INFRARED FACTORISATION AND SUBTRACTION BEYOND NLO

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Outline

- Introduction
- Subtraction
- Factorisation
- Counterterms
- Outlook

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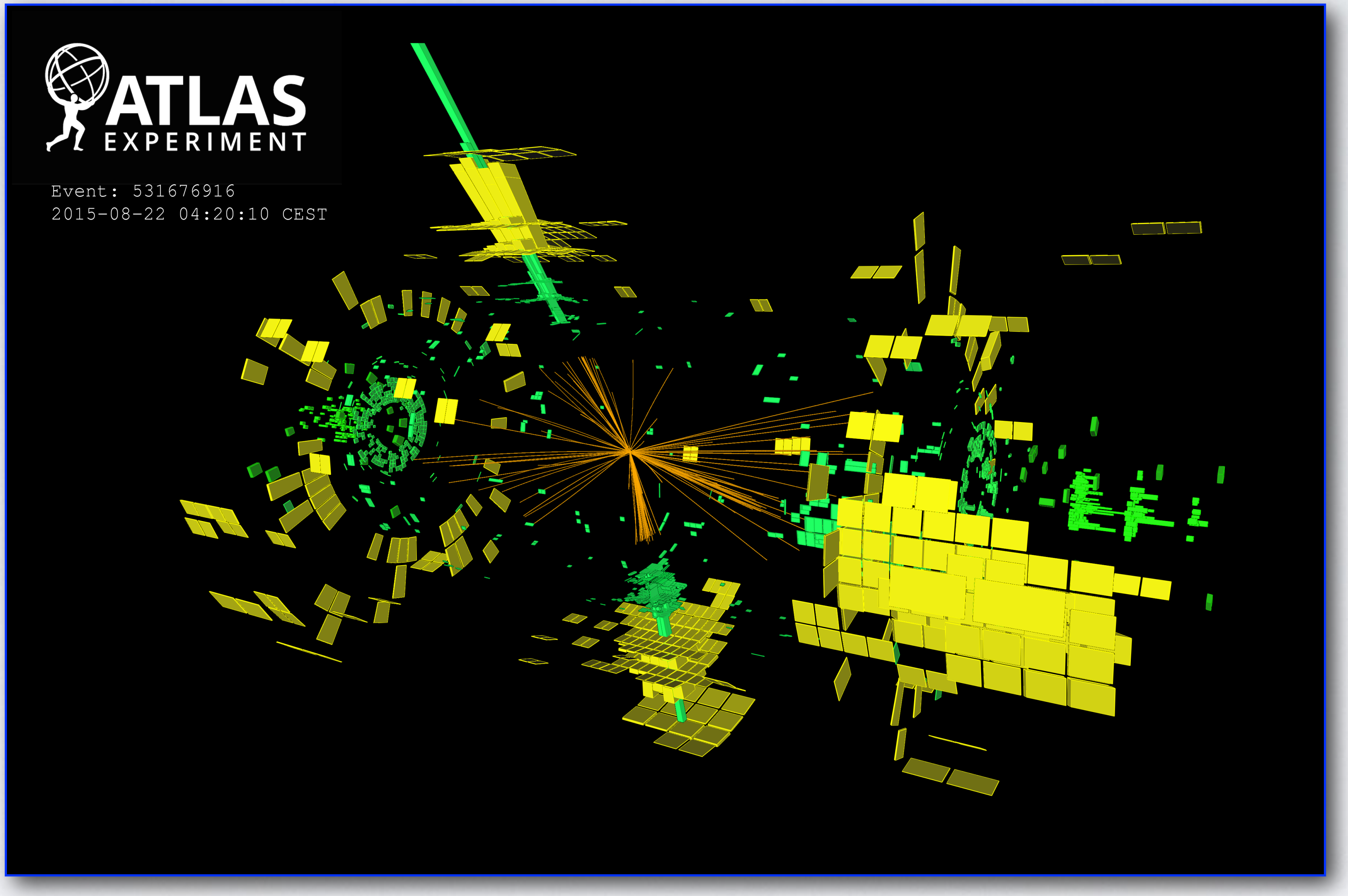
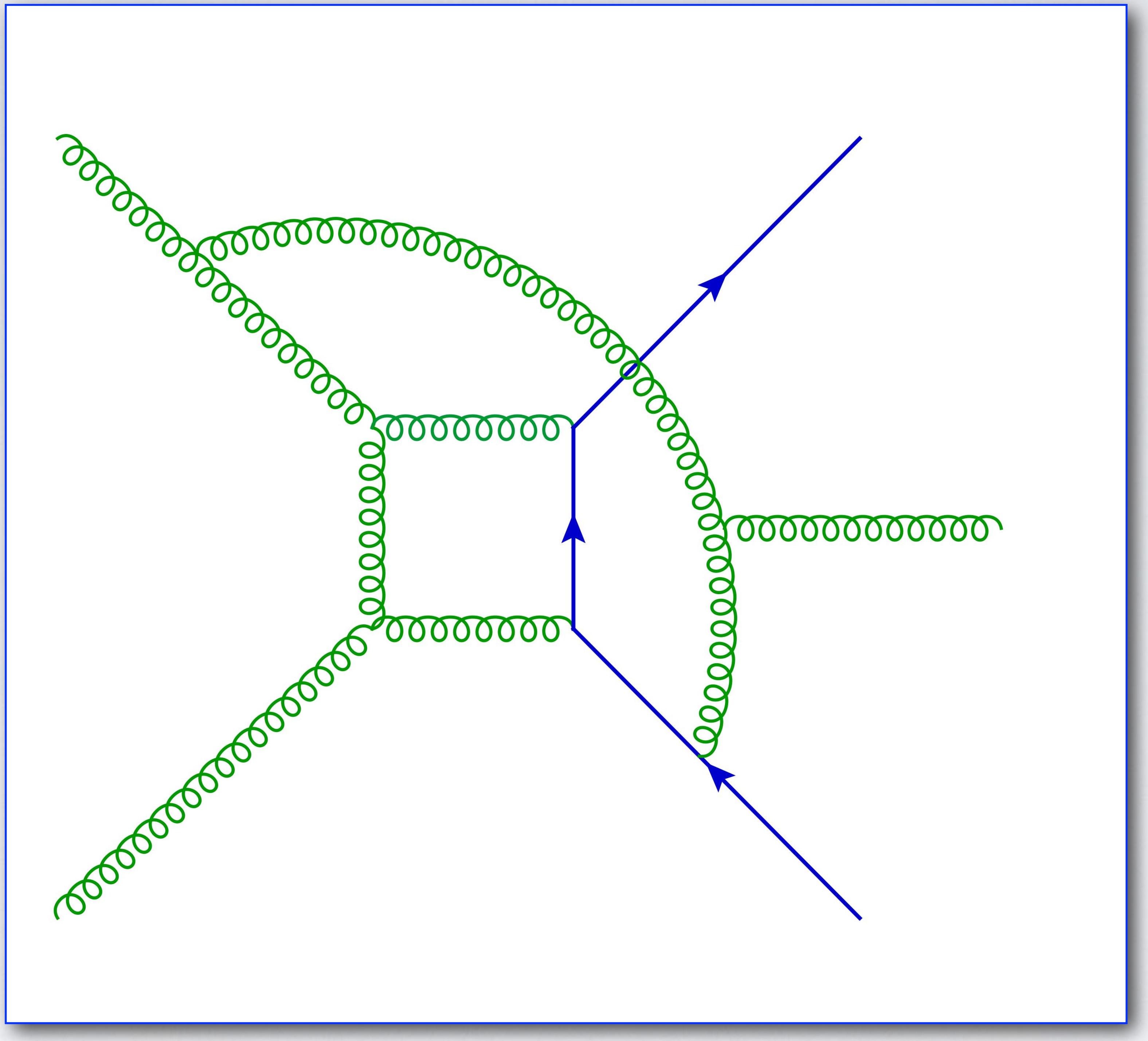
INTRODUCTION

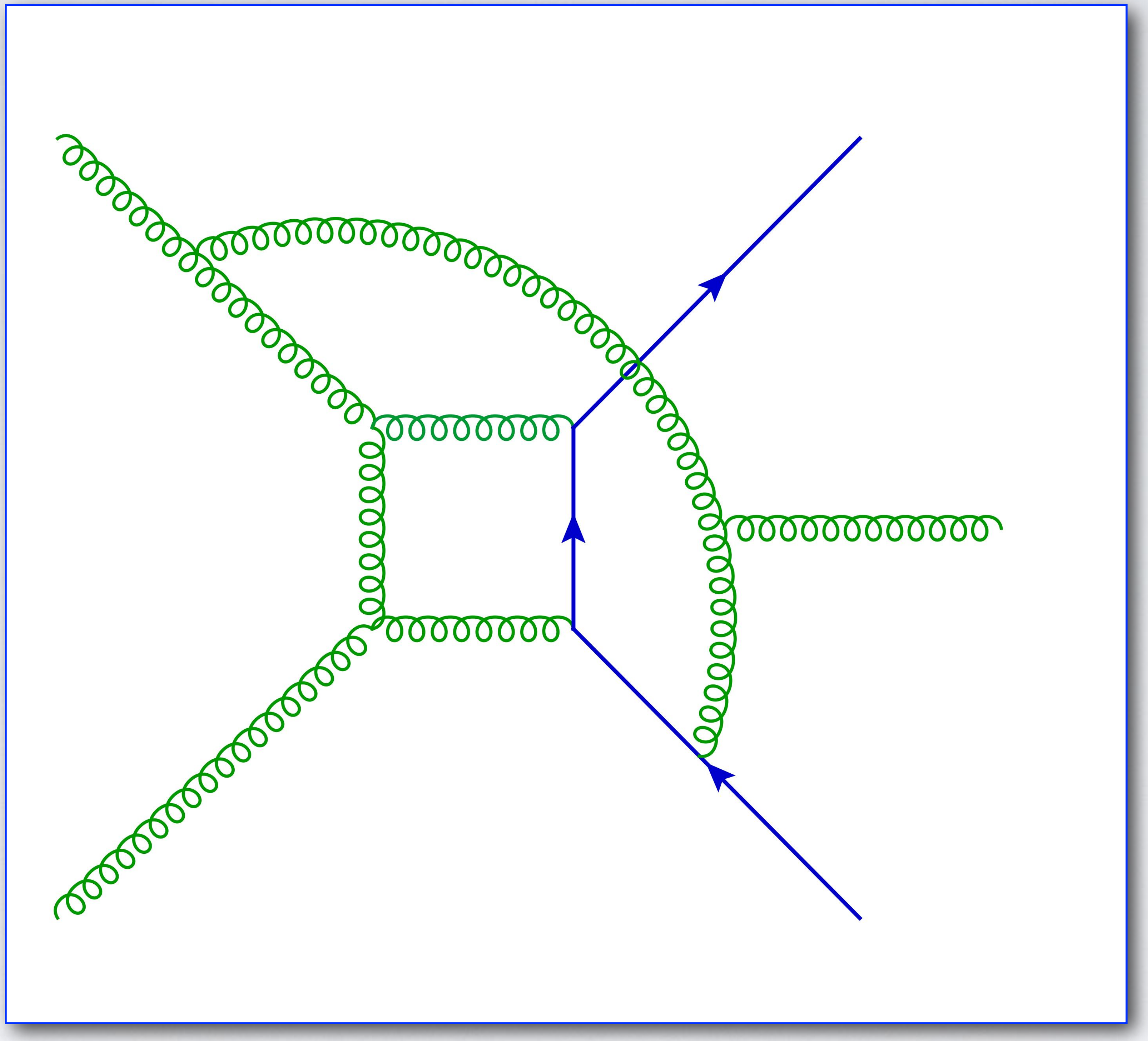


Pictorial infrared



A diagram contributing a double-virtual NNLO correction to t-tbar-jet production

Pictorial infrared



$$\frac{1}{\epsilon^4}$$

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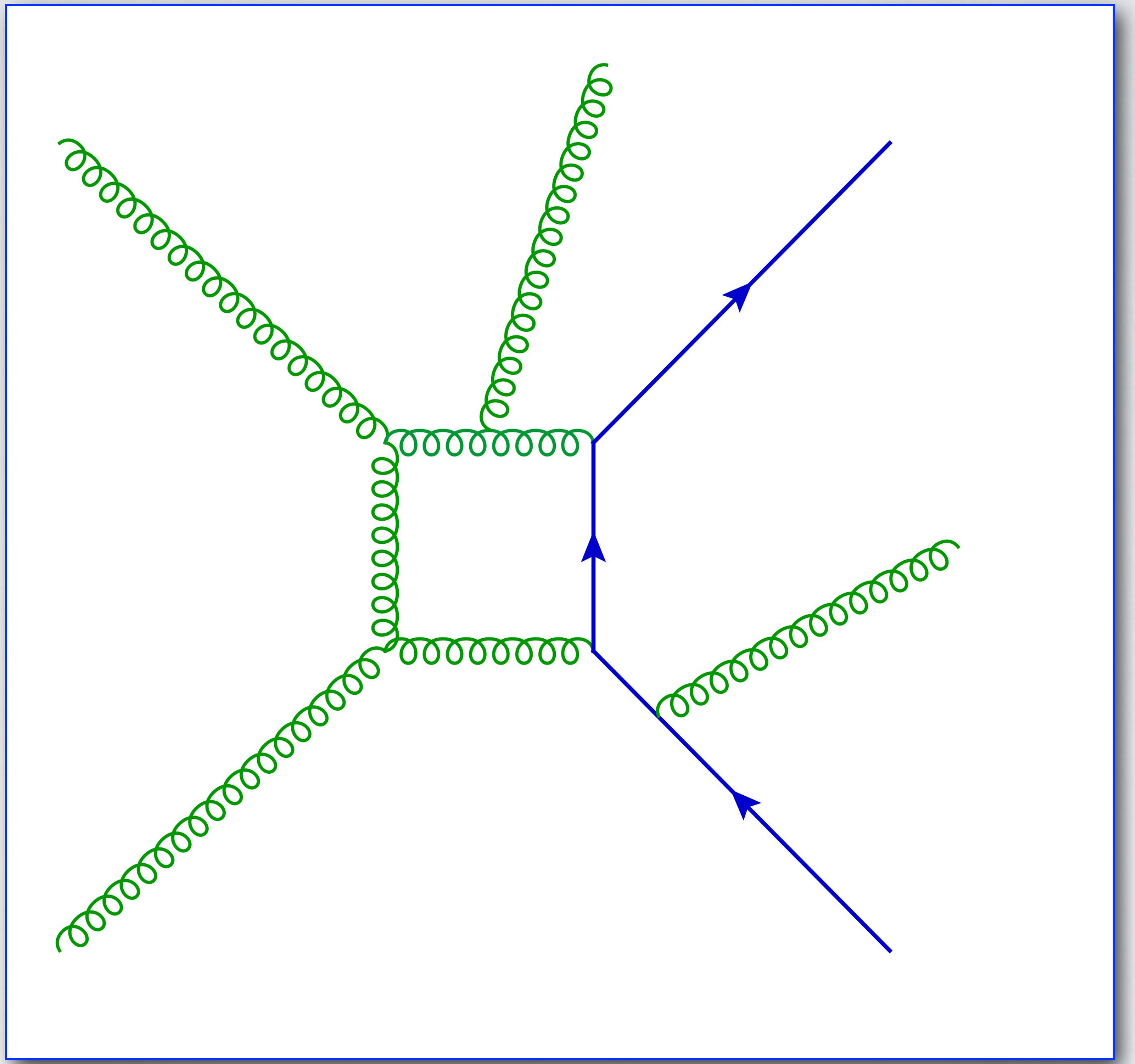

A diagram contributing a real-virtual NNLO correction to t-tbar-jet production

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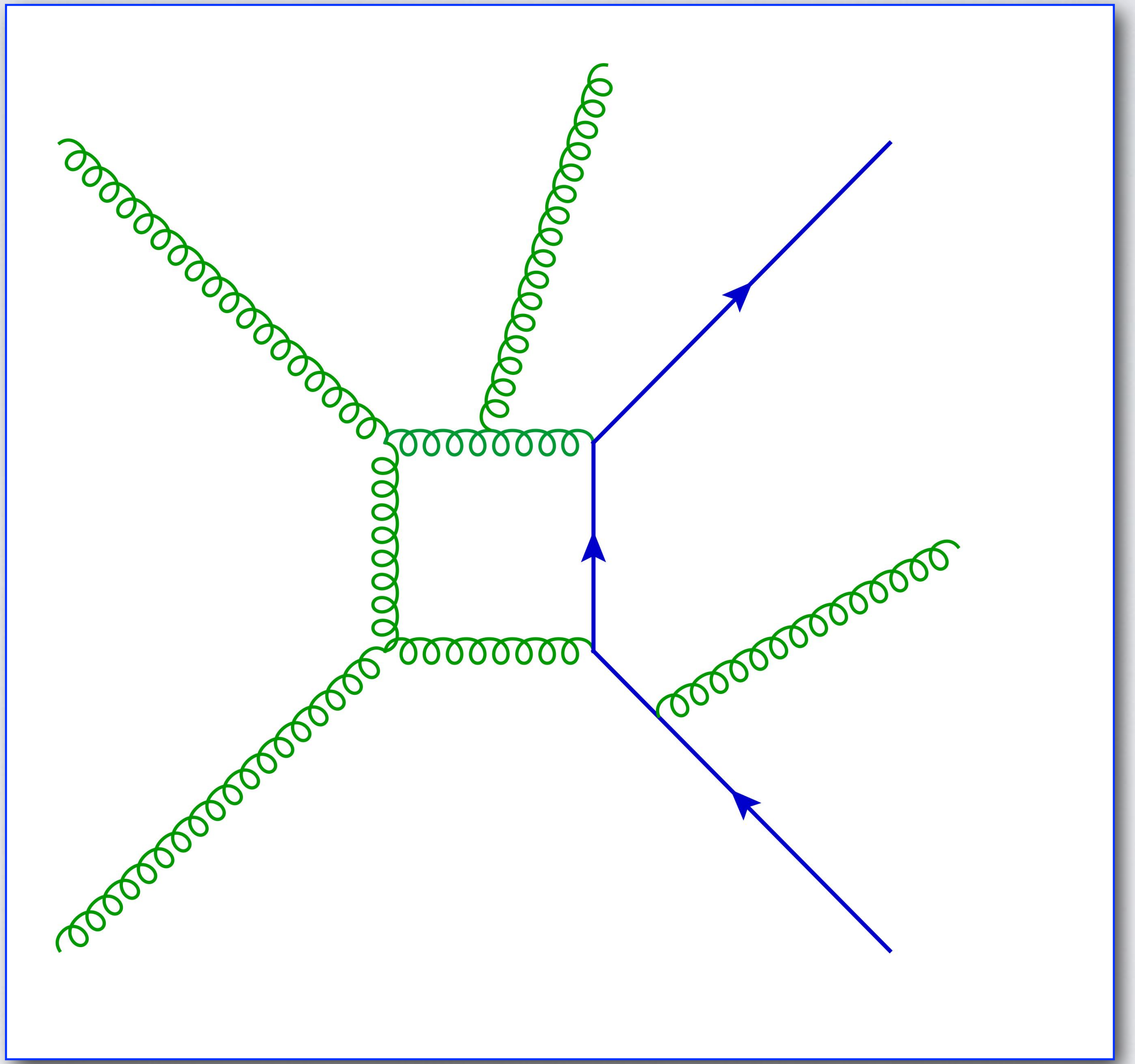

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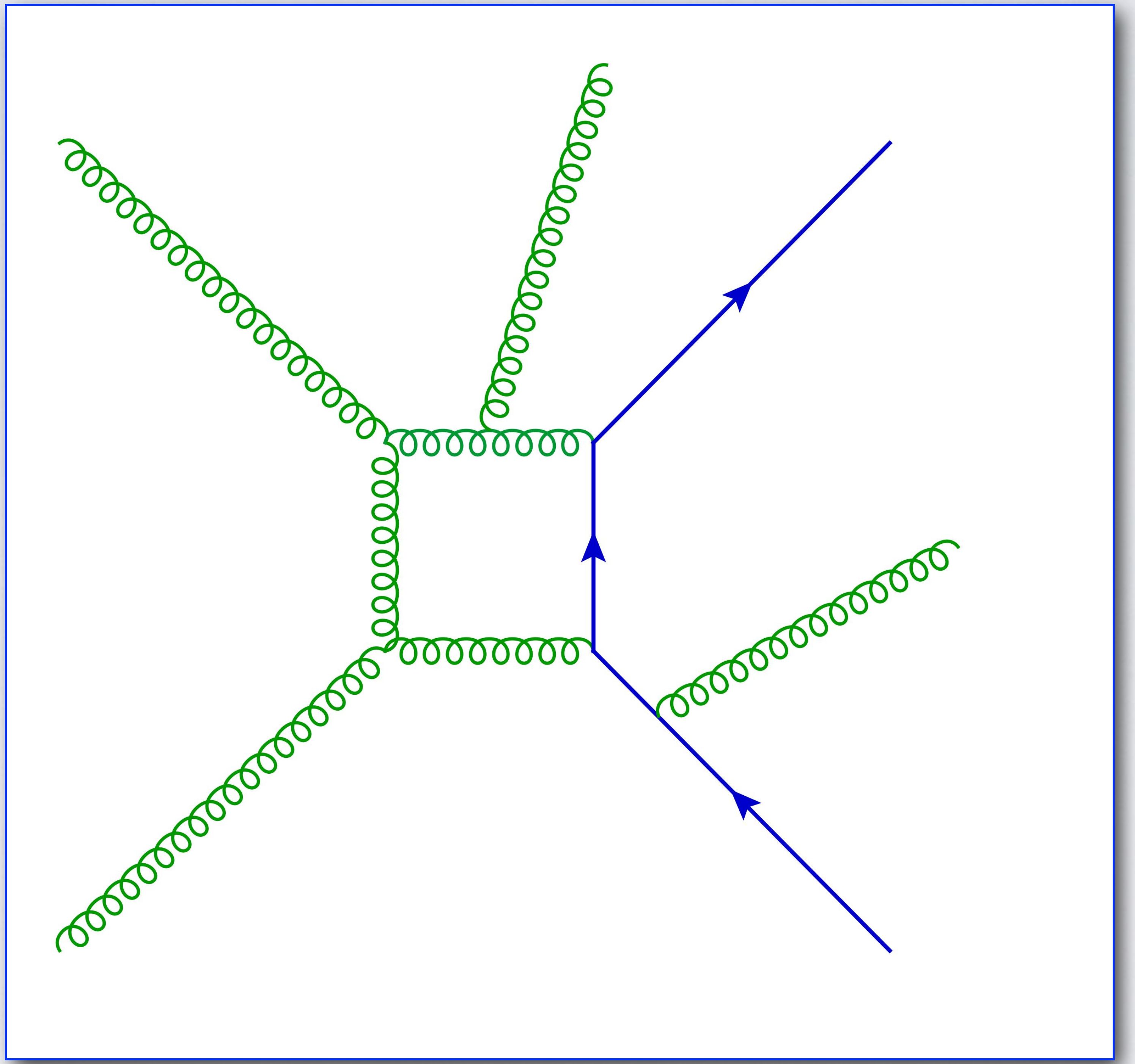

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$$\frac{dE}{E} \frac{dk_\perp}{k_\perp}$$

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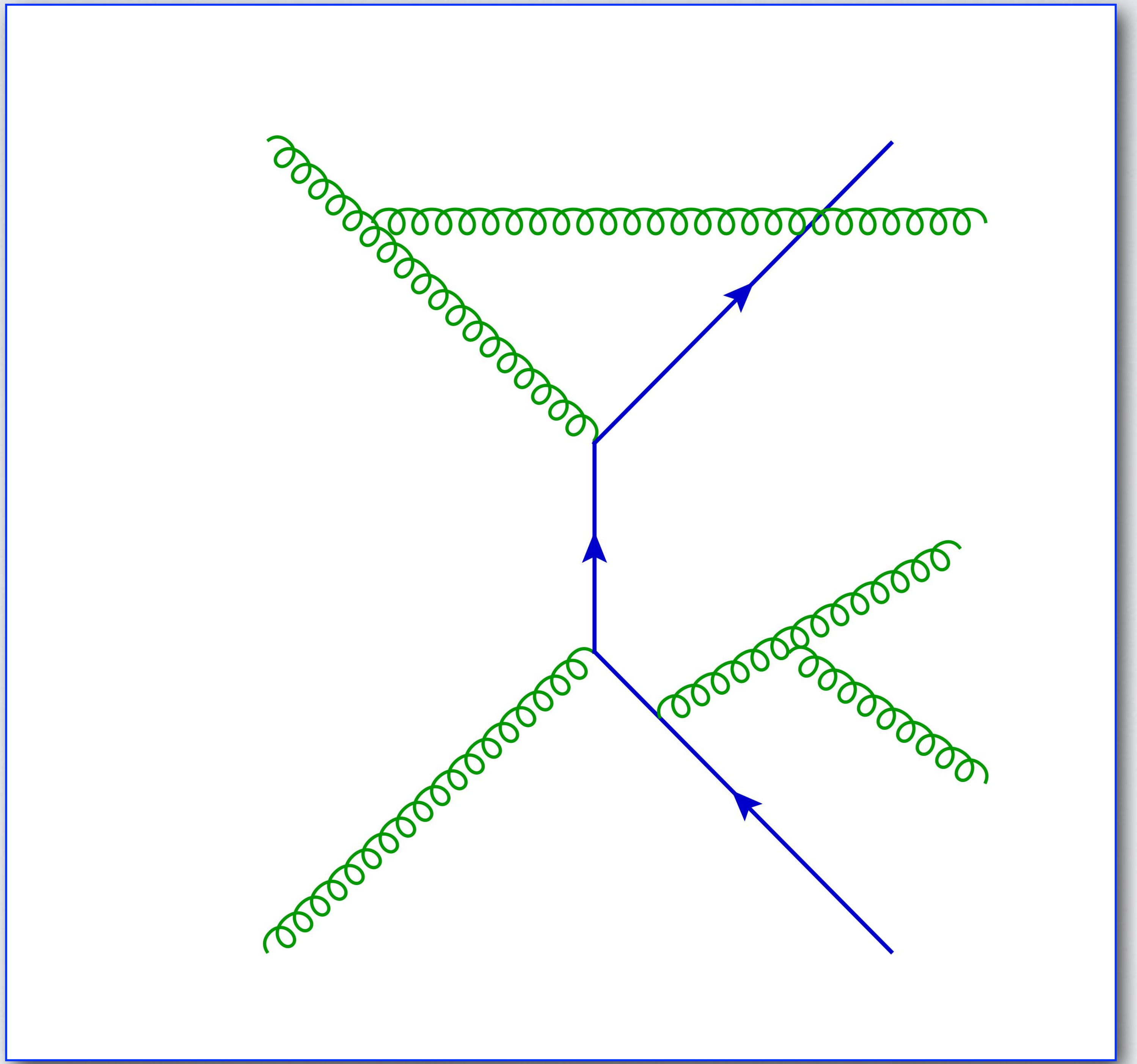
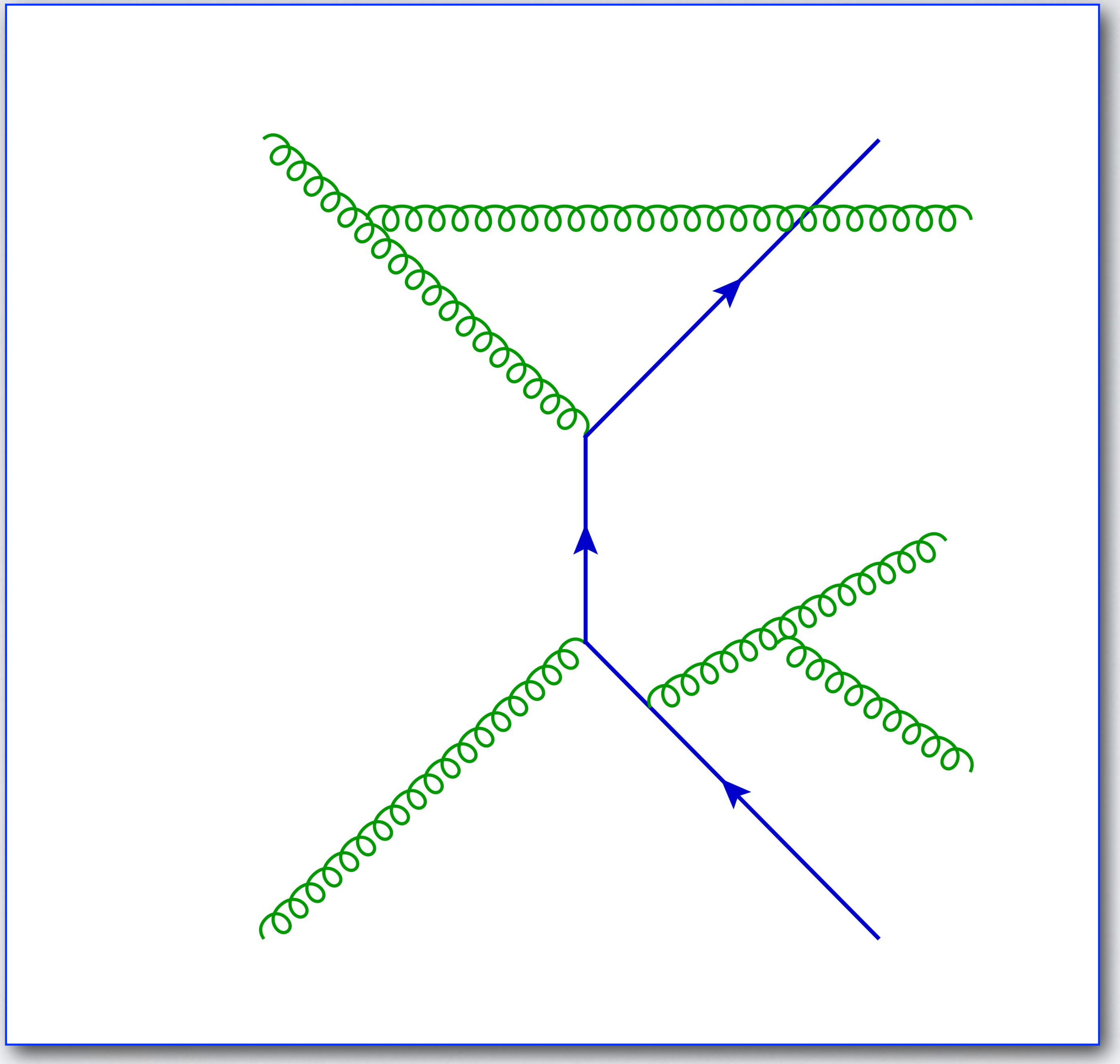


A diagram contributing a double-real NNLO correction to t-tbar-jet production

Pictorial infrared



$$\left(\frac{dE}{E} \frac{dk_\perp}{k_\perp} \right)^2$$

A diagram contributing a double-real NNLO correction to t-tbar-jet production

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- Can one use the structure of virtual singularities as an organising principle for subtraction?
- Can the simplifying features of virtual corrections be exported to real radiation?

A multi-year effort

The subtraction problem at NLO is completely solved, with efficient algorithms applicable to any process for which matrix elements are known.

At NNLO after fifteen years of efforts several groups have working algorithms, successfully applied to `simple' process with up to four legs. Heavy computational costs.

- Antenna Subtraction.
- Stripper
- Nested Soft-Collinear Subtractions.
- ColourfulNNLO.
- N-Jettiness Slicing.
- Q_T Slicing.
- Projection to Born.
- Unsubtraction.
- Geometric Slicing …

GENERAL SUBTRACTION

NLO Subtraction

The computation of a generic IRC-safe observable at NLO requires the combination

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n\, V_n\, \delta_n(X) + \int d\Phi_{n+1}\, R_{n+1}\, \delta_{n+1}(X)\right\},$$

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

$$K^{(\mathbf{1})}_{n+1} = \mathbf{L}^{(\mathbf{1})}\, R_{n+1}\,.$$

$$I^{(\mathbf{1})}_{n} \equiv \int d\Phi^{n+1}_{\mathrm{r},\,1}\, K^{(\mathbf{1})}_{n+1}\,,$$

Add and subtract the same quantity to the observable: each contribution is now finite.

$$\frac{d\sigma_{\mathrm{NLO}}}{dX} = \int d\Phi_n\Big(V_n + I^{(\mathbf{1})}_{n}\Big)\,\delta_n(X) + \int d\Phi_{n+1}\,\Big(R_{n+1}\,\delta_{n+1}(X) - K^{(\mathbf{1})}_{n+1}\,\delta_n(X)\Big)\,,$$

Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

NNLO Subtraction

The pattern of cancellations is more intricate at higher orders

$$\frac{d\sigma_{\rm NNLO}}{dX} = \lim_{d\to 4}\left\{\int d\Phi_n \, VV_n \, \delta_n(X) + \int d\Phi_{n+1} \, RV_{n+1} \, \delta_{n+1}(X) + \int d\Phi_{n+2} \, RR_{n+2} \, \delta_{n+2}(X)\right\},$$

More counterterm functions need to be defined

$$K^{(\mathbf{1})}_{n+2} = \mathbf{L}^{(1)} \, RR_{n+2} \,, \qquad K^{(\mathbf{2})}_{n+2} = \mathbf{L}^{(2)} \, RR_{n+2} \,, \qquad K^{(\mathbf{12})}_{n+2} = \mathbf{L}^{(1)} \, \mathbf{L}^{(2)} \, RR_{n+2} \,, \qquad K^{(\mathbf{RV})}_{n+1} = \mathbf{L}^{(1)} \, RV_{n+1} \,.$$

$$I^{(\mathbf{1})}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},1} \, K^{(\mathbf{1})}_{n+2} \,, \quad I^{(\mathbf{12})}_{n+1} = \int d\Phi^{n+2}_{\mathrm{r},1} \, K^{(\mathbf{12})}_{n+2} \,, \quad I^{(\mathbf{2})}_{n} = \int d\Phi^{n+2}_{\mathrm{r},2} \, K^{(\mathbf{2})}_{n+2} \,, \quad I^{(\mathbf{RV})}_{n} = \int d\Phi^{n+1}_{\mathrm{r},1} \, K^{(\mathbf{RV})}_{n+1} \,.$$

A finite expression for the observable in d=4 must combine several ingredients

$$\begin{aligned}\frac{d\sigma_{\rm NNLO}}{dX} &= \int d\Phi_n \Big[VV_n + I^{(\mathbf{2})}_n + I^{(\mathbf{RV})}_n\Big] \, \delta_n(X) \\ &+ \int d\Phi_{n+1}\Big[\Big(RV_{n+1} + I^{(\mathbf{1})}_{n+1}\Big) \, \delta_{n+1}(X) - \Big(K^{(\mathbf{RV})}_{n+1} + I^{(\mathbf{12})}_{n+1}\Big) \, \delta_n(X)\Big] \\ &+ \int d\Phi_{n+2}\Big[RR_{n+2} \, \delta_{n+2}(X) - K^{(\mathbf{1})}_{n+2} \, \delta_{n+1}(X) - \Big(K^{(\mathbf{2})}_{n+2} - K^{(\mathbf{12})}_{n+2}\Big) \, \delta_n(X)\Big]\end{aligned}$$

N^{3}LO Subtraction

A systematic generalisation to higher orders is possible. At three loops one finds

$$
\begin{aligned}
\frac{d\sigma_{\mathrm{N3LO}}}{dX} = & \int d\Phi_n \left[VVV_n + I_n^{(\mathbf{3})} + I_n^{(\mathbf{RVV})} + I_n^{(\mathbf{RRV},\,\mathbf{2})} \right] \delta_n(X) \\
+ & \int d\Phi_{n+1} \left[\left(RVV_{n+1} + I_{n+1}^{(\mathbf{2})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{1})} \right) \delta_{n+1}(X) \right. \\
& \qquad \left. - \left(K_{n+1}^{(\mathbf{RVV})} + I_{n+1}^{(\mathbf{23})} + I_{n+1}^{(\mathbf{RRV},\,\mathbf{12})} \right) \delta_n(X) \right] \\
+ & \int d\Phi_{n+2} \left\{ \left(RRV_{n+2} + I_{n+2}^{(\mathbf{1})} \right) \delta_{n+2}(X) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{1})} + I_{n+2}^{(\mathbf{12})} \right) \delta_{n+1}(X) \right. \\
& \qquad \left. - \left[\left(K_{n+2}^{(\mathbf{RRV},\,\mathbf{2})} + I_{n+2}^{(\mathbf{13})} \right) - \left(K_{n+2}^{(\mathbf{RRV},\mathbf{12})} + I_{n+2}^{(\mathbf{123})} \right) \right] \delta_n(X) \right\} \\
+ & \int d\Phi_{n+3} \left[RRR_{n+3}\, \delta_{n+3}(X) - K_{n+3}^{(\mathbf{1})}\, \delta_{n+2}(X) - \left(K_{n+3}^{(\mathbf{2})} - K_{n+3}^{(\mathbf{12})} \right) \delta_{n+1}(X) \right. \\
& \qquad \left. - \left(K_{n+3}^{(\mathbf{3})} - K_{n+3}^{(\mathbf{13})} - K_{n+3}^{(\mathbf{23})} + K_{n+3}^{(\mathbf{123})} \right) \delta_n(X) \right],
\end{aligned}
$$

A general formula for N^{k}LO subtraction is available, involving $p = 2^{(k+1)} - 2 - k$ counterterms.

NLO Sectors

Minimize complexity: split phase space in sectors with sector function $\mathcal{W}_{ij}$ in order to have at most one soft (i) and one collinear (ij) singularity in each sector (FKS).

- Sector functions must form a partition of unity.
- In order not to appear in analytic integrations, sector functions must obey sum rules. Denoting with $\mathbf{S}_i$ the soft limit for parton i and $\mathbf{C}_{ij}$ the collinear limit for the ij pair,

$$\mathbf{S}_i \sum_{k \neq i} \mathcal{W}_{ik} = 1\,, \qquad \mathbf{C}_{ij} \sum_{ab \,\in\, \mathrm{perm}(ij)} \mathcal{W}_{ab} = 1\,, \qquad \longleftarrow \qquad \text{sum rules}$$

- Sector functions are defined in terms of Lorentz invariants before choosing an explicit parametrisation of phase space. A possible choice is

$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum\limits_{k,\, l \neq k} \sigma_{kl}}\,, \qquad \text{with} \qquad \sigma_{ij} = \frac{1}{e_i\, w_{ij}}\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad w_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,.$$

- With the help of sector functions, one can now define a candidate counterterm

$$\mathbf{L}^{(1)} R_{n+1} = \sum_i \sum_{j \neq i} \Big(\mathbf{S}_i + \mathbf{C}_{ij} - \mathbf{S}_i \mathbf{C}_{ij} \Big) R_{n+1} \mathcal{W}_{ij}\,.$$

Phase-space mappings at NLO

In order to factorise a Born matrix element B_n with n on-shell particles conserving momentum, we need a mapping from the (n+1)-particle to the Born phase spaces. We use (CS)

$$\bar{k}_i^{(abc)} = k_i\,, \quad \text{if } i \neq a, b, c,$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}}\, k_c\,, \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}}\, k_c\,,$$

We can now redefine soft and collinear limits to include the re-parametrisation. Explicitly

$$\begin{aligned}
\overline{\mathbf{S}}_i\, R(\{k\}) &= -\mathcal{N}_1 \sum_{l,\,m} \delta_{f_i g}\, \frac{s_{lm}}{s_{il}\, s_{im}}\, B_{lm}\Big(\{\bar{k}\}^{(ilm)}\Big)\,, \\
\overline{\mathbf{C}}_{ij}\, R(k) &= \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, B\Big(\{\bar{k}\}^{(ijr)}\Big) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\Big]\,, \\
\overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\, R(\{k\}) &= 2\mathcal{N}_1\, C_{f_j}\, \delta_{f_i g}\, \frac{s_{jr}}{s_{ij}\, s_{ir}}\, B\Big(\{\bar{k}\}^{(ijr)}\Big)\,,
\end{aligned}$$

Note that we have assigned parametrisation triplets differently in different terms. Then

$$\overline{K} = \sum_{i,j\neq i} \overline{K}_{ij}\,, \qquad \overline{K}_{ij} \equiv \left(\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij} - \overline{\mathbf{S}}_i\, \overline{\mathbf{C}}_{ij}\right) R\, \mathcal{W}_{ij}\,,$$

NNLO status

- So far we have applied the formalism to massless final state radiation.
- For this case, at NLO we have a full-fledged subtraction formalism, and simple integrals.
- A simple proof-of-concept case (double-quark-pair production) has been completed.
- A complete set of NNLO sector functions with the desired sum rules is available.
- Flexible phase space mappings for single and double unresolved limits exist.
- Phase space mappings have been checked not to misalign nested limits.
- All integrals for final state radiation are done analytically, with no need for IBP techniques.
- The development of a differential code for NNLO subtraction is under way.
- Generalisation to initial state radiation requires work but no new concepts.
- More `interesting' integrals may arise with massive partons.

Integration at NNLO: an example

$$I^{(\mathbf{2})} \quad = \quad \boxed{I^{(\mathbf{2})}_{\mathrm{ss}}} + I^{(\mathbf{2})}_{\mathrm{hcc}} + I^{(\mathbf{2})}_{\mathrm{cc4}} + I^{(\mathbf{2})}_{\mathrm{sc3}}$$

$$\begin{aligned}
\boxed{I^{(\mathbf{2})}_{\mathrm{ss}}} = \left(\frac{\alpha_{\mathrm{S}}}{4\pi}\right)^2 \left(\frac{\mu^2}{s}\right)^{2\epsilon} \Bigg\{ & \Bigg[2\Big(\sum_{a,b} C_{f_a} C_{f_b}\Big) \boxed{I^{\mathrm{ss}}_{C_f C_f}} + 8\Big(\sum_a C^2_{f_a}\Big) \boxed{I^{\mathrm{ss}}_{C_f^2}} \\
& - \Big(\sum_a C_{f_a}\Big)\Big(N_f\, T_R \boxed{I^{\mathrm{ss}}_{C_f T_R}} - \frac{C_A}{2} \boxed{I^{\mathrm{ss}}_{C_f C_A}}\Big) \Bigg] B(\{\bar{k}\}) \\
& + 2 \sum_{c,d\neq c} \Bigg[- 2\Big(\sum_a C_{f_a}\Big) \boxed{I^{\mathrm{ss}}_{C_f B_{cd}}} + N_f\, T_R \boxed{I^{\mathrm{ss}}_{T_R B_{cd}}} - \frac{C_A}{2} \boxed{I^{\mathrm{ss}}_{C_A B_{cd}}} \Bigg] B_{cd}(\{\bar{k}\}) \\
& + 2 \sum_{c,d\neq c} \boxed{I^{\mathrm{ss}}_{B_{cdcd}}} B_{cdcd}(\{\bar{k}\}) + 4 \sum_{c,d\neq c} \sum_{e\neq d} \boxed{I^{\mathrm{ss}}_{B_{cded}}} B_{cded}(\{\bar{k}\}) \\
& + \sum_{c,d\neq c} \sum_{e,f\neq e} \boxed{I^{\mathrm{ss}}_{B_{cdef}}} B_{cdef}(\{\bar{k}\}) \; + \; \mathcal{O}(\epsilon) \Bigg\}
\end{aligned}$$

$$I^{(2)}_{\mathrm{ss}}$$

$$
\begin{aligned}
I^{\mathrm{ss}}_{C_f C_f} &= \frac{1}{\epsilon^4} + \frac{4}{\epsilon^3} + \left(16 - \frac{7}{6}\pi^2\right)\frac{1}{\epsilon^2} + \left(60 - \frac{14}{3}\pi^2 - \frac{50}{3}\zeta(3)\right)\frac{1}{\epsilon} + 216 - \frac{56}{3}\pi^2 - \frac{200}{3}\zeta(3) + \frac{29}{120}\pi^4 \\
I^{\mathrm{ss}}_{C_f^2} &= \left(1 - \frac{\pi^2}{6}\right)\frac{1}{\epsilon^2} + \left(10 - \frac{2}{3}\pi^2 - 6\,\zeta(3)\right)\frac{1}{\epsilon} + 68 - 4\,\pi^2 - 24\,\zeta(3) - \frac{7}{72}\pi^4 \\
I^{\mathrm{ss}}_{C_f T_R} &= \frac{2}{3}\frac{1}{\epsilon^3} + \frac{34}{9}\frac{1}{\epsilon^2} + \left(\frac{464}{27} - \frac{7}{9}\pi^2\right)\frac{1}{\epsilon} + \frac{5896}{81} - \frac{131}{27}\pi^2 - \frac{76}{9}\zeta(3) \\
I^{\mathrm{ss}}_{C_f C_A} &= \frac{2}{\epsilon^4} + \frac{35}{3}\frac{1}{\epsilon^3} + \left(\frac{487}{9} - \frac{8}{3}\pi^2\right)\frac{1}{\epsilon^2} \\
&\quad + \left(\frac{6248}{27} - \frac{269}{18}\pi^2 - \frac{154}{3}\zeta(3)\right)\frac{1}{\epsilon} + \frac{77404}{81} - \frac{3829}{54}\pi^2 - \frac{2050}{9}\zeta(3) - \frac{23}{60}\pi^4 \\
I^{\mathrm{ss}}_{C_f B_{cd}} &= \ln\frac{\bar{s}_{cd}}{s}\left[-\frac{1}{\epsilon^3} - \frac{4}{\epsilon^2} - \left(20 - \frac{11}{6}\pi^2\right)\frac{1}{\epsilon} - 100 + \frac{22}{3}\pi^2 + \frac{122}{3}\zeta(3)\right. \\
&\quad \left. + \frac{1}{2}\ln\frac{\bar{s}_{cd}}{s}\left(\frac{1}{\epsilon^2} + \frac{4}{\epsilon} + 20 - \frac{11}{6}\pi^2\right) - \frac{1}{6}\ln^2\frac{\bar{s}_{cd}}{s}\left(\frac{1}{\epsilon} + 4\right) + \frac{1}{24}\ln^3\frac{\bar{s}_{cd}}{s}\right] \\
I^{\mathrm{ss}}_{T_R B_{cd}} &= \ln\frac{\bar{s}_{cd}}{s}\left[-\frac{2}{3}\frac{1}{\epsilon^2} - \frac{34}{9}\frac{1}{\epsilon} - \frac{464}{27} + \frac{7}{9}\pi^2 + \ln\frac{\bar{s}_{cd}}{s}\left(\frac{2}{3}\frac{1}{\epsilon} + \frac{34}{9}\right) - \frac{4}{9}\ln^2\frac{\bar{s}_{cd}}{s}\right. \\
I^{\mathrm{ss}}_{C_A B_{cd}} &= \ln\frac{\bar{s}_{cd}}{s}\left[-\frac{2}{\epsilon^3} - \frac{35}{3}\frac{1}{\epsilon^2} - \left(\frac{487}{9} - \frac{8}{3}\pi^2\right)\frac{1}{\epsilon} - \frac{6248}{27} + \frac{269}{18}\pi^2 + \frac{154}{3}\zeta(3)\right. \\
&\quad \left. + \ln\frac{\bar{s}_{cd}}{s}\left(\frac{2}{\epsilon^2} + \frac{35}{3}\frac{1}{\epsilon} + \frac{487}{9} - \frac{8}{3}\pi^2\right) - \frac{2}{3}\ln^2\frac{\bar{s}_{cd}}{s}\left(\frac{2}{\epsilon} + \frac{35}{3}\right) + \frac{2}{3}\ln^3\frac{\bar{s}_{cd}}{s}\right] \\
I^{\mathrm{ss}}_{B_{cdcd}} &= -4\Big(1 - \zeta(3)\Big)\left(\frac{1}{\epsilon} - 2\ln\frac{\bar{s}_{cd}}{s}\right) - 40 - \frac{\pi^2}{3} + 12\,\zeta(3) + \frac{13}{36}\pi^4 \\
I^{\mathrm{ss}}_{B_{cded}} &= \ln\frac{\bar{s}_{cd}}{s}\ln\frac{\bar{s}_{ed}}{s}\left(1 - \frac{\pi^2}{6}\right), \\
I^{\mathrm{ss}}_{B_{cdef}} &= \ln\frac{\bar{s}_{cd}}{s}\ln\frac{\bar{s}_{ef}}{s}\left[\frac{1}{\epsilon^2} + \frac{4}{\epsilon} + 16 - \frac{7}{6}\pi^2 - \frac{1}{2}\left(\ln\frac{\bar{s}_{cd}}{s} + \ln\frac{\bar{s}_{ef}}{s}\right)\left(\frac{1}{\epsilon} + 4\right)\right. \\
&\quad \left. + \frac{1}{6}\left(\ln^2\frac{\bar{s}_{cd}}{s} + \ln^2\frac{\bar{s}_{ef}}{s}\right) + \frac{1}{4}\ln\frac{\bar{s}_{cd}}{s}\ln\frac{\bar{s}_{ef}}{s}\right]
\end{aligned}
$$

$$(\{\bar{k}\})$$

NNLO Tripoles?

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A dipole formula

$$\mathbf{\Gamma}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda), \epsilon\right) = \frac{1}{2}\widehat{\gamma}_K\left(\alpha_s(\lambda,\epsilon)\right)\sum_{i<j=1}^{n}\ln\left(\frac{2p_i\cdot p_j e^{i\pi\sigma_{ij}}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \sum_{i=1}^{n}\gamma_i\left(\alpha_s(\lambda,\epsilon)\right) .$$

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$$\mathcal{M} \sim \left[\exp \int \mathbf{\Gamma}\right] \mathcal{H}$$

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Tripoles for NNLO single poles?

$$\hat{\mathbf{H}}_{\mathrm{f}}^{(2)} = i \sum_{(i,j,k)} f_{a_1 a_2 a_3} \mathbf{T}_i^{a_1} \mathbf{T}_j^{a_2} \mathbf{T}_k^{a_3} \ln\left(\frac{-s_{ij}}{-s_{jk}}\right) \ln\left(\frac{-s_{jk}}{-s_{ki}}\right) \ln\left(\frac{-s_{ki}}{-s_{ij}}\right) ,$$

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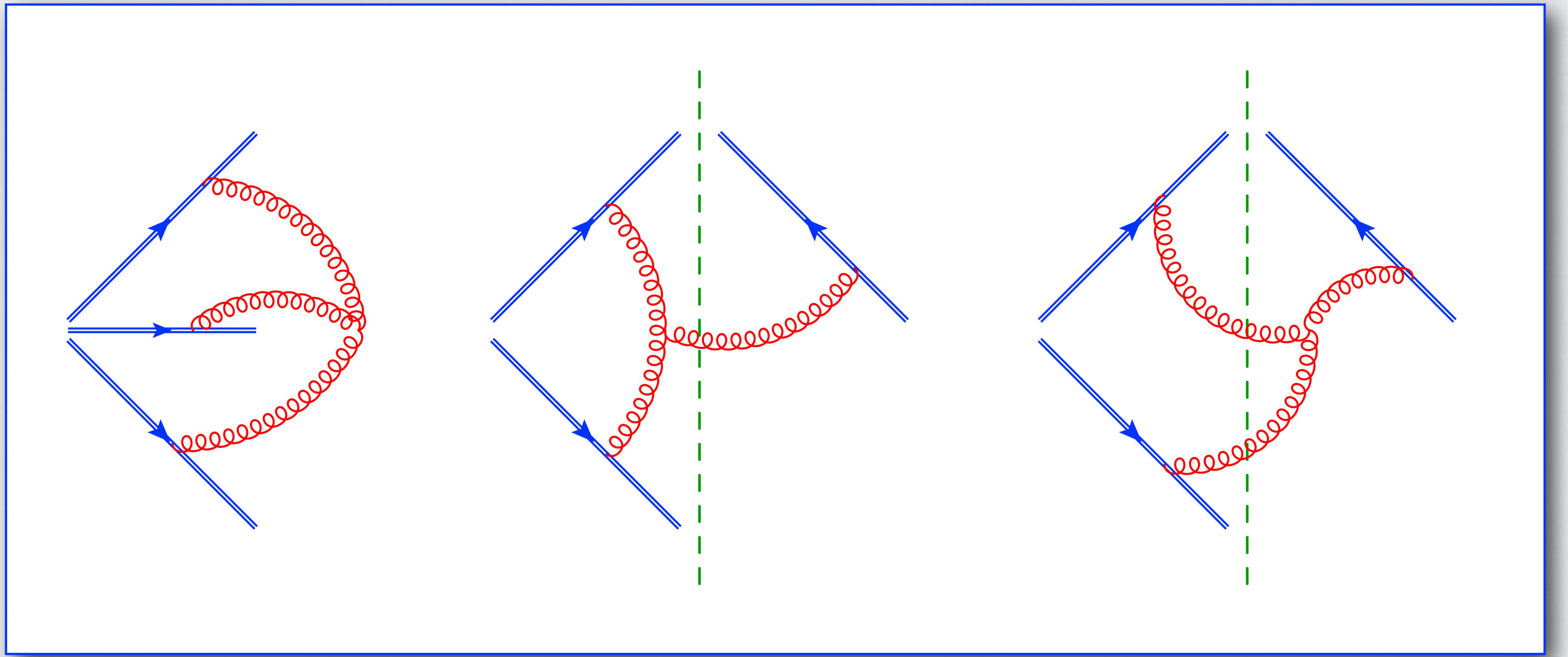

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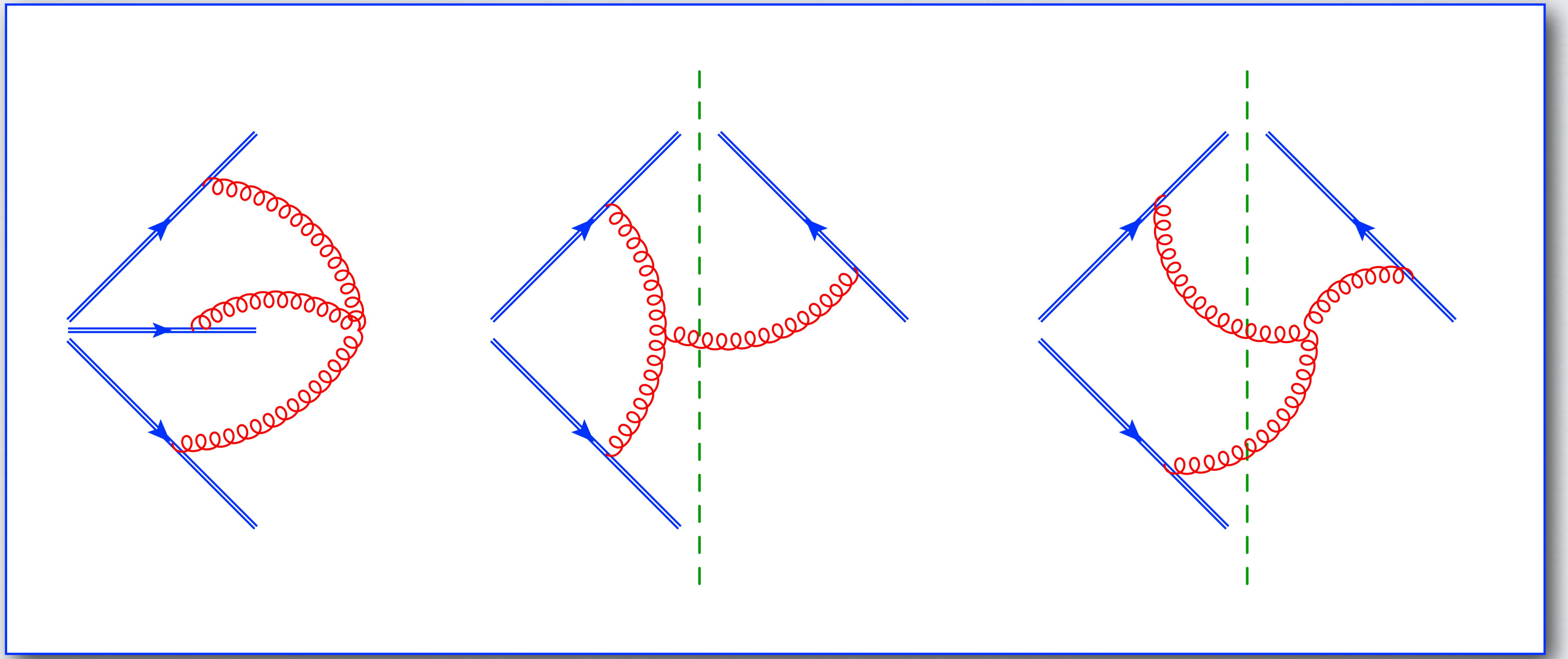
$$\mathbf{\Gamma}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda), \epsilon\right) = \frac{1}{2} \widehat{\gamma}_K \left(\alpha_s(\lambda, \epsilon)\right) \sum_{i<j=1}^{n} \ln\left(\frac{2 p_i \cdot p_j e^{i\pi\sigma_{ij}}}{\lambda^2}\right) \mathbf{T}_i \cdot \mathbf{T}_j - \sum_{i=1}^{n} \gamma_i \left(\alpha_s(\lambda, \epsilon)\right) .$$

Tripoles for NNLO single poles?

$$\hat{\mathbf{H}}_{\mathrm{f}}^{(2)} = i \sum_{(i,j,k)} f_{a_1 a_2 a_3} \mathbf{T}_i^{a_1} \mathbf{T}_j^{a_2} \mathbf{T}_k^{a_3} \ln\left(\frac{-s_{ij}}{-s_{jk}}\right) \ln\left(\frac{-s_{jk}}{-s_{ki}}\right) \ln\left(\frac{-s_{ki}}{-s_{ij}}\right) ,$$



$$= 0$$

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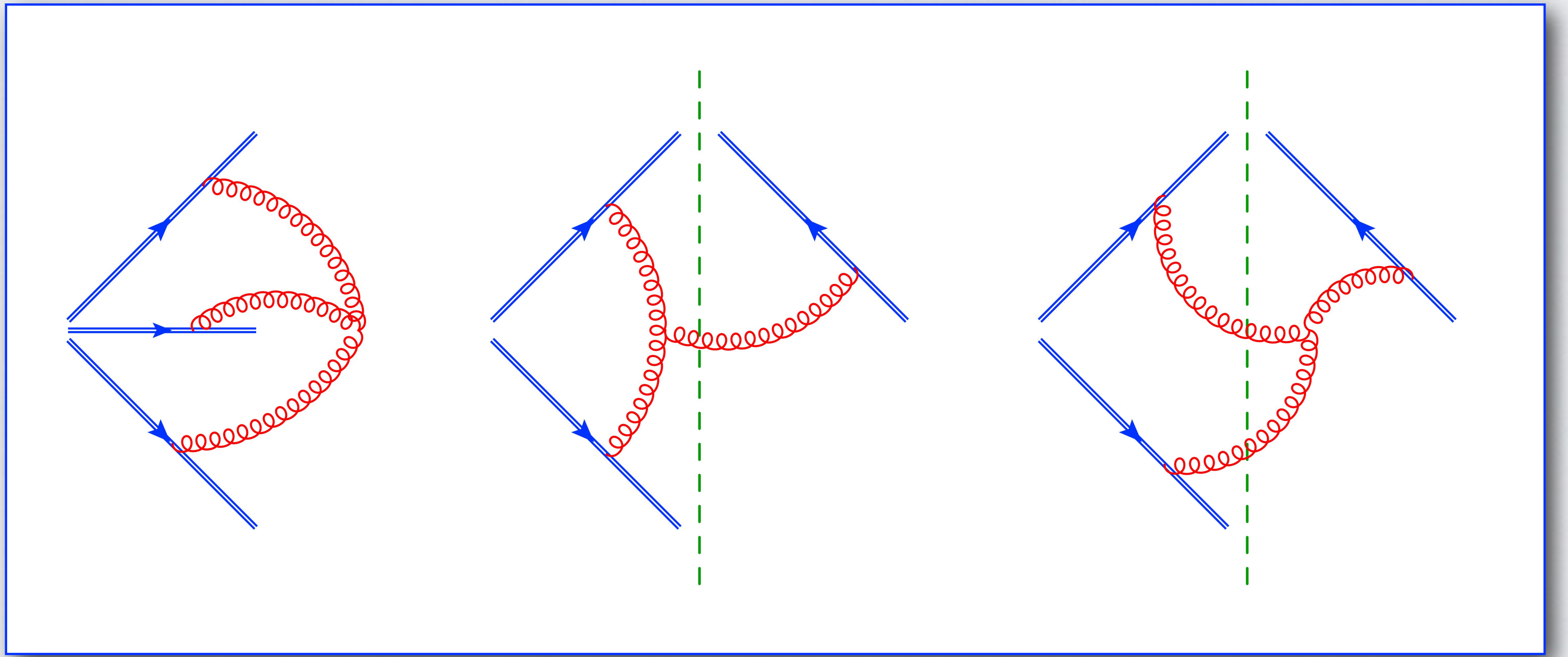

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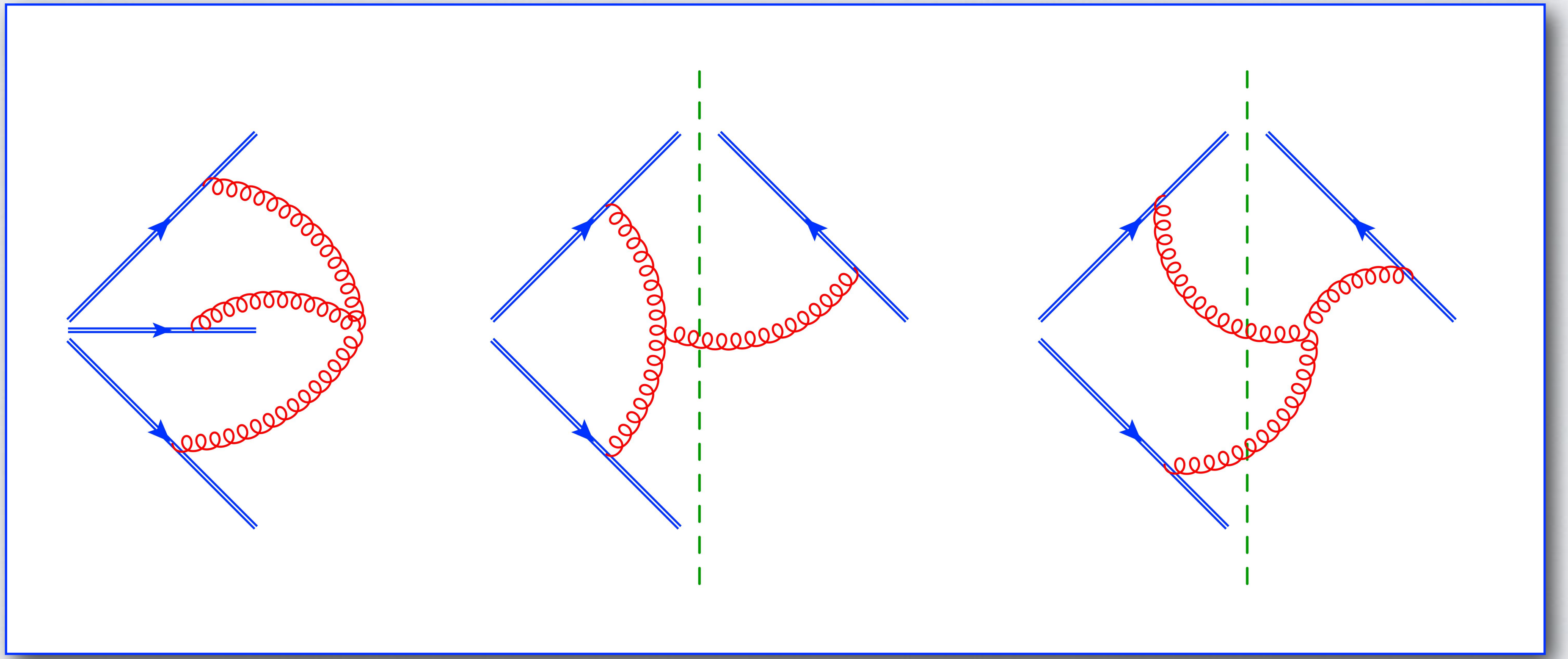
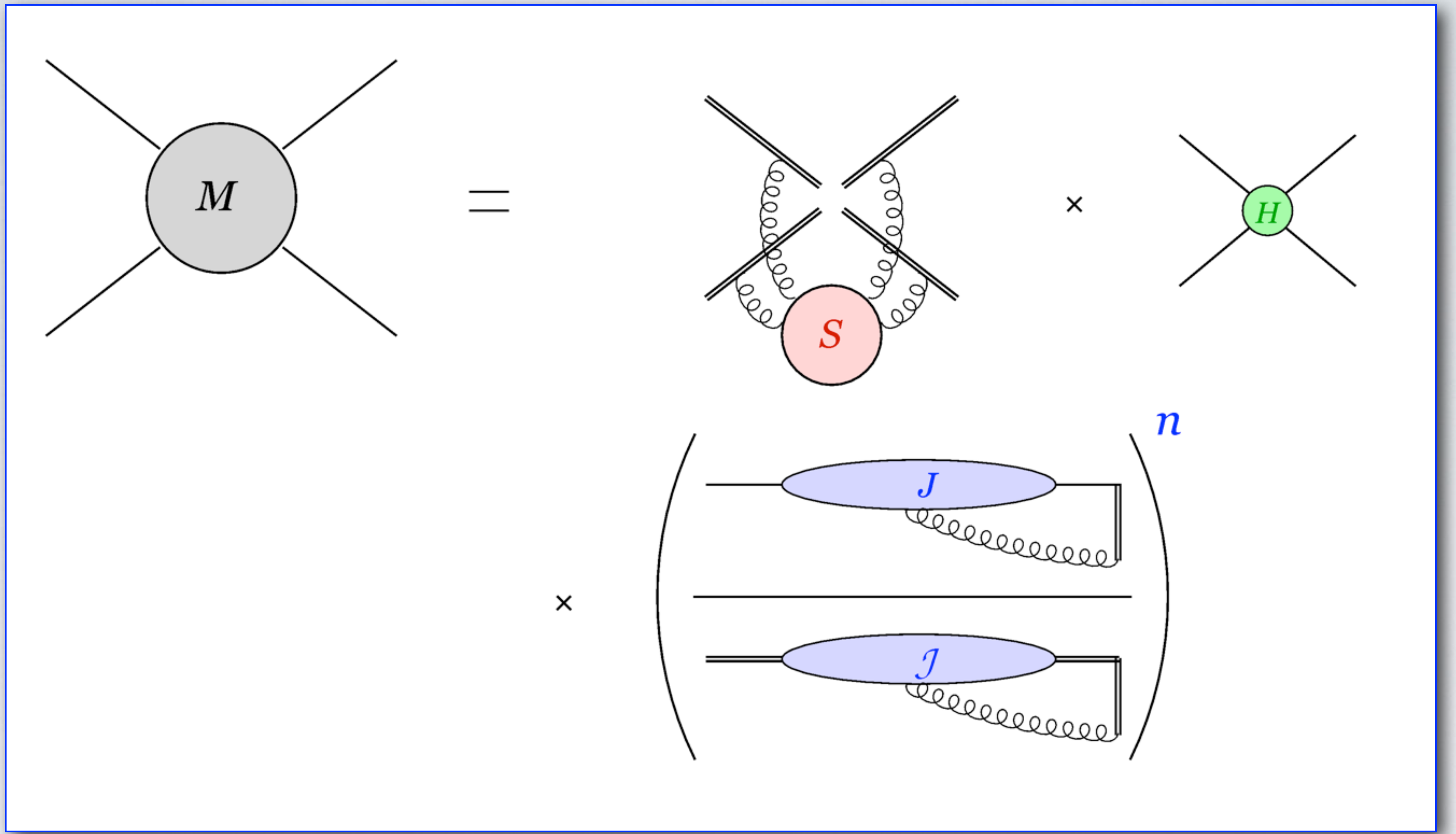


$= 0$ $\quad = 0$ $\quad = 0$

FACTORISATION

FACTORISATION

Virtual factorisation: pictorial



A pictorial representation of soft-collinear factorisation for fixed-angle scattering amplitudes

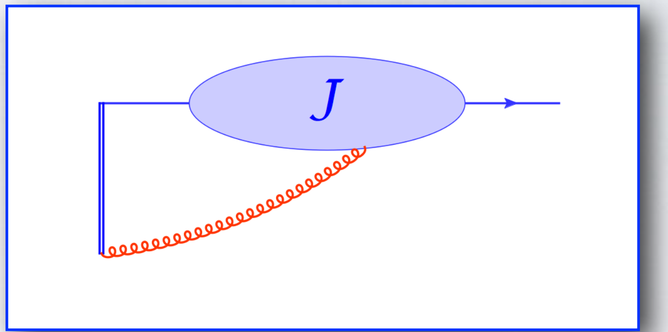
Operator Definitions

The precise functional form of this graphical factorisation is

$$\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i\Big((p_i\cdot n_i)^2/(n_i^2\mu^2)\Big)}{\mathcal{J}_{E,\,i}\Big((\beta_i\cdot n_i)^2/n_i^2\Big)}\right]\mathcal{S}_n\left(\beta_i\cdot\beta_j\right)\;\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right)$$

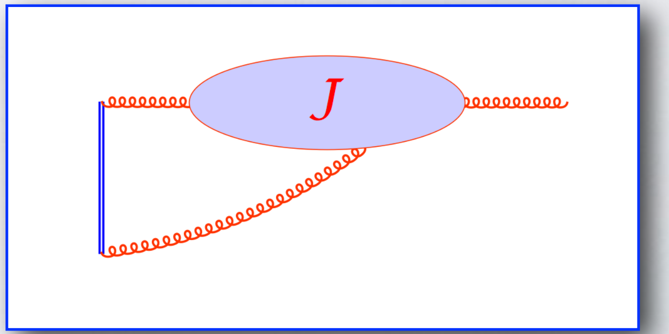
Here we introduced dimensionless four-velocities $\beta_i = p_i/Q$, and factorisation vectors n_i^μ, $n_i^2 \neq 0$ to define the jets in a gauge-invariant way. For outgoing quarks

$$\overline{u}_s(p)\,\mathcal{J}_q\left(\frac{(p\cdot n)^2}{n^2\mu^2}\right) = \langle p,s\,|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle$$



where Φ_n is the Wilson line operator along the direction n. For outgoing gluons

$$g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\mathcal{J}_g^{\mu\nu}\left(\frac{(k\cdot n)^2}{n^2\mu^2}\right) \equiv \langle k,\lambda|\,\Big[\Phi_n(\infty,0)\,iD^\nu\,\Phi_n(0,\infty)\Big]\,|0\rangle\;,$$

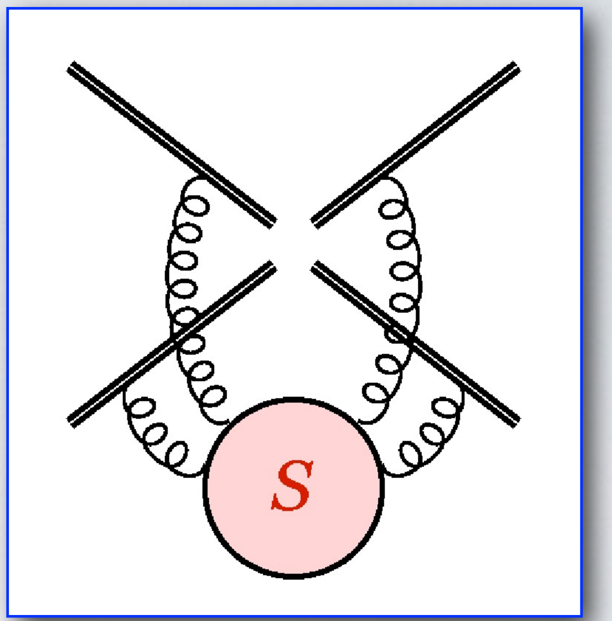


Becher, Bell 2010

Wilson line correlators

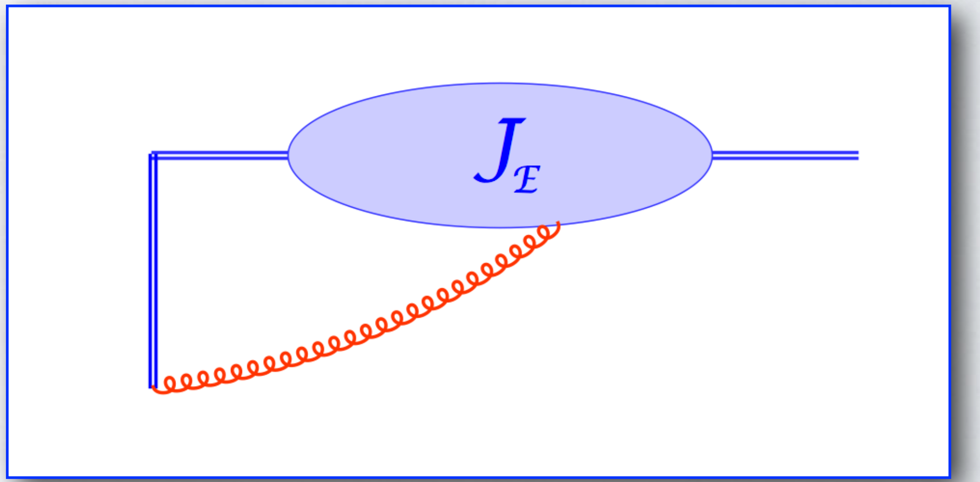
The soft function S is a color operator, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$\mathcal{S}_n\left(\beta_i \cdot \beta_j\right) = \langle 0 | \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0) \, | 0 \rangle$$



The soft jet function $J_{\mathcal{E}}$ contains soft-collinear poles: it is defined by replacing the field in the ordinary jet J with a Wilson line in the appropriate color representation.

$$\mathcal{J}_{\mathrm{E}}\left(\frac{(\beta \cdot n)^2}{n^2}\right) = \langle 0 | \, \Phi_\beta(\infty, 0) \, \Phi_n(0, \infty) \, | 0 \rangle$$



Wilson-line matrix elements exponentiate non-trivially and have tightly constrained functional dependence on their arguments. They are known to three loops.

COUNTERTERMS

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Soft cross sections: pictorial

Consider first the (academic) case of purely soft final state divergences.

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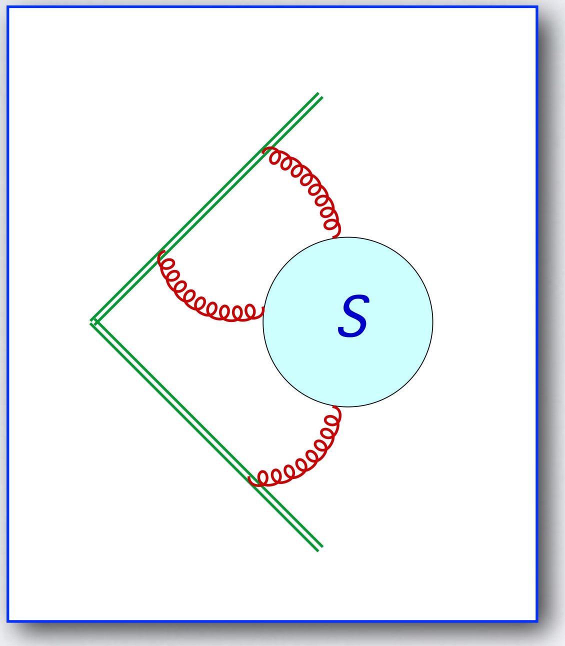
At amplitude level poles factorise and exponentiate.



Soft cross sections: pictorial

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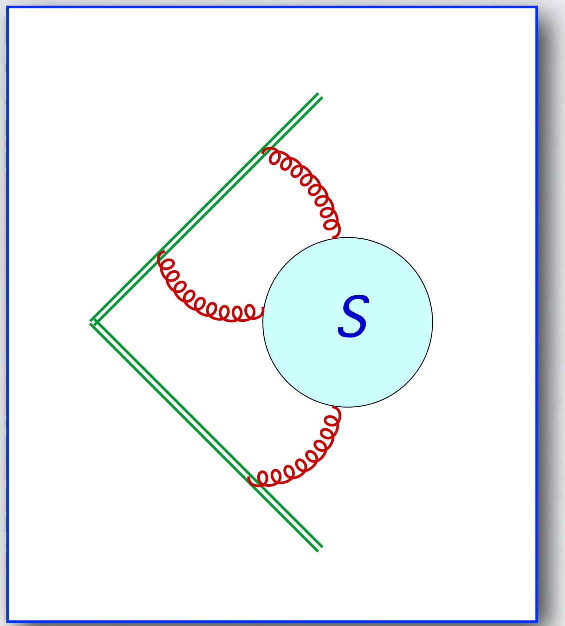
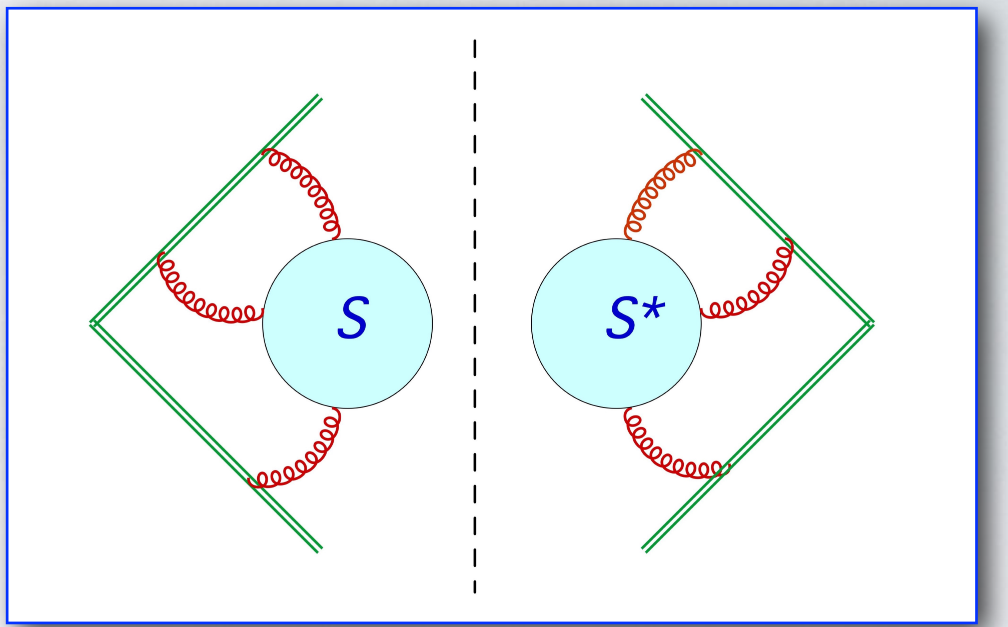


We need to build cross-section level quantities.



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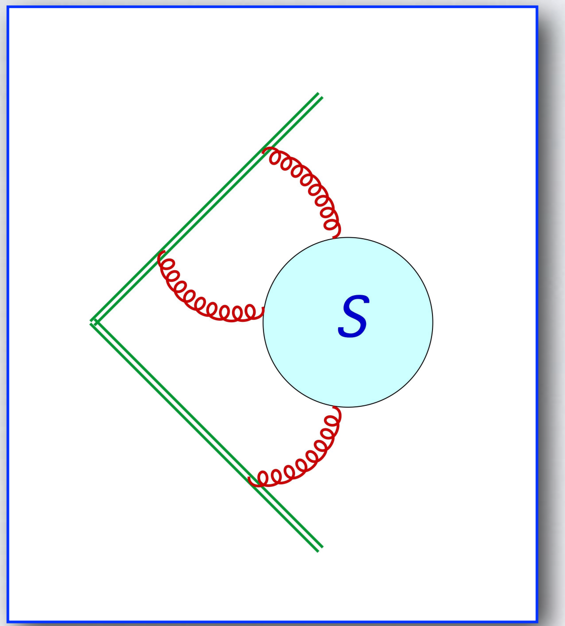

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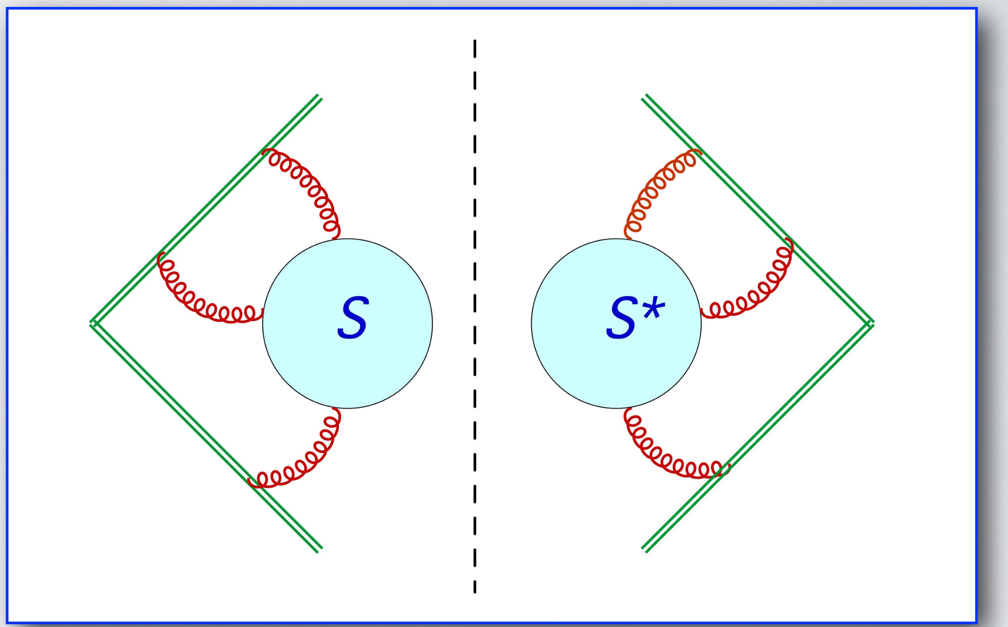
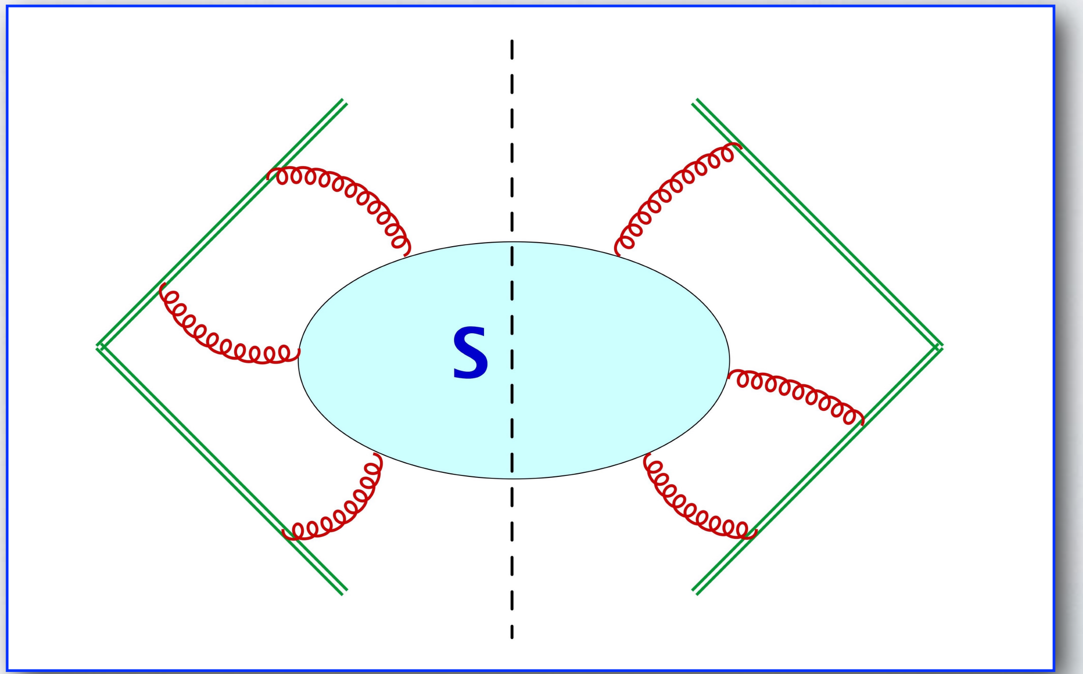


- Inclusive eikonal cross sections are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local IR counterterms.



Collinear cross sections: pictorial

Consider next collinear final state divergences. They are associated with individual partons.

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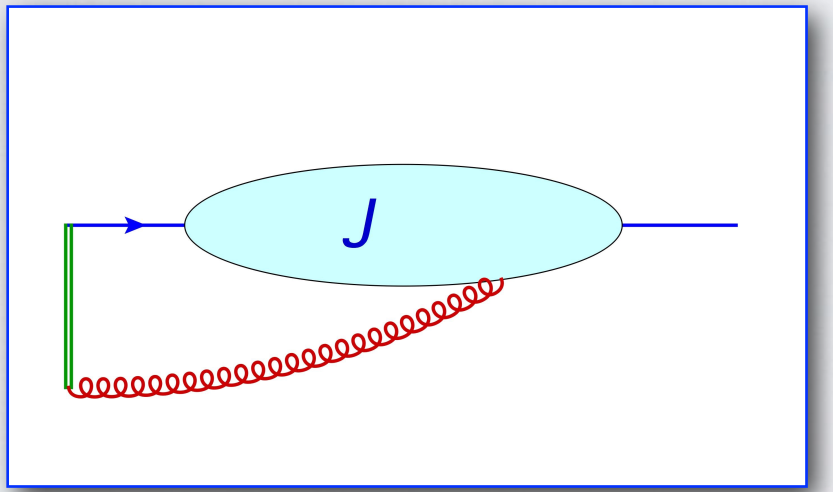

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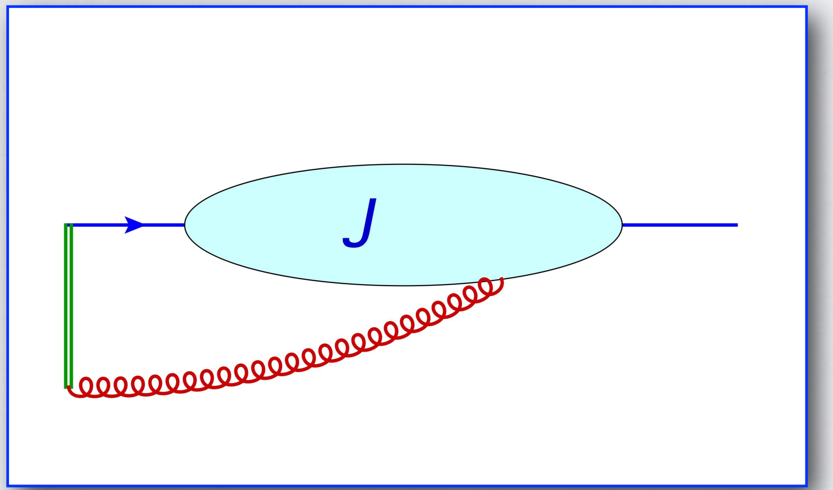
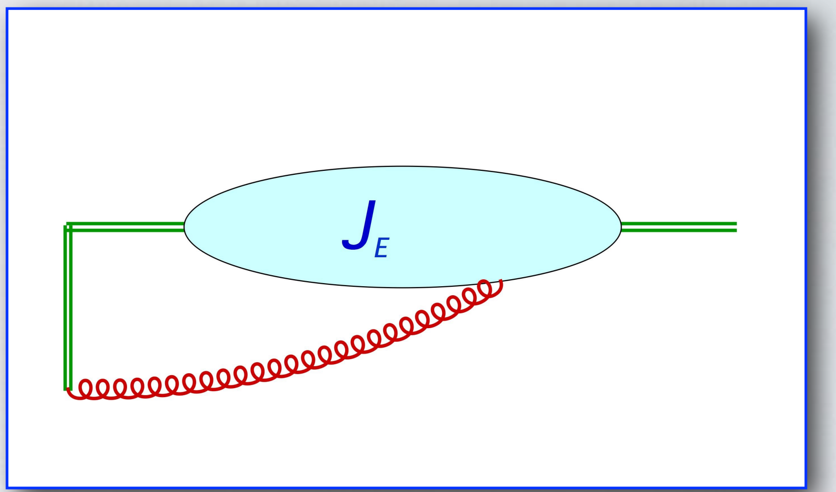


Soft-collinear poles can be subtracted



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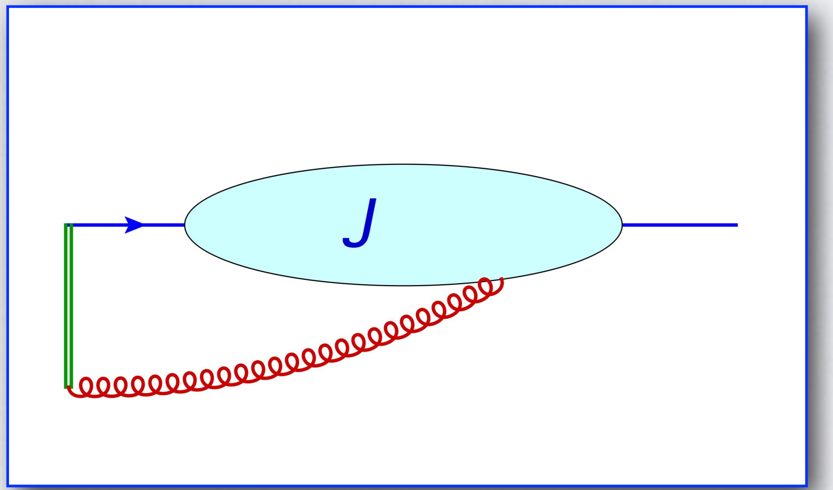

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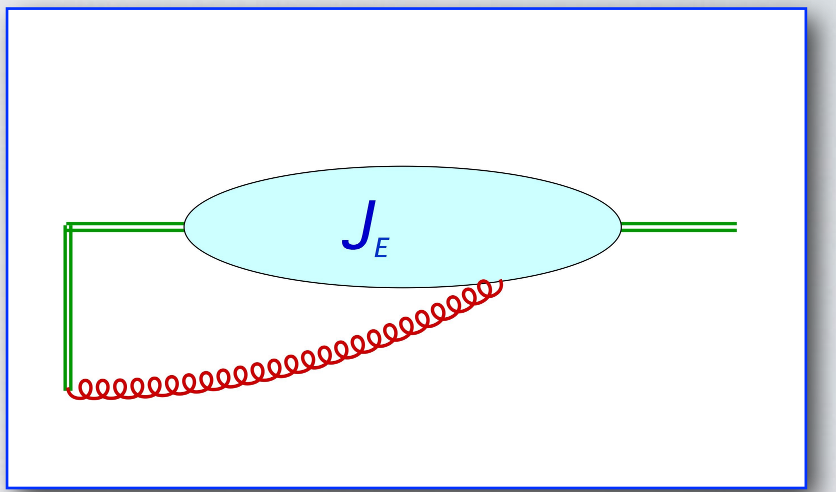
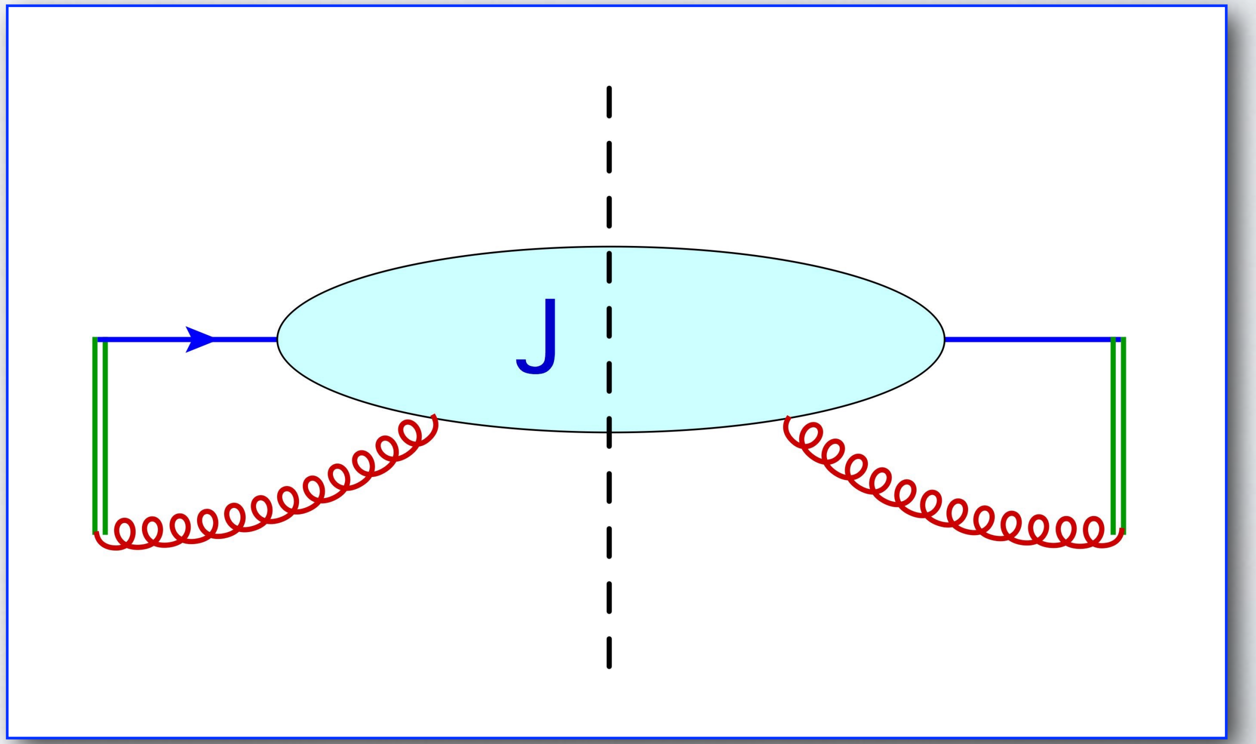


- Inclusive `jet cross sections' are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local collinear counterterms.
- Eikonal jet cross sections subtract the soft-collinear double counting.



Soft counterterms: all orders

Introduce eikonal form factors for the emission of m soft partons from n hard ones.

$$
\begin{aligned}
\mathcal{S}_{n,\,m}\left(k_1, \ldots, k_m; \beta_i\right) &\equiv \langle k_1, \lambda_1; \ldots; k_m, \lambda_m | \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0)\ |0\rangle \\
&\equiv \epsilon_{\mu_1}^{*\,(\lambda_1)}(k_1) \ldots \epsilon_{\mu_m}^{*\,(\lambda_m)}(k_m)\ J_{\mathcal{S}}^{\mu_1 \ldots \mu_m}\left(k_1, \ldots, k_m; \beta_i\right) \\
&\equiv \sum_{p=0}^{\infty} \mathcal{S}_{n,\,m}^{(p)}\left(k_1, \ldots, k_m; \beta_i\right)
\end{aligned}
$$

These matrix elements define soft gluon multiple emission currents. They are gauge invariant and they contain loop corrections to all orders.

Existing finite order calculations and all-order arguments are consistent with the factorisation

$$
\mathcal{A}_{n,\,m}\left(k_1, \ldots, k_m; p_i\right) = \mathcal{S}_{n,\,m}\left(k_1, \ldots, k_m; \beta_i\right)\, \mathcal{H}_n(p_i) + \mathcal{R}_{n,\,m}\left(k_1, \ldots, k_m; p_i\right)
$$

with corrections that are finite in dimensional regularisation, and integrable in the soft gluon phase space. It is a working assumption: a formal all-order proof is still lacking.

Soft counterterms: all orders

The factorisation is reflected at cross-section level, for fixed final state quantum numbers.

$$\sum_{\lambda_i} \left| \mathcal{A}_{n,\,m} \left(k_1, \ldots, k_m; p_i \right) \right|^2 \; \simeq \; \mathcal{H}_n^\dagger(p_i) \, S_{n,\,m} \left(k_1, \ldots, k_m; \beta_i \right) \, \mathcal{H}_n(p_i)$$

The *cross-section level* "radiative soft functions" are Wilson-line squared matrix elements

$$\begin{aligned} S_{n,\,m} \left(k_1, \ldots, k_m; \beta_i \right) \, &\equiv \, \sum_{p=0}^{\infty} S_{n,\,m}^{(p)} \left(k_1, \ldots, k_m; \beta_i \right) \\ &\equiv \, \sum_{\lambda_i} \langle 0 | \prod_{i=1}^{n} \Phi_{\beta_i}(0, \infty) \, | k_1, \lambda_1; \ldots; k_m, \lambda_m \rangle \, \langle k_1, \lambda_1; \ldots; k_m, \lambda_m | \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) \, | 0 \rangle \; . \end{aligned}$$

These functions provide a complete list of local soft subtraction counterterms, to all orders. Indeed, summing over particle numbers and integrating over the soft phase space one finds

$$\sum_{m=0}^{\infty} \int d\Phi_m \, S_{n,\,m} \left(k_1, \ldots, k_m; \beta_i \right) \, = \, \langle 0 | \prod_{i=1}^{n} \Phi_{\beta_i}(0, \infty) \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0) \, | 0 \rangle$$

This is a finite fully inclusive soft cross section, order by order in perturbation theory.

Soft current at tree level

At NLO, only the tree-level single-emission current is required, simply defined by

$$\epsilon^{*\,(\lambda)}(k)\cdot J_{\mathcal{S}}^{(0)}(k,\beta_i) \;=\; \mathcal{S}_{n,\,1}^{(0)}\,(k;\beta_i) \;=\; \langle k,\lambda|\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\;|0\rangle\Big|_{\rm tree}$$

One obviously recovers all the well-known results for the leading-order soft gluon current

$$\mathcal{A}_{n,\,1}^{(0)}(k,p_i) \;=\; \epsilon^{*\,(\lambda)}(k)\cdot J_{\mathcal{S}}^{(0)}(k,\beta_i)\,\mathcal{H}_n^{(0)}(p_i) + \mathcal{O}(k^0)$$

$$J_{\mathcal{S}}^{\mu\,(0)}(k,\beta_i) \;=\; g\sum_{i=1}^{n}\frac{\beta_i^\mu}{\beta_i\cdot k}\,\mathbf{T}_i\;.$$

For the cross-section, the tree-level single-radiation soft function acts as a local counterterm.

$$\sum_\lambda\left|\mathcal{A}_{n,\,1}^{(0)}(k,p_i)\right|^2 \simeq \mathcal{H}^{(0)\,\dagger}(p_i)\,S_{n,\,1}^{(0)}\,(k;\beta_i)\;\mathcal{H}_n^{(0)}(p_i)$$
$$= -\,4\pi\alpha_s\sum_{i,j=1}^{n}\frac{\beta_i\cdot\beta_j}{\beta_i\cdot k\,\beta_j\cdot k}\,\mathcal{A}_n^{(0)\dagger}(p_i)\,\mathbf{T}_i\cdot\mathbf{T}_j\,\mathcal{A}_n^{(0)}(p_i)$$

- The single-radiative soft function acts as a color operator on the color-correlated Born.
- Beyond NLO, tree-level multiple gluon emission currents also follow from this definition.

Soft currents at NLO

At one loop, for single radiation, our definition of the soft currents gives

$$\begin{aligned}\mathcal{A}_{n,\,1}\,(k;p_i) &\simeq \mathcal{S}_{n,\,1}\,(k;\beta_i)\;\mathcal{H}_n(p_i)\\ &= \mathcal{S}^{(0)}_{n,\,1}\,(k;\beta_i)\;\mathcal{H}^{(1)}_n(p_i)\,+\,\mathcal{S}^{(1)}_{n,\,1}\,(k;\beta_i)\;\mathcal{H}^{(0)}_n(p_i)\end{aligned}$$

The factorisation proposed in the classic work by Catani-Grazzini appears different

$$\mathcal{A}_{n,\,1}\,(k;p_i)\;\simeq\;\epsilon^{*\,(\lambda)}(k)\cdot J_{\rm CG}\,(k,\beta_i)\;\mathcal{A}_n(p_i)$$

but it is easily matched using the factorisation of the non-radiative amplitude

$$\mathcal{A}_n\,(p_i)\;\simeq\;\mathcal{S}_n\,(\beta_i)\;\mathcal{H}_n(p_i)\quad\longrightarrow\quad\mathcal{H}^{(1)}_n(p_i)\;=\;\mathcal{A}^{(1)}_n\,(p_i)-\mathcal{S}^{(1)}_n\,(\beta_i)\;\mathcal{A}^{(0)}_n(p_i)$$

Recombining, we get an explicit eikonal expression for the CG one-loop soft current

$$\epsilon^{*\,(\lambda)}(k)\cdot J^{(1)}_{\rm CG}(k,\beta_i)\;=\;\mathcal{S}^{(1)}_{n,\,1}\,(k;\beta_i)\;-\;\mathcal{S}^{(0)}_{n,\,1}\,(k;\beta_i)\;\mathcal{S}^{(1)}_n\,(\beta_i)$$

The two calculations are easily matched: same diagrammatic content, cancellations and result.

Collinear counterterms: all orders

For collinear poles, introduce jet matrix elements for the emission of m partons. For quarks

$$\overline{u}_s(p)\,\mathcal{J}_{q,\,m}\left(k_1,\ldots,k_m;p,n\right) \,\equiv\, \langle p,s;k_1,\lambda_1;\ldots;k_m,\lambda_m|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle$$

At *cross-section level*, "radiative jet functions" can be defined as Fourier transforms of squared matrix elements, to account for the non-trivial momentum flow. We propose

$$J_{q,\,m}\left(k_1,\ldots k_m;l,p,n\right) \,\equiv\, \sum_{p=0}^{\infty} J^{(p)}_{q,\,m}\left(k_1,\ldots k_m;l,p,n\right)$$
$$\equiv \int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\sum_{\{\lambda_j\}}\langle 0|\,\Phi_n(\infty,x)\,\psi(x)\,|p,s;k_j,\lambda_j\rangle\,\langle p,s;k_j,\lambda_j|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle$$

These functions provide a complete list of local collinear counterterms, to all orders. Summing over particle numbers and integrating over the collinear phase space one finds

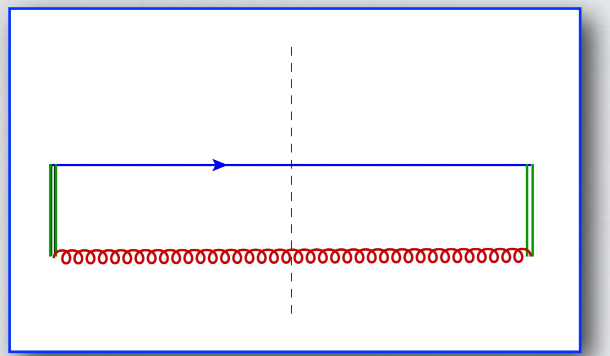
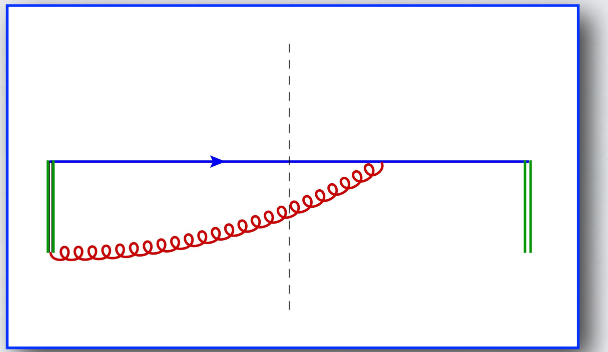
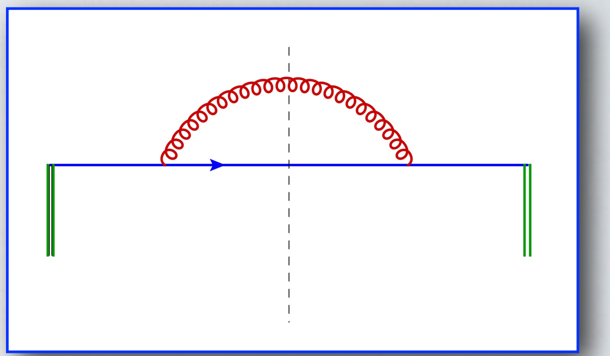
$$\sum_{m=0}^{\infty}\int d\Phi_{m+1}\,J_{q,\,m}\left(k_1,\ldots,k_m;l,p,n\right) \,=\, \mathrm{Disc}\left[\int d^dx\,\mathrm{e}^{\mathrm{i}l\cdot x}\,\langle 0|\,\Phi_n(\infty,x)\psi(x)\overline{\psi}(0)\Phi_n(0,\infty)\,|0\rangle\right]$$

A "two-point function", finite order by order in perturbation theory. Note however

- The collinear limit must still be taken (as $l^2 \to 0$), unlike the case of radiative soft functions.
- $n^2 \neq 0$ avoids spurious collinear poles, but is cumbersome ➔ use SCET-like anti-collinear n^μ.

Collinear counterterms: NLO

At NLO, only tree-level single-emission contributes, resulting (for quarks) in three diagrams

Summing over helicities, and taking the $n^2 \to 0$ limit, one finds a spin-dependent kernel

$$\sum_s J_{q,1}(k; l, p, n) = \frac{4\pi\alpha_s C_F}{l^2} (2\pi)^d \, \delta^d(l - p - k) \left[-\not{l}\gamma_\mu \not{p}\gamma^\mu \not{l} + \frac{1}{k \cdot n} \left(\not{l}\not{n}\not{p} + \not{p}\not{n}\not{l} \right) \right]$$

With a Sudakov decomposition

$$p^\mu = z l^\mu + \mathcal{O}(l_\perp) \, , \qquad k^\mu = (1 - z) l^\mu + \mathcal{O}(l_\perp) \, , \qquad n^2 = 0$$

and taking $l_\perp \to 0$, one recovers the full unpolarised DGLAP LO splitting kernel.

$$\sum_s J_{q,1}(k; l, p, n) = \frac{8\pi\alpha_s C_F}{l^2} (2\pi)^d \, \delta^d(l - p - k) \left[\frac{1 + z^2}{1 - z} - \epsilon (1 - z) + \mathcal{O}(l_\perp) \right]$$

- The three diagrams map precisely to the axial gauge calculation by Catani, Grazzini.
- All LO DGLAP kernels are easily reproduced, multiple collinear limits are under way.

NLO subtraction

The outlines of a subtraction procedure emerge. Begin by expanding the virtual matrix element

$$\mathcal{A}_n(p_i) = \Big[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) + \sum_{i=1}^{n}\Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big)\,\mathcal{S}_n^{(0)}(\beta_i)\,\mathcal{H}_n^{(0)}(p_i)\Big]\Big(1 + \mathcal{O}\left(\alpha_s^2\right)\Big)$$

From the master formula, get the virtual poles of the cross section in terms of virtual kernels

$$V_n \equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}\Big] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i)S_{n,\,0}^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \sum_i \Big(J_{i,\,0}^{(1)}(p_i) - J_{\mathrm{E},\,i,\,0}^{(1)}(\beta_i)\Big)\,\Big|\mathcal{A}_n^{(0)}(p_i)\Big|^2$$

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1\, S_{n,\,1}^{(0)}(k,\beta_i) = \text{finite}$$

$$J_{i,\,0}^{(1)}(l,p,n) + \int d\Phi_1\, J_{i,\,1}^{(0)}(k;l,p,n) = \text{finite}$$

Construct the appropriate local functions.

$$K_{n+1}^{\mathrm{NLO,\,s}} = \mathcal{H}_n^{(0)\,\dagger}(p_i)\,S_{n,\,1}^{(0)}(k,\beta_i)\,\mathcal{H}_n^{(0)}(p_i)$$

$$K_{n+1}^{\mathrm{NLO,\,c}} = \sum_{i=1}^{n} J_{i,\,1}^{(0)}(k_i;l,p_i,n_i)\,\Big|\mathcal{A}_n^{(0)}\,(p_1,\ldots,p_{i-1},l,p_{i+1},\ldots,p_n)\Big|^2$$

with a similar expression for the anti-subtraction of the soft-collinear region in terms of $J_{\mathcal{E}}$.

NNLO subtraction

Let us follow the same procedure at NNLO. Collect the poles of the virtual amplitude

$$
\begin{aligned}
\mathcal{A}_n^{(2)}(p_i) = \; & \mathcal{S}_n^{(2)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(2)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) \\
& + \sum_{i=1}^{n} \left[\mathcal{J}_i^{(2)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(2)}(\beta_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \right) \right] \mathcal{A}_n^{(0)}(p_i) \\
& + \sum_{i<j=1}^{n} \left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \right) \left(\mathcal{J}_j^{(1)}(p_j) - \mathcal{J}_{\mathrm{E},\,j}^{(1)}(\beta_j) \right) \mathcal{A}_n^{(0)}(p_i) \\
& + \sum_{i=1}^{n} \left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i) \right) \left[\mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) \right]
\end{aligned}
$$

Cross-section level soft and jet functions have non-trivial structure starting at NNLO

$$
S_n^{(2)} = \mathcal{S}_n^{(0)\,\dagger}\mathcal{S}_n^{(2)} + \mathcal{S}_n^{(2)\,\dagger}\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)\,\dagger}\mathcal{S}_n^{(1)}
$$

$$
J_{q,\,m}^{(2)} = \int d^d x \, \mathrm{e}^{\mathrm{i}l\cdot x} \sum_{\{\lambda_j\}} \left[\mathcal{J}_{q,\,m}^{(1)\,\dagger}(x) \, \not{n} \, \mathcal{J}_{q,\,m}^{(1)}(0) + \mathcal{J}_{q,\,m}^{(0)\,\dagger}(x) \, \not{n} \, \mathcal{J}_{q,\,m}^{(2)}(0) + \mathcal{J}_{q,\,m}^{(0)}(x) \, \not{n} \, \mathcal{J}_{q,\,m}^{(2)\,\dagger}(0) \right]
$$

All poles of the squared virtual amplitude can nonetheless be expressed in terms of squared jets and eikonal correlators, which leads to the identification of local NNLO counterterms.

Tracing soft and collinear at NNLO

As an example of the detailed structure of soft and collinear subtractions at high orders, consider the “jet factor” in the factorised virtual matrix element.

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$$\frac{\prod_{i=1}^{n} \mathcal{J}_0\left(p_i\right)}{\prod_{i=1}^{n} \mathcal{J}_0^E\left(\beta_i\right)} = 1 + g^2 \sum_{i=1}^{n} \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right]$$

$$+ g^4 \sum_{i=1}^{n} \left[\mathcal{J}_0^{(2)}\left(p_i\right) - \mathcal{J}_0^{E\,(2)}\left(\beta_i\right)\right]$$

$$+ g^4 \sum_{i<j=1}^{n} \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right] \left[\mathcal{J}_0^{(1)}\left(p_j\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_j\right)\right]$$

$$- g^4 \sum_{i=1}^{n} \mathcal{J}_0^{E\,(1)}\left(\beta_i\right) \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right]$$

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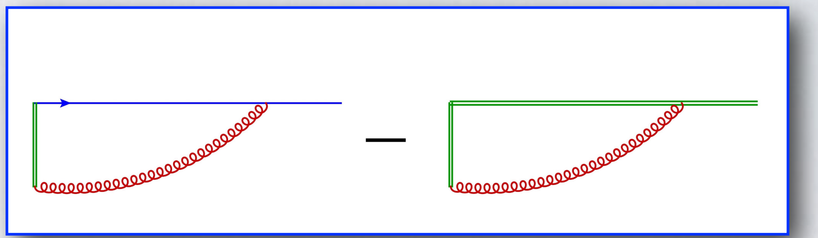
$$- g^4 \sum_{i=1}^{n} \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right]$$



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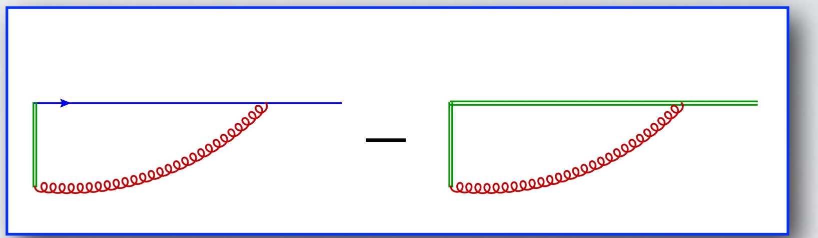
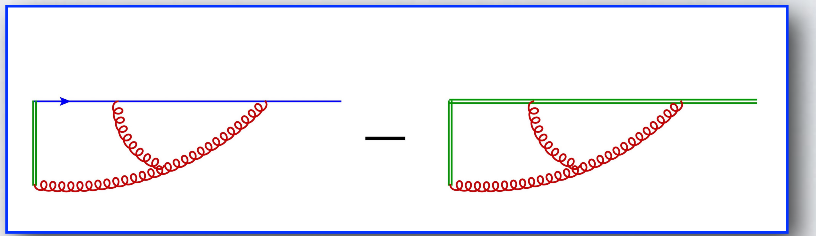
$$
\begin{aligned}
\frac{\prod_{i=1}^{n} \mathcal{J}_0\left(p_i\right)}{\prod_{i=1}^{n} \mathcal{J}_0^{E}\left(\beta_i\right)} \;=\; 1 \;&+\; g^2 \sum_{i=1}^{n} \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right] \\
&+ g^4 \sum_{i=1}^{n} \left[\mathcal{J}_0^{(2)}\left(p_i\right) - \mathcal{J}_0^{E\,(2)}\left(\beta_i\right)\right] \\
&+ g^4 \sum_{i<j=1}^{n} \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right] \left[\mathcal{J}_0^{(1)}\left(p_j\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_j\right)\right] \\
&- g^4 \sum_{i=1}^{n} \mathcal{J}_0^{E\,(1)}\left(\beta_i\right) \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right]
\end{aligned}
$$

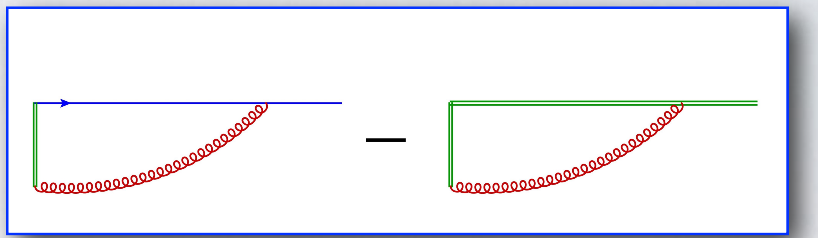


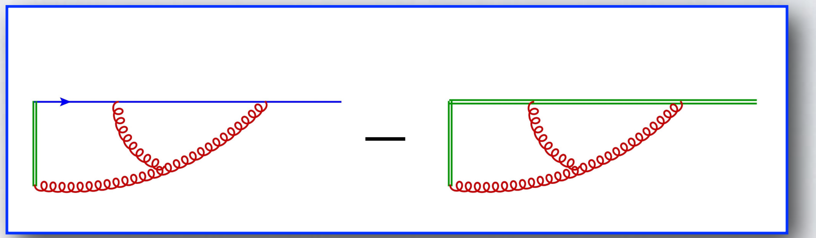


Tracing soft and collinear at NNLO

As an example of the detailed structure of soft and collinear subtractions at high orders, consider the "jet factor" in the factorised virtual matrix element.

$$
\begin{aligned}
\frac{\prod_{i=1}^{n} \mathcal{J}_0\left(p_i\right)}{\prod_{i=1}^{n} \mathcal{J}_0^{E}\left(\beta_i\right)} = \; & 1 + g^2 \sum_{i=1}^{n} \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right] \\
& + g^4 \sum_{i=1}^{n} \left[\mathcal{J}_0^{(2)}\left(p_i\right) - \mathcal{J}_0^{E\,(2)}\left(\beta_i\right)\right] \\
& + g^4 \sum_{i<j=1}^{n} \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right] \left[\mathcal{J}_0^{(1)}\left(p_j\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_j\right)\right] \\
& - g^4 \sum_{i=1}^{n} \mathcal{J}_0^{E\,(1)}\left(\beta_i\right) \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right]
\end{aligned}
$$



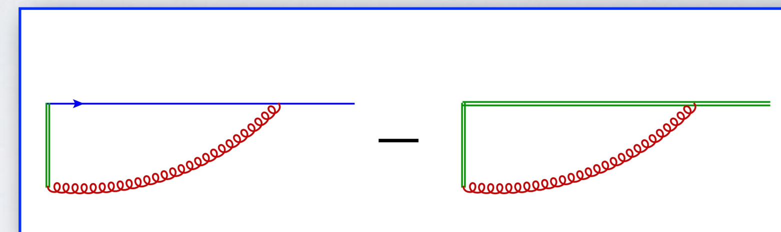


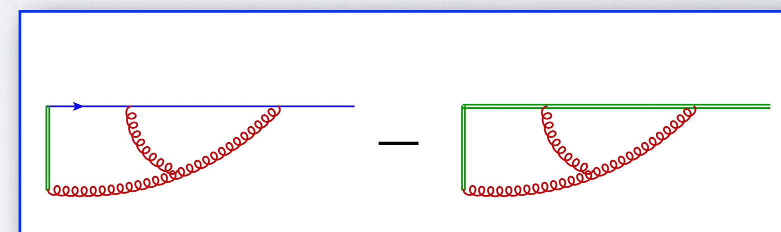
Independent hard collinear poles

Tracing soft and collinear at NNLO

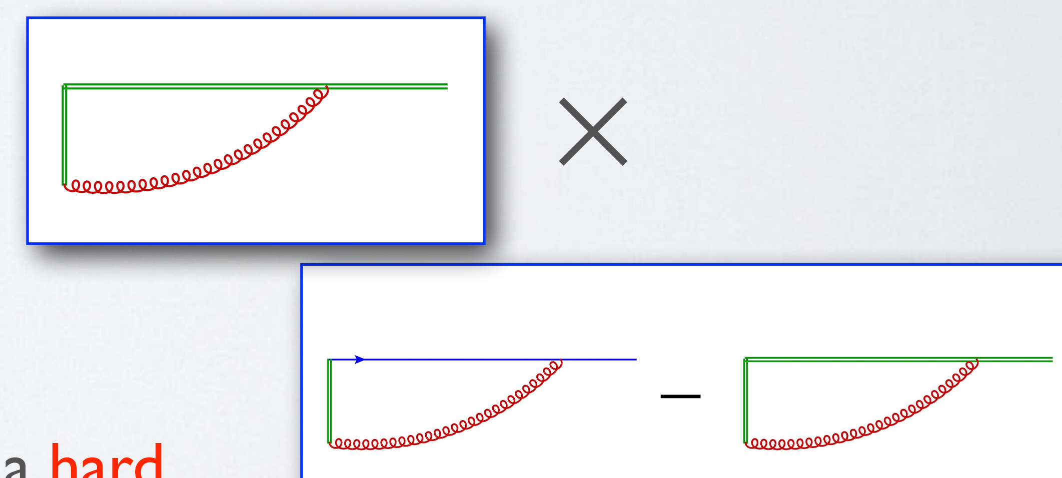
As an example of the detailed structure of soft and collinear subtractions at high orders, consider the "jet factor" in the factorised virtual matrix element.

$$
\begin{aligned}
\frac{\prod_{i=1}^{n} \mathcal{J}_0\left(p_i\right)}{\prod_{i=1}^{n} \mathcal{J}_0^{E}\left(\beta_i\right)} = 1 &+ g^2 \sum_{i=1}^{n} \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right] \\
&+ g^4 \sum_{i=1}^{n} \left[\mathcal{J}_0^{(2)}\left(p_i\right) - \mathcal{J}_0^{E\,(2)}\left(\beta_i\right)\right] \\
&+ g^4 \sum_{i<j=1}^{n} \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right] \left[\mathcal{J}_0^{(1)}\left(p_j\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_j\right)\right] \\
&- g^4 \sum_{i=1}^{n} \mathcal{J}_0^{E\,(1)}\left(\beta_i\right) \left[\mathcal{J}_0^{(1)}\left(p_i\right) - \mathcal{J}_0^{E\,(1)}\left(\beta_i\right)\right]
\end{aligned}
$$





Independent hard collinear poles



The contributions of a single soft gluon accompanied by a hard collinear one factor out and are automatically taken into account.

NNLO subtraction: double collinear

Cross-section level double-virtual poles originate from a number of different configurations

$$(VV)_n \equiv (VV)_n^{(2s)} + (VV)_n^{(1s)} + \sum_{i=1}^{n} (VV)_{n,i}^{(2hc)} + \sum_{i<j=1}^{n} (VV)_{n,ij}^{(2hc)} + \sum_{i=1}^{n} (VV)_{n,i}^{(1hc,1s)} + \sum_{i=1}^{n} (VV)_{n,i}^{(1hc)}$$

Focus on double collinear radiation along the direction of a selected hard particle. One finds

$$(VV)_{n,i}^{(2hc)} = \left[J_{i,0}^{(2)} - J_{\mathrm{E},i,0}^{(2)} - J_{\mathrm{E},i,0}^{(1)} \left(J_{i,0}^{(1)} - J_{\mathrm{E},i,0}^{(1)} \right) \right] \left| \mathcal{A}_n^{(0)} \right|^2$$

It is easy to identify finite combinations of virtual and real (hard) collinear radiation

$$J_{i,0}^{(2)} + \int d\Phi_1 \, J_{i,1}^{(1)} + \int d\Phi_2 \, J_{i,2}^{(0)} = \text{finite}$$

$$\left[J_{\mathrm{E},i,0}^{(1)} + \int d\Phi_1 \, J_{\mathrm{E},i,1}^{(0)} \right] \left[J_{i,0}^{(1)} - J_{\mathrm{E},i,0}^{(1)} + \int d\Phi_1' \left(J_{i,1}^{(0)} - J_{\mathrm{E},i,1}^{(0)} \right) \right] = \text{finite}$$

Real radiation naturally organises into single and double unresolved, and real-virtual terms

$$\begin{aligned}
K_{n+2,i}^{\mathrm{NNLO},\,(\mathbf{2},\,\mathrm{hc})} &= \left[J_{i,2}^{(0)} - J_{\mathrm{E},i,2}^{(0)} - J_{\mathrm{E},i,1}^{(0)} \left(J_{i,1}^{(0)} - \mathcal{J}_{\mathrm{E},i,1}^{(0)} \right) \right] \left| \mathcal{A}_n^{(0)} \right|^2 \\
K_{n+2,i}^{\mathrm{NNLO},\,(\mathbf{1},\,\mathrm{hc})} &= \left(J_{i,1}^{(0)} - \mathcal{J}_{\mathrm{E},i,1}^{(0)} \right) \left| \mathcal{A}_{n+1}^{(0)} \right|^2 \\
K_{n+1,i}^{\mathrm{NNLO},\,(\mathbf{RV},\,\mathrm{hc})} &= \left[J_{i,1}^{(1)} - \mathcal{J}_{\mathrm{E},i,1}^{(1)} - J_{i,0}^{(1)} J_{\mathrm{E},i,1}^{(0)} - J_{\mathrm{E},i,0}^{(1)} J_{i,1}^{(0)} + 2 J_{\mathrm{E},i,0}^{(1)} J_{\mathrm{E},i,1}^{(0)} \right] \left| \mathcal{A}_n^{(0)} \right|^2 .
\end{aligned}$$

NNLO subtraction: soft

Cross-section level double-virtual poles originate from a number of different configurations

$$\boxed{(VV)_n \equiv (VV)_n^{(2s)} + (VV)_n^{(1s)} + \sum_{i=1}^{n} (VV)_{n,i}^{(2hc)} + \sum_{i<j=1}^{n} (VV)_{n,ij}^{(2hc)} + \sum_{i=1}^{n} (VV)_{n,i}^{(1hc,1s)} + \sum_{i=1}^{n} (VV)_{n,i}^{(1hc)}}$$

Focus on double soft and single soft radiation. One finds

$$\boxed{(VV)_n^{(2s)} = \mathcal{H}_n^{(0)\dagger} \, S_{n,0}^{(2)} \, \mathcal{H}_n^{(0)}}$$

$$\boxed{(VV)_n^{(1s)} = \mathcal{H}_n^{(0)\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\dagger} S_{n,0}^{(1)} \mathcal{H}_n^{(0)}}$$

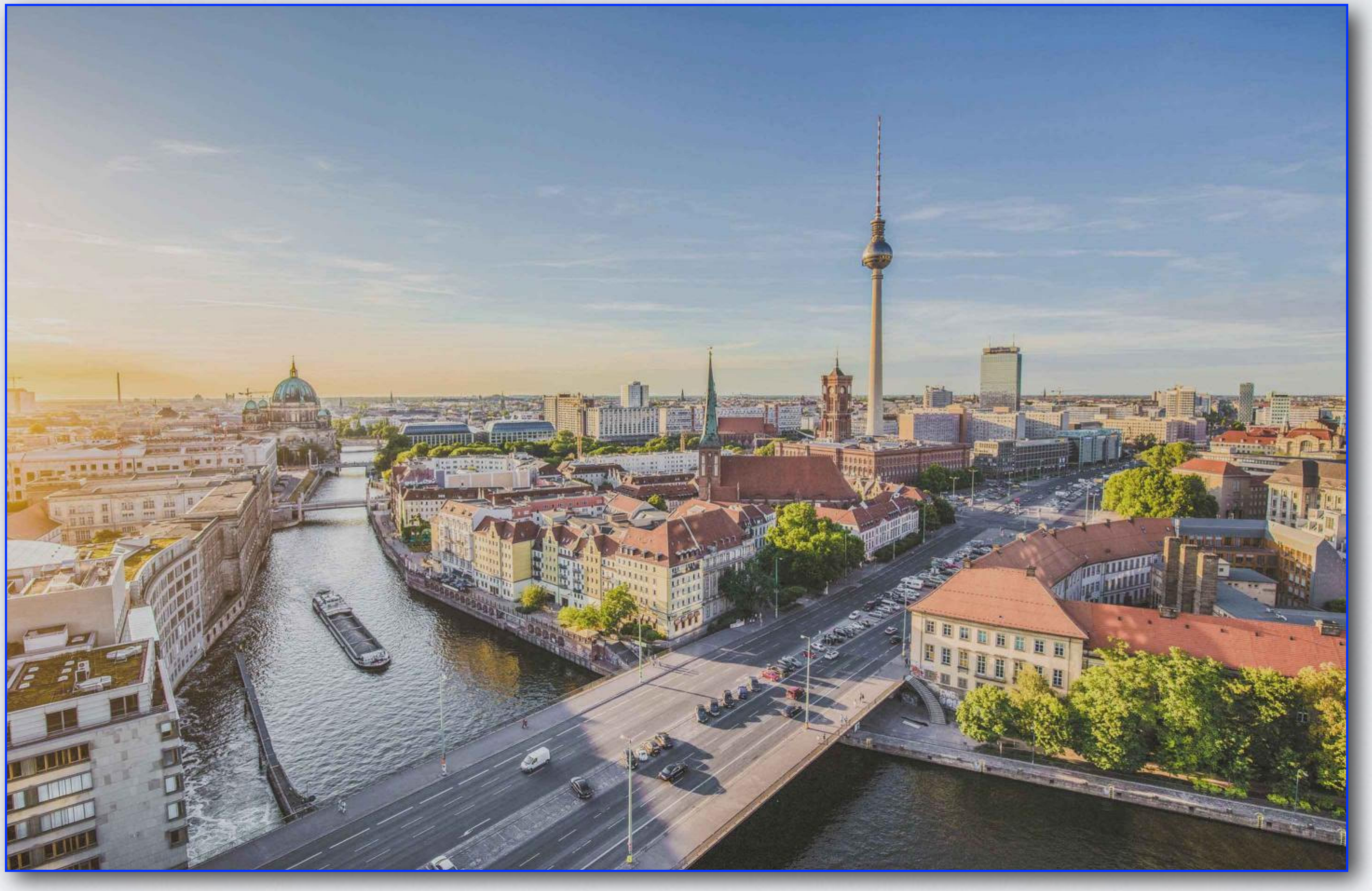
It is easy to identify finite combinations of virtual, real-virtual and double real soft radiation

$$\boxed{S_{n,0}^{(2)}(\beta_i) + \int d\Phi_1 \, S_{n,1}^{(1)}(k, \beta_i) + \int d\Phi_2 \, S_{n,2}^{(0)}(k_1, k_2, \beta_i) = \text{finite} \, .}$$

Real radiation naturally organises into single and double unresolved, and real-virtual terms

$$\boxed{\begin{aligned} K_{n+2}^{\mathrm{NNLO},\,(\mathbf{2},\,\mathrm{s})} &= \mathcal{H}_n^{(0)\dagger} \, S_{n,2}^{(0)} \, \mathcal{H}_n^{(0)} \\ K_{n+2}^{\mathrm{NNLO},\,(\mathbf{1},\,\mathrm{s})} &= \mathcal{H}_{n+1}^{(0)\dagger} \, S_{n+1,1}^{(0)} \, \mathcal{H}_{n+1}^{(0)} \\ K_{n+1}^{\mathrm{NNLO},\,(\mathbf{RV},\,\mathrm{s})} &= \mathcal{H}_n^{(0)\dagger} \, S_{n,1}^{(0)} \, \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\dagger} \, S_{n,1}^{(0)} \, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(0)\dagger} \, S_{n,1}^{(1)} \, \mathcal{H}_n^{(0)} \end{aligned}}$$

OUTLOOK



Outlook

- A number of successful NNLO subtraction algorithms are available.
- They are computationally expensive, either analytically, or numerically, or both.
- Extensions to multi-leg processes or higher orders is expected to be useful but hard.
- Work on refining existing tools to find the `minimal toolbox’ is necessary and under way.
- The factorisation of soft and collinear virtual amplitudes contains important information.
- The formal structure of subtraction has been clarified to all orders.
- A general all-order definition of soft and/or collinear counterterms has been proposed.
- Existing results at NLO and beyond are reproduced and systematised.
- Tracing the real emission counterterms starting from virtual poles is a useful strategy.
- A parallel effort to construct a detailed analytic subtraction algorithm is under way.
- A detailed analysis of counterterms at N^3LO and of strongly ordered regions is under way.

THANK YOU