On the computation of two-loop five-point amplitudes in QCD

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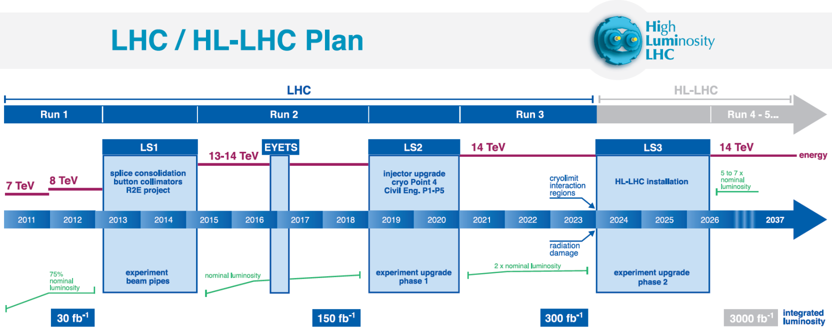
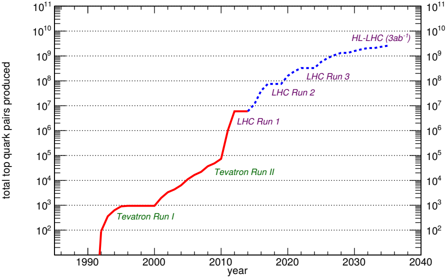


data collected from Run2

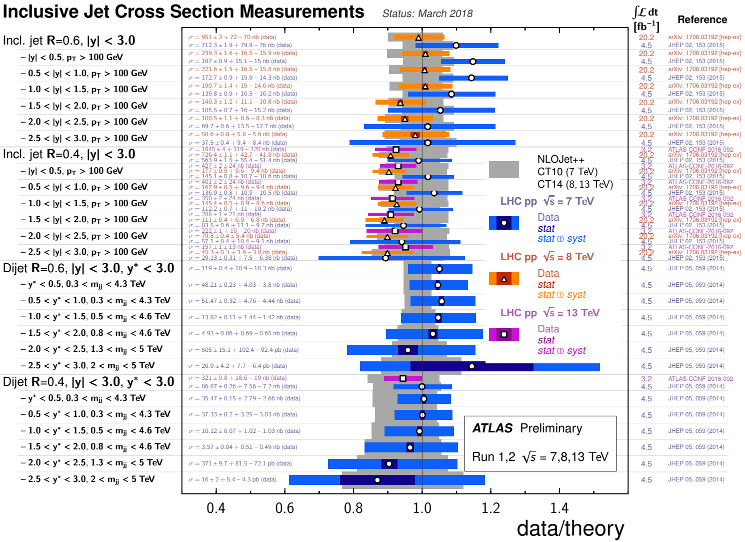
$\sim 150\ \mathrm{fb}^{-1}$

$\sim$ 120 millions of $t\bar{t}$ events produced

more data to come!!!



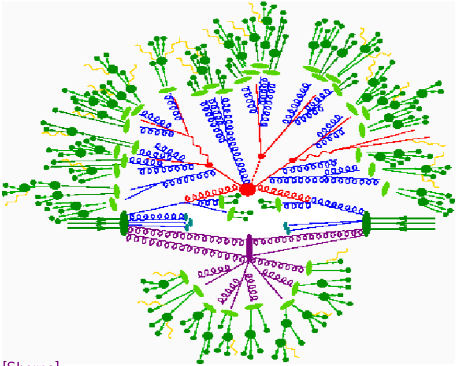
[John Campbell, Top@Twenty]



experimental uncertainties ∼ theoretical uncertainties

New Physics search at the LHC: small deviation from SM predictions

⇒ need to get theoretical uncertainties well under control



[Sherpa]

Predictions for LHC

- Hard interaction
- Parton showers
- Underlying events
- Hadronization
- Decay of hadron
- QED Bremsstrahlung

Factorization of short and long distance part of the QCD interactions

$$d\sigma(pp \to X) = \sum_{ij} \int dx_1 dx_2 \; f_{i/p}(x_1, \mu_F) f_{j/p}(x_2, \mu_F) \; d\hat{\sigma}(ij \to X; \mu_F, \mu_R, \hat{s}) \qquad (\hat{s} = x_1 x_2 s)$$

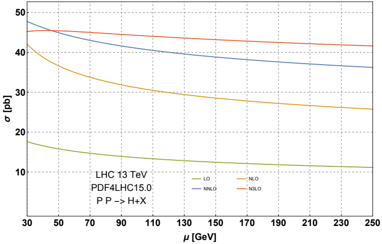
Perturbative expansion of partonic cross section:

$$d\hat{\sigma} = \underbrace{d\hat{\sigma}^{(0)}}_{\mathrm{LO}} + \underbrace{\alpha \; d\hat{\sigma}^{(1)}}_{\mathrm{NLO}} + \underbrace{\alpha^2 \; d\hat{\sigma}^{(2)}}_{\mathrm{NNLO}} + \ldots$$

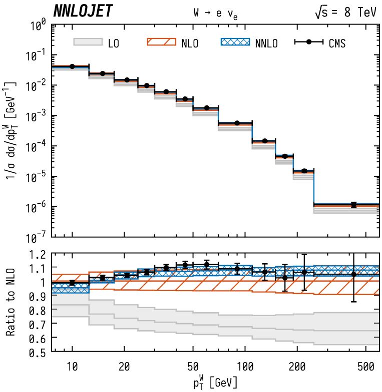
expansion in strong coupling α_s and/or electroweak coupling α_{EW}

Theoretical uncertainties: residual scale dependence, PDFs, α_s, ...

What do we gain?



[Anastasiou,etal,2016][Mistlberger,2018]



[Gehrmann-De Ridder,etal,2017]

- ✓ reduced scale dependence
- ✓ reliable normalization
- ✓ better agreement with data
- ✓ kinematic-dependent corrections

State of the art:

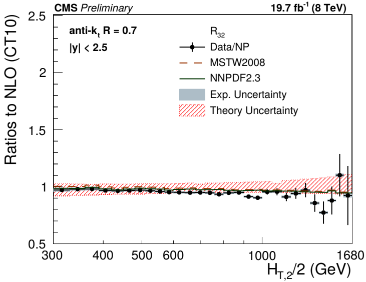
- ▶ NLO QCD and EW: today's standard & largely automized
- ▶ NNLO: $2 \to 2$ processes (jj, Vj, VV, Hj, HH, $t\bar{t}$, $t\bar{t}$+decay, ...)
- ▶ N3LO: $pp \to H$, DIS, Drell-Yan

Precision frontier: NNLO for $2 \to 3$ ($pp \to \gamma\gamma\gamma$ [Chawdry,Czakon,Mitov,Poncelet 2019])

$2 \to 3$ scattering at the LHC

- $pp \to jjj$: $R_{3/2}$, $m_{jjj} \Rightarrow \alpha_s$ determination at multi-TeV range

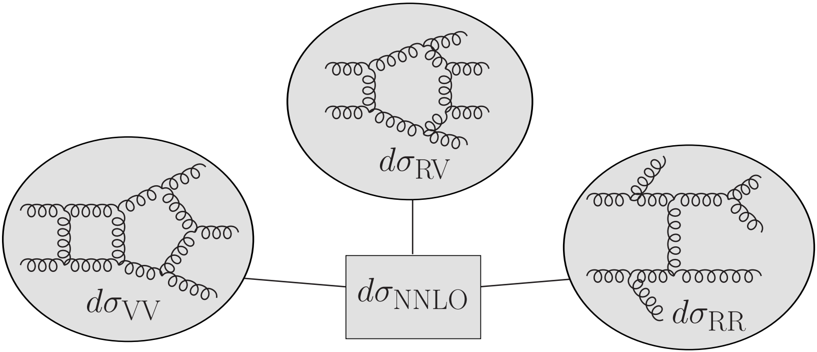
$R_{3/2} \sim \alpha_s \to$ cancellation of uncertainties



$$\alpha_s(M_Z) = 01150 \;\; \pm 0.0010(\text{exp})$$
$$\pm 0.0013(\text{PDF})$$
$$\pm 0.0015(\text{NP})$$
$$^{+0.0050}_{-0.0000}(\text{scale})$$

[CMS-PAS-SMP-16-008]

- $pp \to \gamma\gamma j$: background to Higgs p_T, signal/background interference effects
- $pp \to Hjj$: Higgs p_T, background to VBF (probes Higgs coupling)
- $pp \to Vjj$: Vector boson p_T, W^+/W^- ratios, multiplicity scaling
- $pp \to VVj$: background for new physics



q_T subtraction N-jettiness subtraction   projection to Born

antenna subtraction   CoLoRFulNNLO   STRIPPER

Nested Soft-Collinear Subtraction   geometric subtraction

$+ \ldots$

Two-Loop Calculation: General Strategy

$$
\begin{aligned}
A^{(2)} &= \int [dk_1][dk_2] \sum_d \frac{N_d(k_i \cdot p_j, k_i \cdot \varepsilon_j, k_i \cdot k_j)}{(\text{propagators})_d} \\
&= \sum_i c_i(\epsilon)\, G_i \\
&= \sum_i d_i(\epsilon)\, \mathrm{MI}_i \\
&= \frac{e_4}{\epsilon^4} + \frac{e_3}{\epsilon^3} + \frac{e_2}{\epsilon^2} + \frac{e_1}{\epsilon} + e_0 \\
&= I^{(2)} A^{(0)} + I^{(1)} A^{(1)} + F^{(2)}
\end{aligned}
$$

Feynman diagrams
colour decomposition

Interference with tree-level
Projector method, Integrand reduction

IBP reduction to Master Integrals

$e_i \to \{s_{ij}, \pi, \ln, \mathrm{Li}_i, \ldots\}$

subtract universal pole structures

IBP identities: relations between integrals → reduce to independent set of integrals

$$\int [dk] \frac{\partial}{\partial k_\mu} \frac{v_\mu(k,p)}{(\text{propagators})} = 0$$

[Chetyrkin,Tkachov]

Public software: AIR [Anastasiou,Lazopoulos], FIRE [Smirnov,Smirnov], Reduze [Studerus,Manteuffel], KIRA [Maierhoefer,Usovitsch,Uwer], LiteRed [Lee]

Master Integrals

- Differential equation [Gehrmann,Remiddi,Henn]
- Sector decomposition+numerical integration:SecDec[Borowka, etal],FIESTA[Smirnov,etal]

Multi-scale process ⇒ algebraic and analytic complexities

bottleneck: large intermediate expressions but simple final results

Numerical calculation ⇒ many 1-loop calculation with high multiplicity
(Njet, BlackHat, GoSam, ...)

What kind of numerical evaluation?

- floating-point evaluation ($x = 4.744955523489933 \times 10^6$)
  ✓ fast ✗ limited precision
- evaluation over rational field $\mathcal{Q}$ ($x = 706998373/149$)
  ✓ exact ✗ can be slow and expensive
- evaluation over finite fields $\mathcal{Z}_p$ ($x \bmod_{11} = 8$)
  $\mathcal{Z}_p$ ⇒ the field of integer numbers modulo a prime p
  ✓ exact+fast ✗ some information lost
  → need to reconstruct $\mathcal{Q}$ over several finite fields

Strategy ⇒ reconstruct analytic expressions from finite-field evaluations [Peraro 2016]

Analytics results: fast and stable for pheno applications

Computational Framework

Colour ordered amplitude:

$$\mathcal{A}^{(2)}(\{p\}) = \int [dk_1][dk_2] \, \frac{\mathcal{N}(\{k\},\{p\})}{\mathcal{D}_1 \cdots \mathcal{D}_n}$$

Integrand Reduction: construct irreducible numerators $\Delta_i(\{k\},\{p\})$
[Ossola,Papadopoulos,Pittau,Mastrolia,Badger,Frellesvig,Zhang,Peraro,Mirabella, ...]

$$\frac{\mathcal{N}(\{k\},\{p\})}{\mathcal{D}_1 \cdots \mathcal{D}_n} = \sum_{i \in T} \frac{\Delta_i(\{k\},\{p\})}{(\text{propagators})_i}$$

- $\mathcal{N}(\{k\},\{p\})$: process dependent numerator function

  $\Rightarrow$ generalized unitarity cuts $\rightarrow$ product of tree amplitudes
  [Bern,Dixon,Dunbar,Kosower,1994;Britto,Cachazo,Feng,2004;Ellis,Giele,Kunszt,Melnikov,2007-2008; ...]

  $\Rightarrow$ Feynman diagram input (`QGRAF` [Nogueira], `FeynArts` [Hahn], ...)

- Integrand parameterisation

$$\Delta_i(\{k\},\{p\}) = \sum_j c_{i,j}(\{p\}) \, \mathbf{m}_j(\{k\})$$

- Fit coefficients $c_{i,j}(\{p\})$ on multiple cuts $\{\mathcal{D}_i = 0\}_{i \in T}$
  $\Rightarrow$ solve linear system of equation

Interlude: 4D Spinor Helicity Formalism

For massless four-vector p_i, define spinor products:

$$\langle ij\rangle = \bar{u}_-(p_i)u_+(p_j), \quad [ij] = \bar{u}_+(p_i)u_-(p_j), \quad \langle ij\rangle[ji] = 2p_i \cdot p_j.$$

where

$$u_+(p) = P_R u(p) \qquad u_-(p) = P_L u(p)$$

Spinor sandwiches

$$\begin{aligned}
\langle i|m\cdots n|j] &= \bar{u}_-(p_i)\, \not{p}_m \cdots \not{p}_n\, u_-(p_j) && (\text{odd } \# \text{ of } p) \\
\langle i|m\cdots n|j\rangle &= \bar{u}_-(p_i)\, \not{p}_m \cdots \not{p}_n\, u_+(p_j) && (\text{even } \# \text{ of } p) \\
[i|m\cdots n|j] &= \bar{u}_+(p_i)\, \not{p}_m \cdots \not{p}_n\, u_-(p_j) && (\text{even } \# \text{ of } p)
\end{aligned}$$

Massless polarization vectors

$$\epsilon_+^\mu(k,q) = \frac{\langle q|\gamma^\mu|k]}{\sqrt{2}\langle qk\rangle}, \quad \epsilon_-^\mu(k,q) = \frac{[q|\gamma^\mu|k\rangle}{\sqrt{2}[qk]}.$$

Working with helicity amplitudes: $\mathcal{A}(\langle ij\rangle, [ij], \langle i|k_\ell|j])$

Example: 1-loop *gggg*

$$\mathcal{A}^{(1)}_{gggg} = \int [dk] \left(\frac{\Delta(\square)}{D_1 D_2 D_3 D_4} + \sum_{i_1<i_2<i_3} \frac{\Delta(\triangleleft)}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{i_1<i_2} \frac{\Delta(\circ)}{D_{i_1} D_{i_2}} \right)$$

$$k = k_{[4d]} + k_{[-2\epsilon]}, \quad \mu_{11} = k^2_{[-2\epsilon]}$$

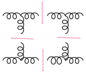
Box coefficient

$$\text{Spanning vectors} = \{p_1, p_2, p_4, \omega\}, \quad \omega \to \text{spurious vector}, \ p_{1,2,3} \cdot \omega = 0$$

$$\text{RSPs} = \{k^2, k \cdot p_i\}, \ \text{ISPs} = \{\mu_{11}, k \cdot \omega\}$$

$$\Delta(\square) = c_0 + c_1 k \cdot \omega + c_2 \mu_{11} + c_3 k \cdot \omega \mu_{11} + c_4 \mu_{11}^2$$

Quadruple cut: $\{D_1 = 0, D_2 = 0, D_3 = 0, D_4 = 0\}$



$$\Big|_{\text{4xcut}} = \Delta(\square)\Big|_{\text{4xcut}}$$

- substitute loop momentum solution
- solve linear system to determine c_i

Example: 1-loop $gggg$

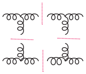
$\mathcal{A}^{(1)}_{gggg} = \int [dk] \Big(\frac{\Delta(}{}$  $\frac{)}{D_1 D_2 D_3 D_4} + \sum_{i_1<i_2<i_3} \frac{\Delta(}{}$  $\frac{)}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{i_1<i_2} \frac{\Delta(}{}$  $\frac{)}{D_{i_1} D_{i_2}} \Big)$

$$k = k_{[4d]} + k_{[-2\epsilon]}, \quad \mu_{11} = k^2_{[-2\epsilon]}$$

Box coefficient

$\Delta($  $) = c_0 + c_1 k\cdot\omega + c_2 \mu_{11} + c_3 k\cdot\omega\mu_{11} + c_4 \mu_{11}^2$

Quadruple cut: $\{D_1 = 0, D_2 = 0, D_3 = 0, D_4 = 0\}$

 $\Big|_{\text{4xcut}} = \Delta($  $)\Big|_{\text{4xcut}} \quad \Rightarrow \quad \sum_{i=0}^{4} d_i \tau^i = \sum_{i=0}^{4} f_i(c_0, c_1, c_2, c_3, c_4)\, \tau^i$

On-shell solution

$$\bar{k}^\mu = \frac{s(1+\tau)}{4\langle 4|2|1]}\langle 4|\gamma^\mu|1] + \frac{s(1-\tau)}{4\langle 1|2|4]}\langle 1|\gamma^\mu|4]$$

$$k\cdot\omega = \frac{st}{2}\tau \qquad \mu_{11} = -\frac{st}{4u}(1-\tau^2)$$

$$\begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{st}{4u} & 0 & \frac{s^2t^2}{16u^2} \\ 0 & \frac{st}{2} & 0 & -\frac{s^2t^2}{8u} & 0 \\ 0 & 0 & \frac{st}{4u} & 0 & -\frac{s^2t^2}{8u^2} \\ 0 & 0 & 0 & \frac{s^2t^2}{8u} & 0 \\ 0 & 0 & 0 & 0 & \frac{s^2t^2}{16u^2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}$$

Example: 1-loop *gggg*

$$\mathcal{A}^{(1)}_{gggg} = \int [dk] \left(\frac{\Delta(\square)}{D_1 D_2 D_3 D_4} + \sum_{i_1<i_2<i_3} \frac{\Delta(\triangleleft)}{D_{i_1} D_{i_2} D_{i_3}} + \sum_{i_1<i_2} \frac{\Delta(\bigcirc)}{D_{i_1} D_{i_2}} \right)$$

$$k = k_{[4d]} + k_{[-2\epsilon]}, \quad \mu_{11} = k^2_{[-2\epsilon]}$$

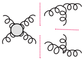
Triangle coefficients

$$\text{Spanning vectors} = \{p_1, p_2, \omega_1, \omega_2\}, \quad \omega_{1,2} \to \text{spurious vectors, } p_{1,2} \cdot \omega_{1,2} = 0$$

$$\text{RSPs} = \{k^2, k \cdot p_i\}, \text{ ISPs} = \{\mu_{11}, k \cdot \omega_1, k \cdot \omega_2\}$$

$$\Delta(\triangleleft) = c_0 + c_1 k \cdot \omega_1 + c_2 k \cdot \omega_2 + c_3 k \cdot \omega_1 \mu_{11} + c_4 k \cdot \omega_2 \mu_{11} + \cdots$$

Triple cut: $\{D_1 = 0, D_2 = 0, D_3 = 0\}$



$$\Big|_{\text{3xcut}} = \frac{\Delta(\square)}{D_4}\Big|_{\text{3xcut}} + \Delta(\triangleleft)\Big|_{\text{3xcut}}$$

- top-down approach: subtract box contribution
- **or** simultaneously determine box and triangle coefficients

$$\mathcal{A}^{(2)}(\{p\}) = \int [dk_1][dk_2] \sum_{i\in T} \frac{\sum_j c_{i,j}(\{p\})\, \mathbf{m}_j(\{k\})}{(\text{propagators})_i} = \sum_i \tilde{c}_i G_i \underset{\text{IBP}}{=} \sum_i d_i \text{MI}_i$$

- use IBP-compatible basis (without μ_{ij})
- **IBP identities:** reduce $G_i \to \text{MI}_i$: use LiteRed[Lee] + Laporta approach
- allows for extraction of the coefficients of master integrals d_i

$$d_i = \underbrace{\begin{pmatrix} \quad \dots \quad \end{pmatrix}}_{\text{all integrand coefficients}} \underbrace{\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}}_{\text{IBP reduction}}$$

- All steps evaluated numerically over finite fields within `FiniteFlow` [Peraro,2019]
  Can be parallelized!! reconstruct analytic expressions (if possible)

$$\mathcal{A}^{(2)}(\{p\}) = \int [dk_1][dk_2] \sum_{i\in T} \frac{\sum_j c_{i,j}(\{p\})\, \mathbf{m}_j(\{k\})}{(\text{propagators})_i} = \sum_i \tilde{c}_i G_i \underset{\text{IBP}}{=} \sum_i d_i \mathrm{MI}_i$$

- use IBP-compatible basis (without μ_{ij})
- **IBP identities:** reduce $G_i \to \mathrm{MI}_i$: use LiteRed[Lee] + Laporta approach
- allows for extraction of the coefficients of master integrals d_i
  or special functions e_i (+ pole subtraction + Laurent expansion in ϵ)

$$e_i = \underbrace{\begin{pmatrix} & \dots & \end{pmatrix}}_{\text{all integrand coefficients}} \underbrace{\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}}_{\text{IBP reduction}} \underbrace{\begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix}}_{\text{map MIs to special functions}} - \underbrace{\begin{pmatrix} & \dots & \end{pmatrix}}_{\epsilon\text{-pole in special functions basis}}$$

- All steps evaluated numerically over finite fields within `FiniteFlow` [Peraro,2019]
  Can be parallelized!! reconstruct analytic form of **finite remainder**
- Not all coefficients are independent: find linear relations between coefficients
  $\sum_i y_i\, e_i = 0$ $\quad\to$ rewrite the more complex coeffs in terms of simpler ones

Momentum Twistor Variables

[Hodges,Badger,Frellesvig, Zhang]

- redundancy: spinor components ($\langle ij\rangle$, $[ij]$) are not all independent
- rational parametrization of the n-point phase-space and the spinor components using $3n-10$ momentum-twistor variables
- 5-point parameterization:

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |1] = \begin{pmatrix}1\\ \frac{x_4-x_5}{x_4}\end{pmatrix},$$

$$|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \qquad |2] = \begin{pmatrix}0\\ x_1\end{pmatrix},$$

$$|3\rangle = \begin{pmatrix}\frac{1}{x_1}\\1\end{pmatrix}, \qquad |3] = \begin{pmatrix}x_1 x_4\\ -x_1\end{pmatrix},$$

$$|4\rangle = \begin{pmatrix}\frac{1}{x_1}+\frac{1}{x_1 x_2}\\1\end{pmatrix}, \qquad |4] = \begin{pmatrix}x_1(x_2 x_3 - x_3 x_4 - x_4)\\ -\frac{x_1 x_2 x_3 x_5}{x_4}\end{pmatrix},$$

$$|5\rangle = \begin{pmatrix}\frac{1}{x_1}+\frac{1}{x_1 x_2}+\frac{1}{x_1 x_2 x_3}\\1\end{pmatrix}, \qquad |5] = \begin{pmatrix}x_1 x_3(x_4 - x_2)\\ \frac{x_1 x_2 x_3 x_5}{x_4}\end{pmatrix}.$$

- phase information is lost: $|i\rangle \to t_i^{-1}|i\rangle$, $|i] \to t_i|i]$

$$A = A^{\text{phase}} \cdot \underbrace{\tilde{A}(x_1,x_2,x_3,x_4,x_5)}_{\text{phase}-\text{free}}$$

Momentum Twistor Variables

- Example: MHV amplitudes

$$\mathcal{A}^{(0)}(1_g^-, 2_g^-, 3_g^+, 4_g^+, 5_g^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} = x_1^3 x_2^2 x_3$$

$$\mathcal{A}^{(0)}(1_g^-, 2_g^+, 3_g^-, 4_g^+, 5_g^+) = \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} = x_1^3 x_2^2 x_3$$

- Example: Momentum conservation

$$\langle 1|2|5] + \langle 1|3|5] + \langle 1|4|5] = 0$$

$$-x_1^2 x_3(x_2 - x_4) - x_1^2 x_3(-x_2 + x_4 + x_2 x_5) + x_1^2 x_3 x_2 x_5 = 0$$

- Example: Schouten identity

$$\langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 42\rangle + \langle 14\rangle\langle 23\rangle = 0$$

$$-\frac{1}{x_1 x_2} + \frac{1 + x_2}{x_1 x_2} - \frac{1}{x_1} = 0$$

Two-loop five-point amplitudes

Five-point master integrals

[Papadopoulos,Tommasini,Wever 2015], [Gehrmann,Henn,Lo Presti 2015,2018], [Abreu,Page,Zeng 2018], [Chicherin,Gehrmann,Henn,Lo Presti,Mitev,Wasser 2018], [Abreu,Dixon,Herrmann,Page,Zeng 2018,2019] [Chicherin,Gehrmann,Henn,Wasser,Zhang,Zoia 2018,2019]

Numerical evaluation of two-loop five-point amplitudes in QCD:

- planar five-gluon [Badger,Brønnum-Hansen,HBH,Peraro 2017] [Abreu,Ita,Febres Cordero,Page,Zeng 2017]
- planar five-parton [Badger,Brønnum-Hansen,HBH,Peraro 2018] [Abreu,Ita,Febres Cordero,Page,Sotnikov 2018]
- planar $W+4$ parton [Badger,Brønnum-Hansen,HBH,Peraro arXiv:1906.11862]

Analytic form of two-loop five-point amplitudes in QCD

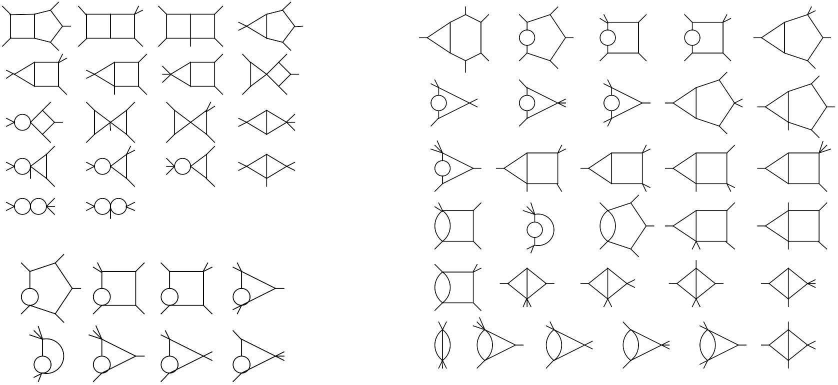
- planar five-gluon all-plus [Gehrmann,Henn,Lo Presti 2015]
- planar five-gluon single-minus [Badger,Brønnum-Hansen,HBH,Peraro arXiv:1811.11699]
- planar five-gluon MHV [Abreu,Dormans,Ita,Febres Cordero,Page 2018]
- planar five-parton MHV [Abreu,Dormans,Ita,Febres Cordero,Page,Sotnikov 2019]
- full-colour five-gluon all-plus [Badger,Chicherin,Gehrmann,Heinrich,Henn,Peraro,Wasser,Zhang,Zoia 2019]
- planar $q\bar{q} \to \gamma\gamma\gamma$ [Chawdry,Czakon,Mitov,Poncelet 2019]

Leading colour two-loop five-gluon amplitudes

[Badger,Brønnum-Hansen,HBH,Peraro,arXiv:1712.02229[hep-ph]]

$$\mathcal{A}^{(L)}(1,2,3,4,5) = n^L g_s^3 \sum_{\sigma\in S_5/Z_5} \mathrm{tr}\left(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(5)}}\right) A^{(L)}\left(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\right)$$

Leading Colour ⇒ coefficient of N_c^2 term ⇒ planar, no closed fermion loop



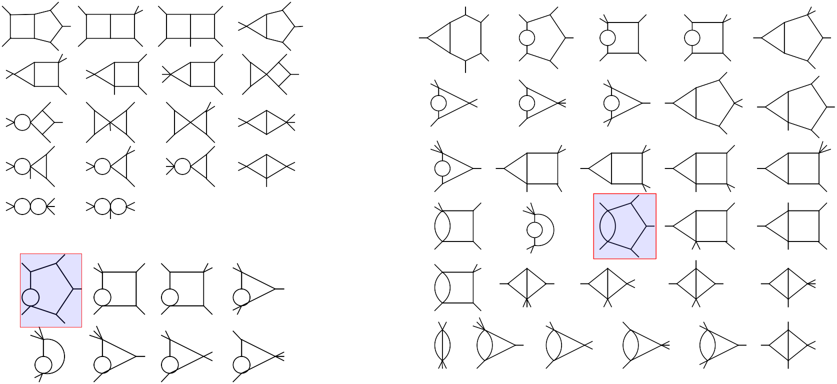
57 distinct topologies, 425 Δ (all permutations)

Leading colour two-loop five-gluon amplitudes

[Badger,Brønnum-Hansen,HBH,Peraro,arXiv:1712.02229[hep-ph]]

$$\mathcal{A}^{(L)}(1,2,3,4,5) = n^L g_s^3 \sum_{\sigma \in S_5/Z_5} \mathrm{tr}\left(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(5)}}\right) A^{(L)}\left(\sigma(1),\sigma(2),\sigma(3),\sigma(4),\sigma(5)\right)$$

Leading Colour $\Rightarrow$ coefficient of N_c^2 term $\Rightarrow$ planar, no closed fermion loop



57 distinct topologies, 425 Δ (all permutations)

Integrand Parameterisation

Reminder: recipe for integrand reduction

$$\frac{\mathcal{N}(\{k\},\{p\})}{\mathcal{D}_1 \cdots \mathcal{D}_n} = \sum_{i\in T} \frac{\Delta_i(\{k\},\{p\})}{(\text{propagators})_i}, \qquad \Delta_i(\{k\},\{p\}) = \sum_j c_{i,j}(\{p\})\, \mathbf{m}_j(\{k\})$$

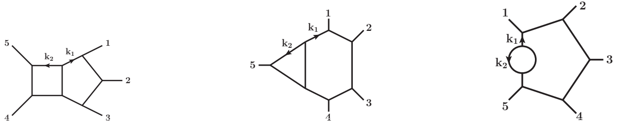
IBP compatible integrand basis

- Define integral family

$$G_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}} = \int [dk_1][dk_2] \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7} D_8^{a_8} D_9^{a_9} D_{10}^{a_{10}} D_{11}^{a_{11}}}$$

$D_1 = k_1^2$, $D_2 = (k_1 - p_1)^2$, $D_3 = (k_1 - p_1 - p_2)^2$, $D_4 = (k_1 - p_1 - p_2 - p_3)^2$, $D_5 = k_2^2$, $D_6 = (k_2 - p_5)^2$, $D_7 = (k_2 - p_4 - p_5)^2$, $D_8 = (k_1 + k_2)^2$, $D_9 = (k_1 + p_5)^2$, $D_{10} = (k_2 + p_1)^2$, $D_{11} = (k_2 + p_1 + p_2)^2$

Master topologies:



$G_{11111111 a_9 a_{10} a_{11}}$ $\quad$ $G_{111111 a_7 11 a_{10} a_{11}}$ $\quad$ $G_{21111 a_6 a_7 11 a_{10} a_{11}}$

Integrand Parameterisation

Reminder: recipe for integrand reduction

$$\frac{\mathcal{N}(\{k\},\{p\})}{\mathcal{D}_1 \cdots \mathcal{D}_n} = \sum_{i\in T} \frac{\Delta_i(\{k\},\{p\})}{(\text{propagators})_i}, \qquad \Delta_i(\{k\},\{p\}) = \sum_j c_{i,j}(\{p\})\,\mathbf{m}_j(\{k\})$$

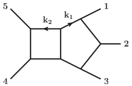
IBP compatible integrand basis

- Define integral family

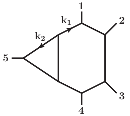
$$G_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}} = \int [dk_1][dk_2] \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7} D_8^{a_8} D_9^{a_9} D_{10}^{a_{10}} D_{11}^{a_{11}}}$$

$D_1 = k_1^2$, $D_2 = (k_1 - p_1)^2$, $D_3 = (k_1 - p_1 - p_2)^2$, $D_4 = (k_1 - p_1 - p_2 - p_3)^2$, $D_5 = k_2^2$, $D_6 = (k_2 - p_5)^2$,
$D_7 = (k_2 - p_4 - p_5)^2$, $D_8 = (k_1 + k_2)^2$, $D_9 = (k_1 + p_5)^2$, $D_{10} = (k_2 + p_1)^2$, $D_{11} = (k_2 + p_1 + p_2)^2$

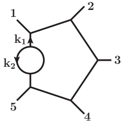
Master topologies:



$G_{11111111 a_9 a_{10} a_{11}}$



$G_{111111 a_7 11 a_{10} a_{11}}$



$G_{21111 a_6 a_7 11 a_{10} a_{11}}$

Integrand Parameterisation

Reminder: recipe for integrand reduction

$$\frac{\mathcal{N}(\{k\},\{p\})}{\mathcal{D}_1 \cdots \mathcal{D}_n} = \sum_{i\in T} \frac{\Delta_i(\{k\},\{p\})}{(\text{propagators})_i}, \qquad \Delta_i(\{k\},\{p\}) = \sum_j c_{i,j}(\{p\})\,\mathbf{m}_j(\{k\})$$

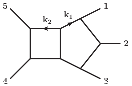
IBP compatible integrand basis

- Define integral family

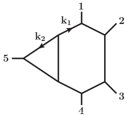
$$G_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}} = \int [dk_1][dk_2] \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7} D_8^{a_8} D_9^{a_9} D_{10}^{a_{10}} D_{11}^{a_{11}}}$$

$D_1 = k_1^2$, $D_2 = (k_1 - p_1)^2$, $D_3 = (k_1 - p_1 - p_2)^2$, $D_4 = (k_1 - p_1 - p_2 - p_3)^2$, $D_5 = k_2^2$, $D_6 = (k_2 - p_5)^2$, $D_7 = (k_2 - p_4 - p_5)^2$, $D_8 = (k_1 + k_2)^2$, $D_9 = (k_1 + p_5)^2$, $D_{10} = (k_2 + p_1)^2$, $D_{11} = (k_2 + p_1 + p_2)^2$

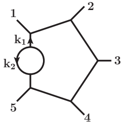
Master topologies:



$G_{11111111 a_9 a_{10} a_{11}}$



$G_{111111 a_7 11 a_{10} a_{11}}$

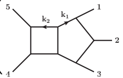


$G_{21111 a_6 a_7 11 a_{10} a_{11}}$

Integrand Basis

$$G_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11}} = \int [dk_1][dk_2] \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7} D_8^{a_8} D_9^{a_9} D_{10}^{a_{10}} D_{11}^{a_{11}}}$$

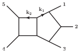
$D_1 = k_1^2$, $D_2 = (k_1 - p_1)^2$, $D_3 = (k_1 - p_1 - p_2)^2$, $D_4 = (k_1 - p_1 - p_2 - p_3)^2$, $D_5 = k_2^2$, $D_6 = (k_2 - p_5)^2$, $D_7 = (k_2 - p_4 - p_5)^2$, $D_8 = (k_1 + k_2)^2$, $D_9 = (k_1 + p_5)^2$, $D_{10} = (k_2 + p_1)^2$, $D_{11} = (k_2 + p_1 + p_2)^2$

$\Delta\Big($ 

$\Big) = \sum_{a_9, a_{10}, a_{11}} c_{11111111 a_9 a_{10} a_{11}} D_9^{-a_9} D_{10}^{-a_{10}} D_{11}^{-a_{11}}$

$$\begin{aligned} -5 &\le a_9 &&\le 0, \\ -4 &\le a_{10} + a_{11} &&\le 0, \\ -7 &\le a_9 + a_{10} + a_{10} &&\le 0. \end{aligned}$$

$\int [dk_1][dk_2] \frac{\Delta\Big(\text{diagram}\Big)}{D_1 D_2 D_3 D_4 D_5 D_6 D_7 D_8} = \sum_{a_9, a_{10}, a_{11}} c_{11111111 a_9 a_{10} a_{11}} G_{11111111 a_9 a_{10} a_{11}}$



Analytic results: single minus amplitude

[Badger,Brønnum-Hansen,HBH,Peraro arXiv:1811.11699]

Unitarity cuts (6D tree amplitudes via Berends-Giele recursion) | integrand reduction

IBP reduction | finite-field sampling | analytic reconstruction

Master integrals in terms of **pentagon functions** f: $\quad \mathrm{MI}_x = \sum_{y,z} c_{xyz}\, \epsilon^y m_{yz}(f)$
[Gehrmann,Henn,Lo Presti 2018]

Subtract IR poles, reconstruct finite remainder $\mathcal{F}$

$$\mathcal{A}^{(2)}_{-++++} = I^{(1)} \mathcal{A}^{(1)}_{-++++} + \mathcal{F}_{-++++}, \qquad \mathcal{A}^{(0)}_{-++++} = 0$$

Obtain compact expressions for $\mathcal{F}$ (decomposition in $(d_s-2)^i$, $d_s = g^\mu_\mu$)

$$F^{(2),[i]}\left(1^-,2^+,3^+,4^+,5^+\right) = \frac{[25]^2}{[12]\langle 23\rangle\langle 34\rangle\langle 45\rangle[51]}\left(F^{(2),[i]}_{\mathrm{sym}}(1,2,3,4,5) + F^{(2),[i]}_{\mathrm{sym}}(1,5,4,3,2)\right)$$

$$\begin{aligned} F^{(2),[1]}_{\mathrm{sym}}(1,2,3,4,5) = \; & c^{(2)}_{51} F^{(2)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) + c^{(1)}_{51} F^{(1)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) + c^{(0)}_{51} F^{(0)}_{\mathrm{box}}(s_{23},s_{34},s_{15}) \\ & + c^{(2)}_{34} F^{(2)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) + c^{(1)}_{34} F^{(1)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) + c^{(0)}_{34} F^{(0)}_{\mathrm{box}}(s_{12},s_{15},s_{34}) \\ & + c_{45} F^{(0)}_{\mathrm{box}}(s_{12},s_{23},s_{45}) + c_{34;51} \hat{L}_1(s_{34},s_{15}) + c_{51;23} \hat{L}_1(s_{15},s_{23}) + c_{\mathrm{rat}} \end{aligned}$$

$\Rightarrow$ analytic expressions derived also for $--+++$ and $-+-++$

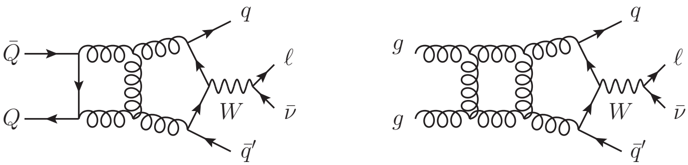
Two-loop $W+4$ parton amplitudes

[Badger,Brønnum-Hansen,HBH,Peraro arXiv:1906.11862]

Leading colour $q\bar{Q}Q\bar{q}'\bar{\nu}\ell$ and $qgg\bar{q}'\bar{\nu}\ell$ amplitudes

$$\mathcal{A}^{(2)}(1_q,2_{\bar{Q}},3_Q,4_{\bar{q}'},5_{\bar{\nu}},6_\ell) \sim g_s^6 g_W^2\ N_c^2\ \delta_{i_1}^{\ \bar{i}_2}\delta_{i_3}^{\ \bar{i}_4}\ A^{(2)}(1_q,2_{\bar{Q}},3_Q,4_{\bar{q}'},5_{\bar{\nu}},6_\ell)$$

$$\mathcal{A}^{(2)}(1_q,2_g,3_g,4_{\bar{q}'},5_{\bar{\nu}},6_\ell) \sim g_s^6 g_W^2 \left[N_c^2\,(\,T^{a_2}T^{a_3})_{i_1}^{\ \bar{i}_4}\,A^{(2)}(1_q,2_g,3_g,4_{\bar{q}'},5_{\bar{\nu}},6_\ell) + (2\leftrightarrow 3)\right]$$

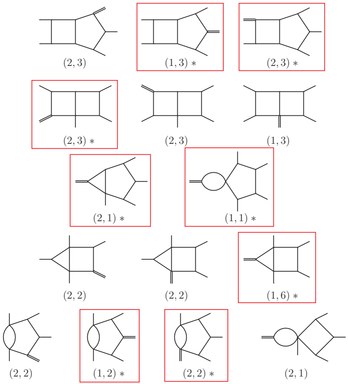


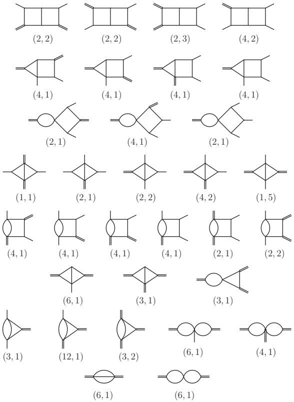
Feynman diagrams | integrand reduction | IBP reduction | finite-field sampling

- Master integrals are not fully known analytically
- Coefficient of master integrals are very complicated to be reconstructed analytically
- First step towards analytical form: numerical benchmark

Two-loop $W + 4$ parton amplitudes

- Coefficient of master integrals are computed numerically over finite fields
- Use momentum twistor parametrisation for $2 \to 4$ massless scattering: $x_1, \cdots, x_8$
- only some of MIs known analytically[Papadopoulos,Tomassini,Wever 2015],[Gehrmann,von Manteuffel,Tancredi 2015], [Henn,Melnikov,Smirnov 2014],[Gehrman,Remiddi 2000]
- unknown MIs are evaluated numerically using `pySecDec`/`Fiesta`





Two-loop $W + 4$ parton amplitudes

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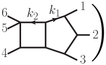
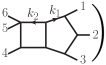
Example: one-mass pentagon-box

$I\Big($ 

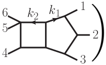
$\Big)\,[1]$

$$\frac{\#}{\epsilon^4} + \frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon} + \#$$

$I\Big($ 

$\Big)\,\left[(k_1 + p_{56})^2\right]$

$$\frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon} + \#$$

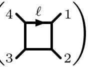
$I\Big($ 

$\Big)\,\left[(k_2 + p_1)^2\right]$

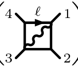
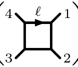
$$\frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon} + \#$$

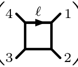
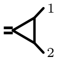
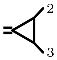
Interlude: local numerator insertion

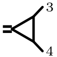
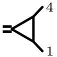
One-loop Box

I^D  $\propto \left(\frac{2}{\epsilon^2}((-s)^{-\epsilon}+(-t)^{-\epsilon}) - \ln^2\left(\frac{s}{t}\right) - \pi^2\right) + \mathcal{O}(\epsilon)$

introduce local numerator insertion

I^D  $\equiv I^D$  $\left[\mathrm{tr}_+(1(\ell - p_1)(\ell - p_{12})3)\right]$

$= \frac{st}{2} I^D$  $+ \frac{s}{2} I^D$  $+ \frac{t}{2} I^D$ 

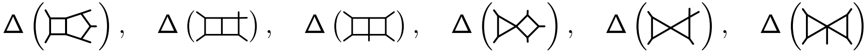
$+ \frac{s}{2} I^D$  $+ \frac{t}{2} I^D$ 

$\propto \ln^2\left(\frac{s}{t}\right) + \pi^2$

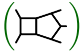
Interlude: local numerator insertion

Local Integrand for +++++ planar 2-loop 5-gluon

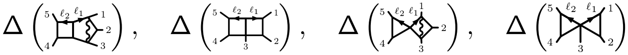
- Integrands contributing to $\mathcal{A}^{(2)}_{+++++}$ in [Badger,Frellesvig,Zhang,arXiv:1310.1051]



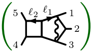
$$\Delta\left(\;\cdot\;\right) \propto F(d_s,\mu)\left(\mathrm{tr}_+(1235)(\ell_1+p_5)^2 + s_{12}s_{34}s_{45}\right)$$



- Local integrand reps. of $\mathcal{A}^{(2)}_{+++++}$ [Badger,Mogull,Peraro,arXiv:1606.02244]



$$\Delta\left(\;\cdot\;\right) \propto F(d_s,\mu)\mathrm{tr}_+\left(1\,(\ell_1-p_1)\,(\ell_1-p_{12})\,345\right)$$



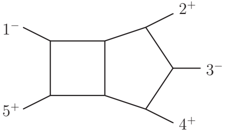
$$F(d_s,\mu) = (d_s-2)(\mu_{11}\mu_{22} + (\mu_{11}+\mu_{22})^2 + 2\mu_{12}(\mu_{11}+\mu_{22})) + 16(\mu_{12}^2 - \mu_{11}\mu_{22})$$

$$\mathrm{tr}_+(ijkl) = [i|j|k|l|i\rangle$$

Interlude: local numerator insertion

Local Integrand for −+−++ planar 2-loop 5-gluon

Pentagon-box topology



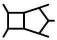
<u>Old Basis</u>

54 non-zero integrand coeffs
28 non-zero coeffs at $\mathcal{O}(\epsilon)$
worst case:
$(k_2 \cdot p_2)^4 \mu_{11}$ (rank 4, 1 dot)

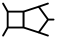
<u>New Basis</u>

49 non-zero integrand coeffs
4 non-zero coeffs at $\mathcal{O}(\epsilon)$
$k_1 \cdot n_1 \; k_2 \cdot n_2$ (rank 2)
$k_1 \cdot n_1 \; \mu_{11}$ (rank 1, 1 dot)

$$\mathrm{ISP} = \{k_1 \cdot p_1, k_2 \cdot p_2, k_2 \cdot p_3, \mu_{11}, \mu_{12}, \mu_{22}\}$$

$I\Big($  $\Big)\,[(k_2 \cdot p_2)^4 \mu_{11}] = \frac{\#}{\epsilon} + \#$

but

$I\Big($  $\Big)\,[\langle 1|k_2|5]^4 \mu_{11}] = \mathcal{O}(\epsilon)$

$$\langle 1|k_2|5] = A + B\; k_2 \cdot p_2 + C\; k_2 \cdot p_3 + \mathrm{props}$$

New ISP basis

$$\{k_1 \cdot n_1, k_1 \cdot n_1^*, k_2 \cdot n_2, k_2 \cdot n_2^*, \mu_{11}, \mu_{12}, \mu_{22}\}$$

$$k_2 \cdot n_2 = \langle 1|k_2|5]$$

$$k_1 \cdot n_1 = \mathrm{tr}_+\left(2\,(k_1 - p_2)\,(k_1 - p_{23})\,4\right)$$

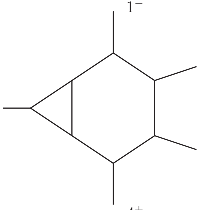
$$\mathrm{tr}_+(ijkl) = [i|j|k|l|i\rangle$$

⇒ Build in as many $\mathcal{O}(\epsilon)$ monomials as possible

Interlude: local numerator insertion

Local Integrand for −+−++ planar 2-loop 5-gluon

Hexagon-triangle topology



Old Basis

16 non-zero integrand coeffs

16 non-zero coeffs at $\mathcal{O}(\epsilon)$

worst case:
$(k_2 \cdot p_1)^3 \mu_{11}$ (rank 3, 1 dot)

$$\mathrm{ISP} = \{k_2 \cdot p_1, k_2 \cdot p_2, k_2 \cdot p_3, \mu_{11}, \mu_{12}, \mu_{22}\}$$

New Basis

10 non-zero integrand coeffs

0 non-zero coeffs at $\mathcal{O}(\epsilon)$

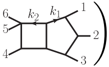
New ISP basis
$$\{k_2 \cdot n_1, k_2 \cdot n_1^*, k_2 \cdot p_1, \mu_{11}, \mu_{12}, \mu_{22}\}$$

$$k_2 \cdot n_1 = \langle 1|k_2|5]$$

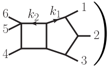
simplify 8- and 7- propagator topologies
lower topology integrands are still too complicated ☹

Two-loop $W + 4$ parton amplitudes

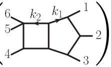
Changing master integral basis: one-mass pentagon-box

I () $[1]$

$$\frac{\#}{\epsilon^4} + \frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon} + \#$$

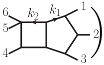
I () $\left[(k_1 + p_{56})^2\right]$

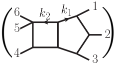
$$\frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon} + \#$$

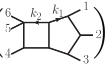
I () $\left[(k_2 + p_1)^2\right]$

$$\frac{\#}{\epsilon^3} + \frac{\#}{\epsilon^2} + \frac{\#}{\epsilon} + \#$$

Introduce local numerator insertions [Arkani-Hamed,Bourjaily,Cachazo,Trnka 2012]

I () $\left[\langle 4|k_2|p_{56}|4\rangle \mu_{11}\right] \sim \underbrace{\mathcal{O}(\epsilon)}_{\text{not needed}}$

I () $\left[[4|k_2|p_{56}|4]\mu_{11}\right] \sim \underbrace{\mathcal{O}(\epsilon)}_{\text{not needed}}$

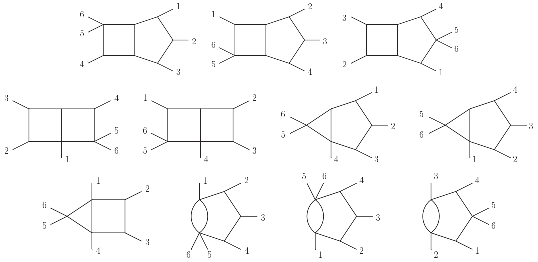
I () $\left[\mathrm{tr}_-(1(k_1 - p_1)(k_1 - p_{12})3)\langle 4|k_2|p_{56}|4\rangle\right] \sim \underbrace{\mathcal{O}(1)}_{\text{finite}} \rightarrow$ directly evaluated in pySecDec

$$k_1^2 = \bar{k}_1^2 - \mu_{11} \qquad \mathrm{tr}_-\,(ijkl) = \langle i|j|k|l|i]$$

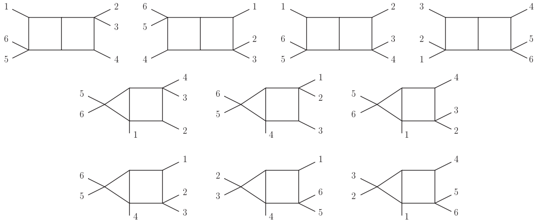
Another approach $\Rightarrow$ quasi-finite basis [von Manteuffel, Panzer, Schabinger, 2014]

Two-loop $W + 4$ parton amplitudes

5-point master integral topologies with local numerator



4-point master integral topologies with local numerator



Two-loop $W + 4$ parton amplitudes

Numerical benchmark: Euclidean phase-space point

$$x_1 = -1, \quad x_2 = \frac{79}{270}, \quad x_3 = \frac{64}{61}, \quad x_4 = -\frac{37}{78}, \quad x_5 = \frac{83}{102}, \quad x_6 = \frac{4723}{9207}, \quad x_7 = -\frac{12086}{7451}, \quad x_8 = \frac{3226}{2287}.$$

$qgg\bar{q}'\bar{\nu}\ell$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2)}_{-++++-}$	4.50000	-3.63577(3)	-277.2182(7)	-344.56(1)	2051.1(2)
$P^{(2)}_{-++++-}$	4.5	-3.63576	-277.2186	-344.569(6)	—
$\widehat{A}^{(2)}_{-+-++-}$	4.50000	-3.63581(9)	-13.6826(2)	6.143(5)	66.21(7)
$P^{(2)}_{-+-++-}$	4.5	-3.63576	-13.6824	6.145(1)	—
$\widehat{A}^{(2)}_{--+++-}$	4.50000	-3.63579(5)	-18.79219(7)	-6.633(6)	79.02(4)
$P^{(2)}_{--+++-}$	4.5	-3.63576	-18.79212	-6.6303(5)	—

$q\bar{Q}Q\bar{q}'\bar{\nu}\ell$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$\widehat{A}^{(2)}_{-+-++-}$	2.00000	-7.16949(9)	-9.9055(2)	39.922(6)	154.79(7)
$P^{(2)}_{--+++-}$	2	-7.16944	-9.9054	39.9245(8)	—
$\widehat{A}^{(2)}_{--+++-}$	2.00000	-7.16948(8)	-12.9371(1)	41.432(8)	189.53(6)
$P^{(2)}_{--+++-}$	2	-7.16944	-12.9370	41.4353(6)	—

Comparison against universal pole structures in t'Hooft-Veltman scheme

Summary

- ✓ Framework to compute two-loop five-point amplitudes
  - $\Rightarrow$ integrand reduction + IBP relations
  - $\Rightarrow$ numerical evaluation over finite field
- ✓ Analytical reconstruction from finite field sampling
  - $\Rightarrow$ avoid large intermediate expressions encountered in traditional approach
  - $\Rightarrow$ know the complexity of the problem before reconstruction
- ✗ More processes to explore
- ✗ From amplitudes to cross sections?

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THANK YOU!!!

Back-up Slides

Two-loop five-gluon: single minus

$$F^{(-1)}_{\text{box}}(s,t,m^2) = \text{Li}_2\left(1-\frac{s}{m^2}\right) + \text{Li}_2\left(1-\frac{t}{m^2}\right) + \log\left(\frac{s}{m^2}\right) + \log\left(\frac{t}{m^2}\right) - \frac{\pi^2}{6},$$

$$F^{(0)}_{\text{box}}(s,t,m^2) = \frac{1}{u(s,t,m^2)} F^{(-1)}_{\text{box}}(s,t,m^2),$$

$$F^{(1)}_{\text{box}}(s,t,m^2) = \frac{1}{u(s,t,m^2)}\left[F^{(0)}_{\text{box}}(s,t,m^2) + \hat{L}_1(s,m^2) + \hat{L}_1(m^2,t)\right],$$

$$F^{(2)}_{\text{box}}(s,t,m^2) = \frac{1}{u(s,t,m^2)}\left[F^{(1)}_{\text{box}}(s,t,m^2) + \frac{s-m^2}{2t}\hat{L}_2(s,m^2) + \frac{m^2-t}{2s}\hat{L}_2(m^2,t) - \left(\frac{1}{s}+\frac{1}{t}\right)\frac{1}{4m^2}\right],$$

$$u(s,t,m^2) = m^2 - s - t$$

$$L_k(s,t) = \frac{\log(t/s)}{(s-t)^k}$$

$$\hat{L}_0(s,t) = L_0(s,t), \qquad \hat{L}_1(s,t) = L_1(s,t),$$

$$\hat{L}_2(s,t) = L_2(s,t) + \frac{1}{2(s-t)}\left(\frac{1}{s}+\frac{1}{t}\right), \qquad \hat{L}_3(s,t) = L_3(s,t) + \frac{1}{2(s-t)^2}\left(\frac{1}{s}+\frac{1}{t}\right).$$

Notation & Convention

$$n = m_\epsilon N_c \alpha_s/(4\pi)$$

$$m_\epsilon = i(4\pi)^\epsilon e^{-\epsilon\gamma_E}$$

$$[dk_i] = -i\pi^{-d/2} e^{\epsilon\gamma_E} d^{4-2\epsilon} k_i$$

Momentum Twistor Variables

[Hodges]

Momentum twistor variables $Z_i(\lambda_i, \mu_i)$ for each momentum $\tilde{\lambda}_i$ are obtained via

$$\tilde{\lambda}_i = \frac{\langle i, i+1\rangle \mu_{i-1} + \langle i+1, i-1\rangle \mu_i + \langle i-1, i\rangle \mu_{i+1}}{\langle i, i+1\rangle \langle i-1, i\rangle}$$

5-point parameterization:

$$Z = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1} + \frac{1}{x_1 x_2} & \frac{1}{x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_1 x_2 x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{x_4}{x_2} & 1 \\ 0 & 0 & 1 & 1 & 1 - \frac{x_5}{x_4} \end{pmatrix}$$

$$s_{12} = x_1, \; s_{23} = x_1 x_4, \; s_{45} = x_1 x_5$$

$$\langle 12\rangle = 1, \; [12] = -x_1, \; \langle 23\rangle = -\frac{1}{x_1}, \; [23] = x_1^2 x_4, \; \langle 45\rangle = -\frac{1}{x_1 x_2 x_3}, \; [45] = x_1^2 x_2 x_3 x_5$$

μ_{ij} ISP monomials

Integrals containing μ_{ij} can be expressed as higher dimensional integrals

[Bern, Morgan, De Freitas, Dixon, Anastasiou]

At 1-loop:

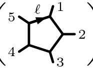
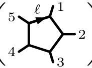
$$\int [d^d k_1] \frac{\mu_{11}^r}{D_1 \cdots D_n} = -\epsilon(1-\epsilon)\cdots(r-1-\epsilon) \int [d^{(d+2r)} k_1] \frac{1}{D_1 \cdots D_n}$$

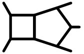
At 2-loop *e.g.*,

$$\int [d^d k_1][d^d k_2] \frac{\mu_{11}}{D_1 \cdots D_n} = \epsilon \sum_{i \in \{2\} \cup \{12\}} \int [d^{(d+2)} k_1][d^{(d+2)} k_2] \frac{1}{D_1 \cdots D_i^2 \cdots D_n}$$

$\Rightarrow$ derived using Schwinger parameterization

μ_{ij} insertion changes the ϵ structure

$I[\mu_{11}]$ () $= \epsilon \cdot I^{6-2\epsilon}$ () $= \mathcal{O}(\epsilon)$

$I[\mu_{12}]$ () $= \mathcal{O}(\epsilon)$