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Two-loop Rational Terms in Yang-Mills Theories

M. F. Zoller

in collaboration with

J.-N. Lang, S. Pozzorini and H. Zhang

based on

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Motivation: Towards automated numerical two-loop calculations

- Theory predictions of $\mathcal{O}(1\%)$ precision for LHC processes require NNLO calculations
  $\Rightarrow$ **Automation of two-loop calculations highly desirable**

- Higher-order calculations are usually performed in $D = 4 - 2\varepsilon$ dimensions
  $\rightarrow$ Regularisation of divergences in Feynman integrals
  But: D-dimensional Arrays cannot be implemented in numerical programs

- Numerical one-loop tools, such as OpenLoops [Buccioni et al], Recola [Actis et al], MadLoop [Hirschi et al] construct the numerator of loop integrals in 4 dimensions
  $\rightarrow$ The $D-4$ dimensional numerator part needs to be restored
  $\rightarrow$ One loop: Universal rational counterterms of type R_2 [Ossola, Papadopoulos, Pittau] from the interplay of $D-4$-dimensional numerator parts with $\frac{1}{\varepsilon}$ UV poles

  $\rightarrow$ **This talk: Two-loop rational terms of UV origin**

Outline

I. Rational terms at one loop

$\rightarrow$ Computation of rational terms from tadpole integrals with one scale

$\rightarrow$ Rational terms of one-loop diagrams and sub-diagrams

II. Rational terms at two loops

$\rightarrow$ Master formula for the computation of renormalised D-dimensional two-loop amplitudes from amplitudes with 4-dimensional numerators and universal rational counterterms

$\rightarrow$ Proof and recipe for the computation of two loop rational terms from tadpole integrals

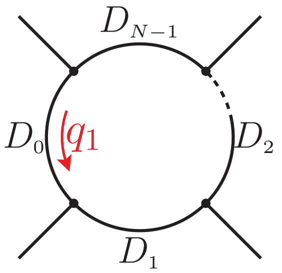
III. Renormalisation scheme dependence of rational terms

IV. Two-loop rational terms in SU(N) and U(1) gauge theories

I. Rational terms at one loop

Generic one-loop diagram γ in $D = 4 - 2\varepsilon$ dimensions

$$\bar{\mathcal{A}}_{1,\gamma} = \int \mathrm{d}\bar{q}_1 \, \frac{\bar{\mathcal{N}}(\bar{q}_1)}{D_0(\bar{q}_1) \cdots D_{N-1}(\bar{q}_1)} \quad =$$



with $D_k(\bar{q}_1) = (\bar{q}_1 + p_k)^2 - m_k^2$,

$$\int \mathrm{d}\bar{q}_1 = \mu_0^{2\varepsilon} \int \frac{\mathrm{d}^D \bar{q}_1}{(2\pi)^D}$$

(D-dim loop momentum $\bar{q}_1$, masses m_k and 4-dim external momenta p_k)

Split D-dimensional numerator

$$\underbrace{\bar{\mathcal{N}}(\bar{q}_1)}_{D-\text{dim}} = \underbrace{\mathcal{N}(q_1)}_{4-\text{dim}} + \underbrace{\tilde{\mathcal{N}}(\bar{q}_1)}_{(D-4)-\text{dim}} \qquad \text{with} \begin{cases} \bar{q}_i & = q_i + \tilde{q}_i \\ \bar{\gamma}^{\bar{\mu}} & = \gamma^{\mu} + \tilde{\gamma}^{\bar{\mu}} \\ \bar{g}^{\bar{\mu}\bar{\nu}} & = g^{\mu\nu} + \tilde{g}^{\bar{\mu}\bar{\nu}} \end{cases}$$

$$\Rightarrow \bar{\mathcal{A}}_{1,\gamma} = \mathcal{A}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma} \qquad \text{with} \qquad \delta\mathcal{R}_{1,\gamma} = \int \mathrm{d}\bar{q}_1 \, \frac{\tilde{N}(\bar{q}_1)}{D_0(\bar{q}_1) \cdots D_{N-1}(\bar{q})},$$

$\Rightarrow$ Interaction of $\tilde{N}$ with $\frac{1}{\varepsilon}$ **UV** poles leads to a **finite set of universal local counterterms $\delta\mathcal{R}_{1,\gamma}$** in any renormalisable theory [Ossola, Papadopoulos, Pittau]

IR divergences do not generate rational terms at one loop [Bredensetein, Denner, Dittmaier, Pozzorini]

Tadpole decomposition

Capture UV divergences via **tadpole decomposition** of propagator denominators $D_k = (\bar{q}_1 + p_k)^2 - m_k^2$ with one auxiliary mass scale M^2 [Chetyrkin, Misiak, Münz]

$$\underbrace{\frac{1}{D_k(\bar{q}_1)}}_{\sim\frac{1}{\bar{q}_1^2}} = \underbrace{\frac{1}{\bar{q}_1^2 - M^2}}_{\sim\frac{1}{\bar{q}_1^2}} + \underbrace{\frac{\Delta_k(\bar{q}_1)}{\bar{q}_1^2 - M^2}\,\frac{1}{D_k(\bar{q}_1)}}_{\sim\frac{1}{\bar{q}_1^3}}$$

with polynomial in external momenta and masses $\Delta_k(\bar{q}_1) = -2\bar{q}_1 \cdot p_k - p_k^2 + m_k^2 - M^2$

Recursive application $\rightarrow$ **Tadpole expansion** up to degree of divergence X and UV-finite remainder:

$$\frac{1}{D_k(\bar{q}_1)} = (\mathbf{S}_X + \mathbf{F}_X)\frac{1}{D_k(\bar{q}_1)}$$

with

$$\mathbf{S}_X \frac{1}{D_k(\bar{q}_1)} = \sum_{\sigma=0}^{X} \frac{[\Delta_k(\bar{q}_1)]^\sigma}{\left(\bar{q}_1^2 - M^2\right)^{\sigma+1}} \quad \text{and} \quad \mathbf{F}_X \frac{1}{D_k(\bar{q})} = \frac{[\Delta_k(\bar{q}_1)]^{X+1}}{\left(\bar{q}_1^2 - M^2\right)^{X+1}} \frac{1}{D_k(\bar{q}_1)} \sim \frac{1}{\bar{q}_1^{3+X}}$$

Decomposition of every propagator denominator in a diagram $\Rightarrow$ **UV-divergences in pure tadpole integrals** with one scale M^2, and **polynomial dependence on external momenta and masses**.

Tadpole expansion of a propagator chain

Extend tadpole expansion operators to chains of propagators with loop momentum $\bar{q}_1$:

$$\mathbf{S}_X^{(1)} \frac{1}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)} = \sum_{\sigma=0}^{X} \frac{\Delta^{(\sigma)}(\bar{q}_1)}{\left(\bar{q}_1^2 - M^2\right)^{N+\sigma}}, \qquad \mathbf{F}_X^{(1)} = 1 - \mathbf{S}_X^{(1)}$$

where $\Delta^{(\sigma)}(\bar{q}_1)$ is a polynomial in masses and external momenta built from the Δ_k $(k = 0, \ldots, N-1)$.

Properties:

- $\mathbf{S}_X^{(1)}$ expands all propagators along the chain with loop momentum $\bar{q}_1$ and picks all powers up to $\frac{1}{\bar{q}_1^{2N+X}}$ $\Rightarrow$ The remainder (higher-power tadpoles and D_k) is attributed to $\mathbf{F}_X^{(1)}$.

- $\mathbf{S}_X^{(1)}, \mathbf{F}_X^{(1)}$ do not act on the numerator of a diagram

$$\mathbf{S}_X^{(1)} \frac{\bar{\mathcal{N}}(\bar{q}_1)}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)} = \bar{\mathcal{N}}(\bar{q}_1)\, \mathbf{S}_X^{(1)} \frac{1}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)}$$

$\Rightarrow$ **Isolate UV divergence in simple tadpole integrals**

$\Rightarrow$ **Only these contribute to UV and rational counterterms**

Computing one-loop UV counterterms via tadpole expansion

Renormalised one-loop amplitude $\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma} = \bar{\mathcal{A}}_{1,\gamma} + \delta Z_{1,\gamma}$

where $\bar{\mathcal{A}}_{1,\gamma} = \int \mathrm{d}\bar{q}_1 \, \frac{\mathcal{N}(q_1) + \tilde{\mathcal{N}}(\bar{q}_1)}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)}$

Extract poles $\frac{1}{\varepsilon^k}$ and possible scale factor $t_Y = \frac{S_Y \mu_0^2}{\mu_R^2}$ with K-operator:

$$f_L(\varepsilon) = (t_Y)^{L\varepsilon} \sum_{k=1}^{L} \frac{f_{L,k}}{\varepsilon^k} + f_{L,\text{finite}} \quad \Rightarrow \quad \mathbf{K} f_L(\varepsilon) = (t_Y)^{L\varepsilon} \sum_{k=1}^{L} \frac{f_{L,k}}{\varepsilon^k}$$

in MS-like renormalisation scheme Y, e.g. $S_{MS} = 1$, $S^{\varepsilon}_{\overline{\text{MS}}} = (4\pi)^{\varepsilon}\,\Gamma(1+\varepsilon)$

(dim-reg scale μ_0 and renormalisation scale μ_R)

$$\begin{aligned} \mathbf{K}\,\bar{\mathcal{A}}_{1,\gamma} &= \mathbf{K}\,\mathbf{S}_X^{(1)}\,\bar{\mathcal{A}}_{1,\gamma} = \mathbf{K} \int \mathrm{d}\bar{q}_1 \sum_{\sigma=0}^{X} \frac{\left(\mathcal{N}(q_1) + \tilde{\mathcal{N}}(\bar{q}_1)\right)\Delta^{(\sigma)}}{(\bar{q}_1 - M^2)^{N+\sigma}} \\ &= \mathbf{K}\left[\int \mathrm{d}\bar{q}_1 \sum_{\sigma=0}^{X} \frac{\mathcal{N}(q_1)\Delta^{(\sigma)}}{(\bar{q}_1 - M^2)^{N+\sigma}} + \mathcal{O}(1)\right] = -\delta Z_{1,\gamma} \end{aligned}$$

$\Rightarrow$ Extend this operator to also capture the interplay of $\tilde{\mathcal{N}}(\bar{q}_1)$ with the same UV poles

Computing one-loop rational terms via tadpole expansion

Decomposition into tensor integrals:

$$\bar{\mathcal{A}}_{1,\gamma} = \sum_{r=0}^{R} \Big(\underbrace{\mathcal{N}_{\mu_1\cdots\mu_r}}_{4\text{-dim}} + \underbrace{\tilde{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r}}_{\varepsilon\text{-dim}} \Big) \underbrace{T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}}_{D\text{-dim}} \quad \text{with} \quad T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \int \mathrm{d}\bar{q}_1 \frac{\bar{q}_1^{\mu_1}\cdots\bar{q}_1^{\mu_r}}{D_0(\bar{q}_1)\cdots D_{N-1}(\bar{q}_1)}$$

$$\mathbf{K}\,\bar{\mathcal{A}}_{1,\gamma} = \sum_{r=0}^{R} \mathcal{N}_{\mu_1\cdots\mu_r}\,\mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \mathbf{K}\,\mathcal{A}_{1,\gamma} = -\delta Z_{1,\gamma}$$

$$\bar{\mathbf{K}}\,\bar{\mathcal{A}}_{1,\gamma} := \sum_{r=0}^{R} \Big(\mathcal{N}_{\mu_1\cdots\mu_r} + \tilde{\mathcal{N}}_{\mu_1\cdots\mu_r} \Big)\,\mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = -\delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}$$

$\bar{\mathbf{K}}$-operator extracts the pole of the tensor integral and contracts it with the full D-dim numerator coefficient $\Rightarrow$ **Capture full UV pole contribution**

Practical calculation:

$$\delta\mathcal{R}_{1,\gamma} = (\bar{\mathbf{K}} - \mathbf{K})\bar{\mathcal{A}}_{1,\gamma} = \sum_{r=0}^{R} \left[\bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r} - \mathcal{N}_{\mu_1\cdots\mu_r}\right] \mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}$$

using the tadpole expansion $\quad \mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \mathbf{K}\,\mathbf{S}_X^{(1)}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \mathbf{K} \sum_{\sigma=0}^{X} \int \mathrm{d}\bar{q}_1 \frac{\bar{q}_1^{\mu_1}\cdots\bar{q}_1^{\mu_r}\,\Delta^{(\sigma)}}{(\bar{q}_1 - M^2)^{N+\sigma}}$

Rational counterterms $\delta\mathcal{R}_{1,\gamma} \neq 0$ only for UV-divergent diagrams
$\Rightarrow$ **finite set in renormalisable theories**

One-loop rational terms from tadpole integrals

One-loop master formula for computing a renormalised D-dim amplitude

$$\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma} = \bar{\mathcal{A}}_{1,\gamma} + \delta Z_{1,\gamma} = \mathcal{A}_{1,\gamma} + \delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}$$

Generic method to compute $\delta\mathcal{R}_{1,\gamma}$ from tadpole integrals with one (auxiliary) scale M^2:

$$\delta\mathcal{R}_{1,\gamma} = \sum_{r=0}^{R} \left[\bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_r} - \mathcal{N}_{\mu_1\cdots\mu_r}\right] \mathbf{K}\,\mathbf{S}_X^{(1)} T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}$$

- Dependence on external momenta and masses resides solely in numerator $\left(\mathcal{N}(q_1) + \tilde{\mathcal{N}}(\bar{q}_1)\right)\Delta^{(\sigma)}$ in polynomial form $\Rightarrow$ **Proof that $\delta\mathcal{R}_{1,\gamma}$ is indeed a rational term**.

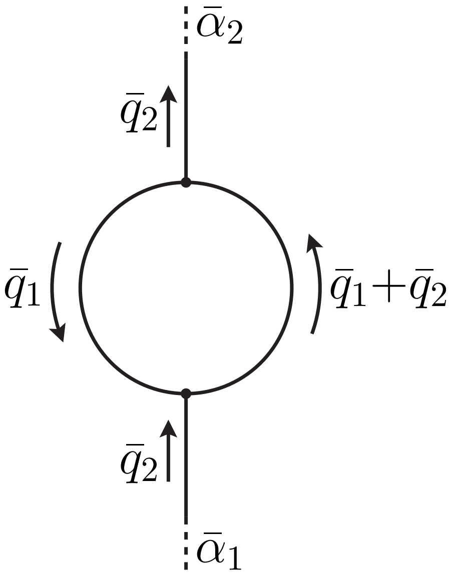
- $\delta\mathcal{R}_{1,\gamma}$ stem from same poles as $\delta Z_{1,\gamma}$ $\Rightarrow$ finite set of rational terms in any renormalisable theory.

- Results for $\delta Z_{1,\gamma}$ and $\delta\mathcal{R}_{1,\gamma}$ **independent of M^2** since $\boxed{\mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r} = \mathbf{K}\,\mathbf{S}_X^{(1)}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}}$ is exact, and the l.h.s. (original denominators $D_k(\bar{q}_1)$) is M^2-independent.

- Note that $\delta\mathcal{R}_{1,\gamma}$ does not correspond to renormalisation of fields and couplings in the Lagrangian, e.g. for the 4-photon-vertex $\delta\mathcal{R}_{1,\gamma} \neq 0$, but $\delta Z_{1,\gamma} = 0$.

Towards two loops: One-loop subdiagrams in two-loop diagrams

First step in renormalisation procedure for two-loop diagrams is the subtraction of subdivergences



- **One-loop diagram with 4-dim q_2:**

$$D_k(\bar{q}_1, q_2) = (\bar{q}_1 + q_2)^2 - m_k^2 = \bar{q}_1^2 + \underbrace{2\bar{q}_1 \cdot q_2 + q_2^2}_{4\text{-dim}} - m_k^2$$

- **One-loop subdiagram with D-dim $\bar{q}_2 = q_2 + \tilde{q}_2$:**

$$D_k(\bar{q}_1, \bar{q}_2) = D_k(\bar{q}_1, q_2) + \underbrace{2\,\bar{q}_1 \cdot \tilde{q}_2 + \tilde{q}_2^2}_{\varepsilon\text{-dim}}$$

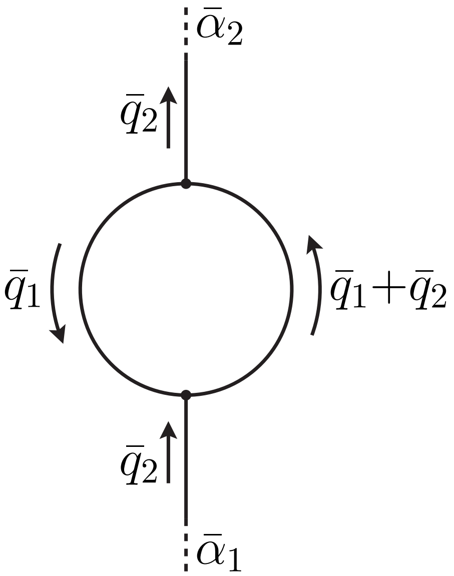
Note that $\bar{q}_1 \cdot q_2 = q_1 \cdot q_2$ and $\bar{q}_1 \cdot \tilde{q}_2 = \tilde{q}_1 \cdot \tilde{q}_2$

The additional term $2\,\tilde{q}_1 \cdot \tilde{q}_2 + \tilde{q}_2^2$ enters Δ_k in the tadpole expansion:

$$\Rightarrow \quad \frac{1}{(\bar{q}_1 + q_2 + \tilde{q}_2)^2 - m^2} = \frac{1}{\bar{q}_1^2 - M^2} + \frac{-\left(q_2^2 + \tilde{q}_2^2\right) - 2\,(q_1 \cdot q_2 + \tilde{q}_1 \cdot \tilde{q}_2) + m^2 - M^2}{(\bar{q}_1^2 - M^2)^2} + \ldots$$

$\Rightarrow$ **Extra pole term $\propto \frac{\tilde{q}_2^2}{\varepsilon}$ possible in amplitude with 4-dim numerator $\mathcal{A}_{1,\gamma}$**
(which has 4-dim q_2 in the numerator, but D-dim $\bar{q}_2$ in the denominator)

One-loop subdiagrams in two-loop diagrams



Numerator in D dimension (including $\bar q_2$ and Lorentz indices $\bar\alpha = (\bar\alpha_1, \bar\alpha_2)$)

$$\begin{aligned}
\bar{\mathcal{A}}^{\bar\alpha}_{1,\gamma}(\bar q_2) &= \sum_{r=0}^{R} \left(\mathcal{N}^{\bar\alpha}_{\mu_1\cdots\mu_r}(\bar q_2) + \tilde{\mathcal{N}}^{\bar\alpha}_{\bar\mu_1\cdots\bar\mu_r}(\bar q_2) \right) T_N^{\bar\mu_1\cdots\bar\mu_r}(\bar q_2) \\
\Rightarrow \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar\alpha}_{1,\gamma} &= \sum_{r=0}^{R} \left(\mathcal{N}^{\bar\alpha}_{\mu_1\cdots\mu_r}(\bar q_2) + \tilde{\mathcal{N}}^{\bar\alpha}_{\bar\mu_1\cdots\bar\mu_r}(\bar q_2) \right) \mathbf{K}\, T_N^{\bar\mu_1\cdots\bar\mu_r}(\bar q_2) \\
&= -\delta Z^{\bar\alpha}_{1,\gamma}(\bar q_2) + \delta\mathcal{R}^{\bar\alpha}_{1,\gamma}(\bar q_2)
\end{aligned}$$

Numerator in 4 dimensions (but D-dim $\bar q_2$ in denominator)

$$\begin{aligned}
\mathcal{A}^{\alpha}_{1,\gamma}(\bar q_2) &= \sum_{r=0}^{R} \mathcal{N}^{\alpha}_{\mu_1\cdots\mu_r}(q_2)\, T_N^{\bar\mu_1\cdots\bar\mu_r}(q_2 + \tilde q_2) \\
\Rightarrow \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma} &= -\delta Z^{\alpha}_{1,\gamma}(q_2) - \delta\tilde Z^{\alpha}_{1,\gamma}(\tilde q_2)
\end{aligned}$$

- $\delta Z^{\alpha}_{1,\gamma}(q_2) = -\sum_{r=0}^{R} \mathcal{N}^{\alpha}_{\mu_1\cdots\mu_r}(q_2)\, \mathbf{K}\, T_N^{\mu_1\cdots\mu_r}(q_2) \Rightarrow$ **Projection of UV counterterm to 4-dim**
- $\delta\tilde Z^{\alpha}_{1,\gamma}(\tilde q_2) = -\sum_{r=0}^{R} \mathcal{N}^{\alpha}_{\mu_1\cdots\mu_r}(q_2) \Big(\mathbf{K}\, T_N^{\mu_1\cdots\mu_r}(q_2+\tilde q_2)) - \mathbf{K}\, T_N^{\mu_1\cdots\mu_r}(q_2) \Big) \propto \frac{\tilde q_2^2}{\varepsilon} \Rightarrow$ **New term**
- $\delta\mathcal{R}^{\bar\alpha}_{1,\gamma}(\bar q_2) = \delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2) + \mathcal{O}(\varepsilon) \Rightarrow$ **Already known one-loop rational terms unchanged**

UV subtracted one-loop subdiagrams

Fully UV subtracted amplitudes in D and 4 dimensions can be identified (proof in Backup)

$$\underbrace{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2)}_{D\text{-dim}} = \underbrace{\mathcal{A}^{\alpha}_{1,\gamma}(q_2) - \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(q_2)}_{4\text{-dim}} + \mathcal{O}(\varepsilon, \tilde{q}_2)$$

$\Rightarrow$ **Master formula for one-loop subdiagrams:**

$$\underbrace{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) + \delta Z^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2)}_{D\text{-dim renormalisation}} = \underbrace{\mathcal{A}^{\alpha}_{1,\gamma}(q_2) + \delta Z^{\alpha}_{1,\gamma}(q_2) + \delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2)}_{4\text{-dim renormalisation}} + \underbrace{\delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2)}_{(D-4)\text{-dim restored}} + \mathcal{O}(\varepsilon, \tilde{q}_2)\,.$$

Extra UV counterterm $\delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2) \propto \frac{\tilde{q}_2^2}{\varepsilon} = \mathcal{O}(1)$ non-zero only for quadratic divergence

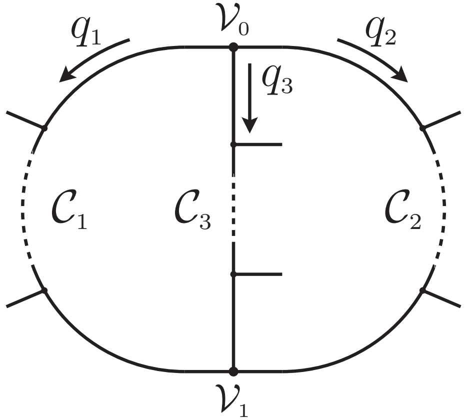
Example: Photon selfenergy (MS scheme)

$$\delta Z^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) = \left(\frac{\mathrm{i}\alpha}{4\pi}\right)\frac{4}{3\,\varepsilon}\left(\bar{q}_2^2 g^{\bar{\alpha}_1\bar{\alpha}_2} - \bar{q}_2^{\bar{\alpha}_1}\bar{q}_2^{\bar{\alpha}_2}\right), \quad \delta Z^{\alpha}_{1,\gamma}(q_2) = \left(\frac{\mathrm{i}\alpha}{4\pi}\right)\frac{4}{3\,\varepsilon}\left(q_2^2 g^{\alpha_1\alpha_2} - q_2^{\alpha_1}q_2^{\alpha_2}\right),$$

$$\delta\tilde{Z}^{\alpha}_{1,\gamma}(\tilde{q}_2) = \left(\frac{\mathrm{i}\alpha}{4\pi}\right)\frac{2}{3}\,\frac{\tilde{q}_2^2}{\varepsilon}\,g^{\alpha_1\alpha_2}, \quad \delta\mathcal{R}^{\alpha}_{1,\gamma}(q_2) = \left(\frac{\mathrm{i}\alpha}{4\pi}\right)\frac{2}{3}\,q_2^2\,g^{\alpha_1\alpha_2}$$

II. Rational terms at two loops

Generic irreducible two-loop diagram Γ consists of three chains $\mathcal{C}_i(\bar{q}_i)$ and two vertices $\mathcal{V}_0, \mathcal{V}_1$

$\bar{\mathcal{A}}_{2,\Gamma} =$ 

$$= \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \, \frac{\bar{\mathcal{N}}(\bar{q}_1, \bar{q}_2)}{\mathcal{D}^{(1)}(\bar{q}_1)\, \mathcal{D}^{(2)}(\bar{q}_2)\, \mathcal{D}^{(3)}(\bar{q}_3)} \Bigg|_{\bar{q}_3 \to -(\bar{q}_1+\bar{q}_2)}$$

$$\mathcal{D}^{(i)}(\bar{q}_i) = D_0^{(i)}(\bar{q}_i) \cdots D_{N_i-1}^{(i)}(\bar{q}_i)$$

$\leftarrow$ denominators $D_a^{(i)}(\bar{q}_i) = (\bar{q}_i + p_{ia})^2 - m_{ia}^2$

$$\bar{\mathcal{N}}(\bar{q}_1, \bar{q}_2) = \left[\bar{\mathcal{V}}_0 \bar{\mathcal{V}}_1\right] \cdot \prod_{i=1}^{3} \bar{\mathcal{N}}^{(i)}(\bar{q}_i) \Big|_{\bar{q}_3 \to -(\bar{q}_1+\bar{q}_2)}$$

$\Rightarrow$ **Factorisation of chains**

Three subdiagrams γ_i from chains $\mathcal{C}_j$ and $\mathcal{C}_k$

$\leftarrow$ $(i|jk)$ is a partition of (123)

$$\bar{\mathcal{A}}_{1,\gamma_i}(\bar{q}_i) = \int \mathrm{d}\bar{q}_j \, \frac{[\bar{\mathcal{V}}_0 \bar{\mathcal{V}}_1] \cdot \bar{\mathcal{N}}^{(j)}(\bar{q}_j)\, \bar{\mathcal{N}}^{(k)}(\bar{q}_k)}{\mathcal{D}^{(j)}(\bar{q}_j)\, \mathcal{D}^{(k)}(\bar{q}_k)} \Bigg|_{\bar{q}_k = -\bar{q}_i - \bar{q}_j}$$

$\leftarrow$ Superficial degree of divergence $X(\gamma_i)$

$X(\gamma_i) \geq 0 \Rightarrow$ **Subdivergence of** Γ

$X(\Gamma) \geq 0 \Rightarrow$ **Global divergence of** Γ

Complements Γ/γ_i

$$\bar{\mathcal{A}}_{2,\Gamma} = \bar{\mathcal{A}}_{1,\gamma_i} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i} = \int \mathrm{d}\bar{q}_i \, \bar{\mathcal{A}}_{1,\gamma_i}(\bar{q}_i) \cdot \frac{\bar{\mathcal{N}}^{(i)}(\bar{q}_i)}{\mathcal{D}^{(i)}(\bar{q}_i)}$$

$\Rightarrow$ **Factorisation of γ_i and Γ/γ_i**

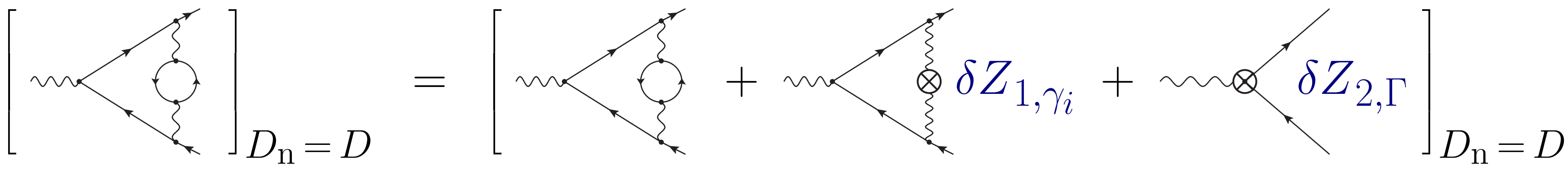
Renormalised D-dimensional amplitude

Renormalisation procedure based on R-operation [Bogoliubov, Parasiuk; Hepp; Zimmermann; Caswell, Kennedy]

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \bar{\mathcal{A}}_{2,\Gamma} \;+\; \underbrace{\sum_{\gamma_i} \delta Z_{1,\gamma_i} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i}}_{\text{subtract subdivergences}} \;+\; \underbrace{\delta Z_{2,\Gamma}}_{\text{subtract remaining local divergence}}$$

All amplitudes here have numerator dimension $D_\mathrm{n} = D$

Example: **R**



R-operation: $\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = (1 - \mathbf{K}_\mathrm{sub} - \mathbf{K}_\mathrm{loc})\,\bar{\mathcal{A}}_{2,\Gamma}$

1. **Subdivergences:** $\mathbf{K}_\mathrm{sub}\,\bar{\mathcal{A}}_{2,\Gamma} = \sum_{\gamma_i} (\mathbf{K}\,\bar{\mathcal{A}}_{1,\gamma_i}) \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i}, \qquad \mathbf{K}\,\bar{\mathcal{A}}_{1,\gamma_i} \;=\; -\delta Z_{1,\gamma_i}(\bar{q}_i)$
2. **Remaining divergence:** $\mathbf{K}_\mathrm{loc}\,\bar{\mathcal{A}}_{2,\Gamma} = \mathbf{K}\,(1 - \mathbf{K}_\mathrm{sub})\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; -\delta Z_{2,\Gamma}$ ←local counterterm

Linear operations wrt sums of diagrams or sums of terms in a single diagram, e.g.

$$\mathbf{K}_\mathrm{sub}\left(\sum_\sigma \bar{\mathcal{A}}_{2,\Gamma_\sigma}\right) = \sum_\sigma \mathbf{K}_\mathrm{sub}\,\bar{\mathcal{A}}_{2,\Gamma_\sigma}$$

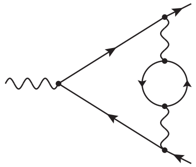
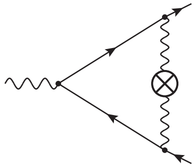
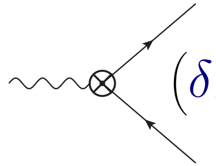
Renormalised D-dimensional amplitude

Goal: Computation from amplitudes with numerator dimension $D_\mathrm{n} = 4$

- Split numerator $\bar{\mathcal{N}}(\bar{q}_1, \bar{q}_2) = \mathcal{N}(q_1, q_2) + \tilde{\mathcal{N}}(\bar{q}_1, \bar{q}_2)$
- Compute amplitudes with $\mathcal{N}(q_1, q_2) = \bar{\mathcal{N}}(\bar{q}_1, \bar{q}_2)\Big|_{\bar{g}^{\bar{\mu}\bar{\nu}} \to g^{\mu\nu},\ \bar{\gamma}^{\bar{\mu}} \to \gamma^{\mu},\ \bar{q}_i \to q_i}$
- Subtract (sub-)divergences and restore $\tilde{\mathcal{N}}$-terms (from subdiagrams and a remaining global)

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\Big(\underbrace{\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma}}_{\text{subtract subdivergences}} + \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\text{restore } \tilde{\mathcal{N}}\text{-terms from subdiagrams}} \Big) \cdot \mathcal{A}_{1,\Gamma/\gamma} + \Big(\underbrace{\delta Z_{2,\Gamma}}_{\text{subtract remaining local divergence}} + \underbrace{\delta\mathcal{R}_{2,\Gamma}}_{\text{restore remaining } \tilde{\mathcal{N}}\text{-term}} \Big)$$

Example:

$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \Big[$  $+$  $\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) +$  $\left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right) \Big]_{D_\mathrm{n} = 4}$

This formula implicitly defines $\delta\mathcal{R}_{2,\Gamma}$. To be shown in the following:

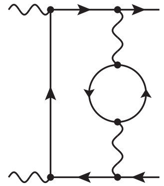
- $\delta\mathcal{R}_{2,\Gamma}$ is a **rational term**.
- In renormalisable theories there is a **finite set** of $\delta\mathcal{R}_{2,\Gamma} \neq 0$.
- **Generic method to compute $\delta\mathcal{R}_{2,\Gamma}$.**

Case 1: Two-loop diagrams with no global divergence

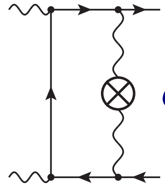
Superficial degree of divergence of two-loop diagram $X(\Gamma) < 0$

$\Rightarrow$ **At most one subdivergence**, i.e. one subdiagram γ with $X(\gamma) \geq 0$

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = (1-\mathbf{K}_{\mathrm{sub}})\bar{\mathcal{A}}_{2,\Gamma} = \underbrace{(1-\mathbf{K})\,\bar{\mathcal{A}}_{1,\gamma_i}}_{\text{divergence subtracted}} \cdot \underbrace{\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}}_{\text{no divergence}} \stackrel{\text{e.g.}}{=} \Big[$$



$$+$$

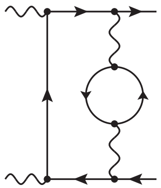


$$\delta Z_{1,\gamma}\Big]_{D_{\mathrm{n}}=D}$$

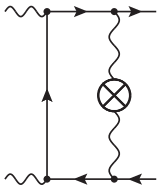
$$= \left(\bar{\mathcal{A}}_{1,\gamma} + \delta Z_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma} + \mathcal{O}(\varepsilon)$$ $\leftarrow$ Project finite chain to $D_{\mathrm{n}} = 4$

$$= \left(\mathcal{A}_{1,\gamma} + \delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma} + \mathcal{O}(\varepsilon)$$ $\leftarrow$ Express UV subtracted subdiagram in $D_{\mathrm{n}} = 4$

$$\stackrel{\text{e.g.}}{=} \Big[$$



$$+$$



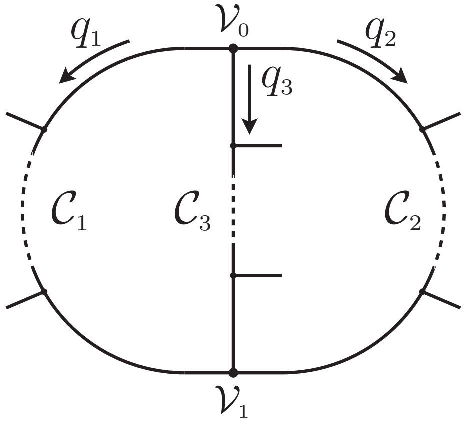
$$\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\Big]_{D_{\mathrm{n}}=4}$$

$\Rightarrow$ **Two-loop $\tilde{\mathcal{N}}$-contribution $\delta\mathcal{R}_{2,\Gamma} = 0$ and UV counterterm $\delta Z_{2,\Gamma} = 0$ for $X(\Gamma) < 0$.**

$\Rightarrow$ **Only globally divergent diagrams contribute to $\delta\mathcal{R}_{2,\Gamma}$ and $\delta Z_{2,\Gamma}$**

$\Rightarrow$ **Finite set of $\delta\mathcal{R}_{2,\Gamma}$ and $\delta Z_{2,\Gamma}$ counterterms in any renormalisable theory**

Case 2: Two-loop diagrams with a global divergence



$$\bar{\mathcal{A}}_{2,\Gamma} = [\text{diagram}] = \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \, \frac{\left[\bar{\mathcal{V}}_0 \bar{\mathcal{V}}_1\right] \prod_{i=1}^{3} \bar{\mathcal{N}}^{(i)}(\bar{q}_i)}{\mathcal{D}^{(1)}(\bar{q}_1)\,\mathcal{D}^{(2)}(\bar{q}_2)\,\mathcal{D}^{(3)}(\bar{q}_3)}\Big|_{\bar{q}_3 \to -(\bar{q}_1+\bar{q}_2)}$$

Exploit factorisation of chains and hence subdiagrams $\bar{\mathcal{A}}_{2,\Gamma} = \bar{\mathcal{A}}_{1,\gamma_i} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i}$

Isolate all (sub)divergences via tadpole decomposition for every chain $\mathcal{C}_i$ ($i = 1, 2, 3$)

- Define **for each chain $\mathcal{C}_i$** the **maximum degree of divergence** of the full diagram ($X(\Gamma) \leq 0$) and the two sub-diagrams γ_j, γ_k involving this chain ← $(i|jk)$ is a partition of (123)

$$X_i = \text{Max}\left\{X(\Gamma), X(\gamma_j), X(\gamma_k)\right\}$$

- Decompose the diagram using the tadpole expansion operators (acting on individual chains)

$$\bar{\mathcal{A}}_{2,\Gamma} = \left(\mathbf{S}^{(1)}_{X_1} + \mathbf{F}^{(1)}_{X_1}\right)\left(\mathbf{S}^{(2)}_{X_2} + \mathbf{F}^{(2)}_{X_2}\right)\left(\mathbf{S}^{(3)}_{X_3} + \mathbf{F}^{(3)}_{X_3}\right)\bar{\mathcal{A}}_{2,\Gamma}$$

Tadpole decomposition of propagator chains

Tadpole expansion operators to chains of propagators with loop momentum $\bar{q}_i$:

$$\mathbf{S}_X^{(i)} \frac{1}{D_0(\bar{q}_i)\cdots D_{N-1}(\bar{q}_i)} = \sum_{\sigma=0}^{X} \frac{\Delta_i^{(\sigma)}(\bar{q}_i)}{\left(\bar{q}_i^2 - M^2\right)^{N+\sigma}}, \qquad \mathbf{F}_X^{(i)} = 1 - \mathbf{S}_X^{(i)}$$

where $\Delta^{(\sigma)}(\bar{q}_i)$ is a polynomial in masses and external momenta built from the Δ_k $(k = 0, \ldots, N-1)$.

Properties:

- $\mathbf{S}_X^{(i)}$ expands all propagators along the chain with loop momentum $\bar{q}_i$ and picks all powers up to $1/\left(\bar{q}_i^2 - M^2\right)^{N+X}$ $\Rightarrow$ The remainder (higher-power tadpoles and D_k) is attributed to $\mathbf{F}_X^{(i)}$.

- $\mathbf{S}_X^{(i)}, \mathbf{F}_X^{(i)}$ do not act on the numerator of a diagram

$$\mathbf{S}_X^{(i)} \frac{\bar{\mathcal{N}}(\bar{q}_1, \bar{q}_2, \bar{q}_3)}{D_0(\bar{q}_i)\cdots D_{N-1}(\bar{q}_i)} = \bar{\mathcal{N}}(\bar{q}_1, \bar{q}_2, \bar{q}_3)\, \mathbf{S}_X^{(i)} \frac{1}{D_0(\bar{q}_i)\cdots D_{N-1}(\bar{q}_i)}$$

- $\mathbf{S}_X^{(i)}, \mathbf{F}_X^{(i)}$ do not act on propagators of other chains with loop momentum $\bar{q}_j$ $(i \neq j)$.

Case 2: Two-loop diagrams with a global divergence

$$\bar{\mathcal{A}}_{2,\Gamma} = \underbrace{\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma}}_{\text{Global divergence}} + \underbrace{\sum_{i=1}^{3}\mathbf{F}^{(i)}_{X_i}\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{2,\Gamma}}_{\text{No global and at most one subdivergence}} + \underbrace{\left(\sum_{i=1}^{3}\mathbf{S}^{(i)}_{X_i}\mathbf{F}^{(j)}_{X_j}\mathbf{F}^{(k)}_{X_k} + \mathbf{F}^{(1)}_{X_1}\mathbf{F}^{(2)}_{X_2}\mathbf{F}^{(3)}_{X_3}\right)\bar{\mathcal{A}}_{2,\Gamma}}_{\text{No divergences}}$$

Exploit the linearity of the R-operation (i.e. of the operators $\mathbf{K}, \mathbf{K}_{\mathbf{sub}}, \mathbf{K}_{\mathbf{loc}}$)

$$\mathbf{R}\bar{\mathcal{A}}_{2,\Gamma} = \mathbf{R}\left(\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma}\right) + \sum_{i=1}^{3}\mathbf{R}\left(\mathbf{F}^{(i)}_{X_i}\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{2,\Gamma}\right) + \ldots$$

and apply to each term our master formula which implicitly defines $\delta\mathcal{R}_{2,\Gamma}$:

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\Big(\underbrace{\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma}}_{\text{subtract subdivergences}} + \underbrace{\delta\mathcal{R}_{1,\gamma}}_{\text{restore }\tilde{\mathcal{N}}\text{-terms from subdiagrams}}\Big)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \Big(\underbrace{\delta Z_{2,\Gamma}}_{\text{subtract remaining local divergence}} + \underbrace{\delta\mathcal{R}_{2,\Gamma}}_{\text{restore remaining }\tilde{\mathcal{N}}\text{-term}}\Big)$$

$$\Rightarrow \delta\mathcal{R}_{2,\Gamma} = \underbrace{\left(\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma_i}\delta Z_{1,\gamma_i}\cdot\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\right)}_{\text{computed in } D_{\mathrm{n}} = D} - \underbrace{\left(\mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot\mathcal{A}_{1,\Gamma/\gamma}\right)}_{\text{computed in } D_{\mathrm{n}} = 4}$$

Case 2: Two-loop diagrams with a global divergence

$$\mathbf{R}\bar{\mathcal{A}}_{2,\Gamma} \;=\; \underbrace{\mathbf{R}\left(\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma}\right)}_{\bar{\mathcal{A}}_{2,\Gamma_{\text{tad}}}} + \sum_{i=1}^{3} \underbrace{\mathbf{R}\left(\mathbf{F}^{(i)}_{X_i}\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{2,\Gamma}\right)}_{\bar{\mathcal{A}}_{2,\Gamma_i}} + \ldots$$

Note:

- $\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}$ captures the full subdivergence stemming from subdiagram γ_i
- All terms involving an $\mathbf{F}^{(i)}$ are (by design) covered by case 1

Example:

$$\begin{aligned}
\mathbf{R}\left(\mathbf{F}^{(i)}_{X_i}\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{2,\Gamma}\right) &= (1-\mathbf{K})\left(\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{1,\gamma_i}\right)\cdot\left(\mathbf{F}^{(i)}_{X_i}\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\right) \\
&= \left(\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{1,\gamma_i} + \delta Z_{1,\gamma_i}\right)\cdot\left(\mathbf{F}^{(i)}_{X_i}\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\right) \\
&= \left(\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\mathcal{A}_{1,\gamma_i} + \delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma_i} + \delta\mathcal{R}_{1,\gamma_i}\right)\cdot\left(\mathbf{F}^{(i)}_{X_i}\mathcal{A}_{1,\Gamma/\gamma_i}\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \delta\mathcal{R}_{2,\Gamma_i} \;=\; &\left(\bar{\mathcal{A}}_{2,\Gamma_i} + \sum_{\gamma_i}\delta Z_{1,\gamma_i}\cdot\bar{\mathcal{A}}_{1,\Gamma_i/\gamma_i}\right) \\
-&\left(\mathcal{A}_{2,\Gamma_i} + \sum_{\gamma_i}\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot\mathcal{A}_{1,\Gamma_i/\gamma}\right) = 0
\end{aligned}$$

$\Rightarrow$ Only the pure tadpole term $\boxed{\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma}}$ **contributes to $\delta\mathcal{R}_{2,\Gamma}$ and $\delta Z_{2,\Gamma}$**

Case 2: Two-loop diagrams with a global divergence

Two-loop $\tilde{\mathcal{N}}$-contribution after subtraction of subdivergences:

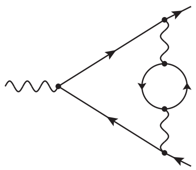
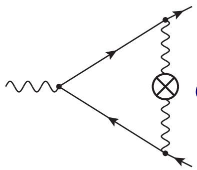
$$\delta\mathcal{R}_{2,\Gamma} = \underbrace{\Big(\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma_i} \delta Z_{1,\gamma_i}\cdot\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\Big)}_{\text{computed in } D_\mathrm{n}=D} - \underbrace{\Big(\mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\Big(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\Big)\cdot\mathcal{A}_{1,\Gamma/\gamma}\Big)}_{\text{computed in } D_\mathrm{n}=4}$$

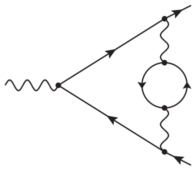
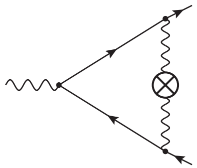
$$= \Big(\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma_i}\delta Z_{1,\gamma_i}\cdot\mathbf{S}^{(i)}_{X_i}\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\Big)$$

$$- \Big(\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\Big(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\Big)\cdot\mathbf{S}^{(i)}_{X_i}\mathcal{A}_{1,\Gamma/\gamma}\Big)$$

using that he full UV divergence in every subdiagram is captured by the tadpole expansion, i.e.

$$\bar{\mathbf{K}}\Big(\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\bar{\mathcal{A}}_{1,\gamma_i}\Big) = -\delta Z_{1,\gamma_i} + \delta\mathcal{R}_{1,\gamma_i}, \qquad \mathbf{K}\Big(\mathbf{S}^{(j)}_{X_j}\mathbf{S}^{(k)}_{X_k}\mathcal{A}_{1,\gamma_i}\Big) = -\delta Z_{1,\gamma_i}(q_2) - \delta\tilde{Z}_{1,\gamma_i}(\tilde{q}_2)$$

Example: $\delta\mathcal{R}_{2,\Gamma} = \Big[\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}$  $+\ \mathbf{S}^{(1)}_{X_1}$  $\delta Z_{1,\gamma_1}\Big]_{D_\mathrm{n}=D}$

$-\ \Big[\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}$  $+\ \mathbf{S}^{(1)}_{X_1}$  $\big(\delta Z_{1,\gamma_1} + \delta\tilde{Z}_{1,\gamma_1} + \delta\mathcal{R}_{1,\gamma_1}\big)\Big]_{D_\mathrm{n}=4}$

Calculation of rational counterterms at two loops

$$\delta\mathcal{R}_{2,\Gamma} = \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \underbrace{\left[\bar{\mathcal{N}}(\bar{q}_1,\bar{q}_2) - \mathcal{N}(q_1,q_2)\right]}_{\text{2-loop numerator difference}} \underbrace{\prod_{i=1}^{3} \left[\sum_{\sigma_i=0}^{X_i} \frac{\Delta_i^{(\sigma_i)}(\bar{q}_i)}{(\bar{q}_i^2 - M^2)^{N_i+\sigma_i}}\right]_{q_3=-q_1-q_2}}_{\text{2-loop tadpole integral}} + \ldots$$

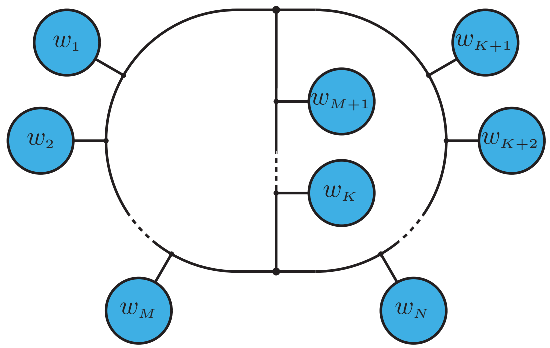
- **Generic method to compute $\delta\mathcal{R}_{2,\Gamma}$ from tadpole integrals with one scale M^2**
- **Result completely independent of M^2** (since definition of $\delta\mathcal{R}_{2,\Gamma}$ is M^2-independent)
- **Only numerators depend on external momenta and masses in polynomial form**
  $\Rightarrow$ $\delta\mathcal{R}_{2,\Gamma}$ **is indeed a rational term**
- **Linearity of the R-operation allows generalisation to sets of diagrams**
  $\Rightarrow$ **Compute finite set of rational counterterms** $\delta\mathcal{R}_{1,\Gamma}$, $\delta\mathcal{R}_{2,\Gamma}$
  and UV counterterms $\delta\tilde{Z}_{1,\Gamma}$, $\delta Z_{2,\Gamma}$, $\delta Z_{2,\Gamma}$ **only for the 1PI UV-divergent vertex functions** Γ **of any renormalisable model once and for all!**

Reducible two-loop amplitudes

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot \mathcal{A}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)$$

which was derived for 1PI amplitudes is **applicable to any two-loop process Γ** due to the factorisation of external subtrees. Full diagram or process (sum of diagrams) Γ:

$\bar{\mathcal{M}}_{2,\Gamma} =$

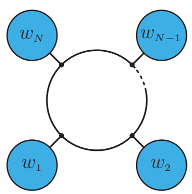


$$= \underbrace{\bar{\mathcal{A}}_{2,\Gamma}^{\sigma_1\cdots\sigma_N}}_{\text{1PI amputated amplitude}} \cdot \underbrace{\prod_{i=1}^{N}\,[w_i]_{\sigma_i}}_{\text{External subtrees (blue bubbles)}}\,,$$

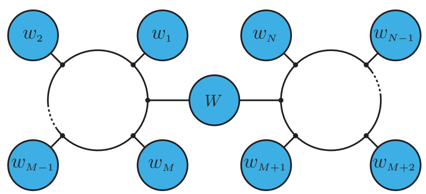
$$\mathbf{R}\,\bar{\mathcal{M}}_{2,\Gamma} \;=\; \left(\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma}^{\sigma_1\cdots\sigma_N}\right)\,\prod_{i=1}^{N}\,[w_i]_{\sigma_i}\,.$$

$\Rightarrow$ **Tree structures do not generate rational terms**

Similar for amplitudes composed of 1PI one-loop subdiagrams,



and



$\Rightarrow$ **Finite set of rational counterterms stemming from 1PI UV-divergent vertex functions allow for two-loop computation of all processes**

Optimisations of the calculation of rational terms

The tadpole expansion of a single propagator

$$\mathbf{S}^{(i)}_{X}\frac{1}{D_k(\bar{q}_i)} = \sum_{\sigma=0}^{X} \frac{[\Delta_k(\bar{q}_i)]^\sigma}{(\bar{q}_1^2 - M^2)^{\sigma+1}} \quad \text{with} \quad \Delta_k(\bar{q}_i) = -p_k^2 - 2\bar{q}_i \cdot p_k + m_k^2 - M^2$$

is designed such that $(1-\mathbf{S}^{(i)}_{X})\frac{1}{D_k(\bar{q}_i)} \leq \mathcal{O}\left(\frac{1}{\bar{q}_i^{X+1}}\right)$. But it contains different orders of $\bar{q}_i$

⇒ Potentially many finite terms generated, which cancel in the difference

$$\delta\mathcal{R}_{2,\Gamma} = \mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\left(\bar{\mathcal{A}}_{2,\Gamma} - \mathcal{A}_{2,\Gamma}\right) + \ldots$$

Optimisations (for details see **JHEP 10 (2020) 016** [arXiv:2007.03713]) [Lang, Pozzorini, Zhang, MZ]

- Power counting in external momenta and masses ⇒ Restriction to mass dimension of the result
- Power counting in loop momenta $\bar{q}_i$ ⇒ Discard terms without UV (sub)divergences
- Taylor expansion trick:
  - Add the auxiliary mass M^2 in every propagator denominator D_k by hand
  - Generate the relevant terms of the tadpole expansion through a Taylor expansion in external masses and propagators
  - Perform a separate M^2-expansion **or** use the M^2-independence of the result to construct auxiliary M^2-counterterms order by order.

Our calculation of two-loop rational terms

- Two independent frameworks for the calculation
  - GEFICOM [Chetyrkin, M.Z.] framework: QGRAF [Noguira] $\to$ Q2E+EXP [Seidesticker, Harlander, Steinhauser] $\to$ FORM [Vermaseren] code $\to$ MATAD [Steinhauser]
  - In-house framework for the reduction of tensor integrals with tadpole topologies, based on IBP identities [Chetyrkin]

- The tadpole expansion and these frameworks can also be used to calculate UV counterterms

$$\delta Z_{2,\Gamma} = \mathbf{K}\left(\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}\bar{\mathcal{A}}_{2,\Gamma} - \sum_{\gamma_i}\delta Z_{1,\gamma_i}\cdot\mathbf{S}^{(i)}_{X_i}\bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\right)$$

$\to$ Agreement with literature.

Note: Similar methods have been used for higher-order UV counterterm calculations [Misiak, Münz; Chetyrkin; MZ]

Results: Two-loop rational counterterms for QED

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\,(1-\eta)}(\partial^\mu A_\mu)^2 \qquad \leftarrow \eta = 0 : \text{Feynman gauge}$$

Structure of rational term for fermion propagator in $\overline{MS}$-like scheme Y:

$$= \mathrm{i}\sum_{k=1}^{2}\left(\frac{\alpha\, t_Y^\varepsilon}{4\pi}\right)^k \left[\delta\hat{\mathcal{R}}^{(\mathrm{P})}_{k,ee}\, \not{p}_{\alpha\beta} + \delta\hat{\mathcal{R}}^{(\mathrm{m})}_{k,ee}\, m\,\delta_{\alpha\beta}\right], \qquad t_Y = \frac{S_Y \mu_0^2}{\mu_R^2}$$

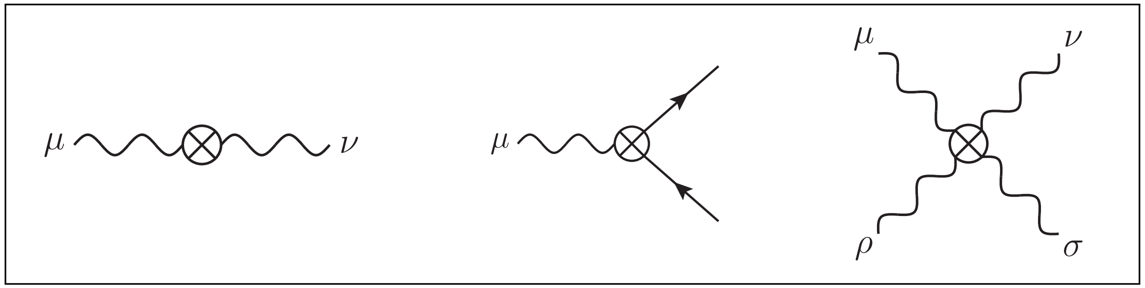
(dim-reg scale μ_0 and renormalisation scale μ_R, scale factor e.g. $S^\varepsilon_{\overline{\text{MS}}} = (4\pi)^\varepsilon\,\Gamma(1+\varepsilon)$)

$$\delta\hat{\mathcal{R}}^{(\mathrm{P})}_{1,ee} = -1 + \frac{2}{3}\eta\,, \qquad \delta\hat{\mathcal{R}}^{(\mathrm{P})}_{2,ee} = \left(\frac{19}{18} - \frac{143}{72}\eta + \frac{11}{30}\eta^2\right)\frac{1}{\varepsilon} + \left(\frac{247}{108} + \frac{293}{864}\eta + \frac{391}{14400}\eta^2\right),$$

$$\delta\hat{\mathcal{R}}^{(\mathrm{m})}_{1,ee} = 2 - \frac{1}{2}\eta\,, \qquad \delta\hat{\mathcal{R}}^{(\mathrm{m})}_{2,ee} = \left(-11 + \frac{41}{9}\eta - \frac{1}{4}\eta^2\right)\frac{1}{\varepsilon} + \left(-\frac{5}{6} - \frac{13}{54}\eta - \frac{7}{288}\eta^2\right)$$

- **Rational terms are gauge-dependent**
- **Interaction of $\tilde{\mathcal{N}}$ with $\frac{1}{\varepsilon^2}$ poles leads to rational terms $\propto \frac{1}{\varepsilon}$ at two loops**

All QED results in **JHEP 05 (2020) 077**
[arXiv:2001.11388] [Pozzorini, Zhang, MZ]



Checks on our calculation

- M^2-independence of the $\delta\mathcal{R}_2$

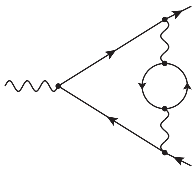
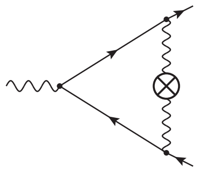
- Pole cancellation in Master formula (including $\delta\mathcal{R}_{2,\Gamma}$ pole)

$$\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} = \mathcal{A}_{2,\Gamma} + \sum_{\gamma_i}\left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)$$

- $\delta Z^{\gamma}_{,1}, \delta\tilde{Z}^{\gamma}_{,1}$ and $\delta\mathcal{R}_{1,\gamma}$ insertions at the level of full one-loop vertex functions and at the level of individual diagrams using the tadpole expansion

- All optimisations checked independently. Different optimisations lead to different finite parts in (expanded) amplitudes which cancel in $\delta\mathcal{R}_{2,\Gamma}$ $\rightarrow$ check of the Master formula

- With the Taylor expansion trick the results become independent of the parametrisation of the loop momenta [Lang, Pozzorini, Zhang, MZ] $\Rightarrow$ A relative shift between loop momentum $\bar{q}_1$ in e.g.

$\mathbf{S}^{(1)}_{X_1}\mathbf{S}^{(2)}_{X_2}\mathbf{S}^{(3)}_{X_3}$  and $\mathbf{S}^{(1)}_{X_1}$  does not change $\delta\mathcal{R}_{2,\Gamma}$.

(All other variants of our method require the same parametrisation in all diagrams)

III Renormalisation scheme dependence

Goal: Generalisation of our master formulas to a generic renormalisation scheme Y

$$\mathbf{R}^{(Y)}\bar{\mathcal{A}}_{1,\Gamma} = \mathcal{A}_{1,\Gamma} + \delta Z^{(Y)}_{1,\Gamma} + \delta\mathcal{R}^{(Y)}_{1,\Gamma}\,,$$

$$\mathbf{R}^{(Y)}\bar{\mathcal{A}}_{2,\Gamma} = \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\left(\delta Z^{(Y)}_{1,\gamma} + \delta\tilde{Z}^{(Y)}_{1,\gamma} + \delta\mathcal{R}^{(Y)}_{1,\gamma}\right)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \delta Z^{(Y)}_{2,\Gamma} + \delta\mathcal{R}^{(Y)}_{2,\Gamma}$$

with counterterms $\delta Z^{(Y)}$ stemming from the **multiplicative renormalisation** of parameters and fields in the Lagrangian:

$$\underbrace{\varphi_{j,0}}_{\text{bare field}} = \left(\mathcal{Z}^{(Y)}_{\varphi_j}\right)^{1/2}\underbrace{\varphi_{i,Y}}_{\text{renormalised field in scheme } Y}\,,\qquad \underbrace{\theta_{i,0}}_{\text{bare parameter}} = \mathcal{Z}^{(Y)}_{\theta_i}\underbrace{\theta_{i,Y}}_{\text{renormalised parameter in } Y}\,,\quad \theta_i = \alpha,\lambda,m$$

Perturbative expansion in gauge coupling $\alpha = \alpha_Y(\mu_{\mathrm{R}}^2)$ and scale factor $t_Y = \frac{S_Y\mu_0^2}{\mu_{\mathrm{R}}^2}$:

$$\mathcal{Z}^{(Y)}_{\chi} = 1 + \sum_{k=1}^{\infty}\delta\mathcal{Z}^{(Y)}_{k,\chi} = 1 + \sum_{k=1}^{\infty}\underbrace{(t_Y^{\varepsilon})^k}_{\text{Explicit scale dependence}}\left(\underbrace{\delta\mathcal{Z}^{(\mathrm{MS})}_{k,\chi}}_{\text{Pole part (MS counterterm)}} + \underbrace{\delta\mathcal{Z}^{(\Delta Y)}_{k,\chi}}_{\text{Finite part}}\right)\quad \text{for } \chi = \alpha,\lambda,m,\varphi_j$$

Idea: Express multiplicative renormalisation through (derivative) operators generating the appropriate counterterm insertions into unrenormalised amplitudes → Similar to **R**-operation

Multiplicative renormalisation through derivative operators

Renormalisation formula for a scattering amplitude

$$\underbrace{\mathbf{R}^{(Y)}\bar{\mathcal{A}}_{\Gamma}\left(\{\theta_{i,Y}\}\right)}_{\substack{\text{renormalised amplitude}\\ \text{expressed through}\\ \text{renormalised parameters}}} = \underbrace{\left[\prod_j \left(\mathcal{Z}^{(Y)}_{\varphi_j}\right)^{1/2}\right]}_{\substack{\text{renormalisation of}\\ \text{external legs } j}} \underbrace{\bar{\mathcal{A}}_{\Gamma}\left(\{\theta_{i,0}\}\right)}_{\substack{\text{amplitude computed}\\ \text{with bare quantities}}} =: \sum_{k=0}^{\infty} D^{(Y)}_k \underbrace{\bar{\mathcal{A}}_{\Gamma}\left(\{\theta_{i,Y}\}\right)}_{\substack{\text{unrenormalised amplitude}\\ \text{expressed through}\\ \text{renormalised parameters}}}$$

with operators $D^{(Y)}_0 = 1\,,\quad D^{(Y)}_1 = \sum_i \delta\mathcal{Z}^{(Y)}_{1,\theta_i}\,\theta_i \frac{\partial}{\partial\theta_i} + \sum_j \frac{1}{2}\delta\mathcal{Z}^{(Y)}_{1,\varphi_j}\,,\quad D^{(Y)}_2 = \sum_i \delta\mathcal{Z}^{(Y)}_{2,\theta_i}\,\theta_i \frac{\partial}{\partial\theta_i} + \dots$

For the first orders in perturbation theory:

$$\begin{aligned}
\mathbf{R}^{(Y)}\bar{\mathcal{A}}_{1,\Gamma} &= \bar{\mathcal{A}}_{1,\Gamma} + D^{(Y)}_1 \mathcal{A}_{0,\Gamma}\,,\\
\mathbf{R}^{(Y)}\bar{\mathcal{A}}_{2,\Gamma} &= \bar{\mathcal{A}}_{2,\Gamma} + D^{(Y)}_1 \bar{\mathcal{A}}_{1,\Gamma} + D^{(Y)}_2 \mathcal{A}_{0,\Gamma}\,.
\end{aligned}$$

Split the operators

$$D^{(Y)}_1 = \underbrace{\left(t^{\varepsilon}_Y\right)}_{\text{scale factor}} \left(\underbrace{D^{(\mathrm{MS})}_1}_{\substack{\text{generates MS}\\ \text{counterterms}}} + \underbrace{D^{(\Delta Y)}_1}_{\substack{\text{generates finite}\\ \text{counterterms}}} \right)$$

One-loop: Finite counterterms $\delta\mathcal{Z}^{(\Delta Y)}_{1,\chi}$ inserted into tree-level diagrams do not create rational terms

Two-loop: Insertion of $\delta\mathcal{Z}^{(\Delta Y)}_{1,\chi}$ into one-loop diagram $\bar{\mathcal{A}}_{1,\Gamma}$ will lead to non-trivial rational terms stemming from its D-dimensional numerator

Renormalisation scheme dependence of one-loop rational counterterms

Strategy: Split renormalised amplitudes in MS and ΔY contributions $\Rightarrow$ **Apply master formula**

$$\mathbf{R}^{(Y)}\bar{\mathcal{A}}_{1,\Gamma} = \underbrace{\bar{\mathcal{A}}_{1,\Gamma} + (t_Y^{\varepsilon})\,\delta Z_{1,\Gamma}^{(\mathrm{MS})}\bar{\mathcal{A}}_{1,\Gamma}}_{=:\mathbf{R}^{(\mathrm{MS}_Y)}\bar{\mathcal{A}}_{1,\Gamma}} + (t_Y^{\varepsilon})\,D_1^{(\Delta Y)}\mathcal{A}_{0,\Gamma}$$

$\leftarrow$ renormalised amplitude in $\overline{\mathrm{MS}}$-like scheme scale factor t_Y

and the **master formula for a one-loop scattering amplitude** in this scheme MS_Y

$$\mathbf{R}^{(\mathrm{MS}_Y)}\bar{\mathcal{A}}_{1,\Gamma} = \mathcal{A}_{1,\Gamma} + (t_Y^{\varepsilon})\left(\delta Z_{1,\Gamma}^{(\mathrm{MS})} + \delta\mathcal{R}_{1,\Gamma}^{(\mathrm{MS})}\right)$$

we derive

$$\begin{aligned}\mathbf{R}^{(Y)}\bar{\mathcal{A}}_{1,\Gamma} &= \mathcal{A}_{1,\Gamma} + t_Y^{\varepsilon}\Big(\delta Z_{1,\Gamma}^{(\mathrm{MS})} + \delta\mathcal{R}_{1,\Gamma}^{(\mathrm{MS})} + \underbrace{D_1^{(\Delta Y)}\mathcal{A}_{0,\Gamma}}_{=\delta Z_{1,\Gamma}^{(\Delta Y)}}\Big)\\ &\equiv \mathcal{A}_{1,\Gamma} + \delta Z_{1,\Gamma}^{(Y)} + \delta\mathcal{R}_{1,\Gamma}^{(Y)}\end{aligned}$$

$\Rightarrow$ **One-loop rational counterterms are scheme-independent apart from a trivial scale dependence** (same for $\delta\tilde{Z}$)

$$\delta\mathcal{R}_{1,\Gamma}^{(Y)} = t_Y^{\varepsilon}\,\mathcal{R}_{1,\Gamma}^{(\mathrm{MS})} = \mathcal{R}_{1,\Gamma}^{(\mathrm{MS})} + \mathcal{O}(\varepsilon)$$

Renormalisation scheme dependence of two-loop rational counterterms

Master formula for two-loop amplitudes:

$$\mathbf{R}^{(Y)}\,\bar{\mathcal{A}}_{2,\Gamma} \;=\; \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\left(\delta Z_{1,\gamma}^{(Y)} + \delta\tilde{Z}_{1,\gamma}^{(Y)} + \delta\mathcal{R}_{1,\gamma}^{(Y)}\right)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma}^{(Y)} + \delta\mathcal{R}_{2,\Gamma}^{(Y)}\right)$$

with

$$\delta\mathcal{R}_{2,\Gamma}^{(Y)} \;=\; \underbrace{(t_Y^{\varepsilon})^2\,\delta\mathcal{R}_{2,\Gamma}^{(\mathrm{MS})}}_{\substack{\text{rescaling of two-loop}\\ \text{rational term (with } \frac{1}{\varepsilon}\text{-poles)}}} + \underbrace{(t_Y^{\varepsilon})^2\,D_1^{\Delta Y}\,\delta\mathcal{R}_{1,\Gamma}^{(\mathrm{MS})}}_{\substack{\text{multiplicative renormalisation}\\ \text{of one-loop rational term}}} + \underbrace{\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)}}_{\substack{\text{non-trivial remainder}\\ \text{from 4-dim numerator}}}$$

where $\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)}$ stems from the subtlety that the multiplicative renormalisation of a one-loop amplitude $\mathcal{A}_{1,\Gamma}$ **after projection** to numerator dimension $D_\mathrm{n} = 4$ does not give the same result as a counterterm insertion with **subsequent projection** to $D_\mathrm{n} = 4$:

$$\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)} \;=\; (t_Y^{\varepsilon})\left(D_1^{(\Delta Y)}\mathcal{A}_{1,\Gamma} - \sum_{\gamma}\delta Z_{1,\gamma}^{(\Delta Y)}\cdot\mathcal{A}_{1,\Gamma/\gamma}\right) \neq 0$$

But $\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)}$ can be expressed through one-loop renormalisation constants and a small set of universal scheme-independent counterterms (presented in **JHEP 10 (2020) 016** [arXiv:2007.03713]):

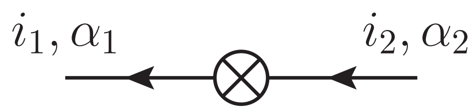
$$\delta\mathcal{K}_{2,\Gamma}^{(\Delta Y)} \;=\; \sum_{\chi}\delta\mathcal{Z}_{1,\chi}^{(\Delta Y)}\,\delta\hat{\mathcal{K}}_{1,\Gamma}^{(\chi)}$$

$\Rightarrow$ **Full renormalisation scheme dependence of two-loop rational terms available**

IV. Two-loop rational terms for SU(N) and U(1) in a generic scheme

- Rational terms for a 1PI vertex function Γ depend on the scale factor t^{ε} and the renormalisation constants $\mathcal{Z}_{\chi} = 1 + \Sigma_{k=1}^{\infty} \left(\frac{\alpha\, t^{\varepsilon}}{4\pi}\right)^k \delta\hat{\mathcal{Z}}_{k,\chi}$ for parameters $\chi = \alpha,\, m_f, \lambda$ and fields $\chi = f,\, A,\, u$
- Set the gauge parameter $\lambda = 1$ (Feynman gauge), but keep generic renormalisation $\mathcal{Z}_{\rm gp} = \frac{\mathcal{Z}_A}{\mathcal{Z}_\lambda}$
- Express result in terms of Casimirs $C_{\rm F}, C_{\rm A}$ and fundamental trace $T_{\rm F}$ and dimension N

Two-point function of a fermion f



$$= \quad \mathrm{i} \underbrace{\delta_{i_1 i_2}}_{\text{gauge group structure}} \left\{ \sum_{k=1}^{2} \left(\frac{\alpha_s\, t^{\varepsilon}}{4\pi}\right)^k \left[\delta\hat{\mathcal{R}}^{(\rm P)}_{k,\rm ff}\, \not{p}_{\alpha_1\alpha_2} + \delta\hat{\mathcal{R}}^{(\rm m)}_{k,\rm ff}\, m_f\, \delta_{\alpha_1\alpha_2}\right]\right\},$$

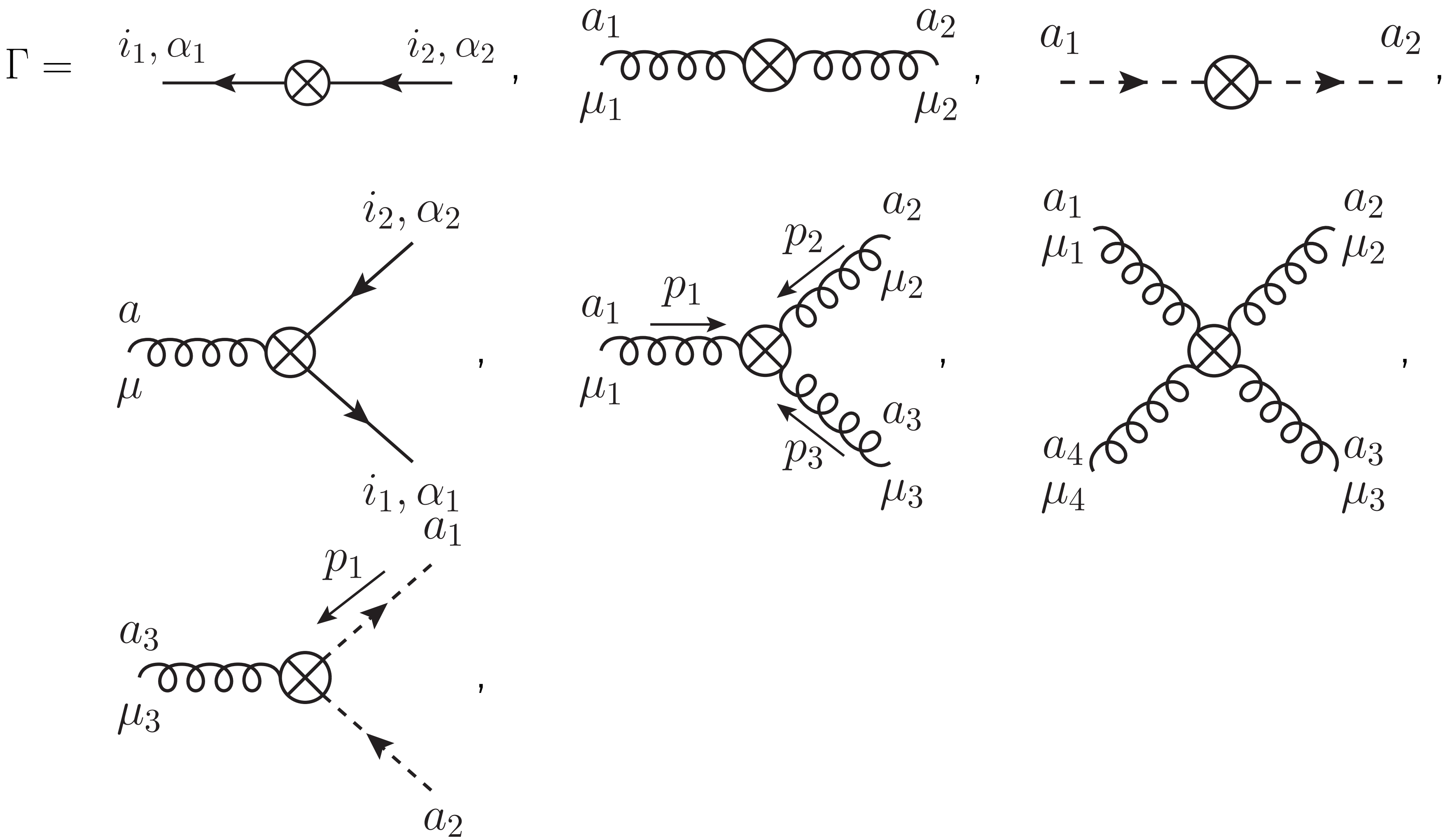
$$\delta\hat{\mathcal{R}}^{(\rm P)}_{1,\rm ff} = -\,C_{\rm F}\,,$$

$$\delta\hat{\mathcal{R}}^{(\rm P)}_{2,\rm ff} = \left(\frac{7}{6} C_{\rm F}^2 - \frac{61}{36} C_{\rm A}\, C_{\rm F} + \frac{5}{9} T_{\rm F}\, n_{\rm f}\, C_{\rm F}\right) \frac{1}{\varepsilon} + \left(\frac{43}{36} C_{\rm F}^2 - \frac{1087}{216} C_{\rm A}\, C_{\rm F} + \frac{59}{54} T_{\rm F}\, n_{\rm f}\, C_{\rm F}\right)$$
$$-\, C_{\rm F} \left(\underbrace{\delta\hat{\mathcal{Z}}_{1,\alpha_s} + \frac{2}{3}\,\delta\hat{\mathcal{Z}}_{1,f} - \frac{2}{3}\,\delta\hat{\mathcal{Z}}_{1,\rm gp}}_{\text{Renormalisation scheme dependendent}} \right)$$

Similarly for $\delta\hat{\mathcal{R}}^{(\rm m)}_{1,\rm ff}, \mathcal{R}^{(\rm m)}_{2,\rm ff}$

Two-loop rational terms for SU(N) and U(1) in a generic scheme

Complete set of $\delta\mathcal{R}_{L,\Gamma}$, $\delta Z_{L,\Gamma}$ $(L = 1, 2)$ and $\delta\tilde{Z}_{1,\Gamma}$, i.e.



available in **JHEP 10 (2020) 016** [arXiv:2007.03713] [Lang, Pozzorini, Zhang, MZ]

Summary

- Renormalised amplitudes in D-dimensions can be computed from amplitudes with 4-dimensional numerators and a **finite set of universal UV and rational counterterm insertions**:

$$
\begin{aligned}
\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma} &= \mathcal{A}_{1,\gamma} + \delta Z_{1,\gamma} + \delta\mathcal{R}_{1,\gamma} \\
\mathbf{R}\,\bar{\mathcal{A}}_{2,\Gamma} &= \mathcal{A}_{2,\Gamma} + \sum_{\gamma}\left(\delta Z_{1,\gamma} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right)\cdot\mathcal{A}_{1,\Gamma/\gamma} + \left(\delta Z_{2,\Gamma} + \delta\mathcal{R}_{2,\Gamma}\right)
\end{aligned}
$$

  $\Rightarrow$ **Numerical implementation in automated tools, e.g. OpenLoops, possible**

- **Generic method** to compute $\delta\mathcal{R}_{L,\gamma}$, $\delta\tilde{Z}_{1,\gamma}$ and $\delta Z_{L,\gamma}$ from simple tadpole integrals, which also serves as a **proof that they are rational**

- **Complete renormalisation scheme dependence presented:**
  - One-loop: $\delta\mathcal{R}_{1,\gamma}$, $\delta\tilde{Z}_{1,\gamma}$ only have a trivial scale-dependence
  - Two-loop: $\delta\mathcal{R}_{2,\Gamma}$ contains multiplicative renormalisation of one-loop rational terms and additional universal one-loop counterterms (which we presented)

- **Full set of rational terms at two-loop level for $SU(N)$ and $U(1)$** in any renormalisation scheme, for QED with full dependence on the gauge parameter

Outlook

- Complete the study of two-loop rational terms of UV origin
  - Connection between rational terms in symmetric theories and their spontaneously broken counterparts (to be published soon)
  - Two-loop rational terms of the full Standard Model
- Computation (to order ε) of the modified one-loop tensor integrals

$$T_{N,k}^{\bar{\mu}_1\cdots\bar{\mu}_r} = \int \mathrm{d}\bar{q}_1 \, \frac{(\tilde{q}^2)^k \, \bar{q}_1^{\mu_1} \cdots \bar{q}_1^{\mu_r}}{D_0(\bar{q}_1) \cdots D_{N-1}(\bar{q}_1)}$$

  for the numerical calculation of one-loop amplitudes with $\delta\tilde{Z}_{1,\gamma}$ insertions.

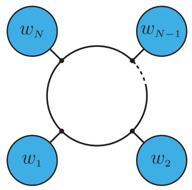
- Study of rational terms of IR origin
- Implementation in OpenLoops

Backup

Reducible one-loop diagrams

Generic unrenormalised amplitude of a one-loop diagram γ

$$\bar{\mathcal{M}}_{1,\gamma} \quad = $$



$$= \bar{\mathcal{A}}_{1,\gamma}^{\sigma_1\ldots\sigma_N} \prod_{i=1}^{N} [w_i]_{\sigma_i} \,,$$

- Amplitude $\bar{\mathcal{A}}_{1,\gamma}$ of the 1PI amputated one-loop subdiagram of γ
- Factorised subtrees w_i (blue bubbles)

$$\mathbf{R}\,\bar{\mathcal{M}}_{1,\gamma} \quad = \quad \left(\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma}^{\sigma_1\ldots\sigma_N}\right) \prod_{i=1}^{N} [w_i]_{\sigma_i} \,.$$

In the 't Hooft–Veltman scheme all tree structures w_i are in 4 dimensions.

$\Rightarrow$ External momenta and indices of the 1PI amplitude $\bar{\mathcal{A}}_{1,\gamma}$ handled as 4-dimensional

$\Rightarrow$ **Tree structures do not generate rational terms** (even in other schemes due to factorisation)

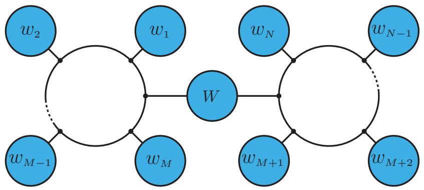
$\Rightarrow$ **Rational terms can be determined at the level of 1PI subdiagrams**

Reducible two-loop diagrams

Reducible after amputation of all external subtrees

Generic unrenormalised amplitude of diagram Γ_{red}

$$\bar{\mathcal{M}}_{2,\Gamma_{\mathrm{red}}} =$$



$$= \bar{\mathcal{A}}_{1,\gamma_1}^{\alpha_1\sigma_1\cdots\sigma_M} \, W_{\alpha_1\alpha_2} \, \bar{\mathcal{A}}_{1,\gamma_2}^{\alpha_2\sigma_{M+1}\cdots\sigma_N} \prod_{i=1}^{N} [w_i]_{\sigma_i} \,.$$

- Amplitudes $\bar{\mathcal{A}}_{1,\gamma_1}$ and $\bar{\mathcal{A}}_{1,\gamma_2}$ of the 1PI amputated two-loop subdiagram of Γ
- Factorised subtrees w_i and connecting tree structure W (blue bubbles) .

$$\mathbf{R}\,\bar{\mathcal{M}}_{2,\Gamma_{\mathrm{red}}} = \left(\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma}^{\alpha_1\sigma_1\cdots\sigma_M}\right) W_{\alpha_1\alpha_2} \left(\mathbf{R}\,\bar{\mathcal{A}}_{1,\gamma}^{\alpha_2\sigma_{M+1}\cdots\sigma_N}\right) \prod_{i=1}^{N} [w_i]_{\sigma_i} \,.$$

In the 't Hooft–Veltman scheme all tree structures w_i and W are in 4 dimensions.

$\Rightarrow$ External momenta and indices of the 1PI amplitude $\bar{\mathcal{A}}_{1,\gamma}$ handled as 4-dimensional

$\Rightarrow$ **Tree structures do not generate rational terms** (even in other schemes due to factorisation)

$\Rightarrow$ **Rational terms can be determined at the level of 1PI subdiagrams**

Proof of master formula for one-loop subdiagrams

Fully UV subtracted amplitude in D dimensions:

$$\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) = \sum_{r=0}^{R} \bar{\mathcal{N}}^{\bar{\alpha}}_{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2)\left[T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2) - \mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2)\right].$$

Fully UV subtracted amplitude in 4 dimensions:

$$\mathcal{A}^{\alpha}_{1,\gamma}(q_2) - \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(q_2) = \sum_{r=0}^{R} \mathcal{N}^{\alpha}_{\mu_1\cdots\mu_r}(q_2)\left[T_N^{\mu_1\cdots\mu_r}(\bar{q}_2) - \mathbf{K}\,T_N^{\mu_1\cdots\mu_r}(\bar{q}_2)\right].$$

Since all UV poles are cancelled in $[\ldots]$ we find

$$\left[T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2) - \mathbf{K}\,T_N^{\bar{\mu}_1\cdots\bar{\mu}_r}(\bar{q}_2)\right] = \left[T_N^{\mu_1\cdots\mu_r}(\bar{q}_2) - \mathbf{K}\,T_N^{\mu_1\cdots\mu_r}(\bar{q}_2)\right] + \mathcal{O}(\varepsilon,\tilde{q}_2).$$

From this follows

$$\boxed{\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) - \bar{\mathbf{K}}\,\bar{\mathcal{A}}^{\bar{\alpha}}_{1,\gamma}(\bar{q}_2) = \mathcal{A}^{\alpha}_{1,\gamma}(q_2) - \bar{\mathbf{K}}\,\mathcal{A}^{\alpha}_{1,\gamma}(q_2) + \mathcal{O}(\varepsilon,\tilde{q}_2)}$$

Practical calculation of two-loop rational terms

$$
\begin{aligned}
\delta\mathcal{R}_{2,\Gamma} &= \left(\bar{\mathcal{A}}_{2,\Gamma} + \sum_{\gamma_i} \delta Z_{1,\gamma_i} \cdot \bar{\mathcal{A}}_{1,\Gamma/\gamma_i}\right) - \left(\mathcal{A}_{2,\Gamma} + \sum_{\gamma_i} \left(\delta Z_{1,\gamma_i} + \delta\tilde{Z}_{1,\gamma} + \delta\mathcal{R}_{1,\gamma}\right) \cdot \mathcal{A}_{1,\Gamma/\gamma}\right) \\
&= \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \left[\bar{\mathcal{N}}(\bar{q}_1,\bar{q}_2) - \mathcal{N}(q_1,q_2)\right] \left[\prod_{i=1}^{3} \mathbf{S}_{X_i}^{(i)} \frac{1}{\mathcal{D}^{(i)}(\bar{q}_i)}\right]_{q_3=-q_1-q_2} \\
&\quad + \sum_{i=1}^{3} \int \mathrm{d}\bar{q}_i \left[\delta Z_{1,\gamma_i}(\bar{q}_i) \cdot \bar{\mathcal{N}}^{(i)}(\bar{q}_i)\right. \\
&\quad \left. - \left(\delta Z_{1,\gamma_i}(q_i) + \delta\tilde{Z}_{1,\gamma_i}(\tilde{q}_i) + \delta\mathcal{R}_{1,\gamma_i}(q_i)\right) \cdot \mathcal{N}^{(i)}(q_i)\right] \mathbf{S}_{X_i}^{(i)} \left(\frac{1}{\mathcal{D}^{(i)}(\bar{q}_i)}\right) \\
&= \int \mathrm{d}\bar{q}_1 \int \mathrm{d}\bar{q}_2 \sum_{r_1=0}^{R_1} \sum_{r_2=0}^{R_2} \left[\bar{\mathcal{N}}_{\bar{\mu}_1\cdots\bar{\mu}_{r_1}\bar{\nu}_1\cdots\bar{\nu}_{r_2}} - \mathcal{N}_{\mu_1\cdots\mu_{r_1}\nu_1\cdots\nu_{r_2}}\right] \times \\
&\quad \times \left[\sum_{\sigma_1=0}^{X_1} \sum_{\sigma_2=0}^{X_2} \sum_{\sigma_3=0}^{X_3} \frac{\bar{q}_1^{\mu_1} \cdots \bar{q}_1^{\mu_{r_1}} \bar{q}_2^{\nu_1} \cdots \bar{q}_2^{\nu_{r_2}} \Delta_1^{(\sigma_1)}(\bar{q}_1)\Delta_2^{(\sigma_2)}(\bar{q}_2)\Delta_3^{(\sigma_3)}(\bar{q}_3)}{(\bar{q}_1^2 - M^2)^{N_1+\sigma_1}(\bar{q}_2^2 - M^2)^{N_2+\sigma_2}(\bar{q}_3^2 - M^2)^{N_3+\sigma_3}}\right]_{q_3=-q_1-q_2} \\
&\quad + \ldots
\end{aligned}
$$

$\Rightarrow$ Computation with D-dim tensor integrals and differences of numerator coefficients in $D_\mathrm{n} = D$ and $D_\mathrm{n} = 4$ dimensions