] = 0, \quad \{\tilde{\gamma}^\mu, \gamma_5\} \neq 0$$

issue

Algebraically consistent, but violates gauge invariance i.e. modifies Ward identities

“Evanescent” -2ε dimensional space

→ additional spurious terms

→ Violation

$$\{\gamma_5, \gamma_\mu\}\{\gamma_5, \gamma^\mu\} = 2\epsilon$$

$$(1+\gamma_5)\gamma_\mu(1+\gamma_5)\gamma_\alpha(1-\gamma_5)\gamma^\mu(1-\gamma_5)$$
$$= -8\epsilon(1+\gamma_5)\gamma_\alpha$$

remedy

Add an extra -2ε dimensional finite counterterms

Larin’s prescription: Avoid dimensional splitting

Our goal

[Larin ’93]

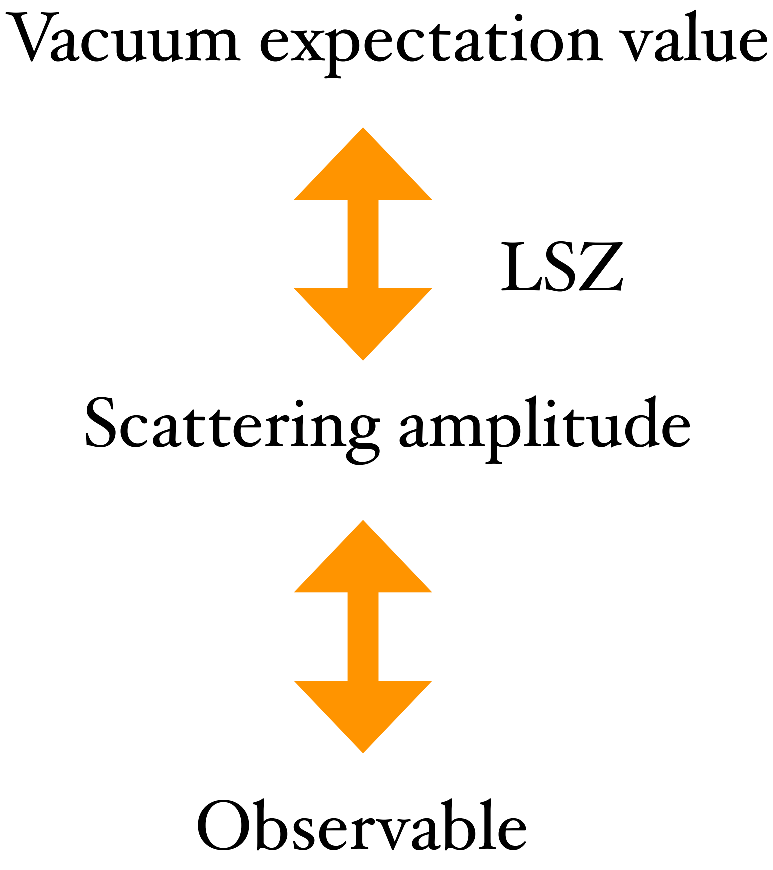
FLAVOUR SINGLET AXIAL CURRENT

Flavour-singlet axial current: $J_5^\mu(x) = \sum \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$

External Local (composite) operator

- May not be Lorentz invariant
- Can be inserted into Green's function

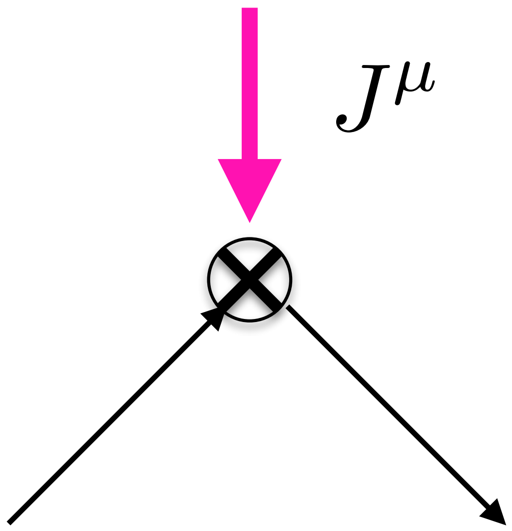
$$\langle\Omega|\, T\{J^\mu(x)\psi(x_1)\bar{\psi}(x_2)\}\,|\Omega\rangle$$


Vacuum expectation value

LSZ

Scattering amplitude

Observable



Prescription

$$\boxed{\gamma_5 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma}$$

All the Lorentz indices are to be treated in d-dimensions!

$\{\gamma^\mu, \gamma_5\} \neq 0$

OPERATOR RENORMALISATION

Green Correlation Function suffers from additional UV divergences

Need Operator Renormalisation Z_J

$$[J_5^\mu]_R = \sum Z_J \, \bar{\psi}_B(x) \, \gamma^\mu \gamma_5 \, \psi_B(x)$$

[Akyeampong, R. Delbourgo]

[Moch, Vermaseren, Vogt, ‘15]

$$\gamma^\mu \gamma_5 \to \frac{1}{2}\Big(\gamma^\mu \gamma_5 - \gamma_5 \gamma^\mu\Big)$$

Symmetrisation to restore the Hermiticity

$$= \sum Z_5^f \; Z_5^{ms} \; \bar{\psi}_B \, \frac{-i}{3!} \epsilon^{\mu\nu\rho\sigma} \gamma_\nu \gamma_\rho \gamma_\sigma \, \psi_B$$

pure $\overline{\mathrm{MS}}$ ren constant

finite ren constant: determined by demanding the restoration of

axial Ward identity or ABJ anomalous WI

Our goal

$$\boxed{[\partial_\mu J_5^\mu]_R = a_s \, n_f \, \mathrm{T}_F \, [F\tilde{F}]_R}$$

$$F\tilde{F} \equiv -\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$$

ANOMALY

Quantum breakdown of a classical symmetry

- Non-gauge symmetries
  - scale invariance
  - global symmetries

At the origin of interesting physical phenomena: $\pi^0 \to \gamma\gamma, \cdots$

- Local/gauge symmetries
  - gauge symmetry
  - gravitational anomaly

These are very dangerous - must be cancelled - else theory would become inconsistent

ANOMALY: SCALE INVARIANCE

Massless Φ^4 theory: invariant under scale transformation

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{g}{4!} \phi^4 \right)$$

$$x^\mu \longrightarrow x'^\mu = \lambda x^\mu$$

$$\phi(x) \longrightarrow \phi'(x) = \lambda^{-\Delta} \phi(\lambda^{-1} x) \quad \text{(with } \Delta = 1\text{)}$$

This invariance is broken by quantum corrections: regularisation & renormalisation introduces an energy scale that breaks scale invariance

Running of coupling constant

$$\beta(g) = \frac{3g^2}{16\pi^2}$$

$$g(\mu) = \frac{g(\mu_0)}{1 - \frac{3}{16\pi^3} g(\mu_0) \log\left(\frac{\mu}{\mu_0}\right)}$$

Physics at different scales do not look same

QCD: asymptotic freedom & confinement

ANOMALY: CHIRAL INVARIANCE

Abelian gauge theory

Lagrangian is invariant under two-independent global symmetries

$$\psi \to e^{i\alpha}\psi, \quad \psi \to e^{i\beta\gamma_5}\psi$$

Noether current $\quad J^{\mu} = \bar{\psi}\gamma^{\mu}\psi, \quad J^{\mu 5} = \bar{\psi}\gamma^{\mu}\gamma^{5}\psi$

Classically $\quad \partial_{\mu}J^{\mu} = 0, \quad \partial_{\mu}J^{\mu 5} = 2im\bar{\psi}\gamma^{5}\psi \overset{m \to 0}{=} 0$

Do these two symmetries remain in quantum theory?

$$\partial_{\mu}\langle J^{\mu}\rangle \overset{?}{=} 0$$

☠ Necessary for consistency (e.g. charge conservation)

$$\partial_{\mu}\langle J^{\mu}_{5}\rangle \overset{?}{=} 0$$

- vector current is exactly conserved
- chiral symmetry is not conserved

Quantum

Using Schwinger-Dyson equation $\partial_{\mu}\langle J^{5\mu}(x)\mathcal{O}(x_1,\ldots,x_n)\rangle = \frac{e^2}{16\pi^2}\langle \varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}(x)F_{\alpha\beta}(x)\mathcal{O}(x_1,\ldots,x_n)\rangle$

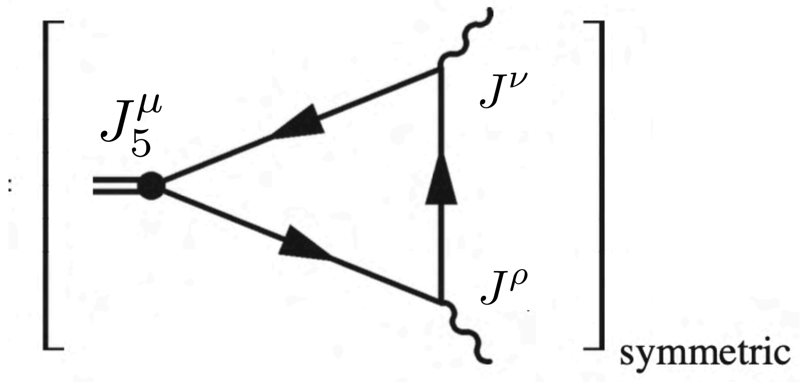
$$\boxed{\partial_{\mu}J^{5}_{\mu} = \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}}$$

ABJ anomaly

Adler ’69, ’70
Bell - Jackiw ’69

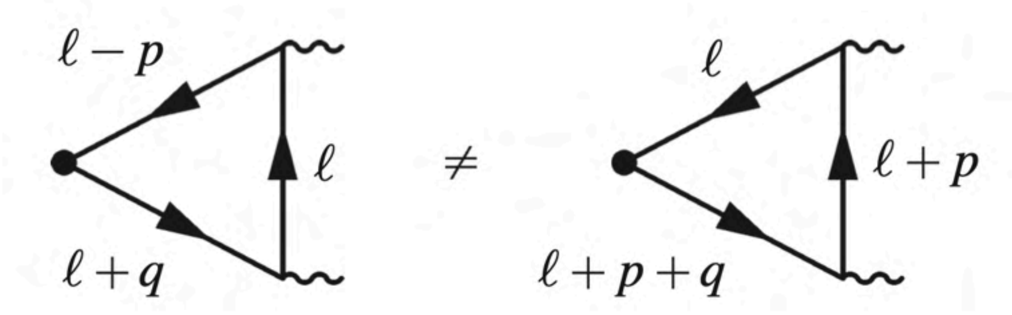
ANOMALY: CHIRAL INVARIANCE

Source



Famous triangle diagram

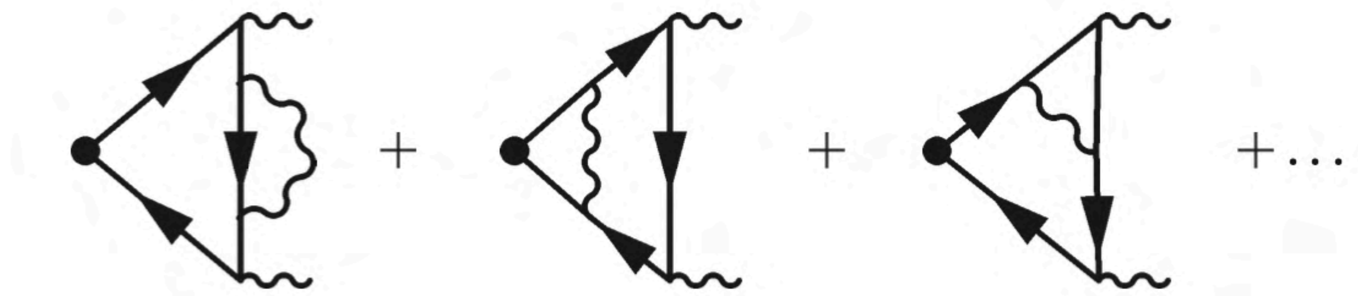
Linearly divergent integral: contribution depends on how we label loop momentum



ambiguous: chose mom routing such that vector Ward identity holds true

True meaning of anomaly: we can't conserve simultaneously both vector and axial WI

- Anomaly arises if a symmetry of action is not a symmetry of functional measure [Fujikawa '79]
- 1-loop exact: Adler-Bardeen theorem '69



do not contribute to anomaly

GREEN CORRELATION FUNCTION

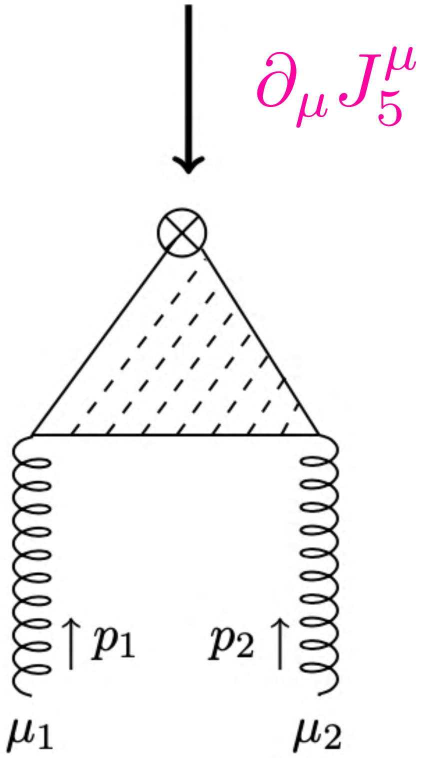
Strategy to fix Z_5^f

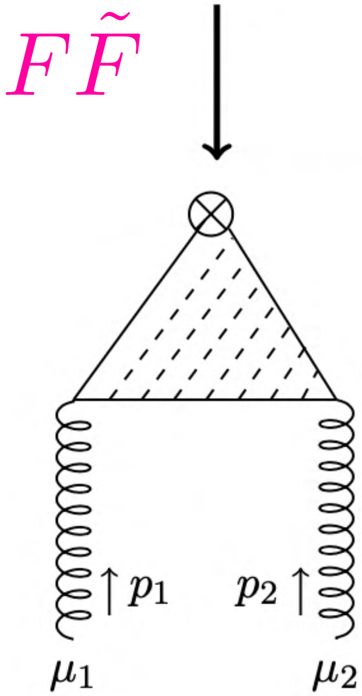
Our goal

- Calculate amputed Green's function with these operators insertion
- Demand Anomaly equation

$$\left[\partial_\mu J_5^\mu\right]_R = a_s\, n_f\, \mathrm{T}_F\, \left[F\tilde{F}\right]_R$$

$$\langle\Omega|\hat{\mathrm{T}}\,[[\partial_\mu J_5^\mu(y)]_R\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0)]\,|\Omega\rangle = a_s n_f T_F \langle\Omega|\hat{\mathrm{T}}\left[[F\tilde{F}]_R\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0)\right]|\Omega\rangle$$





GREEN CORRELATION FUNCTION

Strategy to fix Z_5^f

Our goal

- Calculate amputed Green's function with these operators insertion
- Demand Anomaly equation

Axial gluon current: Chern-Simons 3-form

$$F\tilde{F} = \partial_\mu K^\mu$$
$$= \partial_\mu \left(-4\,\epsilon^{\mu\nu\rho\sigma} \left(A_\nu^a \partial_\rho A_\sigma^a \; + \; g_s \frac{1}{3} f^{abc} A_\nu^a A_\rho^b A_\sigma^c \right) \right)$$

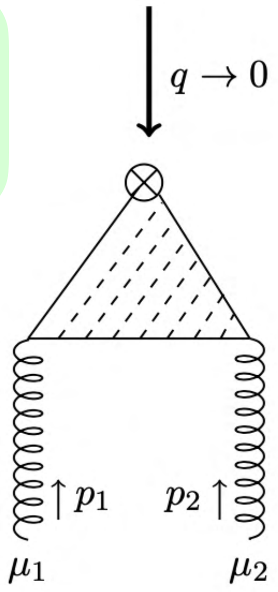
- Not gauge invariant
- Advantage: less number of gauge fields

$$g(p_1) + g(p_2) \to J_5^\mu(q)$$
$$g(p_1) + g(p_2) \to K^\mu(q)$$

Correlation function with operator insertion

$$\Gamma_{lhs}^{\mu\mu_1\mu_2}(p_1, p_2) \equiv \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y} \, \langle \Omega | \hat{\mathrm{T}} \left[J_5^\mu(y)\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0) \right] | \Omega \rangle |_{\mathrm{amp}}$$

$$\Gamma_{rhs}^{\mu\mu_1\mu_2}(p_1, p_2) \equiv \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y} \, \langle \Omega | \hat{\mathrm{T}} \left[K^\mu(y)\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0) \right] | \Omega \rangle |_{\mathrm{amp}}$$



- Simplify by choosing zero momentum insertion

off-shell gluons

COMPUTATION

Projector $$\mathcal{P}_{\mu\mu_1\mu_2} = -\frac{1}{6\, p_1 \cdot p_1}\, \epsilon_{\mu\mu_1\mu_2\nu}\, p_1^{\nu}$$

4-loop calculation!

$$g(p_1) + g(p_2) \to J_5^{\mu}(q)$$
$$g(p_1) + g(p_2) \to K^{\mu}(q)$$

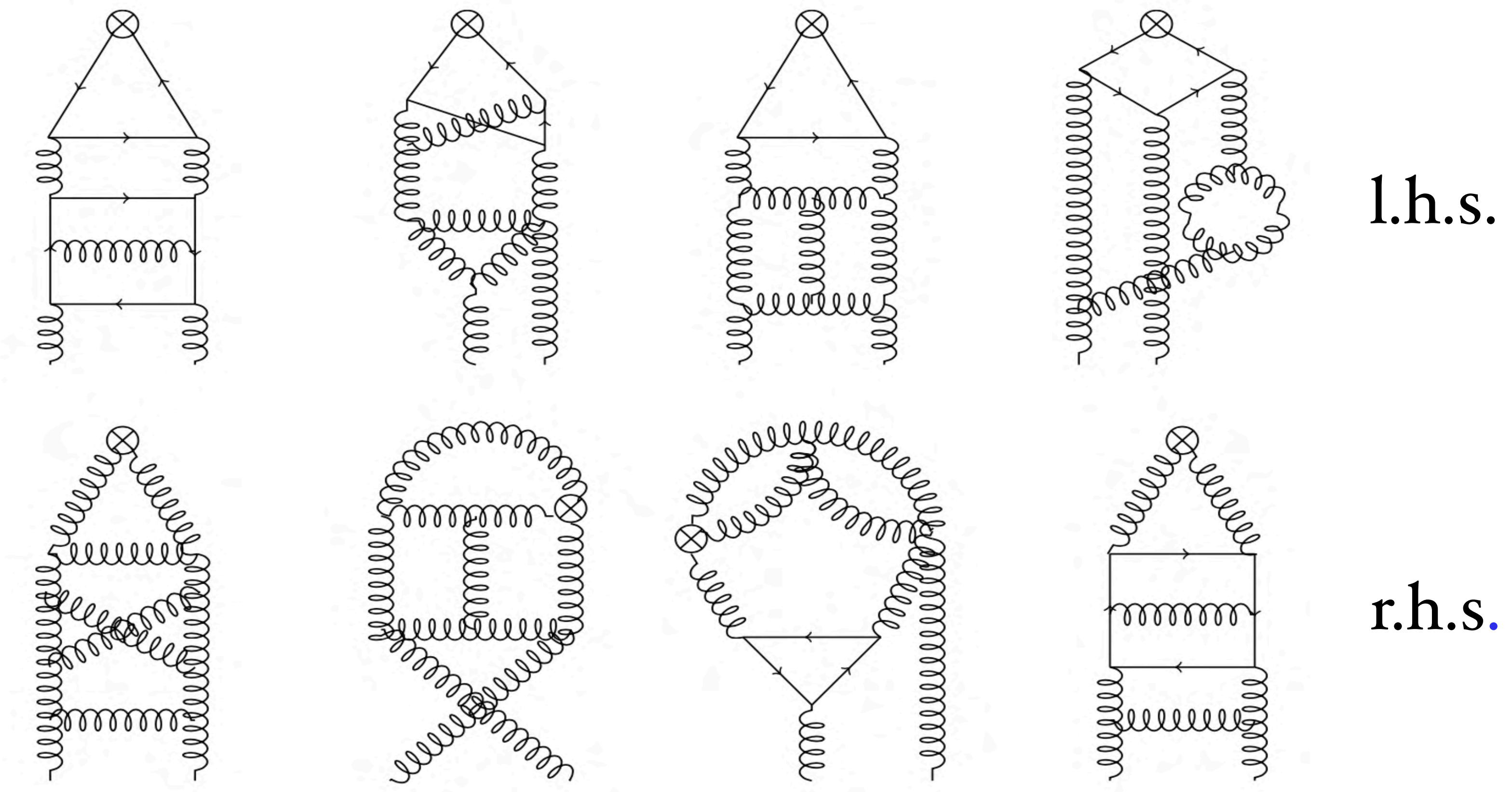
$$\xrightarrow{q \to 0}$$

$$\mathcal{M}_{lhs} = \mathcal{P}_{\mu\mu_1\mu_2}\, \Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1, -p_1)$$
$$\mathcal{M}_{rhs} = \mathcal{P}_{\mu\mu_1\mu_2}\, \Gamma^{\mu\mu_1\mu_2}_{rhs}(p_1, -p_1)$$

Feynman diagrams

[Nogueira]

	Qgraf	
	l.h.s.	r.h.s.
1-loop	2	4
2-loop	21	64
3-loop	447	1488
4-loop	11714	40564



- l.h.s. is loop induced

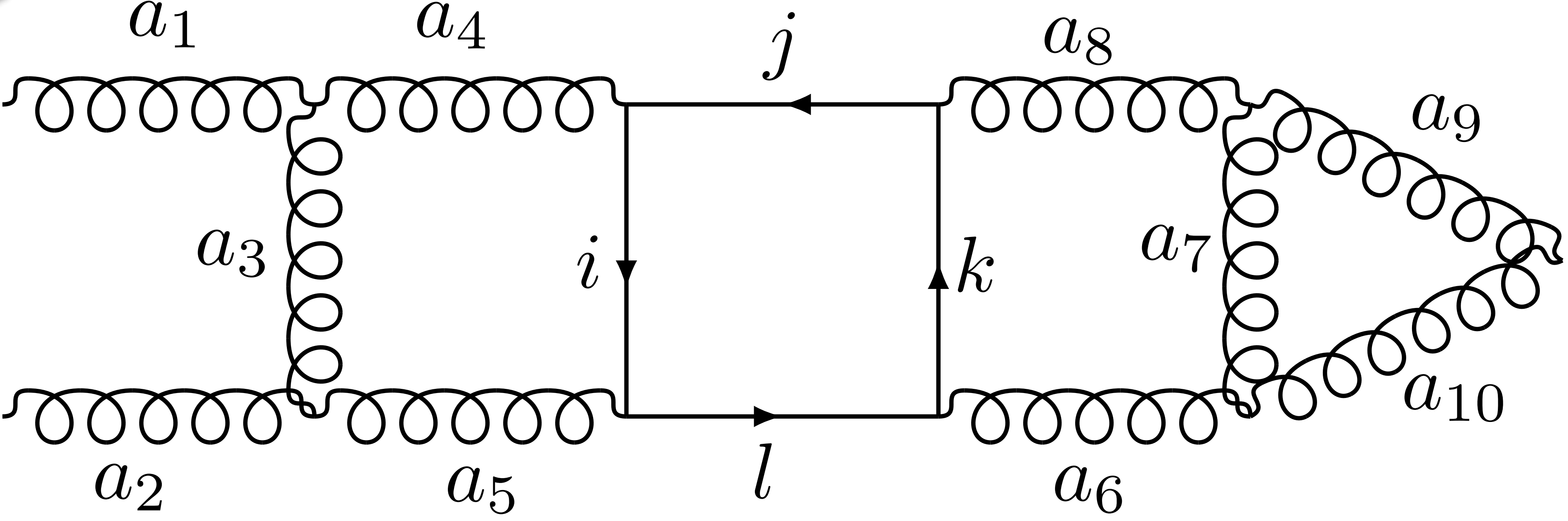
COMPUTATION

- In-house `FORM` code:

`FORM`: computer algebra [Vermaseren]

SU(N) color algebra

Lorentz algebra
Dirac algebra
} in d-spacetime dimensions



$$\sim f^{a_1 a_3 a_4} f^{a_2 a_3 a_5} T^{a_4}_{ij} T^{a_8}_{jk} T^{a_6}_{kl} T^{a_5}_{li} f^{a_7 a_8 a_9} f^{a_6 a_7 a_{10}} \delta_{a_9 a_{10}}$$

$$= \frac{1}{4} \delta_{a_1 a_2} (N - N^3)$$

COMPUTATION

- In-house `FORM` code: `FORM`: computer algebra [Vermaseren]

  SU(N) color algebra

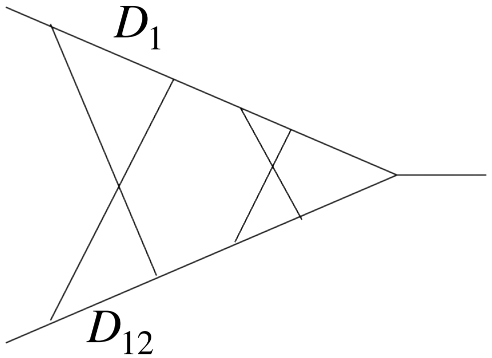
  Lorentz algebra

  Dirac algebra

  $\Big\}$ in d-spacetime dimensions

- Scalar products are converted to inverse of propagators $2k_1 \cdot p_1 = (k_1 + p_1)^2 - k_1^2 - p_1^2$

  Need set of 18 propagators to form a complete basis: 12 real prop + 6 ISP: integral family



46 integral families

Tricky: Upon taking $q \to 0$ limit, should give a valid basis

- Linear transformations on loop momentum: Feynman diagrams are recast to belong to one of the integral families

$$k_i \to \sum_{n=1}^{4} c_n k_n + \sum_{m=1}^{2} l_m p_m$$

COMPUTATION

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p_1-p_2)^2 (k-p_1)^2}$$

$$\Big\downarrow \qquad k \to k + p_1$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p_2)^2 k^2}$$

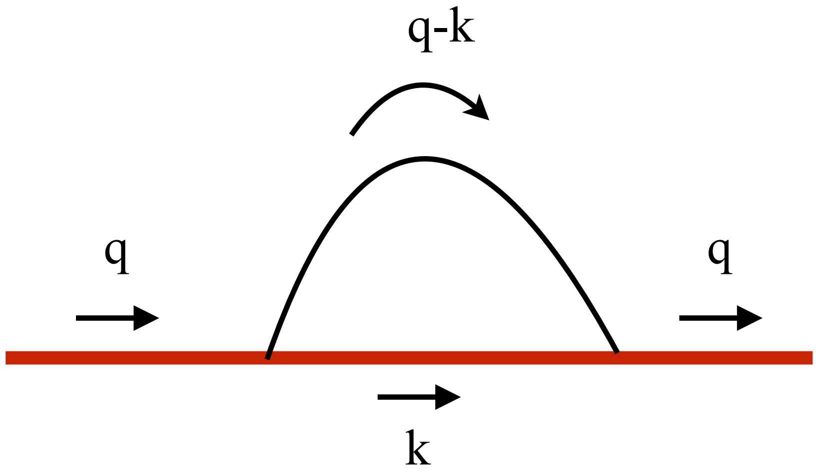
- Linear transformations on loop momentum: Feynman diagrams are recast to belong to one of the integral families

$$k_i \to \sum_{n=1}^{4} c_n k_n + \sum_{m=1}^{2} l_m p_m$$

INTEGRATION-BY-PARTS IDENTITIES

Scalar Feynman integrals are not linearly independent [Chetyrkin, Tkachov]

$$\int \prod_{\alpha=1}^{l} \frac{d^d k_\alpha}{(2\pi)^d} \frac{\partial}{\partial k_{j,\mu}} \left(v^\mu I(a_1, a_2, \cdots, a_n) \right) = 0 \qquad I(a_1, a_2, \cdots, a_n) = \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}$$



$$I(a_1, a_2) = \int d^d k \frac{1}{[k^2 - m^2]^{a_1} [(q-k)^2]^{a_1}}$$

IBP

$$0 = \int d^d k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{[k^2 - m^2]^{a_1} [(q-k)^2]^{a_1}}$$

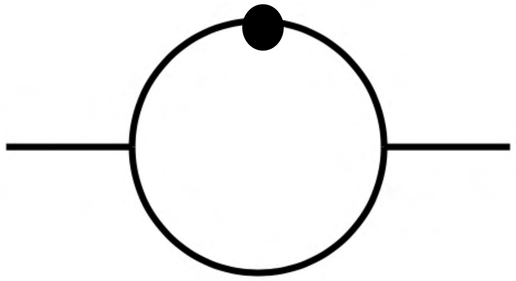
$$\begin{aligned} 0 =& (d - 2a_1 - a_2) I(a_1, a_2) - 2a_1 m^2 I(a_1 + 1, a_2) - a_2 I(a_1 - 1, a_2 + 1) \\ &+ a_2 (q^2 - m^2) I(a_1, a_2 + 1) \end{aligned}$$

INTEGRATION-BY-PARTS IDENTITIES

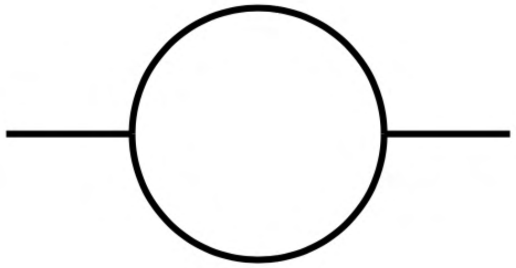
Scalar Feynman integrals are not linearly independent [Chetyrkin, Tkachov]

$$\int \prod_{\alpha=1}^{l} \frac{d^d k_\alpha}{(2\pi)^d} \frac{\partial}{\partial k_{j,\mu}} \left(v^\mu I(a_1, a_2, \cdots, a_n)\right) = 0 \qquad I(a_1, a_2, \cdots, a_n) = \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}$$

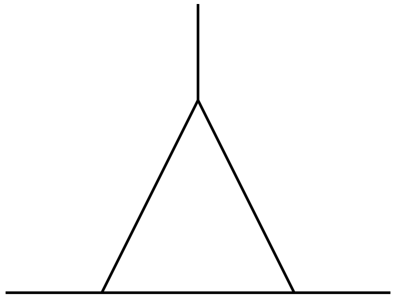
$I(2,1) =$



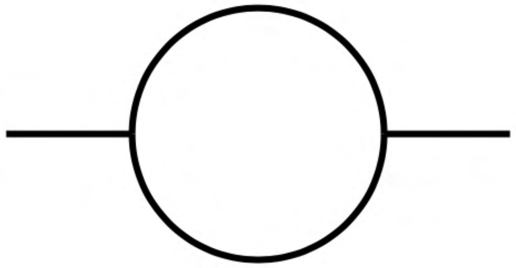
$= \frac{3-d}{s}$



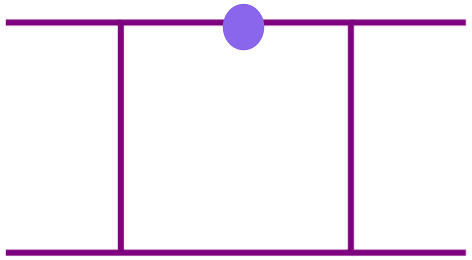
$= \frac{3-d}{s} \quad I(1,1)$



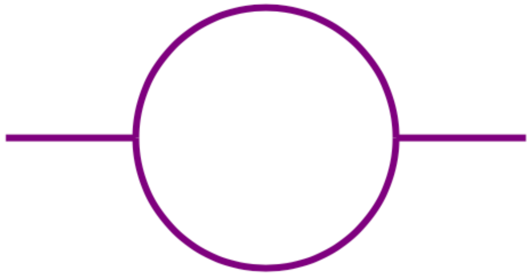
$= -\frac{2}{s}\frac{(d-3)}{(d-4)}$



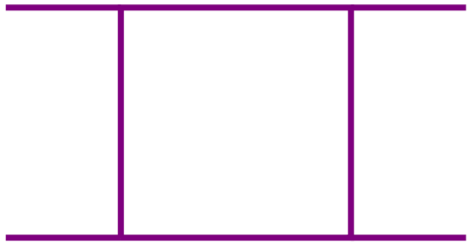
master integrals



$= \frac{1}{st^2}\frac{(d-5)(d-3)}{d-6}$



$+ \frac{1}{s}(d-5)$



Memory and time expensive

For our problem: 500 GB RAM - 64 cores - 6 weeks (roughly)

INTEGRATION-BY-PARTS IDENTITIES

Scalar Feynman integrals are not linearly independent [Chetyrkin, Tkachov]

$$\int \prod_{\alpha=1}^{l} \frac{d^d k_\alpha}{(2\pi)^d} \frac{\partial}{\partial k_{j,\mu}} \left(v^\mu I(a_1, a_2, \cdots, a_n) \right) = 0 \qquad I(a_1, a_2, \cdots, a_n) = \frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_n^{a_n}}$$

Millions of integrals → Handful number of integrals: master integrals

Amplitude is expressed in terms of only a few master integrals

Laporta algorithm

[Laporta]

`FIRE` [Smirnov, Smirnov, Chukharev]

`Kira` [Klappert, Lange, Maierhöfer, Usovitsch]

`Reduze` [Manteuffel, Studerus]

`C++, Fully analytical`

`LiteRed` `Mathematica, Fully analytical` [Lee]

FiniteField Technique

[Manteuffel, Schabinger] [Peraro]

`FiniteFlow` [Peraro]

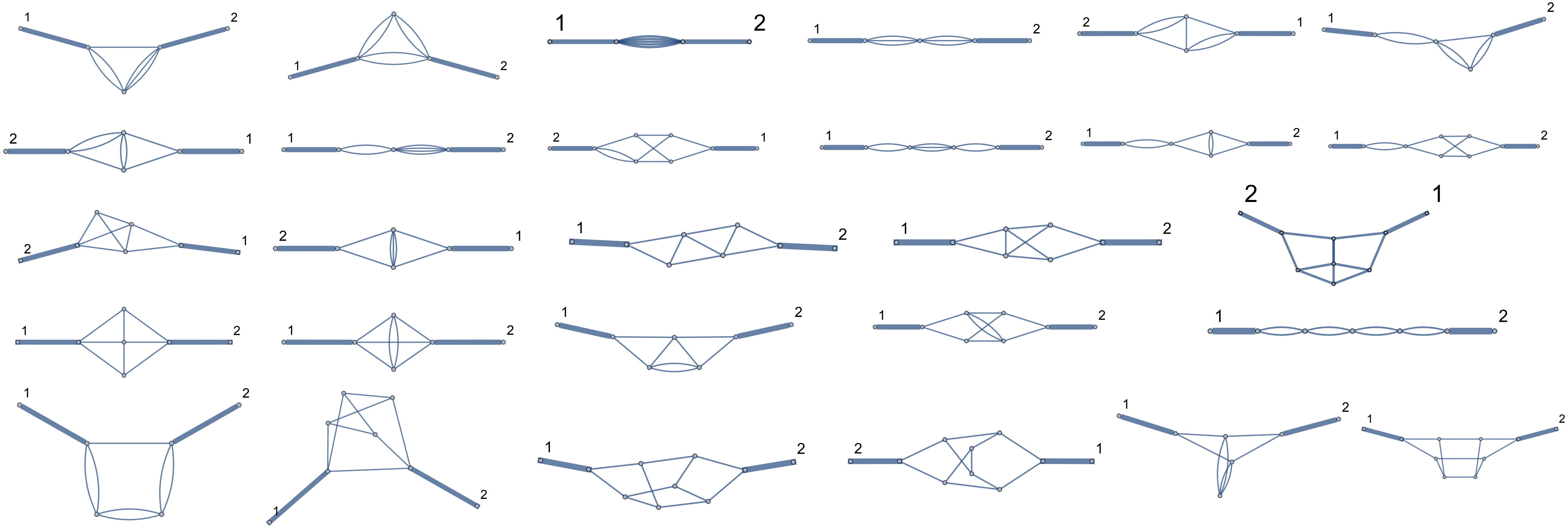
`FireFly+Kira2` [Klappert, Lange, Maierhöfer, Usovitsch]

`Fire6` [Smirnov, Smirnov, Chukharev]

`C++, Numerical -> Analytical`

MASTER INTEGRALS

~1.6 million scalar integrals are expressed in terms of 28 Master integrals



[Baikov, Chetyrkin]

[Lee, Smirnov^2]

[Smirnov, Tentyukov]

- Simplification algebraically: `FORM`, `Mathematica` and `fermat` [Lewis]

- Matrix elements: Just a few lines

RENORMALISATION & OPERATOR MIXING

[Adler '69]
[Espriu, Tarrach '82]
[Breitenlohner, Maison, Stelle '84]

Renormalisation of axial-anomaly operator is not strictly multiplicative

$$[F\tilde{F}]_R = Z_{F\tilde{F}}\,[F\tilde{F}]_B + Z_{FJ}\,[\partial_\mu J_5^\mu]_B$$

$$[\partial_\mu J_5^\mu]_R = Z_J[\partial_\mu J_5^\mu]_B$$

$$\Rightarrow \quad \begin{pmatrix} [\partial_\mu J_5^\mu]_R \\ [F\tilde{F}]_R \end{pmatrix} = \begin{pmatrix} Z_J & 0 \\ Z_{FJ} & Z_{F\tilde{F}} \end{pmatrix} \cdot \begin{pmatrix} [\partial_\mu J_5^\mu]_B \\ [F\tilde{F}]_B \end{pmatrix}$$

Renormalised

Bare

operator ren

Wave-function renormalisation of gauge fields

Gauge fixing parameter

Expressions from Feynman diagram

$$\begin{aligned} \mathcal{M}_{lhs} &= Z_J\, Z_3 \boxed{\hat{\mathcal{M}}_{lhs}\left(\hat{a}_s\,,\,\hat{\xi}\right)} \\ &= Z_5^f \boxed{Z_5^{ms}}\, Z_3\, \hat{\mathcal{M}}_{lhs}\left(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\,\xi\right) \\ &\equiv \boxed{Z_5^f}\, \bar{\mathcal{M}}_{lhs}\,, \end{aligned}$$

strong coupling ren

Goal

Fix by looking at UV poles of matrix element

$$\begin{aligned} \mathcal{M}_{rhs} &= Z_{F\tilde{F}}\, Z_3\, \hat{\mathcal{M}}_{rhs}\left(\hat{a}_s\,,\,\hat{\xi}\right) + Z_{FJ}\, Z_3\, \hat{\mathcal{M}}_{lhs}\left(\hat{a}_s\,,\,\hat{\xi}\right) \\ &= Z_{F\tilde{F}}\, Z_3\, \hat{\mathcal{M}}_{rhs}\left(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\,\xi\right) + Z_{FJ}\, Z_3\, \hat{\mathcal{M}}_{lhs}\left(Z_{a_s}\, a_s\,,\, 1 - Z_3 + Z_3\,\xi\right) \end{aligned}$$

operator ren

RESULTS: 4-LOOP MATRIX ELEMENTS

$$Z_5^f \bar{\mathcal{M}}_{lhs} = Z_5^f \left(a_s \bar{\mathcal{M}}_{lhs}^{(1)} + a_s^2 \bar{\mathcal{M}}_{lhs}^{(2)} + a_s^3 \bar{\mathcal{M}}_{lhs}^{(3)} + a_s^4 \bar{\mathcal{M}}_{lhs}^{(4)} \right)$$

$$a_s n_f T_F Z_5^f \mathcal{M}_{rhs} = \frac{1}{2} a_s n_f \left(\mathcal{M}_{rhs}^{(0)} + a_s \mathcal{M}_{rhs}^{(1)} + a_s^2 \mathcal{M}_{rhs}^{(2)} + a_s^3 \mathcal{M}_{rhs}^{(3)} + a_s^4 \mathcal{M}_{rhs}^{(4)} \right)$$

$$\begin{aligned}
& C_A^3 n_f \left(\frac{2822}{9} \zeta_3 - 500 \zeta_5 + \frac{14896805}{1944} + \frac{\pi^4}{30} \right) + C_A^2 C_F n_f \left(\frac{57125}{27} - 160 \zeta_3 \right) \\
& + C_A^2 n_f^2 \left(- \frac{12752}{9} \zeta_3 + \frac{1600}{3} \zeta_5 - \frac{1063039}{486} - \frac{\pi^4}{5} \right) + C_A C_F^2 n_f \left(640 \zeta_3 - \frac{16952}{27} \right) \\
& + C_A C_F n_f^2 \left(\frac{4864}{9} \zeta_3 + 320 \zeta_5 - \frac{494545}{162} + \frac{4\pi^4}{15} \right) + C_A n_f^3 \left(\frac{368}{3} \zeta_3 + \frac{31021}{243} \right) \\
& + C_F^3 n_f \left(\frac{136}{3} - 384 \zeta_3 \right) + C_F^2 n_f^2 \left(\frac{2048 \zeta_3}{3} - 640 \zeta_5 - \frac{3832}{27} \right) + C_F n_f^3 \left(\frac{19124}{81} - \frac{1024}{9} \zeta_3 \right)
\end{aligned}$$

$\bar{\mathcal{M}}_{lhs}^{(4)}$ No quartic Casimirs since quadratic Casimirs start from 2-loop

$\mathcal{M}_{rhs}^{(4)}$ Presence of quartic Casimirs

New result by us
2101.09479: with Chen, Czakon

RESULTS: FINITE REN CONSTANT

$Z_5^f \, \bar{\mathcal{M}}_{lhs} = a_s \, n_f \, \mathrm{T}_F \, \mathcal{M}_{rhs}$:

- $\mathcal{O}(a_s^2)$ Larin in 1993

$$Z_5^f = 1 + a_s\Big\{-4C_F\Big\} + a_s^2\Big\{C_AC_F\Big(-\frac{107}{9}\Big) + C_F^2\Big(22\Big) + C_Fn_f\Big(\frac{31}{18}\Big)\Big\}$$

$$+a_s^3\Big\{C_A^2C_F\Big(56\zeta_3 - \frac{2147}{27}\Big) + C_AC_F^2\Big(\frac{5834}{27} - 160\zeta_3\Big) + C_AC_Fn_f\Big(\frac{110}{3}\zeta_3 - \frac{133}{81}\Big)$$

$$+C_F^3\Big(96\zeta_3 - \frac{370}{3}\Big) + C_F^2n_f\Big(\frac{497}{54} - \frac{104}{3}\zeta_3\Big) + C_Fn_f^2\Big(\frac{316}{81}\Big)\Big\}$$

New result by us

Difference with non-singlet $\propto C_F n_f$

[Moch, Vermaseren, Vogt '15]

- QCD $\longrightarrow$ QED

$$a_s \to \frac{\alpha}{4\pi}, n_f T_F \to n_f, C_A \to 0, C_F \to 1$$

[Anselm, Johansen '89]

Non-zero contributions only from light-by-light scattering containing famous triangle diagram

Contributions to the decay width of $\pi^0 \to \gamma \, \gamma$

[Adler '69]

- In $\overline{\mathrm{MS}}$, UV ren constants do not depend on masses

applicable to massive scenario

RESULTS: $Z_{F\tilde{F}}$

Determined $Z_{F\tilde{F}}$ to $\mathcal{O}(a_s^4)$ → $Z_{F\tilde{F}} = Z_{a_s}$

Should hold true to all orders to ensure the Adler-Bardeen theorem to non-abelian

[Breitenlohner, Maison, Stelle '84]

[Lüscher, Weisz '21]

Verified to

$\mathcal{O}(a_s)$ [Espriu, Tarrach '82]

$\mathcal{O}(a_s^2)$ [Larin '93]

- Lu¨scher and P. Weisz

[Bos '93]

$\mathcal{O}(a_s^3)$ [Zoller 2013] (modulo other ref.)

[TA, Gehrmann, Mathews, Rana, Ravindran 2015]

$\mathcal{O}(a_s^4)$ [TA, Chen, Czakon 2021]

Consequence of this equality to low-energy region [Chetyrkin, Kniehl, Steinhauser, Bardeen '98]

OPERATOR MIXING AND RG EQUATION

Renormalisation of axial-anomaly operator is not strictly multiplicative [Adler '69]

$$[F\tilde{F}]_R = Z_{F\tilde{F}}\,[F\tilde{F}]_B + Z_{FJ}\,[\partial_\mu J_5^\mu]_B$$

$$[\partial_\mu J_5^\mu]_R = Z_J[\partial_\mu J_5^\mu]_B$$

Renormalised

Bare

$$\begin{pmatrix}[\partial_\mu J_5^\mu]_R \\ [F\tilde{F}]_R\end{pmatrix} = \begin{pmatrix} Z_J & 0 \\ Z_{FJ} & Z_{F\tilde{F}}\end{pmatrix} \cdot \begin{pmatrix}[\partial_\mu J_5^\mu]_B \\ [F\tilde{F}]_B\end{pmatrix}$$

$$\frac{d}{d\ln\mu^2}$$

$$\begin{pmatrix}\gamma_J & 0 \\ \gamma_{FJ} & \gamma_{F\tilde{F}}\end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}\ \ln Z_J}{\mathrm{d}\ \ln\mu^2} & 0 \\ \frac{1}{Z_J}\frac{\mathrm{d}\, Z_{FJ}}{\mathrm{d}\ \ln\mu^2} - \frac{Z_{FJ}}{Z_J}\frac{\mathrm{d}\ \ln Z_{F\tilde{F}}}{\mathrm{d}\ \ln\mu^2} & \frac{\mathrm{d}\ \ln Z_{F\tilde{F}}}{\mathrm{d}\ \ln\mu^2}\end{pmatrix}$$

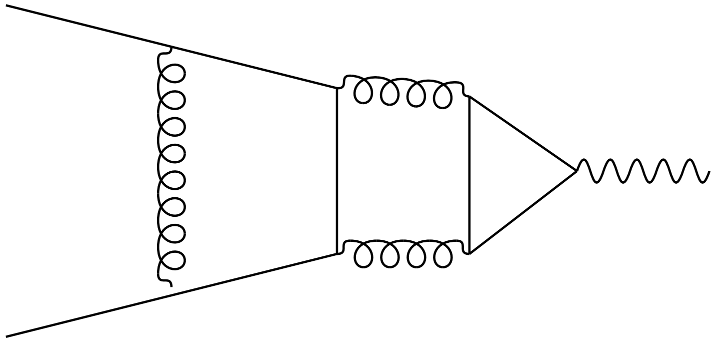
Anomalous dimension matrix

$\gamma_J = n_f T_F a_s \gamma_{FJ}$ verified to $\mathcal{O}(a_s^4)$ New result by us

- Checks: two independent calculations, gauge invariant ren constant, log(p^2/mu^2) indp

TAKE HOME MESSAGES AND OUTLOOK

- Renormalisation for flavour-singlet axial vector current
  - Issue of chiral handling chiral quantity under dimensional regularisation
  - ’t Hooft-Veltman and Breitenlohner-Maison prescription
  - Breakdown of axial Ward identity or ABJ anomalous WI
  - Restoration requires a finite renormalisation
  - We have shown how to calculate at 3rd order in QCD through 4-loop calculation
- Chiral anomaly in Abelian and non-Abelian gauge theory
- 1st application



Z-boson

Gehrmann, Primo
2102.12880

and we missed it, of course!

Thank you!

Questions/Comments?