The energy-momentum tensor as the most fundamental object in gravitational physics

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Outline

- The Energy-Momentum Tensor in Electromagnetism
- Electrodynamics from Effective Field Theory
- General Relativity from Effective Field Theory
- General relativistic Thermodynamics

Of related interest:

G. Schäfer, From Newtonian energy conservation in self-gravitating systems to General Relativity, Physica Scripta **93**, 104007 (2018)

The Energy-Momentum Tensor in Electromagnetism

4-current of point charge $j^\mu = qu^\mu$, $\eta_{\mu\nu}u^\mu u^\nu = c^2$ $[\mu = (0, i), i = (1, 2, 3)]$

linear 4-momentum of point mass $p_\mu = mu_\mu$

force equations with Lorentz force $\frac{dp_\mu}{d\tau} = F_{\mu\nu}j^\nu$, $\quad F_{\mu\nu} = -F_{\nu\mu}$

homogeneous field equations $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$

$F_{\mu\nu} = (\mathbf{E}, \mathbf{B})$

inhomogeneous field equations $\partial_\nu G^{\nu\mu} = J^\mu$, $\quad G^{\mu\nu} = -G^{\nu\mu}$

$G^{\mu\nu} = (\mathbf{D}, \mathbf{H})$

4-current density $J^\mu = \rho_e u^\mu$, $\quad \partial_\mu J^\mu = 0 \quad$ (charge conservation)

structure tensor of medium $G^{\mu\nu} = M^{\mu\nu\alpha\beta}F_{\alpha\beta}$

vacuum $G^{\mu\nu} = \frac{1}{2\mu_0}(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\nu\alpha}\eta^{\mu\beta})F_{\alpha\beta} = \frac{1}{\mu_0}\eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$,
$\epsilon_0\mu_0 c^2 = 1$

force-density equations with Lorentz-force density $\partial_\nu(P_\mu u^\nu) = F_{\mu\nu}J^\nu$

4-momentum density $P_\mu = \rho u_\mu, \quad P_\mu u^\mu = \rho c^2$, proper-mass density ρ

energy-momentum tensor for matter ${}^{(ma)}T_\mu{}^\nu = P_\mu u^\nu$

$${}^{(ma)}T^{\nu\mu} = {}^{(ma)}T^{\mu\nu} \quad \longrightarrow \quad {}^{(ma)}T_\mu{}^\nu = {}^{(ma)}T_\mu^\nu, \quad {}^{(ma)}T_\mu^\mu = \rho c^2$$

energy-momentum tensor for em-field

$${}^{(em)}T_\mu^\nu = \tfrac{1}{4} G^{\alpha\beta} F_{\alpha\beta} \delta_\mu^\nu - G^{\nu\beta} F_{\mu\beta}, \quad {}^{(em)}T_\mu^\mu = 0$$

$$\partial_\nu {}^{(em)}T_\mu^\nu = -F_{\mu\nu} J^\nu \quad \text{(applying field equations)}$$

total energy-momentum tensor $T_\mu^\nu = {}^{(ma)}T_\mu^\nu + {}^{(em)}T_\mu^\nu$

conservation laws $\partial_\nu T_\mu^\nu = 0$ (Lorentz-force-density equations)

Electrodynamics from Effective Field Theory

J. Schwinger, Sources and Gravitons, Phys. Rev. **173**, 1264 (1968):

The fundamental objects in physics are currents, macroscopically

(i), the electric charge vector current J^μ

(ii), the energy-momentum tensor T^μ_ν

Both do fulfill conservation equations,

(i), charge conservation: $\partial_\mu J^\mu = 0$

(ii), energy-momentum conservation: $\partial_\mu T^\mu_\nu = 0$

Let us assume that currents can interact with themselves in a relativistically invariant manner.

Question: If a current stays alone in the distant past, what is the probability that the current will stay alone in the distant future?

Writing

$$< \ 0_+ \mid 0_- \ >^S \ = \ \exp\left(\frac{i}{\hbar}W[S]\right),$$

the probability reads

$$\mid < \ 0_+ \mid 0_- \ >^S \mid^2 \ = \ \exp\left(-\frac{2}{\hbar}\mathrm{Im}W[S]\right) \ \leq \ 1,$$

where W is an effective action ("Wirkung" in German) and S is the source in question. The real part of W is a classical action functional.

$\bar{e} = \sqrt{\hbar c} = \frac{e}{\sqrt{\alpha}} = 1.875 \times 10^{-18}$ C, $\quad \alpha \equiv \frac{e^2}{\hbar c} = \frac{1}{137.036}$

$$\frac{W[J]}{\hbar} = \frac{1}{(\bar{e}c)^2}\frac{4\pi}{2}\int (dx)(dx')\eta_{\mu\nu}J^{\mu}(x)G_F(x-x')J^{\nu}(x')$$

assumption to be made: Lorentz invariance

obviously, a symmetric Green function: $G_F(x-x') = G_F(x'-x)$

consistency with probability strictly implies $\partial_\mu J^\mu = 0$

$$\delta W[J] = \int (dx)\delta J^{\mu}(x)A_{\mu}(x)$$

$$A_\mu(x) = \frac{4\pi}{c}\int (dx')G_F(x-x')J_\mu(x') + \partial_\mu \lambda(x) \quad \text{(gauge field)}$$

$\partial_\mu A^\mu = \partial^2 \lambda, \qquad \partial^2 A_\nu - \partial_\nu \partial_\mu A^\mu = \frac{4\pi}{c}J_\nu$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \partial_\mu F^{\mu\nu} = \frac{4\pi}{c}J^\nu$

$$G_F(x-x') = \frac{1}{4\pi}\delta\left((x-x')^2\right) - \frac{i}{4\pi^2}\frac{1}{(x-x')^2} = \frac{-i}{4\pi^2}\frac{1}{(x-x')^2 - i0}$$

$$G_F = \bar{G} - \frac{i}{2}G^{(1)}\,, \quad \bar{G} = \frac{1}{2}(G_{\rm ret} + G_{\rm adv}) \quad G_F = G_{\rm ret} + G^{(+)}$$

$$G_{\rm ret} = \bar{G} - \frac{1}{2}\tilde{G}\,, \quad \tilde{G} = G_{\rm adv} - G_{\rm ret}\,, \quad G^{(+)} = \frac{1}{2}\left(\tilde{G} - iG^{(1)}\right)$$

classical mechanics: boundary of quantum mechanics

$$(1/i\hbar)[\Phi(x),\Phi(x')] \;=\; \tilde{G}(x,x') \quad \text{classical causality for observables}$$

$$(1/i\hbar) < 0|[\Phi^{(+)}(x),\Phi(x')]|0 > \;=\; G^{(+)}(x-x')$$

$$\Phi(x) \;=\; \Phi^{(+)}(x) \;+\; \Phi^{(-)}(x), \quad \Phi^{(+)}(x)|0 > \;=\; 0$$

$$x^0 > x^{0\prime}: \quad G_F(x-x') = -i \int d\omega_k e^{-ik(x-x')}$$

$$d\omega_k = \frac{(d\mathbf{k})}{(2\pi)^3}\frac{1}{2k^0}, \quad k^0 = |\mathbf{k}|$$

$$| < \ 0_+ \mid 0_- >^J |^2 \ = \ \exp\left(\frac{4\pi}{(e^*c)^2}\int d\omega_k \eta_{\mu\nu} J^\mu(k)^* J^\nu(k)\right)$$

$J^\mu(k)^* J_\mu(k) = |J^0(k)|^2 - |\mathbf{J}(k)|^2$ must be negative!

With $\partial_\mu J^\mu(x) = 0$, i.e. $k_\mu J^\mu(k) = 0$, and $k_3 = k^0$, we get $J_3 = J^0$ and

$$-J^\mu(k)^* J_\mu(k) = |J_1(k)|^2 + |J_2(k)|^2 = \frac{1}{2}|J_1 + iJ_2|^2 + \frac{1}{2}|J_1 - iJ_2|^2 \geq 0$$

Source for transverse linear or circular polarized helicity-1 waves built on massless photons.

If the source is independent of time for a long time interval $t_f - t_i$, then

$$W = -E\,(t_f - t_i)$$

with energy E given by

$$E = \frac{2\pi}{c^2}\int (d\mathbf{x})(d\mathbf{x}')J^{\mu}(\mathbf{x})\frac{\eta_{\mu\nu}}{4\pi|\mathbf{x}-\mathbf{x}'|}J^{\nu}(\mathbf{x}')$$

using

$$\int_{-\infty}^{\infty} d(x^0 - x^{0\prime})G_F(x-x') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

The interaction energy between two quasi-static charge-current distributions J_1^{μ} and J_2^{μ} reads $(\partial_\mu J^\mu = \nabla\cdot\mathbf{J} = 0)$

$$E_{12} = \frac{1}{c^2}\int (d\mathbf{x})(d\mathbf{x}')\frac{J_1^0(\mathbf{x})J_2^0(\mathbf{x}') - \mathbf{J}_1(\mathbf{x})\cdot\mathbf{J}_2(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Coulomb-Ampère laws

General Relativity from Effective Field Theory

$$M_{\mathrm{P}} = \sqrt{\hbar c/G} = 2.176 \times 10^{-8}\ \mathrm{kg}, \quad \text{Planck mass}$$

$$\frac{W[T]}{\hbar} = \frac{1}{(M_{\mathrm{P}}c^2)^2}\frac{8\pi}{2}\int (dx)(dx')\left(T^{\nu}_{\mu}(x)G_F(x-x')T^{\mu}_{\nu}(x') - \frac{1}{2}T(x)G_F(x-x')T(x')\right)$$

$$T^{\mu\nu} = T^{\nu\mu}, \qquad T = T^{\mu}_{\mu}, \qquad \partial_\mu T^{\mu}_{\nu} = 0$$

$$\delta W[T] = \frac{1}{2}\int (dx)\delta T^{\mu\nu}(x) h_{\mu\nu}(x)$$

$$h_{\mu\nu}(x) = \frac{16\pi G}{c^4}\int (dx')G_F(x-x')\left(T_{\mu\nu}(x') - \frac{1}{2}\eta_{\mu\nu}T(x')\right) + \partial_\mu\xi_\nu(x) + \partial_\nu\xi_\mu(x) \quad \text{(gauge field)}$$

or, with $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h$,

$$\bar{h}^{\mu\nu}(x) = \frac{16\pi G}{c^4}\int (dx')G_F(x-x')T^{\mu\nu}(x') \ + \ \partial^\mu\xi^\nu(x) + \partial^\nu\xi^\mu(x) - \eta^{\mu\nu}\partial_\lambda\xi^\lambda$$

$$\partial^2 \bar{h}^{\mu\nu}(x) = \frac{16\pi G}{c^4} T^{\mu\nu}(x) \ + \ \partial^\mu \partial^2 \xi^\nu(x) + \partial^\nu \partial^2 \xi^\mu(x) - \eta^{\mu\nu} \partial_\lambda \partial^2 \xi^\lambda$$

$$\partial_\mu \bar{h}^{\mu\nu} = \partial^2 \xi^\nu$$

$$-\partial^2 h_{\mu\nu} + \partial_\mu \partial^\lambda h_{\lambda\nu} + \partial_\nu \partial^\lambda h_{\lambda\mu} - \partial_\mu \partial_\nu h = -\frac{16\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right)$$

Introducing "Christoffel symbols"

$$\Gamma_{\lambda,\mu\nu} = \frac{1}{2} \left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu} \right), \quad \Gamma^\lambda_{\mu\nu} = \eta^{\lambda\sigma} \Gamma_{\sigma,\mu\nu}$$

$$\partial_\lambda \Gamma^\lambda_{\mu\nu} - \frac{1}{2} (\partial_\mu \Gamma^\lambda_{\nu\lambda} + \partial_\nu \Gamma^\lambda_{\mu\lambda}) = -\frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right)$$

infinitesimal coordinate transformation $\quad x'^{\mu} = x^{\mu} + \delta x^{\mu}(x)$

induces $\delta h_{\mu\nu} = -(\partial_\mu \delta x_\nu + \partial_\nu \delta x_\mu)$, with gauge vector $\xi^\mu = -\delta x^\mu$

and also $\delta \Gamma^\lambda_{\mu\nu} \; = \; - \; \partial_\mu \partial_\nu \delta x^\lambda$ (Γs are not gauge invariant)

But with $g_{\mu\nu}(x) \; = \; \eta_{\mu\nu} \; + \; h_{\mu\nu}(x)$ and $|h_{\mu\nu}(x)| << 1$,

it also holds: $\delta g_{\mu\nu} = \delta h_{\mu\nu} \; = \; - \; (\partial_\mu \delta x_\nu \; + \; \partial_\nu \delta x_\mu)$,

and $\delta \Gamma^\lambda_{\mu\nu} \; = \; - \; \partial_\mu \partial_\nu \delta x^\lambda$, $\quad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} \eta^{\lambda\kappa} \left(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu} \right)$

However, the (linear) field equations

$$\partial_\lambda \Gamma^\lambda_{\mu\nu} - \frac{1}{2}(\partial_\mu \Gamma^\lambda_{\nu\lambda} + \partial_\nu \Gamma^\lambda_{\mu\lambda}) = -\frac{8\pi G}{c^4} \left(T_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} T \right)$$

are gauge invariant

Finite coordinate transformations read

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}$$

The non-linear Einstein field equations do follow then most naturally:

$$R_{\mu\nu}(\partial\Gamma, \Gamma) \;=\; -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

(covariant or gauge invariant equations)

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \frac{1}{2}(\partial_\mu \Gamma^\lambda_{\lambda\nu} + \partial_\nu \Gamma^\lambda_{\mu\lambda}) + \Gamma^\kappa_{\lambda\kappa}\Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\kappa}\Gamma^\kappa_{\nu\lambda}$$

(Ricci tensor)

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$$

The standard form of the field equations is

$$G_{\mu\nu}(\partial^2 g, \partial g, g) \;=\; -\frac{8\pi G}{c^4} T_{\mu\nu}(\eta \to g) \qquad \text{(minimal coupling)}$$

with the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

The Einstein Equivalence Principle holds without to be postulated.

The field equations imply the equations of motion:

$$D_\mu G^\mu_\nu \;\equiv\; 0 \;\Rightarrow\; D_\mu T^\mu_\nu = 0$$

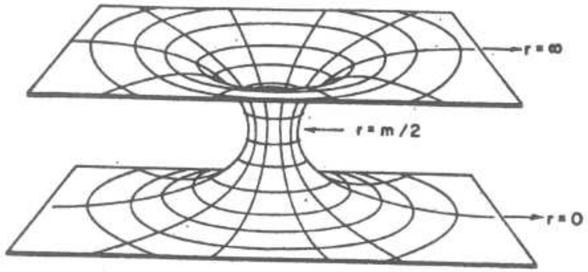


FIG. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2 = (1+m/2r)^4(dr^2+r^2d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold ($r\to 0$, $r\to\infty$).

Brill/Lindquist 1963

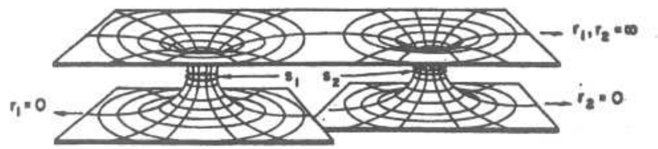


FIG. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

$$ds^2 = (1+m/2r_1+m/2r_2)^4 ds_F^2.$$

$$| < 0_+ \mid 0_- >^T|^2 = \exp\left(-\frac{8\pi}{(M_{\mathrm{P}}c^2)^2}\int d\omega_k (T^\mu_\nu(k)^* T^\nu_\mu(k) - \frac{1}{2}T(k)^* T(k))\right)$$

$$\partial_\mu T^{\mu\nu}(x) = 0,\ k_\mu T^{\mu\nu}(k) = 0,\ k_3 = k^0$$

$$T^\mu_\nu(k)^* T^\nu_\mu(k) - \frac{1}{2}T(k)^* T(k) = \sum_{i,j=1,2} |T_{ij}|^2 - \frac{1}{2}|T_{ii}|^2 \geq 0$$

transversal linear polarisation, or, equal to,

$$\frac{1}{4}|T_{11} - T_{12} + 2iT_{12}|^2 + \frac{1}{4}|T_{11} - T_{12} - 2iT_{12}|^2$$

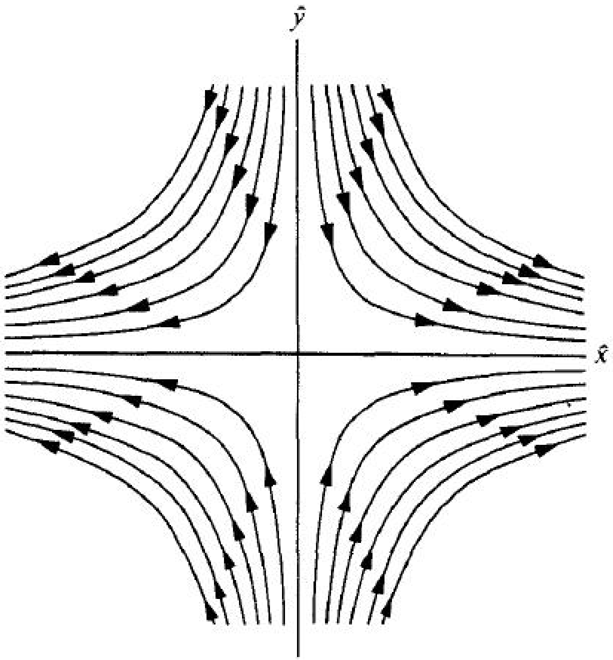
transversal circular polarisations, massless helicity-2 particle, the graviton. Gravitational waves are built on gravitons.

line element $$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

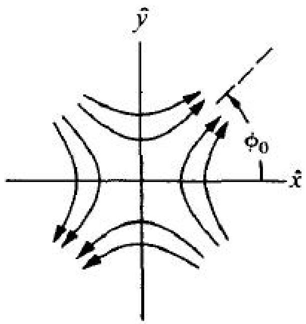
gravitational wave $$\mathrm{ds}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + h_{ij}^{\mathrm{TT}} dx^i dx^j$$

Fermi normal coordinates

$$ds^2 = \left(c^2 - \frac{1}{2}\ddot{h}_{ij}^{\mathrm{TT}} \hat{x}^i \hat{x}^j\right) d\hat{t}^2 - d\hat{x}^i d\hat{x}^i \qquad (|\hat{\mathbf{x}}| << c/f)$$



(a) Force lines for $\ddot{A}_\times = 0$, $\ddot{A}_+ > 0$



(b)

gravitational quadrupole wave

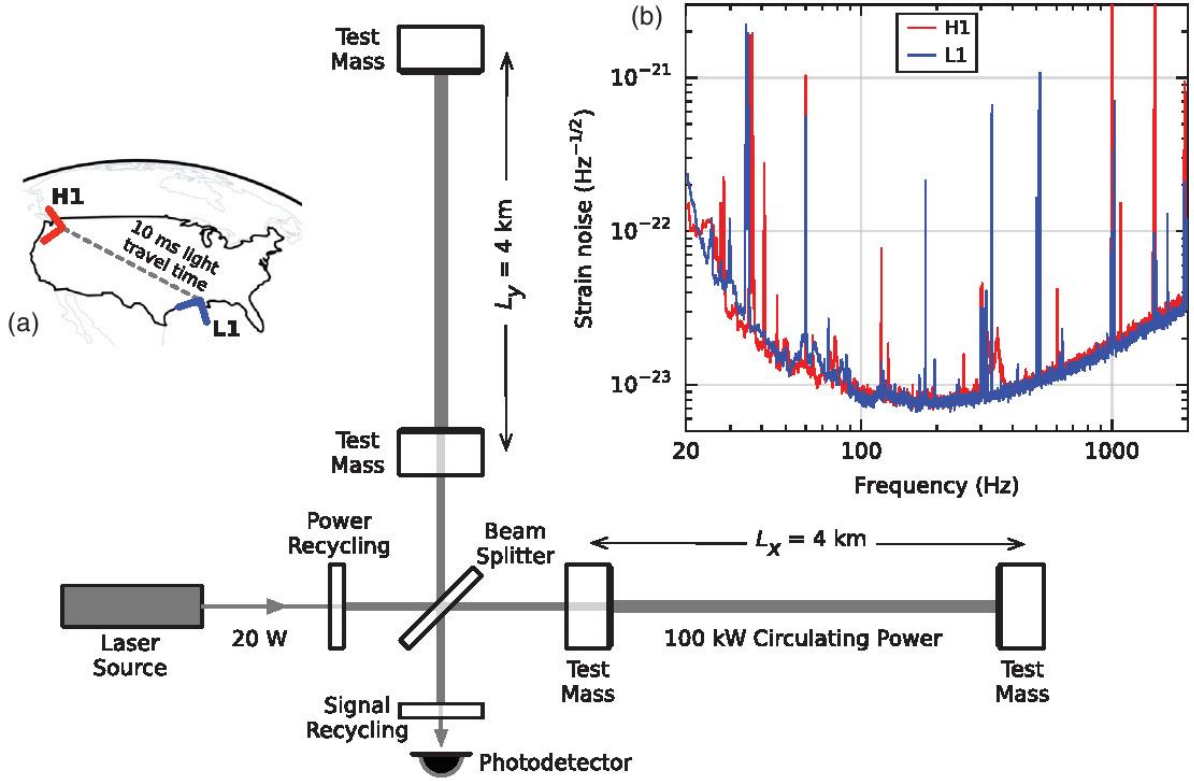
propagating in $\hat{z}$-direction

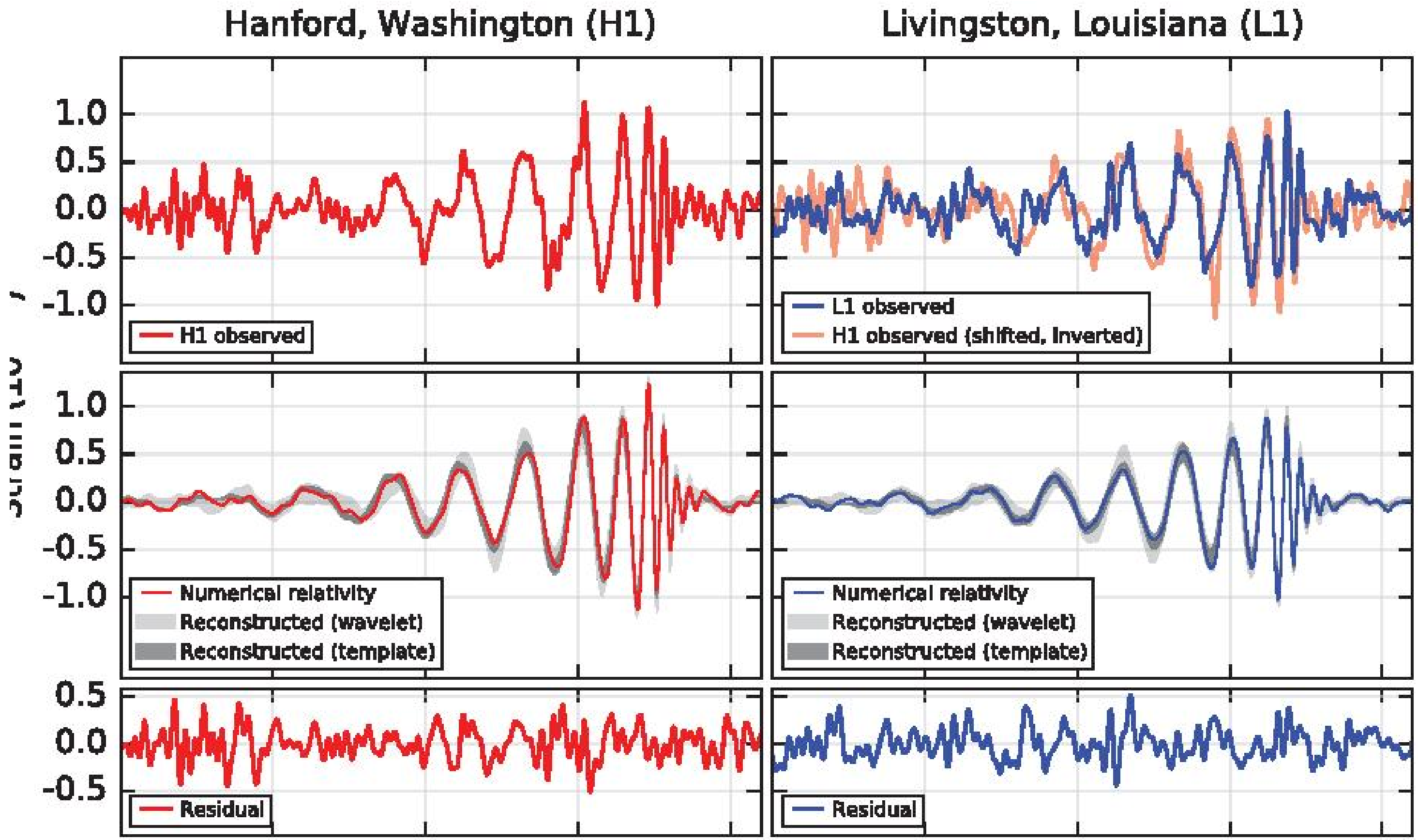
$$d^2\hat{x}/d\hat{t}^2 = \tfrac{1}{2}(\ddot{A}_+\hat{x} + \ddot{A}_\times\hat{y})$$

$$d^2\hat{y}/d\hat{t}^2 = \tfrac{1}{2}(-\ddot{A}_+\hat{y} + \ddot{A}_\times\hat{x})$$

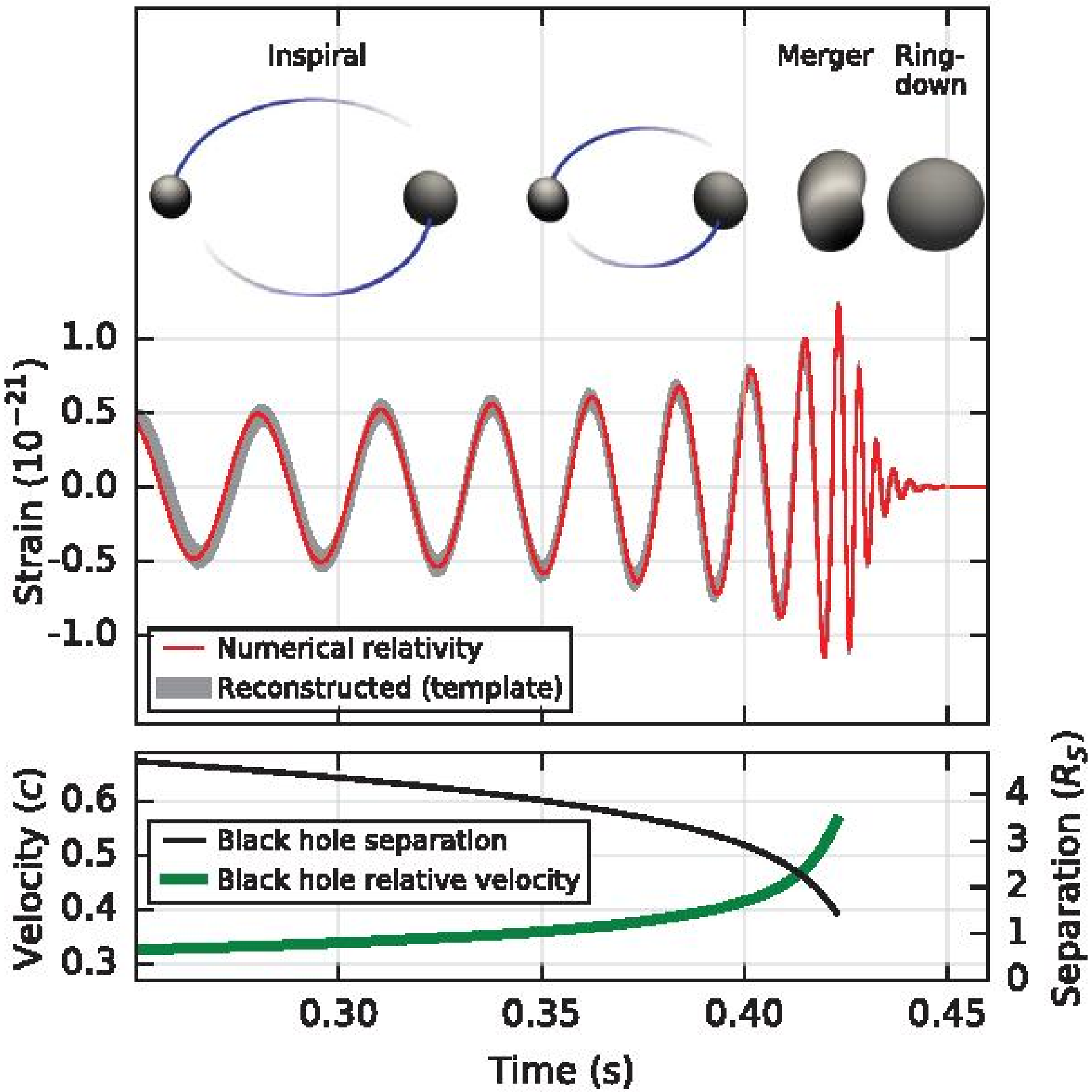
$$d^2\hat{z}/d\hat{t}^2 = 0$$

MTW: Gravitation (1973)


(a)
H1
10 ms light travel time
L1
Test Mass
Test Mass
L_y = 4 km
Power Recycling
Beam Splitter
L_x = 4 km
Laser Source
20 W
100 kW Circulating Power
Test Mass
Test Mass
Signal Recycling
Photodetector
(b)
H1
L1
Strain noise (Hz$^{-1/2}$)
10^{-21}
10^{-22}
10^{-23}
20
100
1000
Frequency (Hz)



Hanford, Washington (H1)
Livingston, Louisiana (L1)
1.0
0.5
0.0
-0.5
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual



$$h(t)L = \Delta L(t) = \delta L_x - \delta L_y$$

Quasi-static source distribution, binding energy

$W = -E\ (t_f - t_i):$

$$E_{12} = -\frac{2G}{c^4}\int (d\mathbf{x})(d\mathbf{x}')\left(\frac{T_1^{\mu\nu}(\mathbf{x})T_{2\mu\nu}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{2}\frac{T_1(\mathbf{x})T_2(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right)$$

With $T_a^{00} = M_a c^2$ ($T_a^{\mu\nu}$ zero otherwise) and r the relative distance of the two masses,

$$E_{12} = -\frac{GM_1M_2}{r} = \frac{M_1}{\sqrt{\hbar c/G}}\frac{-\hbar c}{r}\frac{M_2}{\sqrt{\hbar c/G}}$$

Newton law from Effective Field Theory

General relativistic Thermodynamics

e.g.

L. Rezzolla and O. Zanotti

Relativistic Hydrodynamics

Oxford University Press, 2013

The energy momentum tensor:

$$T^{\mu\nu} = T^{\mu\nu}_{\text{cons}} + T^{\mu\nu}_{\text{diss}}, \qquad D_\nu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = T^{\nu\mu}$$

with

$$T^{\mu\nu}_{\text{cons}} = (e+p)u^\mu u^\nu - pg^{\mu\nu}, \quad u^\mu u_\mu = 1, \quad u_\nu T^{\mu\nu}_{\text{cons}} = eu^\mu$$

e energy density, $e = \rho(c^2 + \Pi)$, Π specific internal energy, p pressure

1st Law of TD: $d\Pi = \frac{p}{\rho^2} d\rho + T ds$ $(dU = -pdV + TdS)$
local equilibria!

equation of state: $\Pi = \Pi(\rho, s)$

$$T^{\mu\nu}_{\text{diss}} = u^\mu q^\nu + u^\nu q^\mu + \pi^{\mu\nu}, \quad q^\mu u_\mu = 0, \quad \pi^{\mu\nu} u_\nu = 0,$$

q^μ heat-flux 4-vector, $\pi^{\mu\nu}$ viscosity tensor energy balance equation (1st Law of TD)

$$u_\mu D_\nu T^{\mu\nu} = 0$$

momentum balance equations (equations of motion)

$$P_{\lambda\mu} D_\nu T^{\mu\nu} = 0$$

$$P_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu}, \qquad P_{\mu\nu} u^\nu = 0$$

(J. Ehlers, Beiträge zur relativistischen Mechanik kontinuierlicher Medien, (1961); translated by G.F.R. Ellis and P.K.S. Dunsby in: Contributions to the relativistic mechanics of continuous media, Gen. Relativ. Gravit. **25** 1225 (1993))

The entropy-density vector

$$S^\mu = s\rho u^\mu + \frac{q^\mu}{T}, \qquad S^\mu u_\mu = s\rho$$

s specific entropy, $s\rho$ entropy density, T temperature

entropy production rate equation $D_\mu S^\mu = T^{\mu\nu}_{\mathrm{diss}} D_\mu \Theta_\nu$

$$\text{temperature vector :} \qquad \boldsymbol{\Theta}^\mu = \frac{u^\mu}{T}$$

2nd Law of TD: $D_\mu S^\mu \geq 0$ $\quad (dS - \frac{\delta Q}{T} \geq 0)$

$\eta \equiv T^{\mu\nu}_{\mathrm{diss}} D_\mu \Theta_\nu \geq 0$

decomposition of u-congruence

$$D_\nu u_\mu = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta P_{\mu\nu} - \dot{u}_\mu u_\nu, \quad \theta = D_\mu u^\mu, \quad \dot{u}^\mu = u^\nu D_\nu u^\mu$$

$\omega_{\mu\nu}$ vortex vel. (antisym.), $\sigma_{\mu\nu}$ shearing vel. (sym.), θ expansion vel.

$$\omega_{\mu\nu} u^\mu = 0, \qquad \sigma_{\mu\nu} u^\mu = 0, \qquad \sigma^\mu_\mu = 0$$

$$\eta = -\frac{1}{T}\left(\pi^{\mu\nu}\sigma_{\mu\nu} + \frac{1}{3}\pi\theta + q^\mu(\dot{u}_\mu + \partial_\mu \ln T)\right)$$

$$\pi^{\mu\nu} = -\lambda_s \sigma^{\mu\nu} - \lambda_b \frac{\theta}{3} P^{\mu\nu}$$

$\lambda_s \geq 0$ shear-viscosity coefficient, $\lambda_b \geq 0$ bulk-viscosity coefficient

$$q^\mu = -\kappa P^{\mu\nu}(\partial_\nu T + T\dot{u}_\nu)$$

$\kappa \geq 0$ heat-conduction coefficient

$$\eta = \frac{1}{T}(\lambda_s \sigma^{\mu\nu}\sigma_{\mu\nu} + \lambda_b \frac{1}{3}\theta^2) + \kappa P^{\mu\nu}(\dot{u}_\mu + \partial_\mu \ln T)(\dot{u}_\nu + \partial_\nu \ln T)$$

$$S^\mu = s\rho u^\mu + T^{\mu\nu}_{\text{diss}}\Theta_\nu = (-f\rho\delta^\mu_\nu + T^\mu_\nu)\Theta^\nu$$

f specific Helmholtz free energy: $f = c^2 + \Pi - Ts$

$df = \frac{p}{\rho^2}d\rho - sdT$, equation of state $f = f(\rho, T)$

Einstein field equations: $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\frac{8\pi G}{c^4}T^{\mu\nu}$

herewith: variational representation of entropy-density 4-vector

$$\sqrt{-g(x)}S^{\mu}(x) = -2\Theta_{\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}\int_{\Omega} d^4y\ \mathcal{L}(y)$$

with Lagrange density $\mathcal{L} = \sqrt{-g}\ \left(f\rho\ -\ \frac{c^4 R}{16\pi G}\right)$

$$-\partial_{\mu}(\sqrt{-g}S^{\mu}) = \frac{\delta \int_{\Omega} d^4y\mathcal{L}}{\delta g_{\mu\nu}}(D_{\mu}\Theta_{\nu} + D_{\nu}\Theta_{\mu}) \leq 0$$

$D_{\mu}\Theta_{\nu} + D_{\nu}\Theta_{\mu} = \mathrm{LieD}_{\Theta^{\lambda}} g_{\mu\nu}$

(G. Neugebauer, Entropy and gravitation, Int. J. Theor. Phys. **16** 241 (1977))