

CUSP AND COLLINEAR ANOMALOUS DIMENSIONS IN FOUR-LOOP QCD FROM FORM FACTORS

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supported by the National Science Foundation

Theorieseminar DESY Zeuthen / HU Berlin
Zoom, 9. Juli 2020

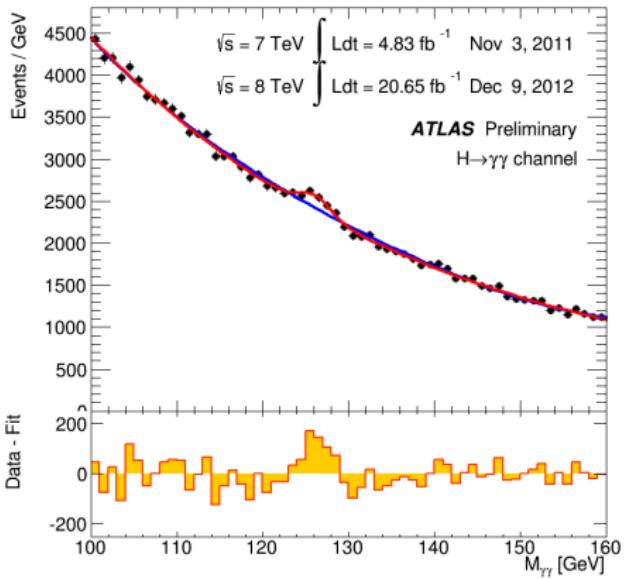
THE NEED FOR PRECISION PHYSICS @ LHC

- precise test of theory:

- ▶ parameters
- ▶ electroweak symmetry breaking
- ▶ quantum effects

- new physics searches:

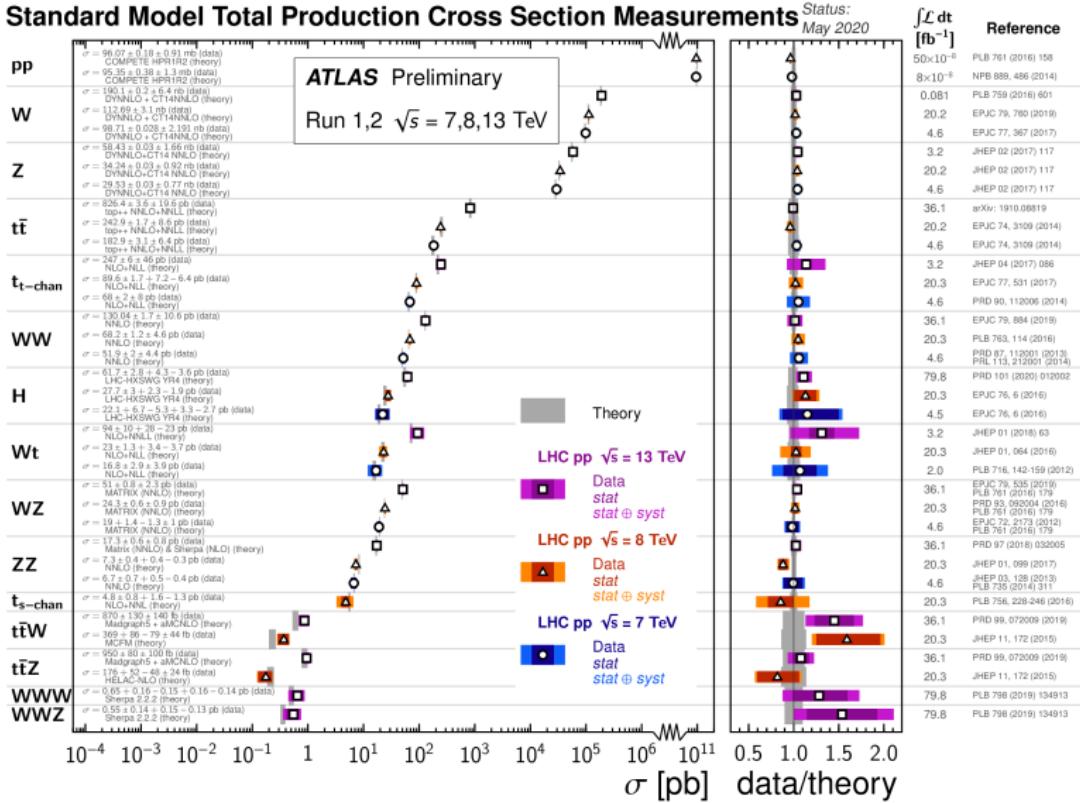
- ▶ don't expect large and simple signals
- ▶ requires precise measurements
- ▶ requires precise theory predictions



STANDARD MODEL PROCESSES @ LHC

Standard Model Total Production Cross Section Measurements

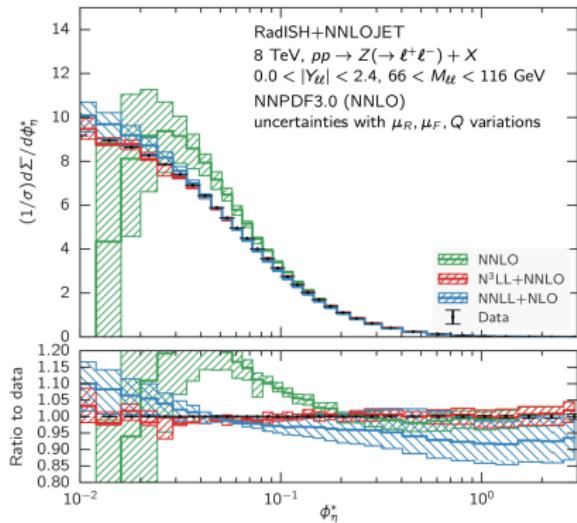
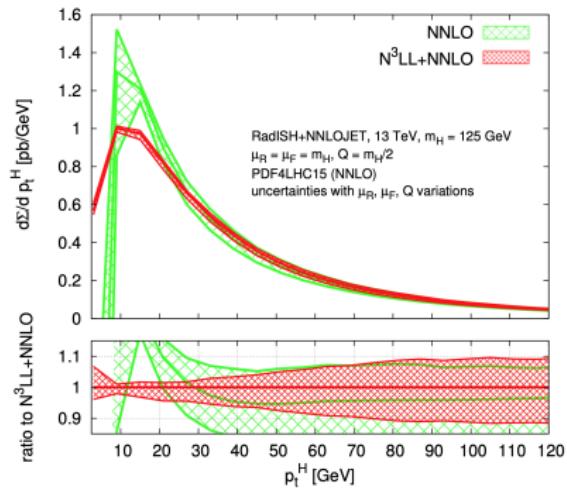
Status:
May 2020



- want fixed-order predictions at high perturbative order

DISTRIBUTIONS @ LHC

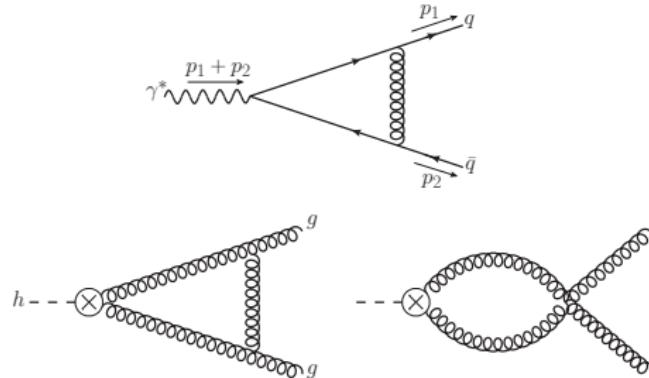
[Bizon, Chen, Gehrmann-De Ridder, Gehrmann, Glover, Huss, Monni, Re, Rottoli, Torrielli 2018]



- low p_T region requires to supplement fixed-order with resummations
- N3LL requires **cusp anomalous dimension at 4 loops**

THE BASIC QUARK AND GLUON FORM FACTORS

- In this talk, I focus on the basic $q\bar{q}\gamma^*$ and ggH vertices
- consider perturbative expansion, e.g. at 1-loop:



- calculate form factors through 4 loop QCD in dim. reg. ($d = 4 - 2\epsilon$)
- they provide virtual corrections to Drell-Yan and Higgs production at N4LO
- after UV renormalization, the amplitudes contain soft and collinear poles in ϵ
- leading poles through to $1/\epsilon^2$ predicted by cusp anomalous dimension Γ_{cusp}
- $1/\epsilon$ pole involves collinear anomalous dimensions

EVOLUTION EQUATION FOR THE FORM FACTOR

- The evolution of the form factor \mathcal{F} can be written as [Magnea, Sterman 1990]

$$Q^2 \frac{\partial}{\partial Q^2} \ln \mathcal{F}(\alpha_s, Q^2/\mu^2, \epsilon) = \frac{1}{2} K(\alpha_s, \epsilon) + \frac{1}{2} G(Q^2/\mu^2, \alpha_s, \epsilon)$$

where

- μ is the renormalization scale
- K contains all infrared poles in ϵ
- G contains all dependence on Q^2 and is finite for $\epsilon \rightarrow 0$
- renormalization scale dependence must cancel between K and G :

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s, \epsilon) \frac{\partial}{\partial \alpha_s} \right) G(Q^2/\mu^2, \alpha_s, \epsilon) = \Gamma_{\text{cusp}}(\alpha_s)$$

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(\alpha_s, \epsilon) \frac{\partial}{\partial \alpha_s} \right) K(\alpha_s, \epsilon) = -\Gamma_{\text{cusp}}(\alpha_s)$$

- $\Gamma_{\text{cusp}} = \sum_{l=1}^{\infty} \left(\frac{\alpha_s}{4\pi} \right)^l \Gamma_l$ is the cusp anomalous dimension
- At l loop order, Γ_l encodes the soft poles which are not just the exponentiated poles from lower loops
- Γ_{cusp} universal quantity, appears also in splitting functions and Wilson loops

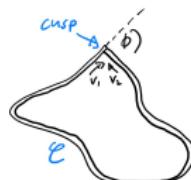
CUSPS IN WILSON LOOPS

- Wilson line:

$$W = \frac{1}{N_R} \langle 0 | \text{tr}_R \mathcal{P} \exp \left(ig \int_C dx^\mu A_\mu(x) \right) | 0 \rangle$$

where $A_\mu(x) = A_\mu^a(x) T^a$ with T^a generators in representation R

- UV renormalization of Wilson loop depends on **cusps** (non-analytic pts) along the path \mathcal{C} :

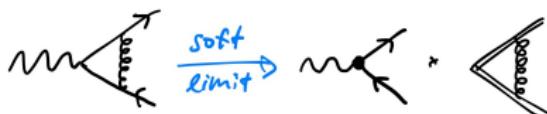


$$\text{where } \cos(\phi) = \frac{v_1 \cdot v_2}{\sqrt{v_1^2} \sqrt{v_2^2}} \text{ (massive path, eucl.)}$$

- Using a multiplicative UV renormalization factor Z such that $Z^{-1}W$ finite, the **angle dependent** cusp anomalous dimension is

$$\Gamma_{\text{cusp}}(\phi, \alpha_s) = \frac{d \ln Z}{d \ln \mu}$$

- light-like cusp anomalous dimension: $\Gamma_{\text{cusp}}(\alpha_s) = \lim_{\phi \rightarrow i\infty} i\Gamma(\phi, \alpha_s)/\phi$
- factorization in **soft limit**:



REPRESENTATION DEPENDENCE

- Cusp anomalous dimension Γ_{cusp}^r is representation r dependent (different for quarks and gluons)
- IR poles of general scattering amplitudes follow simple $T_i T_j$ pattern observed through to 2 loops [Catani '98], considered also for higher loops ("dipole conjecture") [Becher, Neubert '09; Gardi, Magnea '09]
- Fitting this framework, the cusp anomalous dimension was hoped to fulfill quadratic Casimir scaling $\underbrace{\Gamma_{\text{cusp}}^g}_{?} = C_A/C_F \Gamma_{\text{cusp}}^q$ (observed through 3 loops)
- As became clear in the following, picture needs to be generalized to account for more complicated Casimir operators, e.g. [Caron-Huot '15; Almelid et al '16; Henn et al '16; Boels et al '16; Moch et al '17, '18; Lee et al '19; Henn et al '19; Catani et al '19; Becher, Neubert '19]

STATUS @ 4 LOOPS

- Cusp anomalous dimensions ($1/\epsilon^2$, weight 6):

- ▶ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory:

- ★ $\mathcal{N} = 4$ SYM non-planar:

- [Boels, Kniehl, Tarasov, Yang '12, '15], [Boels, Huber, Yang '17], [Henn, Korchemsky, Mistlberger '19], [Huber, AvM, Panzer, Schabinger, Yang '19]

- ★ QCD complete numerical for quark and gluon:
[Moch, Ruijl, Ueda, Vermaseren, Vogt '17, '18]

- ★ QCD analytical:
[Beneke, Braun '95, Grozin, Henn, Korchemsky, Marquard '15, Henn, Smirnov, Smirnov, Steinhauser '16, AvM, Schabinger '16, '19, Grozin '18, Lee, Smirnov, Smirnov, Steinhauser '19, Brüser, Grozin, Henn, Stahlhofen '19, Henn, Peraro, Stahlhofen, Wasser '19], [Henn, Korchemsky, Mistlberger '19]
first complete analytical calculation without any conjectures: [AvM, Panzer, Schabinger '20]

- ★ note: very different approaches (Wilson lines, form factors, . . .)

- Collinear anomalous dimension ($1/\epsilon$, weight 7):

- ▶ $\mathcal{N} = 4$ SYM non-planar: [Dixon '17]

- ▶ QCD full matter dependence for quarks and gluons: [AvM, Panzer, Schabinger '20]

- ▶ QCD numerical for gluons: [Das, Moch, Vogt '20]

- Quark form factors (ϵ^0):

- ▶ n_f^3 , planar, n_f^2 , quartic, $n_f n_{q\gamma}$: [Henn, Smirnov, Smirnov, Steinhauser, Lee '16, '16], [AvM, Schabinger '16, '19, '20], [Lee, Smirnov, Smirnov, Steinhauser, Lee '17, '19]

- Gluon form factors (ϵ^0):

- ▶ n_f^3 , n_f^2 : [AvM, Schabinger '16, '19]

FORM FACTOR APPROACH

[AvM, Erik Panzer, Rober Schabinger]

Analytical calculation of $q\bar{q}\gamma$ and ggH amplitudes to four-loop QCD:

- ① Feynman diagrams (Qgraf) + numerator algebra (Form)
- ② reduction to finite master integrals (Finred)
- ③ solve master integrals by direct integration (Hyperint)
- ④ extract anomalous dimensions from poles

Complexity:

- $O(50000)$ Feynman diagrams
- $O(10^9)$ integrals in diagrams, one scale, up to 6 irreducible scalar products
- $O(100)$ different 12-line topologies, 10 integral families (18 indices)
- $O(25000)$ sectors, $O(2000)$ non-shiftable, max $O(10^8 - 10^9)$ eqs/sector
- $O(300)$ master integrals

Checks:

- use R_ξ gauge (power one for props + ext pol) and show ξ independence
- calculate redundant sets of master integrals, check consistency
- numerical checks of master integrals with Fiesta, ...

Part 1: The Method of Finite Integrals

MULTI-LOOP FEYNMAN INTEGRALS

$$I(\nu_1, \dots, \nu_N) = \int d^d k_1 \cdots d^d k_L \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \quad \nu_i \in \mathbb{Z}, \quad D_1 = k_1^2 - m_1^2 \text{ etc.}$$

family of loop integrals:

- fulfill linear relations: integration-by-parts identities [Chetyrkin, Tkachov '81]
- systematic reduction to master integrals possible [Laporta '00]
- linear vector space with some finite basis
- specific basis choices:
 - ▶ ϵ -basis (uniform weight) for differential equations [Henn '13], [Adams, Weinzierl '17]
 - ▶ basis of finite integrals for direct integration (analyt., numeric.) [A.v.M., Panzer, Schabinger '16]
 - ▶ uniform weight + finite: [Schabinger '18]
- depending on application, ν_i will be different from 0 and 1:
 - ▶ numerators ($\nu_i < 0$): occur naturally in scattering amplitudes, ...
 - ▶ dots ($\nu_i > 1$): occur naturally for finite integrals, ...

observation: always possible to decompose wrt basis of finite integrals

$$\begin{array}{c}
 \text{Diagram 1: } (4-2\epsilon) \\
 \text{Diagram 2: } = -\frac{4(1-4\epsilon)}{\epsilon(1-\epsilon)q^2} \text{ Diagram 3: } (6-2\epsilon) \\
 \\
 \text{Diagram 4: } -\frac{2(2-3\epsilon)(5-21\epsilon+14\epsilon^2)}{\epsilon^4(1-\epsilon)^2(2-\epsilon)^2q^2} \text{ Diagram 5: } (8-2\epsilon) \\
 \\
 \text{Diagram 6: } +\frac{4(2-3\epsilon)(7-31\epsilon+26\epsilon^2)}{\epsilon^4(1-2\epsilon)(1-\epsilon)^2(2-\epsilon)^2q^2} \text{ Diagram 7: } (8-2\epsilon)
 \end{array}$$

basis consists of standard Feynman integrals, but

- in **shifted dimensions**
- with additional **dots** (propagators taken to higher powers)

PRACTICAL ALGORITHM FOR BASIS CONSTRUCTION

ALGORITHM: CONSTRUCTION OF FINITE BASIS

- systematic scan for finite integrals with dim-shifts and dots (with Reduze 2)
- IBP + dimensional recurrence for actual basis change

remarks:

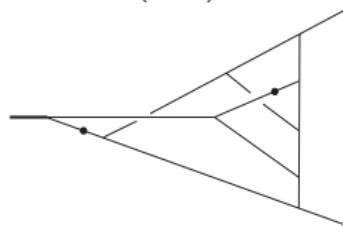
- computationally expensive part shifted to IBP solver
- efficient, easy to automate
- any dim-shift good, e.g. shifts by [Tarasov '96], [Lee '10]
- see [Bern, Dixon, Kosower '93] for dim-shifted one-loop pentagon

ANALYTICAL INTEGRATION @ 4-LOOPS

[AvM, Panzer, Schabinger '15]

a non-planar 12-line topology @ 4-loops:

$(6-2\epsilon)$

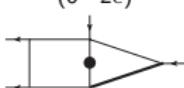
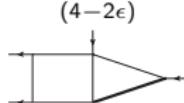
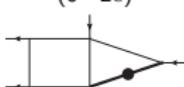
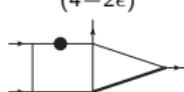
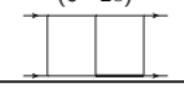
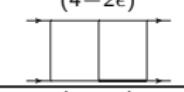
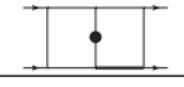
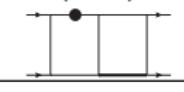


$$\begin{aligned} &= \frac{18}{5} \zeta_2^2 \zeta_3 - 5 \zeta_2 \zeta_5 + \left(24 \zeta_2 \zeta_3 + 20 \zeta_5 - \frac{188}{105} \zeta_2^3 - 17 \zeta_3^2 + 9 \zeta_2^2 \zeta_3 \right. \\ &\quad \left. - 47 \zeta_2 \zeta_5 - 21 \zeta_7 + \frac{6883}{2100} \zeta_2^4 + \frac{49}{2} \zeta_2 \zeta_3^2 + \frac{1}{2} \zeta_3 \zeta_5 - 9 \zeta_{5,3} \right) \epsilon + \mathcal{O}(\epsilon^2) \end{aligned}$$

- numerical result with **Fiesta** [A. Smirnov]: straight-forward confirmation

NUMERICAL PERFORMANCE

- finite integrals render problematic double boxes numerically accessible [AvM, Schabinger '17]

finite	time	rel. err.	conventional	time	rel. err.
$(6-2\epsilon)$ 	201 s	2.34×10^{-4}	$(4-2\epsilon)$ 	384 s	8.12×10^{-4}
$(6-2\epsilon)$ 	150 s	4.83×10^{-4}	$(4-2\epsilon)$ 	56538 s	1.67×10^{-2}
$(6-2\epsilon)$ 	280 s	1.00×10^{-3}	$(4-2\epsilon)$ 	214135 s	8.29×10^{-3}
$(6-2\epsilon)$ 	294 s	1.21×10^{-3}	$(4-2\epsilon)$ 	3484378 s	30.9

timings with SecDec 3 in physical region

- used for top-quark mass dep. in HH [Heinrich et al], Hj [Jones et al],

WEIGHT DROPS IN A CONVENTIONAL BASIS

- consider sector with a conventional choice for the 2 master integrals:

(4-2 ϵ)

$$\begin{aligned}
 &= \frac{1}{\epsilon^8} \left(\frac{1}{144} \right) + \frac{1}{\epsilon^6} \left(-\frac{1}{24} \zeta_2 \right) + \frac{1}{\epsilon^5} \left(-\frac{29}{24} \zeta_3 \right) + \frac{1}{\epsilon^4} \left(-\frac{71}{16} \zeta_2^2 \right) + \frac{1}{\epsilon^3} \left(-\frac{1819}{24} \zeta_5 + \frac{23}{6} \zeta_2 \zeta_3 \right) \\
 &+ \frac{1}{\epsilon^2} \left(\frac{1285}{24} \zeta_3^2 - \frac{80579}{1008} \zeta_2^3 \right) + \frac{1}{\epsilon} \left(-\frac{434203}{192} \zeta_7 + \frac{7139}{24} \zeta_2 \zeta_5 + \frac{54139}{120} \zeta_2^2 \zeta_3 \right) \\
 &+ \frac{2023}{12} \zeta_{5,3} + \frac{30581}{4} \zeta_3 \zeta_5 + \frac{6829}{24} \zeta_2 \zeta_3^2 - \frac{45893321}{100800} \zeta_2^4 + \mathcal{O}(\epsilon) \quad \text{and}
 \end{aligned}$$

(4-2 ϵ)

$$\begin{aligned}
 &= \frac{1}{\epsilon^8} \left(-\frac{1}{144} \right) + \frac{1}{\epsilon^7} \left(-\frac{1}{12} \right) + \frac{1}{\epsilon^6} \left(\frac{1}{24} \zeta_2 - \frac{7}{36} \right) + \frac{1}{\epsilon^5} \left(\frac{29}{24} \zeta_3 + \frac{1}{2} \zeta_2 - \frac{1}{72} \right) \\
 &+ \frac{1}{\epsilon^4} \left(\frac{71}{16} \zeta_2^2 + \frac{29}{2} \zeta_3 + \frac{39}{16} \zeta_2 + \frac{335}{144} \right) + \frac{1}{\epsilon^3} \left(\frac{1819}{24} \zeta_5 - \frac{23}{6} \zeta_2 \zeta_3 + \frac{213}{4} \zeta_2^2 + \frac{1211}{24} \zeta_3 + \frac{431}{48} \zeta_2 \right. \\
 &\left. + \frac{47}{18} \right) + \frac{1}{\epsilon^2} \left(-\frac{1285}{24} \zeta_3^2 + \frac{80579}{1008} \zeta_2^3 + \frac{1819}{2} \zeta_5 - 46 \zeta_2 \zeta_3 + \frac{25787}{160} \zeta_2^2 + \frac{417}{8} \zeta_3 - \frac{1175}{48} \zeta_2 - \frac{7277}{72} \right) \\
 &+ \frac{1}{\epsilon} \left(\frac{434203}{192} \zeta_7 - \frac{7139}{24} \zeta_2 \zeta_5 - \frac{54139}{120} \zeta_2^2 \zeta_3 - \frac{1285}{2} \zeta_3^2 + \frac{80579}{84} \zeta_2^3 + \frac{5571}{2} \zeta_5 - \frac{9005}{24} \zeta_2 \zeta_3 + \frac{967}{480} \zeta_2^2 \right. \\
 &\left. - \frac{4045}{8} \zeta_3 - \frac{733}{24} \zeta_2 + \frac{57635}{72} \right) - \frac{2023}{12} \zeta_{5,3} - \frac{30581}{4} \zeta_3 \zeta_5 - \frac{6829}{24} \zeta_2 \zeta_3^2 + \frac{45893321}{100800} \zeta_2^4 + \frac{434203}{16} \zeta_7 \\
 &- \frac{7139}{2} \zeta_2 \zeta_5 - \frac{54139}{10} \zeta_2^2 \zeta_3 - \frac{10706}{3} \zeta_3^2 + \frac{7987951}{3360} \zeta_2^3 + \frac{1309}{12} \zeta_5 - \frac{30317}{24} \zeta_2 \zeta_3 - \frac{43847}{96} \zeta_2^2 + \frac{32335}{24} \zeta_3 \\
 &+ \frac{2553}{4} \zeta_2 - \frac{334727}{72} + \mathcal{O}(\epsilon).
 \end{aligned}$$

would need to compute 10 terms of epsilon expansion (weight 9) to obtain cusp (weight 6) :

CHOOSING A GOOD FINITE BASIS

- good basis choice avoids substantial “weight drops”
- in our example, using a suitable finite integral basis including

$$\text{(6-2}\epsilon\text{)} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \end{array} = -\frac{3}{2}\zeta_3^2 - \frac{4}{3}\zeta_2^3 + 10\zeta_5 + 2\zeta_2\zeta_3 - \frac{1}{5}\zeta_2^2 - 6\zeta_3 + \mathcal{O}(\epsilon)$$

we need only **1 term (weight 6)** for the cusp (weight 6)

- in many cases, our choice of basis allows us to **avoid the calculation of complicated topologies** altogether (see later)

Part 2: Reductions with Finite Fields and No-Numerator Syzygies

INTEGRATION-BY-PARTS (IBP) IDENTITIES

in dimensional regularisation, integral over total derivative vanishes:

$$0 = \int d^d k_1 \cdots d^d k_L \frac{\partial}{\partial k_i^\mu} \left(k_j^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right)$$
$$0 = \int d^d k_1 \cdots d^d k_L \frac{\partial}{\partial k_i^\mu} \left(p_j^\mu \frac{1}{D_1^{\nu_1} \cdots D_N^{\nu_N}} \right)$$

where p_j are external momenta, $\nu_i \in \mathbb{Z}$, $D_1 = k_1^2 - m_1^2$ etc.

problems of above construction:

- introduces many auxiliary integrals with additional dots and/or numerators
- sparse but still rather coupled system of equations
- new focus: syzygy constructions [Gluza, Kadja, Kosower] and [Bitoun, Böhm, Bogner, Georgoudis, Ita, Klausen, Larsen, Lee, Page, Panzer, Schabinger, Schulze, Zhang, Zeng and others], mostly to avoid squared propagators
- here: speed-up with **finite fields** and **no-numerator syzygies**

IBP REDUCTIONS FROM FINITE FIELD SAMPLES

A FINITE FIELD APPROACH TO IBPs [AvM, SCHABINGER '14]

- ➊ finite field sampling
 - set variables to integer numbers
 - consider coefficients modulo a prime field \mathbb{Z}_p
- ➋ solve finite field system
- ➌ reconstruct rational solution from many such samples

finite field techniques:

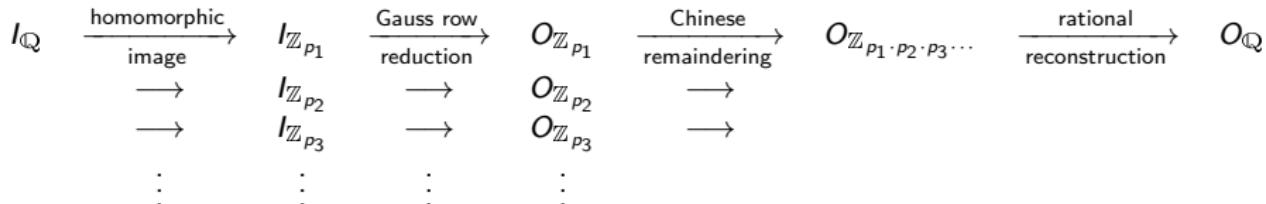
- no intermediate expression swell by construction
- early discarding of redundant and auxiliary quantities
- great potential for parallelisation

established in computer sciences, more and more popular in physics:

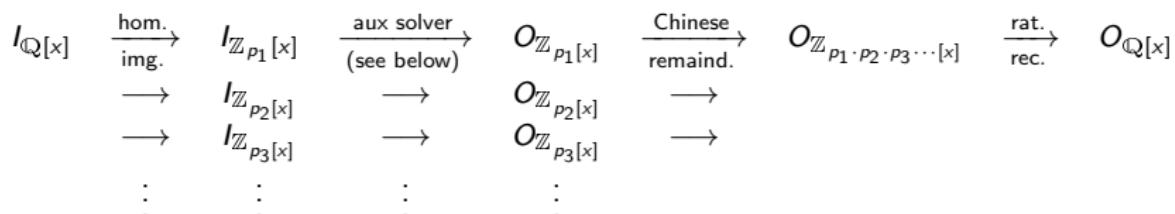
- dense solver: [Kauers], filtering: Ice [Kant '13], integrand construction: [Peraro '16], public IBP codes: Kira [Maierhöfer, Usovitsch, Uwer '17], Fire [Smirnov, Chukharev '19], and private codes...

A FAST UNIVARIATE SOLVER

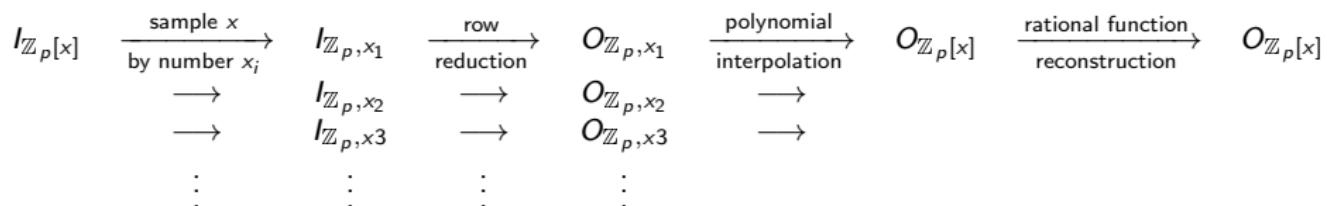
rational solver: reduce matrix $I_{\mathbb{Q}}$ of rational numbers



univariate solver: reduce matrix $I_{\mathbb{Q}[x]}$ of rational functions in x



aux solver: reduce matrix $I_{\mathbb{Z}_p[x]}$ of polynomials in x with finite field coefficients



note: multivariate case by iteration

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Package: finred

Author: Andreas v. Manteuffel

features:

- C++11 implementation for multivariate sparse matrices
- links flint library
- parallelisation: SIMD, threads, batch
- custom file formats with compression and checksums
- equation filtering: eliminate redundant rows
- pivot optimization
- plus lots of IBP specific features
- much faster than Reduze 2
- employed for construction and application of syzygies

SYZYGY BASED IBPs WITHOUT NUMERATORS

[Lee, Pomeransky '13] representation:

$$I(\nu_1, \dots, \nu_N) = \mathcal{N} \left[\prod_{i=1}^N \int_0^\infty dx_i x_i^{\nu_i - 1} \right] G^{-d/2} \quad \text{with } G = \mathcal{U} + \mathcal{F}$$

[Bitoun, Bogner, Klausen, Panzer '17]: define (twisted) Mellin Transform

$$\mathcal{M}\{f\}(\nu) := \left(\prod_{k=1}^N N \int_0^\infty \frac{x_k^{\nu_k - 1} dx_k}{\Gamma(\nu_k)} \right) f(x_1, \dots, x_N)$$

Feynman integrals are Mellin transforms:

$$\tilde{I}(\nu) = \mathcal{M}\left\{G^{-d/2}\right\}(\nu)$$

with $\nu = (\nu_1, \dots, \nu_N)$ and $\tilde{I}(\nu) = \Gamma[(L+1)d/2 - \nu] I(\nu)$

Properties of Mellin transform

- ① $\mathcal{M}\{\alpha f + \beta g\}(\nu) = \alpha \mathcal{M}\{f\}(\nu) + \beta \mathcal{M}\{g\}(\nu)$
- ② $\mathcal{M}\{x_i f\}(\nu) = \nu_i \mathcal{M}\{f\}(\nu + e_i)$
- ③ $\mathcal{M}\{-\partial_i f\}(\nu) = \mathcal{M}\{f\}(\nu - e_i)$ (proof: partial integration)

SHIFT RELATIONS FROM ANNIHILATORS

Define shift operators

$$(\hat{i}^+ F)(\nu_1, \dots, \nu_N) = \nu_i F(\nu_1, \dots, \nu_i + 1, \dots, \nu_N)$$
$$(\hat{i}^- F)(\nu_1, \dots, \nu_N) = F(\nu_1, \dots, \nu_i - 1, \dots, \nu_N)$$

which form Weyl algebra, $[\hat{i}^+, \hat{j}^-] = \delta_{ij}$

[Lee '14; Bitoun, Bogner, Klausen, Panzer '17]: a differential operator P which annihilates $G^{-d/2}$

$$P G^{-d/2}$$

generates via the substitutions

$$x_i \rightarrow \hat{i}^+,$$
$$\partial_i \rightarrow -\hat{i}^-$$

a shift relation according to

$$\mathcal{M}\{P G^{-d/2}\} = 0$$

In fact, *every* shift relation is related in this way.

idea: construct annihilators:

$$\left[c_0 + \sum_{i=1}^N c_i \frac{\partial}{\partial x_i} + \sum_{i,j=1}^N c_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \dots \right] G^{-d/2} = 0$$

new in this talk: annihilators beyond linear order

determine $c_0(x_1, \dots, x_N), \dots$ via syzygy equations:

$$c_0 \left[-\frac{2}{d} G^2 \right] + \sum_{i=1}^N c_i \left[G \frac{\partial G}{\partial x_i} \right] + \sum_{i,j=1}^N c_{ij} \left[G \frac{\partial^2 G}{\partial x_i \partial x_j} + \left(-\frac{d}{2} - 1 \right) \frac{\partial G}{\partial x_i} \frac{\partial G}{\partial x_j} \right] + \dots = 0$$

Syzygies generate linear relations for Feynman integrals:

$$\left(\left[c_0(\hat{1}^+, \dots, \hat{N}^+) - \sum_{i=1}^N c_i(\hat{1}^+, \dots, \hat{N}^+) \hat{i}^- + \sum_{i,j=1}^N c_{ij}(\hat{1}^+, \dots, \hat{N}^+) \hat{i}^- \hat{j}^- + \dots \right] \tilde{I} \right) (\nu_1, \dots, \nu_N) = 0$$

reminder: **syzygies**

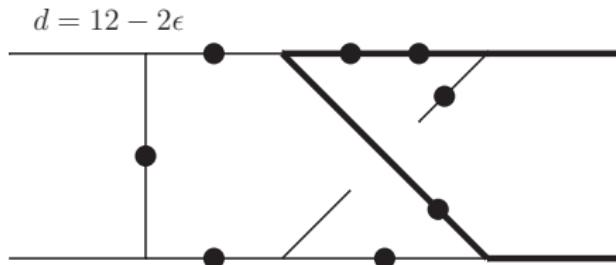
- suppose that for given polynomials $f = (f_1, f_2, \dots)$ one can find polynomials $s = (s_1, s_2, \dots)$ such that $\sum_i f_i s_i = 0$, then s is called a **syzygy**
- if s is a syzygy, then $s \cdot g$ is a syzygy for any polynomial g
- the (infinite) set of syzygies for f is a **syzygy module**

COMPUTATION

- computation of syzygies:
 - ▶ available in CAS like Singular
 - ▶ for complicated Feynman integrals: performance issues
- strategy:
 - ▶ observation: need only generating syzygies of low degree
 - ▶ use **custom syzygy finder** based on linear algebra (w/ Finred)
 - ▶ induced syzygies: via seed integrals
 - ▶ followed by "Laporta's reduction"
- basic linear algebra method: based on LASyz [Carbacas, Ding '11], see also [Schabinger '11]
 - ▶ homogenize system
 - ▶ compute generating syzygies up to some **degree**
 - ▶ represent explicit expressions in matrix
 - ▶ row reduce + nullspace → syzygies
 - ▶ remove syzygies induced by lower degree syzygies]
 - ▶ finite fields: benefit from late reconstruction
- result:
 - ▶ reductions of integrals with large number of dots possible
 - ▶ no auxiliary numerators need to be introduced

EXAMPLE AT 2-LOOPS

reductions like this easily accessible now:



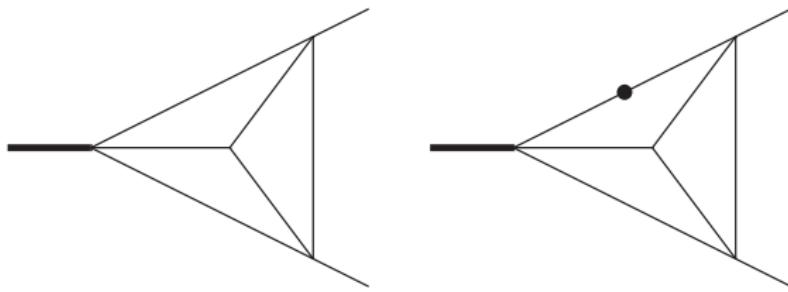
finite integral with improved convergence properties

used in [Becchetti, Bonciani, Casconi, Ferroglio, Lavacca, AvM '19]

QUESTION: ARE FIRST ORDER ANNIHILATORS SUFFICIENT ?

EXAMPLE: PLANAR THREE-POINT FUNCTION, 3 LOOPS

- Relations with 1 derivative: leave two integrals unreduced



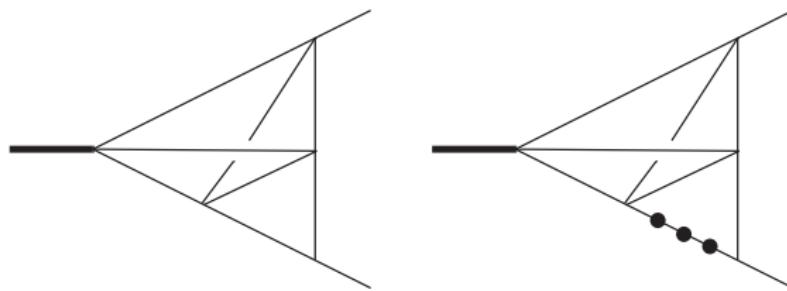
- ▶ counting for integer powers, ignoring subsectors
- ▶ tested up to 15 dots (degree of syzygies) in relations
- Relations with 2 derivatives: reduce one of the two integrals

- ▶ degree 3 syzygies + seeds sufficient
- ▶ in the same sector, in general with a different number of dots
- ▶ in subsectors, possibly with a numerator (inverse propagator)
- ▶ both, inverse propagators and sub-sub-sectors are key to completeness

QUESTION: ARE FIRST ORDER ANNIHILATORS SUFFICIENT ?

EXAMPLE: NON-PLANAR THREE-POINT FUNCTION, 4 LOOPS

- Relations with 1 derivative: a tower of integrals is not reduced:

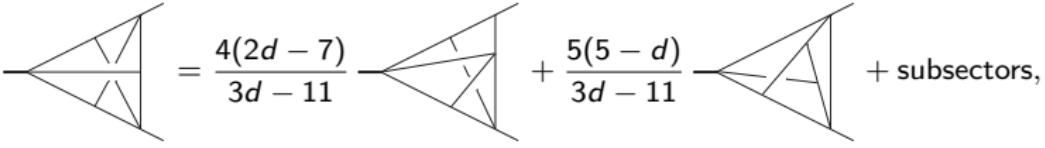


- ▶ dots on a specific propagator unreduced
- ▶ tested up to degree four syzygies + 17 dots for seeds

- Relations with 2 derivatives: reduce all but one integral
 - ▶ degree 3 syzygies + seeds sufficient

FORM FACTOR REDUCTION RESULTS

- successfully completed the reduction of the form factors to 294 finite master integrals
- reconstructed relations from up to $\mathcal{O}(40)$ 64-bit-based prime fields and $\mathcal{O}(600)$ values for the space-time dimension
- compressed reduction tables consume $\mathcal{O}(10 \text{ TB})$ on disk
- gauge parameter ξ drops out (checked for all matter dependent terms)
- there is a cross-sector relation



A Feynman diagram reduction equation. On the left is a complex multi-loop diagram. An equals sign follows, then $\frac{4(2d - 7)}{3d - 11}$, followed by a minus sign and another fraction $\frac{5(5 - d)}{3d - 11}$, followed by a plus sign and the text "subsectors,". To the right of the plus sign is another complex multi-loop diagram.

(required for gauge parameter to cancel)

- would be missed in a bottom-up approach, but can be obtained from a common parent topology

Part 3: Physics Results

RESULTS: 4 LOOP CUSP ANOMALOUS IN $\mathcal{N} = 4$ SYM

- [Boels, Huber, Yang '17] found a compact representation for the $\mathcal{N} = 4$ Sudakov form factor:

$$\begin{aligned}
 F^{(4)} = & 2 \left[8I_{p,1}^{(1)} + 2I_{p,2}^{(2)} - 2I_{p,3}^{(3)} + 2I_{p,4}^{(4)} + \frac{1}{2}I_{p,5}^{(5)} + 2I_{p,6}^{(6)} + 4I_{p,7}^{(7)} + 2I_{p,8}^{(9)} - 2I_{p,9}^{(10)} + I_{p,10}^{(12)} \right. \\
 & + I_{p,11}^{(12)} + 2I_{p,12}^{(13)} + 2I_{p,13}^{(14)} - 2I_{p,14}^{(17)} + 2I_{p,15}^{(17)} - 2I_{p,16}^{(19)} + I_{p,17}^{(19)} + I_{p,18}^{(21)} + \frac{1}{2}I_{p,19}^{(25)} + 2I_{p,20}^{(30)} + 2I_{p,21}^{(13)} \\
 & + 4I_{p,22}^{(14)} - 2I_{p,23}^{(14)} - I_{p,24}^{(14)} + 4I_{p,25}^{(17)} - I_{p,26}^{(17)} - 2I_{p,27}^{(17)} - 2I_{p,28}^{(17)} - I_{p,29}^{(19)} - I_{p,30}^{(19)} + I_{p,31}^{(19)} - \frac{1}{2}I_{p,32}^{(30)} \Big] \\
 & + \frac{48}{N_c^2} \left[\frac{1}{2}I_1^{(21)} + \frac{1}{2}I_2^{(22)} + \frac{1}{2}I_3^{(23)} - I_4^{(24)} + \frac{1}{4}I_5^{(25)} - \frac{1}{4}I_6^{(26)} - \frac{1}{4}I_7^{(26)} + 2I_8^{(27)} + I_9^{(28)} \right. \\
 & + 4I_{10}^{(29)} + I_{11}^{(30)} + I_{12}^{(27)} - \frac{1}{2}I_{13}^{(28)} + I_{14}^{(29)} + I_{15}^{(29)} + I_{16}^{(30)} + I_{17}^{(30)} + I_{18}^{(30)} + I_{19}^{(22)} + I_{20}^{(22)} \\
 & \left. - I_{21}^{(24)} + \frac{1}{4}I_{22}^{(24)} + \frac{1}{2}I_{23}^{(28)} \right], \quad (\text{master integrals } I \text{ conjectured to be uniform weight})
 \end{aligned}$$

- calculation of ϵ expansion suggests I and $F^{(4)}$ are UT [Huber, AvM, Panzer, Schabinger, Yang '19]:

$$\begin{aligned}
 F^{(4)} = & \frac{1}{\epsilon^8} \left(\frac{2}{3} \right) + \frac{1}{\epsilon^6} \left(\frac{2}{3} \zeta_2 \right) + \frac{1}{\epsilon^5} \left(-\frac{38}{9} \zeta_3 \right) + \frac{1}{\epsilon^4} \left(\frac{5}{18} \zeta_2^2 \right) + \frac{1}{\epsilon^3} \left(\frac{1082}{15} \zeta_5 + \frac{23}{3} \zeta_3 \zeta_2 \right) \\
 & + \frac{1}{\epsilon^2} \left(\frac{10853}{54} \zeta_3^2 + \frac{95477}{945} \zeta_2^3 \right) + \frac{1}{N_c^2} \left[\frac{1}{\epsilon^2} \left(18\zeta_3^2 + \frac{372}{35} \zeta_2^3 \right) \right] + \mathcal{O}(\epsilon^{-1}).
 \end{aligned}$$

- this results in the following cusp anomalous dimension:

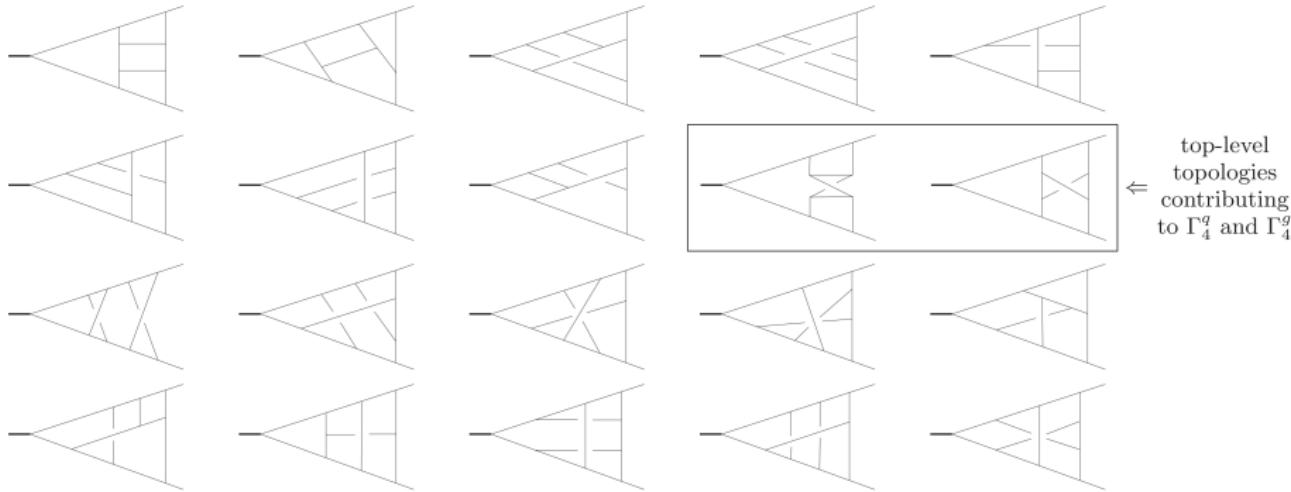
$$\Gamma_4^{\mathcal{N}=4} = \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left(-384\zeta_3^2 - \frac{7936}{35} \zeta_2^3 \right) + \textcolor{blue}{C_A^4} \left(-16\zeta_3^2 - \frac{20032}{105} \zeta_2^3 \right),$$

[Henn, Korchemsky, Mistlberger '19], [Huber, AvM, Panzer, Schabinger, Yang '19]

RESULTS: TOPOLOGIES FOR QCD

[AvM, Panzer, Schabinger '20]

irreducible twelve-propagators topologies:



- total of 100 twelve-propagator topologies
- first two rows are **linearly reducible**
- last two rows **not linearly reducible** out of the box
- use **variable changes** to render non-linearly reducible topologies reducible
- hardest topologies don't contribute to cusp due to our basis choice

RESULTS: REDUCED QCD FORM FACTORS

[AvM, Panzer, Schabinger '20]

- quark form factor:

$$\begin{aligned}\bar{\mathcal{F}}_4^q(\epsilon) = & N_f^3 C_F c_1^q(\epsilon) + N_f^2 C_A C_F c_2^q(\epsilon) + N_f^2 C_F^2 c_3^q(\epsilon) + N_{q\gamma} N_f \frac{d_F^{abc} d_F^{abc}}{N_F} c_4^q(\epsilon) + N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} c_5^q(\epsilon) \\ & + N_f C_A^2 C_F c_6^q(\epsilon) + N_f C_A C_F^2 c_7^q(\epsilon) + N_f C_F^3 c_8^q(\epsilon) + N_{q\gamma} C_A \frac{d_F^{abc} d_F^{abc}}{N_F} c_9^q(\epsilon) + N_{q\gamma} C_F \frac{d_F^{abcd} d_F^{abcd}}{N_F} c_{10}^q(\epsilon) \\ & + C_A^3 C_F c_{11}^q(\epsilon) + C_A^2 C_F^2 c_{12}^q(\epsilon) + C_A C_F^3 c_{13}^q(\epsilon) + C_F^4 c_{14}^q(\epsilon) + \frac{d_A^{abcd} d_F^{abcd}}{N_F} c_{15}^q(\epsilon),\end{aligned}$$

- gluon form factor:

$$\begin{aligned}\bar{\mathcal{F}}_4^g(\epsilon) = & N_f^3 C_A c_1^g(\epsilon) + N_f^3 C_F c_2^g(\epsilon) + N_f^2 C_A^2 c_3^g(\epsilon) + N_f^2 C_A C_F c_4^g(\epsilon) + N_f^2 C_F^2 c_5^g(\epsilon) + N_f \frac{d_F^{abcd} d_F^{abcd}}{N_A} c_6^g(\epsilon) \\ & + N_f C_A^3 c_7^g(\epsilon) + N_f C_A^2 C_F c_8^g(\epsilon) + N_f C_A C_F^2 c_9^g(\epsilon) + N_f C_F^3 c_{10}^g(\epsilon) + N_f \frac{d_A^{abcd} d_F^{abcd}}{N_A} c_{11}^g(\epsilon) \\ & + C_A^4 c_{12}^g(\epsilon) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} c_{13}^g(\epsilon).\end{aligned}$$

RESULTS: 4 LOOP CUSP ANOMALOUS DIMENSIONS IN QCD

[Henn, Korchemsky, Mistlberger '19], [AvM, Panzer, Schabinger '20]

$$\begin{aligned}
\Gamma_4^r = & N_f^3 C_R \left(\frac{64}{27} \zeta_3 - \frac{32}{81} \right) \\
& + N_f^2 C_A C_R \left(-\frac{224}{15} \zeta_2^2 + \frac{2240}{27} \zeta_3 - \frac{608}{81} \zeta_2 + \frac{923}{81} \right) \\
& + N_f^2 C_F C_R \left(\frac{64}{5} \zeta_2^2 - \frac{640}{9} \zeta_3 + \frac{2392}{81} \right) \\
& + N_f C_A^2 C_R \left(\frac{2096}{9} \zeta_5 + \frac{448}{3} \zeta_3 \zeta_2 - \frac{352}{15} \zeta_2^2 - \frac{23104}{27} \zeta_3 + \frac{20320}{81} \zeta_2 - \frac{24137}{81} \right) \\
& + N_f C_A C_F C_R \left(160 \zeta_5 - 128 \zeta_3 \zeta_2 - \frac{352}{5} \zeta_2^2 + \frac{3712}{9} \zeta_3 + \frac{440}{3} \zeta_2 - \frac{34066}{81} \right) \\
& + N_f C_F^2 C_R \left(-320 \zeta_5 + \frac{592}{3} \zeta_3 + \frac{572}{9} \right) \\
& + N_f \frac{d_F^{abcd} d_R^{abcd}}{N_R} \left(-\frac{1280}{3} \zeta_5 - \frac{256}{3} \zeta_3 + 256 \zeta_2 \right) \\
& + \frac{d_A^{abcd} d_R^{abcd}}{N_R} \left(-384 \zeta_3^2 - \frac{7936}{35} \zeta_2^3 + \frac{3520}{3} \zeta_5 + \frac{128}{3} \zeta_3 - 128 \zeta_2 \right) \\
& + C_A^3 C_R \left(-16 \zeta_3^2 - \frac{20032}{105} \zeta_2^3 - \frac{3608}{9} \zeta_5 - \frac{352}{3} \zeta_3 \zeta_2 + \frac{3608}{5} \zeta_2^2 + \frac{20944}{27} \zeta_3 - \frac{88400}{81} \zeta_2 + \frac{84278}{81} \right)
\end{aligned}$$

where $R = F, A$ for $r = q, g$. Note the quadratic Casimir (dd) contributions.

RESULTS: 4 LOOP COLLINEAR ANOMALOUS DIMENSIONS IN QCD

[AvM, Panzer, Schabinger '20]

$$\gamma_4^q = N_f^3 C_F \left(\frac{128}{135} \zeta_2^2 + \frac{1424}{243} \zeta_3 + \frac{16}{27} \zeta_2 - \frac{37382}{6561} \right) + N_f^2 C_F^2 \left(\frac{1040}{9} \zeta_5 - \frac{224}{9} \zeta_3 \zeta_2 - \frac{8032}{135} \zeta_2^2 - \frac{4232}{81} \zeta_3 + \frac{1972}{27} \zeta_2 + \frac{9965}{486} \right) \\ + N_f^2 C_A C_F \left(-\frac{1184}{9} \zeta_5 + \frac{256}{9} \zeta_3 \zeta_2 + \frac{152}{15} \zeta_2^2 + \frac{14872}{243} \zeta_3 + \frac{41579}{729} \zeta_2 - \frac{97189}{17496} \right) \\ + N_f C_A^2 C_F \left(\frac{6916}{9} \zeta_3^2 + \frac{24184}{315} \zeta_2^3 + \frac{6088}{27} \zeta_5 - \frac{3584}{9} \zeta_3 \zeta_2 - \frac{17164}{45} \zeta_2^2 + \frac{140632}{243} \zeta_3 - \frac{445117}{729} \zeta_2 + \frac{326863}{1944} \right) \\ + N_f C_A C_F^2 \left(-\frac{3400}{3} \zeta_3^2 + \frac{5744}{35} \zeta_2^3 - \frac{4472}{3} \zeta_5 + \frac{3904}{9} \zeta_3 \zeta_2 + \frac{105488}{135} \zeta_2^2 - \frac{23518}{81} \zeta_3 + \frac{673}{27} \zeta_2 - \frac{1092511}{972} \right) \\ + N_f C_F^3 \left(368 \zeta_3^2 - \frac{117344}{315} \zeta_2^3 + \frac{3872}{3} \zeta_5 - \frac{512}{3} \zeta_3 \zeta_2 - \frac{668}{5} \zeta_2^2 - \frac{1120}{9} \zeta_3 + 322 \zeta_2 + \frac{27949}{108} \right) \\ + N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} \left(\frac{1216}{3} \zeta_3^2 + \frac{9472}{315} \zeta_2^3 - \frac{21760}{9} \zeta_5 + 128 \zeta_3 \zeta_2 - \frac{320}{3} \zeta_2^2 - \frac{5312}{9} \zeta_3 + \frac{4544}{3} \zeta_2 - 384 \right) + \mathcal{O}(N_f^0)$$

$$\gamma_4^g = N_f^3 C_A \left(\frac{256}{135} \zeta_2^2 - \frac{400}{243} \zeta_3 - \frac{16}{81} \zeta_2 - \frac{15890}{6561} \right) + N_f^3 C_F \left(\frac{308}{243} \right) + N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \left(\frac{1024}{3} \zeta_3 - \frac{1408}{9} \right) \\ + N_f^2 C_A^2 \left(-\frac{1024}{9} \zeta_5 - 32 \zeta_3 \zeta_2 + \frac{3128}{135} \zeta_2^2 + \frac{37354}{243} \zeta_3 - \frac{13483}{729} \zeta_2 + \frac{611939}{17496} \right) + N_f^2 C_A C_F \left(\frac{304}{9} \zeta_5 + \frac{32}{3} \zeta_3 \zeta_2 + \frac{128}{45} \zeta_2^2 \right. \\ \left. - \frac{1688}{81} \zeta_3 - \frac{172}{9} \zeta_2 + \frac{1199}{18} \right) + N_f C_F^2 \left(-\frac{352}{9} \zeta_3 + \frac{676}{27} \right) + N_f C_A^3 \left(-\frac{596}{9} \zeta_3^2 + \frac{148976}{945} \zeta_2^3 + \frac{16066}{27} \zeta_5 + 148 \zeta_3 \zeta_2 \right. \\ \left. - \frac{69502}{135} \zeta_2^2 - \frac{260822}{243} \zeta_3 + \frac{155273}{729} \zeta_2 - \frac{421325}{1944} \right) + N_f C_A^2 C_F \left(152 \zeta_3^2 + \frac{5632}{315} \zeta_2^3 + \frac{8}{9} \zeta_5 - 176 \zeta_3 \zeta_2 - \frac{1196}{45} \zeta_2^2 + \frac{29606}{81} \zeta_3 \right. \\ \left. + \frac{3023}{9} \zeta_2 - \frac{903983}{972} \right) + N_f C_A C_F^2 \left(-80 \zeta_3^2 - \frac{320}{7} \zeta_2^3 - \frac{1600}{3} \zeta_5 + \frac{148}{5} \zeta_2^2 + \frac{1592}{3} \zeta_3 - 2 \zeta_2 + \frac{685}{12} \right) + N_f C_F^3 (46) \\ + N_f \frac{d_A^{abcd} d_F^{abcd}}{N_A} \left(\frac{1216}{3} \zeta_3^2 - \frac{14464}{315} \zeta_2^3 - \frac{30880}{9} \zeta_5 + 1216 \zeta_3 \zeta_2 + \frac{2464}{15} \zeta_2^2 + \frac{2560}{9} \zeta_3 - 64 \zeta_2 + \frac{448}{9} \right) + \mathcal{O}(N_f^0)$$

RESULTS: 4 LOOP FORM FACTORS IN QCD

- all **planar master integrals** available: [Henn, Smirnov, Smirnov, Steinhauser '16], [AvM, Schabinger '19]
- form factor with most **non-planar topologies** so far: n_f^2 terms in ggH (163 master integrals): result in terms of multiple zeta values [AvM, Schabinger '19]

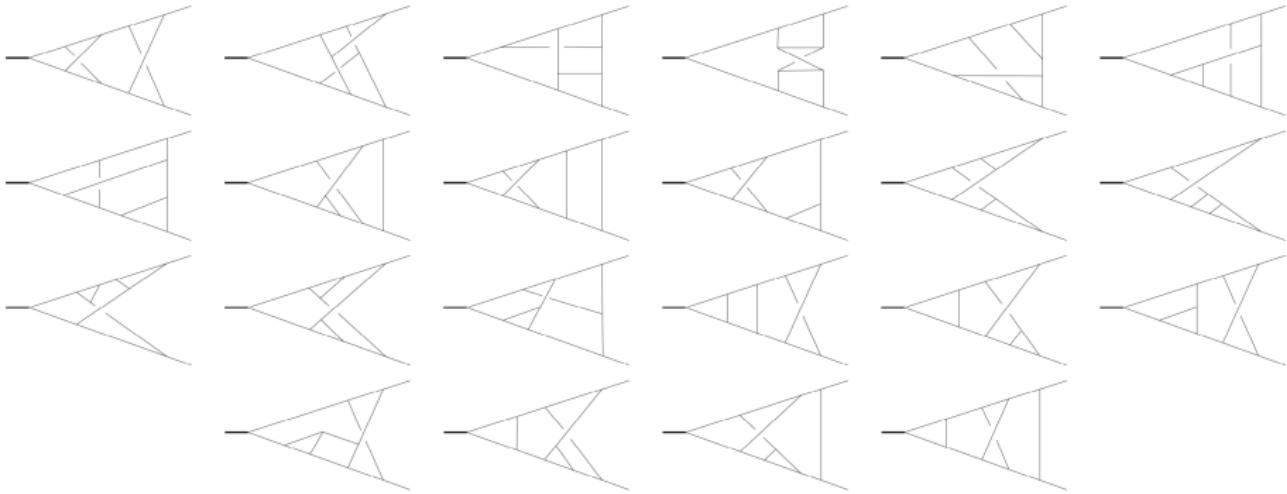


TABLE I. Complexity of various form factor contributions.

	$\bar{\mathcal{F}}_4^q _{N_f^2}$	$\bar{\mathcal{F}}_4^q _{N_q N_f}$	$\bar{\mathcal{F}}_4^g _{N_f^2}$
# diagrams	71	226	2554
# planar twelve-line top.	0	4	21
# nonplanar twelve-line top.	0	3	19
# nonequivalent top. in red.	158	923	1781
# integrals in amp. ($\xi \neq 1$)	$\mathcal{O}(10^5)$	$\mathcal{O}(10^6)$	$\mathcal{O}(10^7)$
max. # inverse propagators	5	5	6

CONCLUSIONS

method of finite integrals:

- simple and efficient method for singularity resolution in multi-loop integrals
- analytical integrations: finite integrals are Feynman integrals (dim-shifted, dotted)
- numerical integrations: faster and more stable evaluations

reductions without numerators:

- syzygy construction using linear algebra (w/ finite fields)
- higher order derivatives required

4 loop results:

- complete ab initio calculation of exact cusp anomalous dimensions in QCD
- complete matter dependence of collinear anomalous dimensions
- progress towards finite parts of virtual amplitudes