

# Hyperasymptotic approximation to the Operator Product Expansion (OPE)

Based on

Ayala, Lobregat, Pineda: 1902.07736; 1909.01370; Montpellier proceedings

Antonio Pineda

Universitat Autònoma de Barcelona & IFAE

DESY-Zeuthen, Berlin, 16-01-2020

## OPE

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) = S_{\text{pert}}(\alpha_X(Q)) + \sum_d C_{O,d}(\alpha_X(Q)) \frac{\langle O_d \rangle}{Q^d}.$$

What is  $S_{\text{pert}}(\alpha_X(Q))$ ?

What is  $C_{O,d}(\alpha_X(Q)) \frac{\langle O_d \rangle}{Q^d}$ ?

$$\begin{aligned} \text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) &= \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q) + \left( K + \sum_{n=0}^{\infty} p_n^{(X,d)} \alpha_X^{n+1}(Q) \right) \alpha_X^{\gamma}(Q) \frac{\Lambda_X^d}{Q^d} + \dots \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q) + \left( K' + \sum_{n=0}^{\infty} p_n^{(X,d)} \alpha_X^{n+1}(Q) \right) \alpha_X^{\gamma-db}(Q) e^{-d \frac{2\pi}{\beta_0 \alpha_X(Q)}} + \dots \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) + \left( K' + \sum_{n=0}^{\infty} p_n^{(X,d)} \left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \right) \alpha_X^{\gamma-db}(\mu) \frac{\mu^d}{Q^d} e^{-d \frac{2\pi}{\beta_0 \alpha_X(\mu)}} \end{aligned}$$

## OPE

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) = \mathcal{S}_{\text{pert}}(\alpha_X(Q)) + \sum_d \mathcal{C}_{O,d}(\alpha_X(Q)) \frac{\langle O_d \rangle}{Q^d}.$$

What is  $\mathcal{S}_{\text{pert}}(\alpha_X(Q))$ ?

What is  $\mathcal{C}_{O,d}(\alpha_X(Q)) \frac{\langle O_d \rangle}{Q^d}$ ?

$$\begin{aligned} \text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) &= \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q) + \left( K + \sum_{n=0}^{\infty} p_n^{(X,d)} \alpha_X^{n+1}(Q) \right) \alpha_X^\gamma(Q) \frac{\Lambda_X^d}{Q^d} + \dots \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q) + \left( K' + \sum_{n=0}^{\infty} p_n^{(X,d)} \alpha_X^{n+1}(Q) \right) \alpha_X^{\gamma-db}(Q) e^{-d \frac{2\pi}{\beta_0 \alpha_X(Q)}} + \dots \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) + \left( K' + \sum_{n=0}^{\infty} p_n^{(X,d)} \left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \right) \alpha_X^{\gamma-db}(\mu) \frac{\mu^d}{Q^d} e^{-d \frac{2\pi}{\beta_0 \alpha_X(\mu)}} \end{aligned}$$

## OPE

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) = S_{\text{pert}}(\alpha_X(Q)) + \sum_d C_{O,d}(\alpha_X(Q)) \frac{\langle O_d \rangle}{Q^d}.$$

What is  $S_{\text{pert}}(\alpha_X(Q))$ ?

What is  $C_{O,d}(\alpha_X(Q)) \frac{\langle O_d \rangle}{Q^d}$ ?

$$\begin{aligned} \text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) &= \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q) + \left( K + \sum_{n=0}^{\infty} p_n^{(X,d)} \alpha_X^{n+1}(Q) \right) \alpha_X^{\gamma}(Q) \frac{\Lambda_X^d}{Q^d} + \dots \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q) + \left( K' + \sum_{n=0}^{\infty} p_n^{(X,d)} \alpha_X^{n+1}(Q) \right) \alpha_X^{\gamma-db}(Q) e^{-d \frac{2\pi}{\beta_0 \alpha_X(Q)}} + \dots \\ &= \sum_{n=0}^{\infty} p_n^{(X)} \left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) + \left( K' + \sum_{n=0}^{\infty} p_n^{(X,d)} \left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \right) \alpha_X^{\gamma-db}(\mu) \frac{\mu^d}{Q^d} e^{-d \frac{2\pi}{\beta_0 \alpha_X(\mu)}} \end{aligned}$$

$\sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q)$  is divergent:  $p_n \sim n!$

One possible way out: Organize the computation using superasymptotic and hyperasymptotic approximations.

It allows for a parametric control of the error.

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) - \sum_{n=0}^N p_n^{(X)} \left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \sim \mathcal{O}(\alpha_X^{N+2})$$

but with large coefficient!!

$\sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q)$  is divergent:  $p_n \sim n!$

One possible way out: Organize the computation using superasymptotic and hyperasymptotic approximations.

It allows for a parametric control of the error.

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) - \sum_{n=0}^N p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \sim \mathcal{O}(\alpha_X^{N+2})$$

but with large coefficient!!

$\sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q)$  is divergent:  $p_n \sim n!$

One possible way out: Organize the computation using superasymptotic and hyperasymptotic approximations.

It allows for a parametric control of the error.

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) - \sum_{n=0}^N p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \sim \mathcal{O}(\alpha_X^{N+2})$$

but with large coefficient!!

$\sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q)$  is divergent:  $p_n \sim n!$

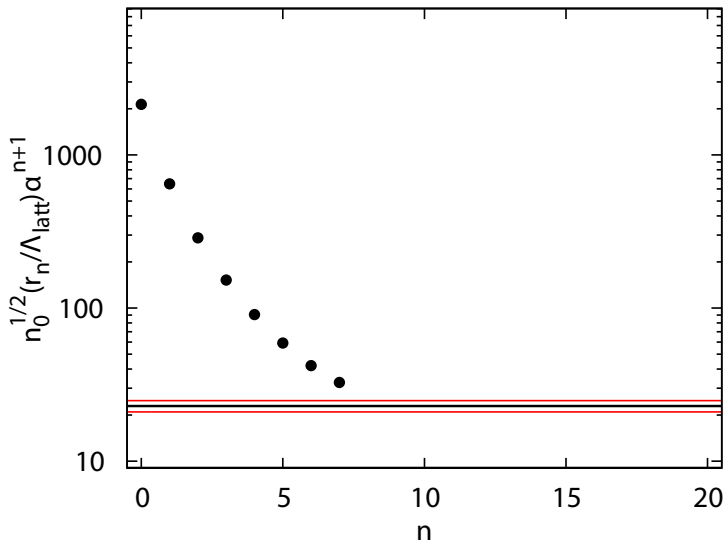
One possible way out: Organize the computation using superasymptotic and hyperasymptotic approximations.

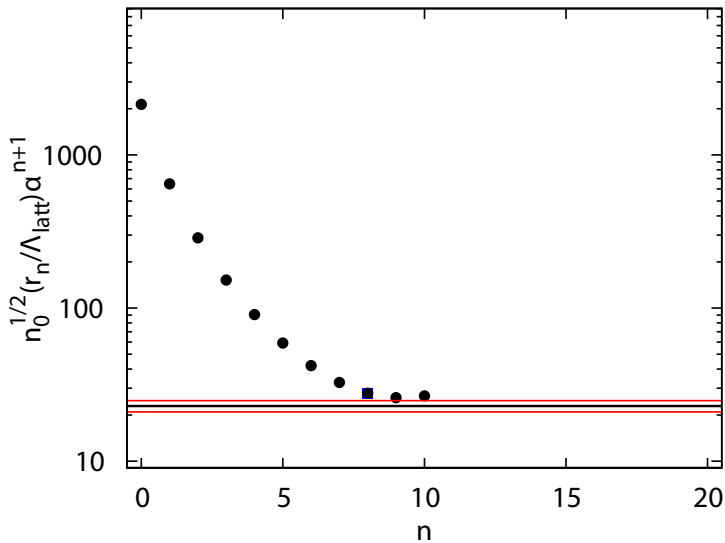
It allows for a parametric control of the error.

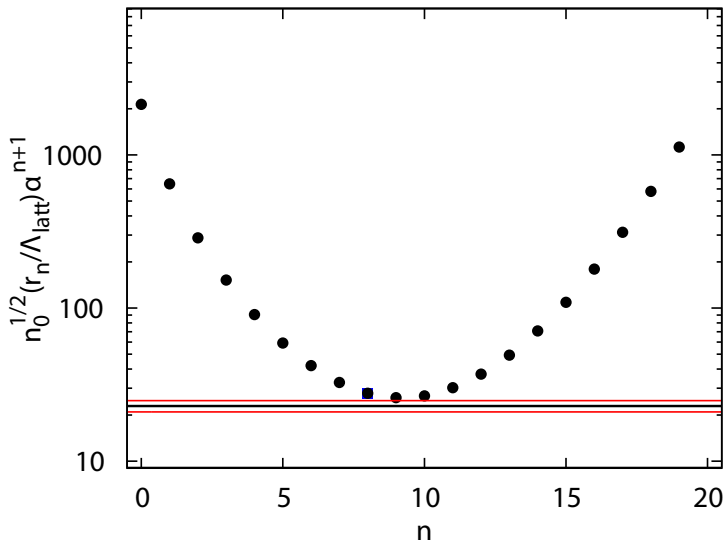
$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) - \sum_{n=0}^N p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \sim \mathcal{O}(\alpha_X^{N+2})$$

but with large coefficient!!









$\sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q)$  is divergent:  $p_n \sim n!$

One possible way out: Organize the computation using supersymptotic and hyperasymptotic approximations.

It allows for a parametric control of the error.

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) - \sum_{n=0}^N p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \sim \mathcal{O}(\alpha_X^{N+2})$$

but with large coefficient!!

Truncate the sum at the minimal term  $\rightarrow$  supersymptotic approximation:

1)  $N$  and  $\mu \sim Q$  large but finite:

$$N = N_P \equiv |d| \frac{2\pi}{\beta_0 \alpha_X(\mu)} (1 - c \alpha_X(\mu)),$$

2)  $N \rightarrow \infty$  and  $\mu \rightarrow \infty$  in a correlated way.

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) - \sum_{n=0}^{N_P} p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \sim \mathcal{O}\left(e^{-|d| \frac{2\pi}{\beta_0 \alpha_X(Q)}}\right)$$

$\sum_{n=0}^{\infty} p_n^{(X)} \alpha_X^{n+1}(Q)$  is divergent:  $p_n \sim n!$

One possible way out: Organize the computation using superasymptotic and hyperasymptotic approximations.

It allows for a parametric control of the error.

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) - \sum_{n=0}^N p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \sim \mathcal{O}(\alpha_X^{N+2})$$

but with large coefficient!!

Truncate the sum at the minimal term  $\rightarrow$  **superasymptotic** approximation:

1)  $N$  and  $\mu \sim Q$  large but finite:

$$N = N_P \equiv |d| \frac{2\pi}{\beta_0 \alpha_X(\mu)} (1 - c \alpha_X(\mu)),$$

2)  $N \rightarrow \infty$  and  $\mu \rightarrow \infty$  in a correlated way.

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) - \sum_{n=0}^{N_P} p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \sim \mathcal{O}\left(e^{-|d| \frac{2\pi}{\beta_0 \alpha_X(Q)}}\right)$$

How to go beyond that?

1. Predict observables with  $e^{-A\frac{2\pi}{\beta_0\alpha(Q)}}$  precision ( $A > |d|$ ).
2. Avoid spurious renormalon problems.

## Non-relativistic Quantum Mechanics (ODE)

$$\langle O \rangle \sim \sum p_n \alpha^{n+1}$$

Asymptotic:  $p_n \sim \# a^n n^\gamma n!$

Supersasymptotic:

$$\langle O \rangle = \sum_{n=0}^{N_P} p_n \alpha^{n+1} + \mathcal{O}\left(e^{-\frac{\#}{\alpha}}\right)$$

Hyperasymptotic (Berry and Howls; Boyd):

$$\langle O \rangle = \sum_{n=0}^{N_P} p_n \alpha^{n+1} + \# e^{-\frac{\#}{\alpha}} + o\left(e^{-\frac{\#}{\alpha}}\right)$$

Situation in QM:

- ▶ Situation, structure (and strength) of the singularities can be deduced from semiclassical analysis.
- ▶ No scale/scheme dependence.
- ▶ One can actually compute the NP corrections.

In reality things are more complicated but still....

## QFT

$$\langle O \rangle \sim \sum p_n^X \alpha_X^{n+1}(\mu)$$

Asymptotic:  $p_n^X \sim \# a^n n^\gamma n!$

Supersasymptotic:

$$\langle O \rangle = \sum_{n=0}^{N_P} p_n^X \alpha_X^{n+1}(\mu) + \mathcal{O}\left(e^{-\frac{\#}{\alpha_X(\mu)}}\right)$$

Hyperasymptotic (Non-perturbative OPE):

$$\langle O \rangle = \sum_{n=0}^{N_P} p_n^X \alpha_X^{n+1}(\mu) + \# e^{-\frac{\#}{\alpha_X(\mu)}} + o\left(e^{-\frac{\#}{\alpha_X(\mu)}}\right)$$

Situation in QFT:

- ▶ Situation, structure (Not the strength) of the singularities assumed from the NP OPE.
- ▶ scale/scheme dependence.
- ▶ One CANNOT compute the NP corrections.

Things get as complicated as they can get...



## How to go beyond the superasymptotic approximation in QFT?

### Borel transform

$$\sum_{n=0}^{\infty} p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \rightarrow B[O](t) = \sum_{n=0}^{\infty} \frac{p_n^{(X)}\left(\frac{\mu}{Q}\right)}{n!} t^n$$

### Inverse Borel transform

$$\int_0^{\infty} dt e^{-t/\alpha_X(Q)} B[O](t)$$

### Singularities in the real axis! $\rightarrow$ PV regularization

$$S_{\text{PV}}(\alpha(Q)) \equiv \int_{0,\text{PV}}^{\infty} dt e^{-t/\alpha_X(Q)} B[O](t)$$

### Scale and Scheme independent

How to go beyond the superasymptotic approximation in QFT?

Borel transform

$$\sum_{n=0}^{\infty} p_n^{(X)}\left(\frac{\mu}{Q}\right) \alpha_X^{n+1}(\mu) \rightarrow B[O](t) = \sum_{n=0}^{\infty} \frac{p_n^{(X)}\left(\frac{\mu}{Q}\right)}{n!} t^n$$

Inverse Borel transform

$$\int_0^{\infty} dt e^{-t/\alpha_X(Q)} B[O](t)$$

Singularities in the real axis!  $\rightarrow$  PV regularization

$$S_{\text{PV}}(\alpha(Q)) \equiv \int_{0,\text{PV}}^{\infty} dt e^{-t/\alpha_X(Q)} B[O](t)$$

Scale and Scheme independent

## Assumption

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) = S_{\text{PV}}(\alpha_X(Q)) \\ + K_X^{(\text{PV})} \alpha_X^\gamma(Q) \frac{\Lambda_X^d}{Q^d} (1 + \mathcal{O}(\alpha_X(Q))) + \mathcal{O}\left(\frac{\Lambda_X^{d'}}{Q^{d'}}\right),$$

$S_{\text{PV}}$  can only be computed in an approximated way  $\rightarrow$  **hyperasymptotics**  
 $B[O](t) \rightarrow$  analytic function in the complex plane plus cuts.

$$\delta B[O](t) = Z_{O_d}^X \frac{\mu^d}{Q^d} \frac{1}{(1 - 2u/d)^{1+db-\gamma}} (1 + b_1(1 - 2u/d) + \dots)$$

where  $u \equiv \beta_0 t / (4\pi)$ .

These singularities are linked to the asymptotic behavior of perturbation theory

$$l(db) \equiv \int_{0, \text{PV}}^{\infty} dt e^{-t/\alpha} \frac{1}{(1 - 2u/d)^{1+db-\gamma}} = \alpha D_{db-\gamma}(-2\pi d / (\beta_0 \alpha)) \\ \sim \sum_{n=0}^{\infty} \frac{\Gamma(1 + db - \gamma + n)}{\Gamma(1 + db - \gamma)} \left(\frac{\beta_0}{2\pi d}\right)^n \alpha^{n+1}(\mu),$$

## Assumption

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) = S_{\text{PV}}(\alpha_X(Q)) \\ + K_X^{(\text{PV})} \alpha_X^\gamma(Q) \frac{\Lambda_X^d}{Q^d} (1 + \mathcal{O}(\alpha_X(Q))) + \mathcal{O}\left(\frac{\Lambda_X^{d'}}{Q^{d'}}\right),$$

$S_{\text{PV}}$  can only be computed in an approximated way  $\rightarrow$  **hyperasymptotics**  
 $B[O](t) \rightarrow$  analytic function in the complex plane plus cuts.

$$\delta B[O](t) = Z_{O_d}^X \frac{\mu^d}{Q^d} \frac{1}{(1 - 2u/d)^{1+db-\gamma}} (1 + b_1(1 - 2u/d) + \dots)$$

where  $u \equiv \beta_0 t / (4\pi)$ .

These singularities are linked to the asymptotic behavior of perturbation theory

$$l(db) \equiv \int_{0, \text{PV}}^{\infty} dt e^{-t/\alpha} \frac{1}{(1 - 2u/d)^{1+db-\gamma}} = \alpha D_{db-\gamma}(-2\pi d / (\beta_0 \alpha)) \\ \sim \sum_{n=0}^{\infty} \frac{\Gamma(1 + db - \gamma + n)}{\Gamma(1 + db - \gamma)} \left(\frac{\beta_0}{2\pi d}\right)^n \alpha^{n+1}(\mu),$$

## Assumption

$$\text{Observable}\left(\frac{Q}{\Lambda_{\text{QCD}}}\right) = S_{\text{PV}}(\alpha_X(Q)) \\ + K_X^{(\text{PV})} \alpha_X^\gamma(Q) \frac{\Lambda_X^d}{Q^d} (1 + \mathcal{O}(\alpha_X(Q))) + \mathcal{O}\left(\frac{\Lambda_X^{d'}}{Q^{d'}}\right),$$

$S_{\text{PV}}$  can only be computed in an approximated way  $\rightarrow$  **hyperasymptotics**  
 $B[O](t) \rightarrow$  analytic function in the complex plane plus cuts.

$$\delta B[O](t) = Z_{O_d}^X \frac{\mu^d}{Q^d} \frac{1}{(1 - 2u/d)^{1+db-\gamma}} (1 + b_1(1 - 2u/d) + \dots)$$

where  $u \equiv \beta_0 t / (4\pi)$ .

These singularities are linked to the asymptotic behavior of perturbation theory

$$l(db) \equiv \int_{0, \text{PV}}^{\infty} dt e^{-t/\alpha} \frac{1}{(1 - 2u/d)^{1+db-\gamma}} = \alpha D_{db-\gamma}(-2\pi d / (\beta_0 \alpha)) \\ \sim \sum_{n=0}^{\infty} \frac{\Gamma(1 + db - \gamma + n)}{\Gamma(1 + db - \gamma)} \left(\frac{\beta_0}{2\pi d}\right)^n \alpha^{n+1}(\mu),$$

$$\begin{aligned}
 \delta S_{\text{PV}} &= Z_{O_d}^X \frac{\mu^d}{Q^d} [I(db) + b_1 I(db-1) + \dots] \\
 &= Z_{O_d}^X \frac{\mu^d}{Q^d} \sum_{n=0}^N \frac{\Gamma(1+db-\gamma+n)}{\Gamma(1+db-\gamma)} \left[ 1 + b_1 \frac{db-\gamma}{db-\gamma+n} \right. \\
 &\quad \left. + b_2 \frac{(db-\gamma)^2}{(n+db-\gamma)(n+db-\gamma-1)} + \dots \right] \left( \frac{\beta_0}{2\pi d} \right)^n \alpha_X^{n+1}(\mu) + \Omega_d,
 \end{aligned}$$

The finite sum is included in

$$S_P \equiv \sum_{n=0}^{N_P(|d_{\min}|)} p_n \alpha^{n+1}(\mu)$$

Terminant (Dingle)

$$\begin{aligned}
 \Omega_d &\sim Z_{O_d}^X \frac{\mu^d}{Q^d} \sum_{n=N+1}^{\infty} \frac{\Gamma(1+db-\gamma+n)}{\Gamma(1+db-\gamma)} [1 + \dots] \left( \frac{\beta_0}{2\pi d} \right)^n \alpha_X^{n+1}(\mu) \\
 \Omega_d &\sim \sqrt{\alpha} \frac{\Lambda_{\text{QCD}}^d}{Q^d}
 \end{aligned}$$

$$\begin{aligned}
\delta S_{\text{PV}} &= Z_{O_d}^X \frac{\mu^d}{Q^d} [I(db) + b_1 I(db-1) + \dots] \\
&= Z_{O_d}^X \frac{\mu^d}{Q^d} \sum_{n=0}^N \frac{\Gamma(1+db-\gamma+n)}{\Gamma(1+db-\gamma)} \left[ 1 + b_1 \frac{db-\gamma}{db-\gamma+n} \right. \\
&\quad \left. + b_2 \frac{(db-\gamma)^2}{(n+db-\gamma)(n+db-\gamma-1)} + \dots \right] \left( \frac{\beta_0}{2\pi d} \right)^n \alpha_X^{n+1}(\mu) + \Omega_d,
\end{aligned}$$

The finite sum is included in

$$S_P \equiv \sum_{n=0}^{N_P(|d_{\min}|)} p_n \alpha^{n+1}(\mu)$$

Terminant (Dingle)

$$\begin{aligned}
\Omega_d &\sim Z_{O_d}^X \frac{\mu^d}{Q^d} \sum_{n=N+1}^{\infty} \frac{\Gamma(1+db-\gamma+n)}{\Gamma(1+db-\gamma)} [1 + \dots] \left( \frac{\beta_0}{2\pi d} \right)^n \alpha_X^{n+1}(\mu) \\
\Omega_d &\sim \sqrt{\alpha} \frac{\Lambda_{\text{QCD}}^d}{Q^d}
\end{aligned}$$

$\Omega_d$  is the terminant (Dingle) of the asymptotic series when we truncate at  $\alpha^{N+1}$ :

$$\Omega_d = \Delta\Omega(db) + b_1\Delta\Omega(db-1) + w_2\Delta\Omega(db-2) + \dots$$

where

$$w_2 = \frac{b_2(db-\gamma)}{db-\gamma-1}$$

and  $\Delta\Omega$  admits the following integral (but not a Borel integral) representation

$$\Delta\Omega(db) \equiv Z_{O_d}^X \frac{\mu^d}{Q^d} \frac{1}{\Gamma(1+db-\gamma)} \left(\frac{\beta_0}{2\pi d}\right)^{N+1} \alpha_X^{N+2}(\mu) \int_{0, \text{PV}}^{\infty} dx \frac{x^{db-\gamma+N+1} e^{-x}}{1-x \frac{\beta_0 \alpha_X(\mu)}{2\pi d}}.$$



$$\Omega_d = \sqrt{\alpha_X(\mu)} K_X^{(P)} \frac{\mu^d}{Q^d} e^{-\frac{d2\pi}{\beta_0 \alpha_X(\mu)}} \left( \frac{\beta_0 \alpha_X(\mu)}{4\pi} \right)^{-db} \alpha_X^\gamma(\mu) \\ \times \left( 1 + \bar{K}_{X,1}^{(P)} \alpha_X(\mu) + \bar{K}_{X,2}^{(P)} \alpha_X^2(\mu) + O(\alpha_X^3(\mu)) \right), \\ \Omega_d = \sqrt{\alpha_X(\mu)} K_X^{(P)} \frac{\Lambda_X^d}{Q^d} \alpha_X^\gamma(\mu) \left( 1 + K_{X,1}^{(P)} \alpha_X(\mu) + K_{X,2}^{(P)} \alpha_X^2(\mu) + O(\alpha_X^3(\mu)) \right),$$

$$K_X^{(P)} = \frac{-Z_{O_d}^X}{\Gamma(1 + bd - \gamma)} \left( \frac{2\pi d}{\beta_0} \right)^{bd - \gamma + 1} \left( \frac{\beta_0}{4\pi} \right)^{bd} \left( \frac{\beta_0}{d} \right)^{1/2} \left[ -\eta_c + \frac{1}{3} \right] \\ \bar{K}_{X,1}^{(P)} = \frac{\beta_0 / (\pi d)}{-\eta_c + \frac{1}{3}} \left[ -b_1 (bd - \gamma) \left( \frac{1}{2} \eta_c + \frac{1}{3} \right) - \frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c - \frac{1}{1080} \right] \\ K_{X,1}^{(P)} = \bar{K}_{X,1}^{(P)} - \frac{b\beta_0 ds_1}{2\pi} \\ \dots$$

where  $\eta_c \equiv -db + \gamma + \frac{2\pi|d|c}{\beta_0} - 1$

## Ultraviolet renormalon $d < 0$

$$p_n^{(as)} = Z_{O_d}^X \frac{\mu^d}{Q^d} \frac{\Gamma(n+b'+1)}{\Gamma(b'+1)} \left( \frac{\beta_0}{2\pi d} \right)^n \left\{ 1 + c_1 \frac{b'}{n+b'} + c_2 \frac{b'^2}{(n+b')(n+b'-1)} + \dots \right\},$$

$\Omega_{d<0}$  reads

$$\Omega_{d<0} = \Delta\Omega_{UV}(db) + c_1 \Delta\Omega_{UV}(db-1) + \dots,$$

where  $b' = db - \gamma$

$$\begin{aligned} \Delta\Omega_{UV}(db) &= Z_{O_d}^X \frac{\mu^d}{Q^d} \frac{(-1)^{N_P+1}}{\Gamma(b'+1)} \left( \frac{\beta_0}{2\pi d} \right)^{N_P+1} \alpha^{N_P+2} \int_0^\infty dx \frac{e^{-x} x^{N_P+1+b'}}{1 + \frac{x\beta_0\alpha}{2\pi|d|}} \\ &= Z_{O_d}^X \frac{\mu^d}{Q^d} (-1)^{N_P+1} \frac{\pi}{\Gamma(b'+1)} \left( \frac{\beta_0}{|d|} \right)^{-b'-1/2} \alpha(\mu)^{1/2-b'} e^{\frac{-2\pi|d|}{\beta_0\alpha(\mu)}} \left\{ 1 \right. \\ &\quad \left. + \frac{\alpha(\mu)}{\pi} \frac{\beta_0}{12|d|} [-1 + 3\eta_c^2] \right. \\ &\quad \left. + \frac{\alpha^2(\mu)}{\pi^2} \frac{\beta_0^2}{1152|d|^2} [13 - 48\eta_c - 60\eta_c^2 + 48\eta_c^3 + 36\eta_c^4] + \mathcal{O}(\alpha^3) \right\}. \end{aligned}$$

Joining all terms together we have

$$\Omega_{d<0} = \sqrt{\alpha(\mu)} K_X^{(P)} \frac{Q^{|d|}}{\mu^{|d|}} e^{\frac{-2\pi|d|}{\beta_0\alpha(\mu)}} \left( \frac{\beta_0\alpha(\mu)}{4\pi} \right)^{-b'} \\ \times \left\{ 1 + \bar{K}_{X,1}^{(P)}\alpha(\mu) + \bar{K}_{X,2}^{(P)}\alpha^2(\mu) + \mathcal{O}(\alpha^3(\mu)) \right\},$$

where

$$K_X^{(P)} \equiv Z_{\mathcal{O}_d}^X (-1)^{N_p+1} \left( \frac{\beta_0}{\pi^2|d|} \right)^{-1/2} \frac{1}{\Gamma(b'+1)} \left( \frac{2}{|d|} \right)^{-b'}, \\ \bar{K}_{X,1}^{(P)} \equiv \left( \frac{2}{\pi} \right)^{1/2} \left( c_1 \frac{\beta_0 b'}{2\sqrt{2\pi}|d|} + \frac{\beta_0}{12|d|\sqrt{2\pi}} (-1 + 3\eta_c^2) \right), \\ \bar{K}_{X,2}^{(P)} \equiv \left( \frac{2}{\pi} \right)^{1/2} \left( c_2 \frac{b'^2 \beta_0^2}{4\sqrt{2}|d|^2 \pi^{3/2}} + c_1 \frac{b' \beta_0^2 (-1 + 3(\eta_c + 1)^2)}{24\sqrt{2}|d|^2 \pi^{3/2}} \right. \\ \left. + \frac{\beta_0^2}{1152|d|^2 2^{1/2} \pi^{3/2}} \left[ 13 - 48\eta_c - 60\eta_c^2 + 48\eta_c^3 + 36\eta_c^4 \right] \right). \\ \Omega_{d<0} \sim \sqrt{\alpha(\mu)} \frac{\Lambda_{\text{QCD}}^{|d|} Q^{|d|}}{\mu^{2|d|}}$$

$S_{\text{PV}}$  will be computed truncating the hyperasymptotic expansion in a systematic way. This means truncating as follows (note that we always define  $D$  to be positive):

$$S_{\text{PV}}(Q) = S_P(Q; \mu) + \Omega(\mu) + \sum_{n=N_P+1}^{N'_P} (p_n - p_n^{(\text{as})}) \alpha_X^{n+1}(\mu) + \Omega'(\mu) + \dots,$$

$$S_{\text{PV}}^{(D, N)}(Q) = \sum_{\{|d|\}} S_{|d| < D} + \sum_{\{|d| \leq D\}} \Omega_d + \sum_{n=N_P(D)+1}^{N_P(D)+N} (p_n - p_n^{(\text{as})}) \alpha^{n+1}(\mu).$$

where

$$S_P \equiv \sum_{n=0}^{N_P(|d_{\min}|)} p_n \alpha^{n+1}(\mu) \equiv S_{|d|=0},$$

and ( $|d| > 0$ )

$$S_{|d|} = \sum_{n=N_P(|d|)+1}^{N_P(|d'|)} (p_n - p_n^{(\text{as})}) \alpha^{n+1}(\mu),$$

$$S_{\text{PV}}(Q) = S_P + \sum_{\{|d|\}} S_{|d|} + \sum_{\{d > 0\}} \Omega_d + \sum_{\{d < 0\}} \Omega_d,$$

$S_{\text{PV}}$  will be computed truncating the hyperasymptotic expansion in a systematic way. This means truncating as follows (note that we always define  $D$  to be positive):

$$S_{\text{PV}}(Q) = S_P(Q; \mu) + \Omega(\mu) + \sum_{n=N_P+1}^{N'_P} (p_n - p_n^{(\text{as})}) \alpha_X^{n+1}(\mu) + \Omega'(\mu) + \dots,$$

$$S_{\text{PV}}^{(D, N)}(Q) = \sum_{\{|d|\}} S_{|d| < D} + \sum_{\{|d| \leq D\}} \Omega_d + \sum_{n=N_P(D)+1}^{N_P(D)+N} (p_n - p_n^{(\text{as})}) \alpha^{n+1}(\mu).$$

where

$$S_P \equiv \sum_{n=0}^{N_P(|d_{\min}|)} p_n \alpha^{n+1}(\mu) \equiv S_{|d|=0},$$

and ( $|d| > 0$ )

$$S_{|d|} = \sum_{n=N_P(|d|)+1}^{N_P(|d'|)} (p_n - p_n^{(\text{as})}) \alpha^{n+1}(\mu),$$

$$S_{\text{PV}}(Q) = S_P + \sum_{\{|d|\}} S_{|d|} + \sum_{\{d > 0\}} \Omega_d + \sum_{\{d < 0\}} \Omega_d,$$

For each value of the couple  $(D, N)$  we can state the parametric accuracy of  $S_{\text{PV}}^{(D, N)}(Q)$ . For  $S^{(0, N_P)}$  the error would be (up to a numerical and a  $\sqrt{\alpha_X}$  factor)

$$\delta S^{(0, N_P)} \sim \mathcal{O} \left( e^{-|d_{\min}| \frac{2\pi}{\beta_0 \alpha_X(Q)}} \right),$$

and for  $S^{(|d_{\min}|, 0)}$  (up to a numerical and a possible  $\alpha_X^{3/2}$  factor):

$$\delta S^{(|d_{\min}|, 0)} \sim \mathcal{O} \left( \left( \left| \frac{d}{d_{\min}} \right| e \right)^{-|d_{\min}| \frac{2\pi}{\beta_0 \alpha_X(Q)}} = e^{-|d_{\min}| \frac{2\pi}{\beta_0 \alpha_X(Q)} (1 + \ln(|d/d_{\min}|))} \right),$$

where  $d$  is the location of the next renormalon closest to the origin. This corresponds to the first term in the hyperasymptotic approximation. The expression for the error in the general case  $S_{\text{PV}}^{(D, N)}(Q)$  reads ( $N \neq N_P$  but large)

$$\delta S^{(D, N)} \sim \mathcal{O} \left( e^{-D \frac{2\pi}{\beta_0 \alpha_X(Q)} (1 + \ln(|d/D|))} \alpha_X^N \right),$$

where  $d$  is the location of the next renormalon closest to the origin after  $D$ .

## Static potential in the large $\beta_0$ approximation

$$V(r) = -\frac{2C_F}{\pi} \int_0^\infty dq \frac{\sin qr}{qr} \alpha_V(q).$$

This defines  $\alpha_V(q)$  in the V-scheme.

In the large- $\beta_0$  approximation

$$\alpha_V(q) = \alpha_X \sum_{n=0}^{\infty} L^n = \alpha_X \frac{1}{1-L},$$

where  $L = \frac{\beta_0 \alpha_X}{2\pi} \ln\left(\frac{\mu e^{-c_X/2}}{q}\right)$ .  $c_{\overline{\text{MS}}} = -5/3$ ;  $c_V = 0$ . If  $X = \text{latt}$ , we take the  $n_f = 0$  number for a Wilson action of  $d_1 = 5.88359$  Hasenfratz:1980 and use  $c_{\text{latt}} = -2\left(\frac{5}{6} + \frac{2\pi d_1}{\beta_0}\right)$ :  $c_{\text{latt}}(n_f = 0) = -8.38807$  and  $c_{\text{latt}}(n_f = 3) = -9.88171$ .  $\tilde{\Lambda} = \Lambda_X e^{-c_X/2}$  and  $\rho = \tilde{\Lambda} r$ . Note that  $\tilde{\Lambda}$  is scheme independent.

$V(r)$  is ill defined but not its Borel transform. It reads Aglietti:1995

$$B[V](t(u)) = B(t(u)) = \frac{-C_F}{\pi^{1/2}} \frac{1}{r} e^{-c_X u} \left( \frac{\mu^2 r^2}{4} \right)^u \frac{\Gamma(1/2 - u)}{\Gamma(1 + u)},$$

We then define (where the single poles of the Borel transform are regulated using the PV prescription)

$$V_{\text{PV}}(r) = \int_{0, \text{PV}}^{\infty} dt e^{-t/\alpha(\mu)} B[V](t(u)).$$

We define

$$V_P \equiv \sum_{n=0}^{N_P} V_n \alpha^{n+1}.$$

Hyperasymptotic yields

$$V_{\text{PV}} = V_P + \frac{1}{r} \Omega_V + \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(\text{as})}) \alpha^{n+1} + \frac{1}{r} \Omega'_V + o(\Lambda_{\text{QCD}}^3 r^2),$$



where  $\Omega_V$  reads for this case

$$\begin{aligned} \Omega_V &= \sqrt{\alpha_X(\mu)} K_X^{(P)} r \mu e^{-\frac{2\pi}{\beta_0 \alpha_X(\mu)}} \\ &\times \left( 1 + \bar{K}_{X,1}^{(P)} \alpha_X(\mu) + \bar{K}_{X,2}^{(P)} \alpha_X^2(\mu) + \mathcal{O}(\alpha_X^3(\mu)) \right), \end{aligned}$$

and  $K_X^{(P)}$  and  $\bar{K}_{X,i}^{(P)}$  read

$$\begin{aligned} K_X^{(P)} &= -2\pi Z_V^X \beta_0^{-1/2} \left[ -\eta_c + \frac{1}{3} \right], \\ \bar{K}_{X,1}^{(P)} &= \frac{\beta_0/(\pi)}{-\eta_c + \frac{1}{3}} \left[ -\frac{1}{12} \eta_c^3 + \frac{1}{24} \eta_c - \frac{1}{1080} \right], \\ \bar{K}_{X,2}^{(P)} &= \frac{\beta_0^2/\pi^2}{-\eta_c + \frac{1}{3}} \left[ -\frac{1}{160} \eta_c^5 \right. \\ &\quad \left. - \frac{1}{96} \eta_c^4 + \frac{1}{144} \eta_c^3 + \frac{1}{96} \eta_c^2 - \frac{1}{640} \eta_c - \frac{25}{24192} \right], \end{aligned}$$

where

$$Z_V^X = -2 \frac{C_F}{\pi} e^{-\frac{c_X}{2}} \quad \text{and} \quad \eta_c = \frac{2\pi c}{\beta_0} - 1.$$

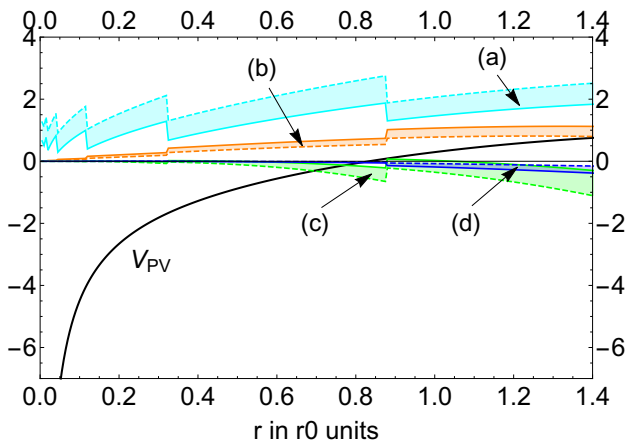


Figure: We plot  $V_{PV}$  (black line) and the differences: (a)  $V_{PV} - V_P$  (cyan), (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  (orange), (c)  $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$  (green), and (d)  $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$  (blue) in the **lattice scheme** with  $n_f = 0$  light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of  $c$  that yields integer values for  $N_P$ .

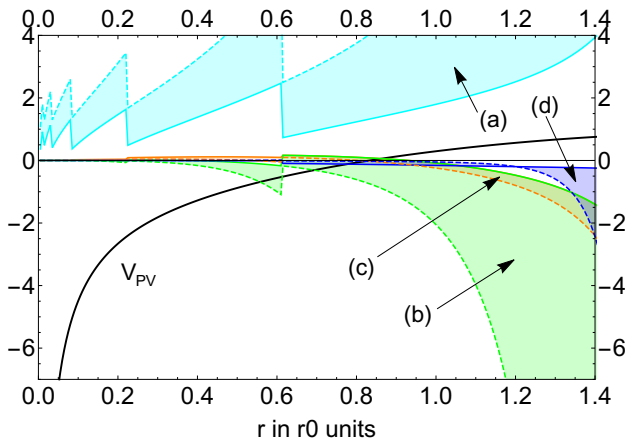


Figure: We plot  $V_{PV}$  (black line) and the differences: (a)  $V_{PV} - V_P$  (cyan), (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  (orange), (c)  $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$  (green), and (d)  $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$  (blue) in the  $\overline{\text{MS}}$  scheme with  $n_f = 0$  light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of  $c$  that yields integer values for  $N_P$ .

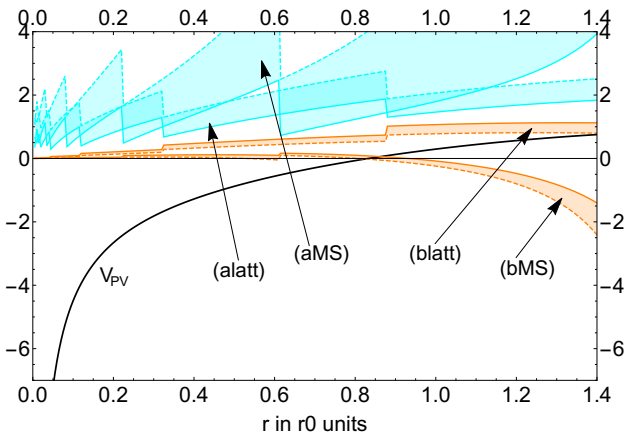


Figure: Comparison of lattice and  $\overline{\text{MS}}$  scheme results for  $n_f = 0$ . We plot  $V_{PV}$  and the differences: (a)  $V_{PV} - V_P$ , and (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  in the lattice and  $\overline{\text{MS}}$  scheme with  $n_f = 0$  light flavours.

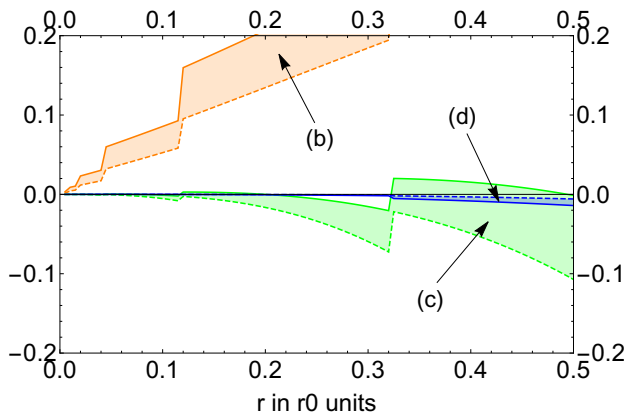


Figure: (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  (orange), (c)

$V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$  (green), and (d)

$V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$  (blue) in the **lattice scheme** with  $n_f = 0$  light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of  $c$  that yields integer values for  $N_P$ .

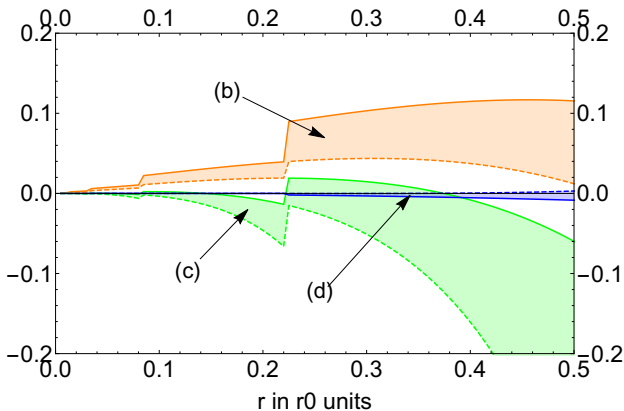


Figure: (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  (orange), (c)

$V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$  (green), and (d)

$V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$  (blue) in the  $\overline{\text{MS}}$  scheme with  $n_f = 0$  light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of  $c$  that yields integer values for  $N_P$ .

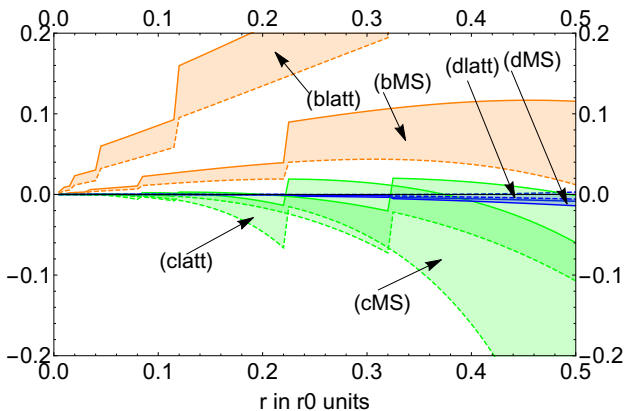


Figure: Comparison of lattice and  $\overline{\text{MS}}$  scheme results for  $n_f = 0$ .

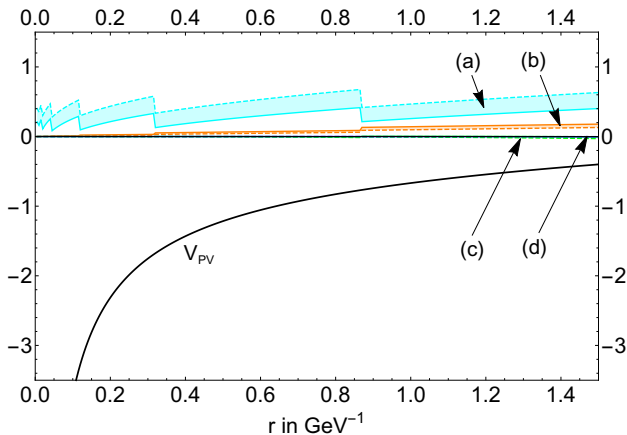


Figure: We plot  $V_{PV}$  (black line) and the differences: (a)  $V_{PV} - V_P$  (cyan), (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  (orange), (c)  $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$  (green), and (d)  $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$  (blue) in the **lattice scheme** with  $n_f = 3$  light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of  $c$  that yields integer values for  $N_P$ .



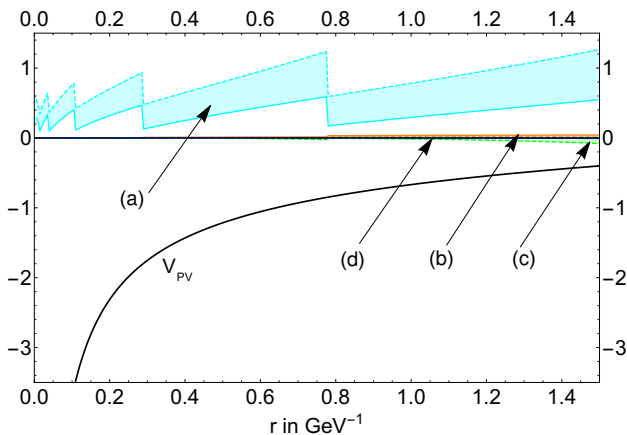


Figure: We plot  $V_{PV}$  (black line) and the differences: (a)  $V_{PV} - V_P$  (cyan), (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  (orange), (c)  $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$  (green), and (d)  $V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$  (blue) in the  $\overline{\text{MS}}$  scheme with  $n_f = 3$  light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of  $c$  that yields integer values for  $N_P$ .

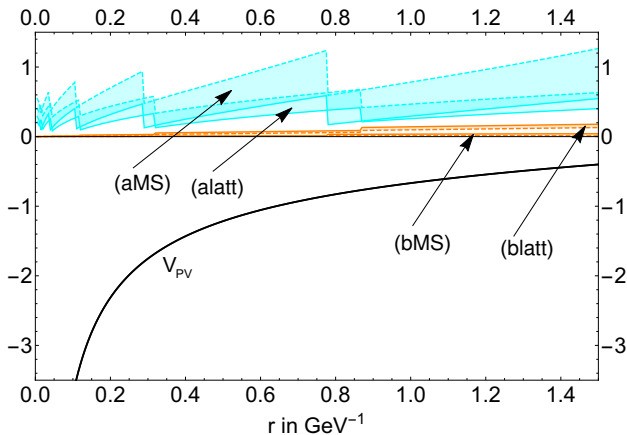


Figure: Comparison of lattice and  $\overline{\text{MS}}$  scheme results for  $n_f = 3$ . We plot  $V_{PV}$  and the differences: (a)  $V_{PV} - V_P$ , and (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  in the **lattice** and  $\overline{\text{MS}}$  scheme with  $n_f = 3$  light flavours.

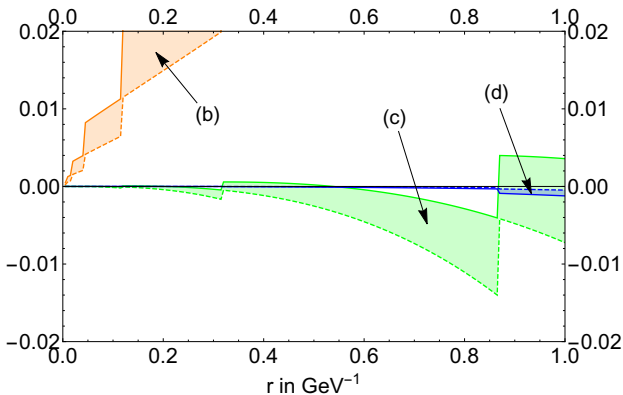


Figure: (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  (orange), (c)

$V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$  (green), and (d)

$V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$  (blue) in the **lattice scheme with  $n_f = 3$  light flavours**. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of  $c$  that yields integer values for  $N_P$ .

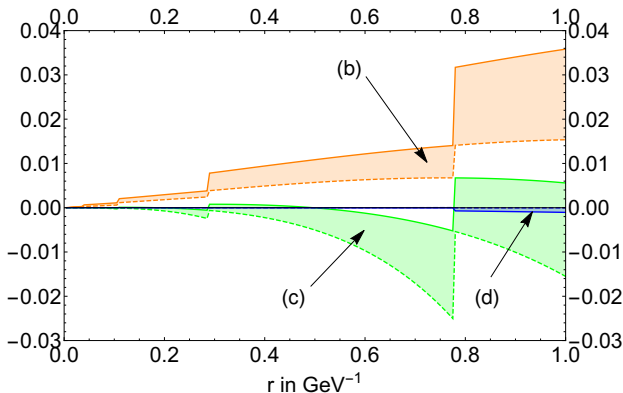


Figure: (b)  $V_{PV} - V_P - \frac{1}{r}\Omega_V$  (orange), (c)

$V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1}$  (green), and (d)

$V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)})\alpha^{n+1} - \frac{1}{r}\Omega'_V$  (blue) in the  $\overline{\text{MS}}$  scheme with  $n_f = 3$  light flavours. For each difference, the bands are generated by the difference of the prediction produced by the smallest positive or negative possible values of  $c$  that yields integer values for  $N_P$ .

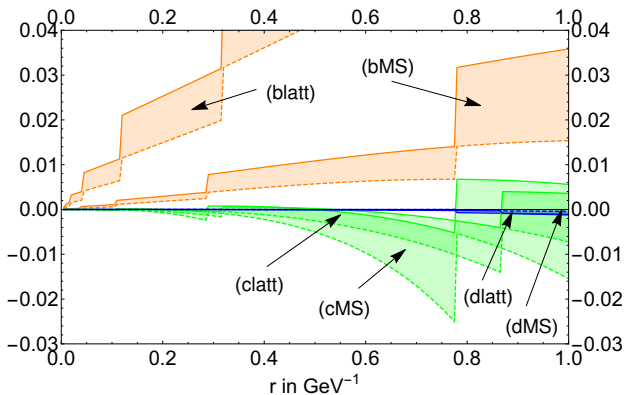


Figure: Comparison of lattice and  $\overline{\text{MS}}$  scheme results for  $n_f = 3$ .

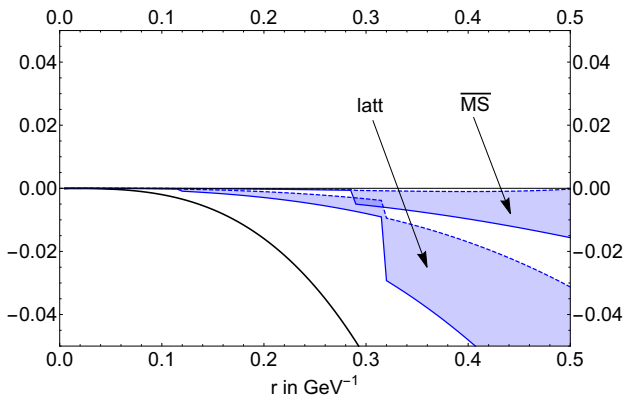


Figure:  $\mathcal{O}(\Lambda_{\text{QCD}}^3)$  accuracy. Comparison of lattice and  $\overline{\text{MS}}$  scheme results for  $n_f = 3$ . Vertical axis in MeV.

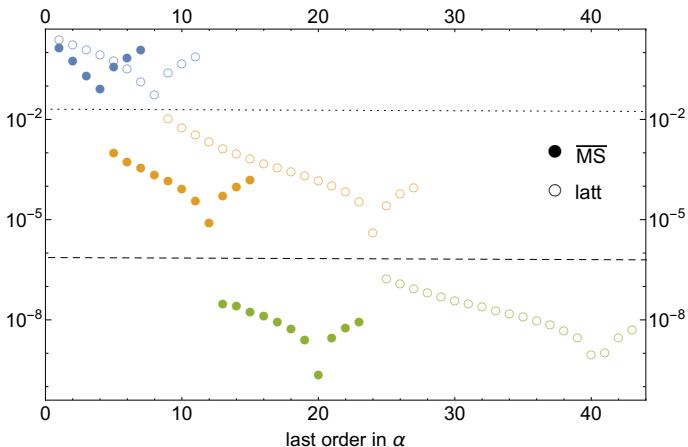


Figure:  $|V_{PV} - V_{PV}^{\text{Hyperasymptotic}}|$  for  $r = 0.1 \text{ GeV}^{-1}$ . Points above the horizontal dotted line are  $|V_{PV} - V_N|$ . Points between the horizontal dotted and horizontal dashed lines are  $|V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^N (V_n - V_n^{(\text{as})})\alpha^{n+1}|$ . Points below the horizontal dashed lines are  $|V_{PV} - V_P - \frac{1}{r}\Omega_V - \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(\text{as})})\alpha^{n+1} - \frac{1}{r}\Omega'_V - \sum_{n=3N_P+1}^N (V_n - V_n^{(\text{as})})\alpha^{n+1}|$ .

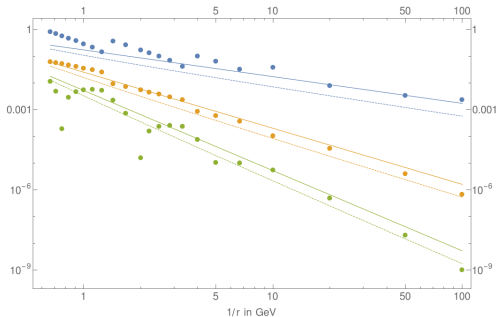


Figure: Blue points are  $|r(V_{PV} - V_P)|$ . Orange points are  $|r(V_{PV} - V_P) - \Omega_V|$ , Green  $|r(V_{PV} - V_P) - \Omega_V - r \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}|$ . They are plotted as functions of  $1/r$  in logarithmic scale (which is equivalent to plotting them in terms of  $1/\alpha$ ). The continuous lines are  $e^{-\frac{2\pi}{\beta_0 \alpha}}$  (blue),  $e^{-(1+\ln 3)\frac{2\pi}{\beta_0 \alpha}}$  (orange),  $e^{-3\frac{2\pi}{\beta_0 \alpha}}$  (green). The dashed lines are the same functions multiplied by  $\sqrt{\alpha}$ . In the notation  $(D, N)$  of they correspond to  $(0, N_P)$ ,  $(1, 0)$  and  $(1, 2N_P)$  respectively. The computation has been done with  $n_f = 3$ , in the  $\overline{\text{MS}}$  scheme, and taking the smallest positive value possible of  $c$  that yields integer values for  $N_P$ .



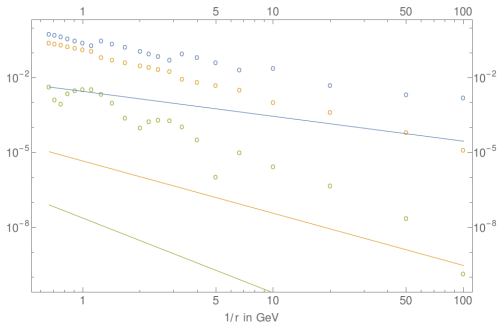


Figure: Blue points are  $|r(V_{PV} - V_P)|$ . Orange points are  $|r(V_{PV} - V_P) - \Omega_V|$ , Green  $|r(V_{PV} - V_P) - \Omega_V - r \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}|$ . They are plotted as functions of  $1/r$ . The continuous lines are  $e^{-\frac{2\pi}{\beta_0 \alpha}}$  (blue),  $e^{-(1+\ln 3)\frac{2\pi}{\beta_0 \alpha}}$  (orange),  $e^{-3\frac{2\pi}{\beta_0 \alpha}}$  (green). In the notation  $(D, N)$  of they correspond to  $(0, N_P)$ ,  $(1, 0)$  and  $(1, 2N_P)$  respectively. The computation has been done with  $n_f = 3$ , in the lattice scheme, and taking the smallest positive value possible of  $c$  that yields integer values for  $N_P$ .

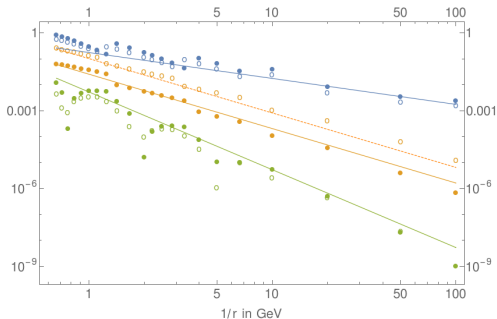


Figure: Blue points are  $|r(V_{PV} - V_P)|$ . Orange points are  $|r(V_{PV} - V_P) - \Omega_V|$ , Green  $|r(V_{PV} - V_P) - \Omega_V - r \sum_{n=N_P+1}^{3N_P} (V_n - V_n^{(as)}) \alpha^{n+1}|$ . They are plotted as functions of  $1/r$ . Full points are computed in the  $\overline{\text{MS}}$  scheme and empty points in the lattice

scheme. The continuous lines are  $e^{-\frac{2\pi}{\beta_0 \alpha_{\overline{\text{MS}}}}} = e^{-5/6} e^{-c_{\text{latt}}/2} e^{-\frac{2\pi}{\beta_0 \alpha_{\text{latt}}}}$  (blue),

$e^{-(1+\ln 3) \frac{2\pi}{\beta_0 \alpha_{\overline{\text{MS}}}}} = e^{-(1+\ln 3)(5/6+c_{\text{latt}}/2)} e^{-(1+\ln 3) \frac{2\pi}{\beta_0 \alpha_{\text{latt}}}}$  (orange),

$e^{-3 \frac{2\pi}{\beta_0 \alpha_{\overline{\text{MS}}}}} = e^{-5/2} e^{-3c_{\text{latt}}/2} e^{-3 \frac{2\pi}{\beta_0 \alpha_{\text{latt}}}}$  (green). In the notation  $(D, N)$  of they correspond to  $(0, N_P)$ ,  $(1, 0)$  and  $(1, 2N_P)$  respectively. The dashed orange line

corresponds to  $4 \times e^{-(1+\ln 3) \frac{2\pi}{\beta_0 \alpha_{\overline{\text{MS}}}}}$ . The computation has been done with  $n_f = 3$  and taking the smallest positive value possible of  $c$  that yields integer values for  $N_P$ .

## Bottom pole mass

$$M_B = m_{\text{PV}} + \bar{\Lambda}_{\text{PV}} + \mathcal{O}\left(\frac{1}{m_{\text{PV}}}\right).$$

$$\bar{\Lambda}_{\text{PV}} = M_B - m_P(\bar{m}_b) - \bar{m}_b \Omega_m - \sum_{N_P+1}^{N'=2N_P} [r_n - r_n^{(\text{as})}] \alpha^{n+1} + \dots.$$

$$m_P = \bar{m} + \sum_{n=0}^{N_P} r_n \alpha^{n+1}(\mu)$$

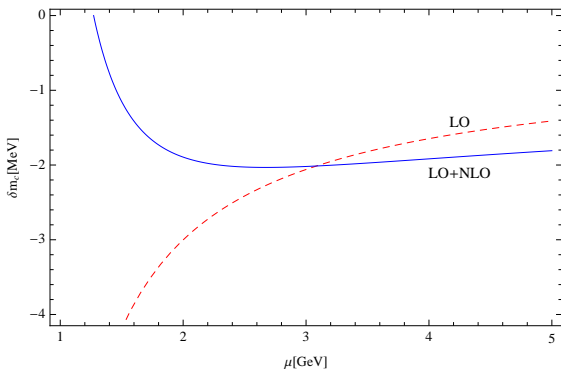
$$m_{b,\text{PV}} = 4836(\mu)_{-17}^{+8} (Z_m)_{+12}^{-11} (\alpha)_{-9}^{+8} \text{ MeV}.$$

$$\bar{\Lambda}_{\text{PV}} = 477(\mu)_{+17}^{-8} (Z_m)_{-12}^{+11} (\alpha)_{+9}^{-8} \text{ MeV}.$$

$\mu \in (\bar{m}_b/2, 2\bar{m}_b)$ . For  $Z_m$  we take  $Z_m^{\overline{\text{MS}}}(n_f = 3) = 0.5626(260)$ .

$n_f = 3$  light fermions in the  $\overline{\text{MS}}$ -pole mass relation

$$m_b e^{-n} < m_c$$



# Top pole mass

$$m_{\text{PV}}(\bar{m}) = \bar{m} + \sum_{n=0}^{N_{\text{max}}} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) + \delta m_b^{(n_f)}(\bar{m}) + \delta m_c^{(n_f)}(\bar{m}) + \delta m_{bc}^{(n_f)}(\bar{m}),$$

$$\begin{aligned} \mathcal{F}(\bar{m}, n_f) &\equiv \frac{d}{d\bar{m}} (m_{\text{PV}}(\bar{m}) - \bar{m}) \simeq \frac{d}{d\bar{m}} \sum_{n=0} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) \\ &\equiv \sum_{n=1}^{N+1} f_n(\bar{m}) \left( \frac{\alpha_{(n_f)}(\bar{m})}{\pi} \right)^n. \end{aligned}$$

$$\begin{aligned} m_{\text{PV}}(\bar{m}_t) &= \bar{m}_t + \int_{\mu_b}^{\bar{m}_t} d\bar{m} \left( \mathcal{F}(\bar{m}, 5) + \frac{d}{d\bar{m}} (\delta m_b^{(5)}(\bar{m}) + \delta m_c^{(5)}(\bar{m}) + \delta m_{(bc)}^{(5)}(\bar{m})) \right) \\ &+ \int_{\mu_c}^{\mu_b} d\bar{m} \left( \mathcal{F}(\bar{m}, 4) + \frac{d}{d\bar{m}} (\delta m_b^{(4)}(\bar{m}) + \delta m_c^{(4)}(\bar{m}) + \delta m_{(bc)}^{(4)}(\bar{m})) \right) \\ &+ m_{\text{PV}}(\mu_c) - \mu_c. \end{aligned}$$

$$m_{\text{PV}}(\mu_c) = m_P(\mu_c) + \mu_c \Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0 \alpha(\mu_c)}(1+\ln(2))})$$

## Top pole mass

$$m_{\text{PV}}(\bar{m}) = \bar{m} + \sum_{n=0}^{N_{\text{max}}} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) + \delta m_b^{(n_f)}(\bar{m}) + \delta m_c^{(n_f)}(\bar{m}) + \delta m_{bc}^{(n_f)}(\bar{m}),$$

$$\begin{aligned} \mathcal{F}(\bar{m}, n_f) &\equiv \frac{d}{d\bar{m}} (m_{\text{PV}}(\bar{m}) - \bar{m}) \simeq \frac{d}{d\bar{m}} \sum_{n=0} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) \\ &\equiv \sum_{n=1}^{N+1} f_n(\bar{m}) \left( \frac{\alpha_{(n_f)}(\bar{m})}{\pi} \right)^n. \end{aligned}$$

$$\begin{aligned} m_{\text{PV}}(\bar{m}_t) = \bar{m}_t &+ \int_{\mu_b}^{\bar{m}_t} d\bar{m} \left( \mathcal{F}(\bar{m}, 5) + \frac{d}{d\bar{m}} (\delta m_b^{(5)}(\bar{m}) + \delta m_c^{(5)}(\bar{m}) + \delta m_{(bc)}^{(5)}(\bar{m})) \right) \\ &+ \int_{\mu_c}^{\mu_b} d\bar{m} \left( \mathcal{F}(\bar{m}, 4) + \frac{d}{d\bar{m}} (\delta m_b^{(4)}(\bar{m}) + \delta m_c^{(4)}(\bar{m}) + \delta m_{(bc)}^{(4)}(\bar{m})) \right) \\ &+ m_{\text{PV}}(\mu_c) - \mu_c. \end{aligned}$$

$$m_{\text{PV}}(\mu_c) = m_P(\mu_c) + \mu_c \Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0 \alpha(\mu_c)}(1+\ln(2))})$$

## Top pole mass

$$m_{\text{PV}}(\bar{m}) = \bar{m} + \sum_{n=0}^{N_{\text{max}}} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) + \delta m_b^{(n_f)}(\bar{m}) + \delta m_c^{(n_f)}(\bar{m}) + \delta m_{bc}^{(n_f)}(\bar{m}),$$

$$\begin{aligned} \mathcal{F}(\bar{m}, n_f) &\equiv \frac{d}{d\bar{m}} (m_{\text{PV}}(\bar{m}) - \bar{m}) \simeq \frac{d}{d\bar{m}} \sum_{n=0} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) \\ &\equiv \sum_{n=1}^{N+1} f_n(\bar{m}) \left( \frac{\alpha_{(n_f)}(\bar{m})}{\pi} \right)^n. \end{aligned}$$

$$\begin{aligned} m_{\text{PV}}(\bar{m}_t) &= \bar{m}_t + \int_{\mu_b}^{\bar{m}_t} d\bar{m} \left( \mathcal{F}(\bar{m}, 5) + \frac{d}{d\bar{m}} (\delta m_b^{(5)}(\bar{m}) + \delta m_c^{(5)}(\bar{m}) + \delta m_{(bc)}^{(5)}(\bar{m})) \right) \\ &+ \int_{\mu_c}^{\mu_b} d\bar{m} \left( \mathcal{F}(\bar{m}, 4) + \frac{d}{d\bar{m}} (\delta m_b^{(4)}(\bar{m}) + \delta m_c^{(4)}(\bar{m}) + \delta m_{(bc)}^{(4)}(\bar{m})) \right) \\ &+ m_{\text{PV}}(\mu_c) - \mu_c. \end{aligned}$$

$$m_{\text{PV}}(\mu_c) = m_P(\mu_c) + \mu_c \Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0 \alpha(\mu_c)}(1+\ln(2))})$$

## Top pole mass

$$m_{\text{PV}}(\bar{m}) = \bar{m} + \sum_{n=0}^{N_{\text{max}}} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) + \delta m_b^{(n_f)}(\bar{m}) + \delta m_c^{(n_f)}(\bar{m}) + \delta m_{bc}^{(n_f)}(\bar{m}),$$

$$\begin{aligned} \mathcal{F}(\bar{m}, n_f) &\equiv \frac{d}{d\bar{m}} (m_{\text{PV}}(\bar{m}) - \bar{m}) \simeq \frac{d}{d\bar{m}} \sum_{n=0} r_n^{(n_f)}(\bar{m}; \nu = \bar{m}) \alpha_{(n_f)}^{n+1}(\bar{m}) \\ &\equiv \sum_{n=1}^{N+1} f_n(\bar{m}) \left( \frac{\alpha_{(n_f)}(\bar{m})}{\pi} \right)^n. \end{aligned}$$

$$\begin{aligned} m_{\text{PV}}(\bar{m}_t) &= \bar{m}_t + \int_{\mu_b}^{\bar{m}_t} d\bar{m} \left( \mathcal{F}(\bar{m}, 5) + \frac{d}{d\bar{m}} (\delta m_b^{(5)}(\bar{m}) + \delta m_c^{(5)}(\bar{m}) + \delta m_{(bc)}^{(5)}(\bar{m})) \right) \\ &+ \int_{\mu_c}^{\mu_b} d\bar{m} \left( \mathcal{F}(\bar{m}, 4) + \frac{d}{d\bar{m}} (\delta m_b^{(4)}(\bar{m}) + \delta m_c^{(4)}(\bar{m}) + \delta m_{(bc)}^{(4)}(\bar{m})) \right) \\ &+ m_{\text{PV}}(\mu_c) - \mu_c. \end{aligned}$$

$$m_{\text{PV}}(\mu_c) = m_P(\mu_c) + \mu_c \Omega_m + \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) + \mathcal{O}(\mu_c e^{-\frac{2\pi}{\beta_0 \alpha(\mu_c)}(1+\ln(2))})$$



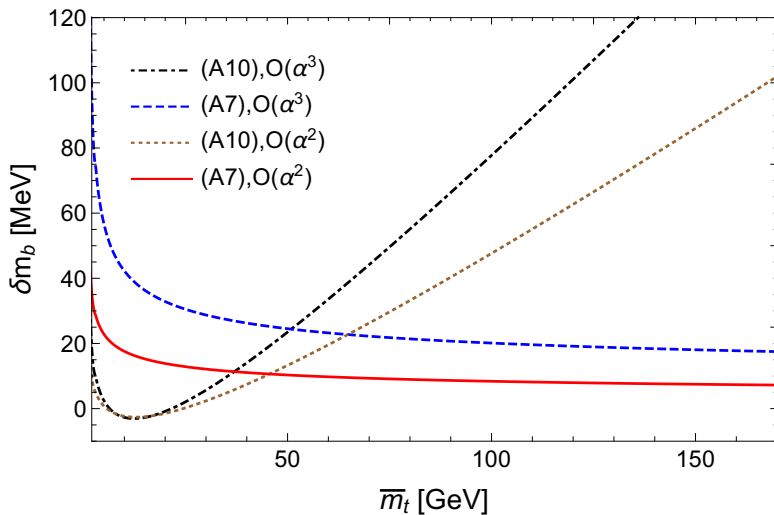


Figure: **Upper panel:** Plot of the correction to the PV mass of a top mass with varying  $\bar{m}_t$  mass due to a heavy quark with  $\overline{MS}$  mass equal to 4.185 GeV (bottom) with and without decoupling (assuming a single heavy quark).

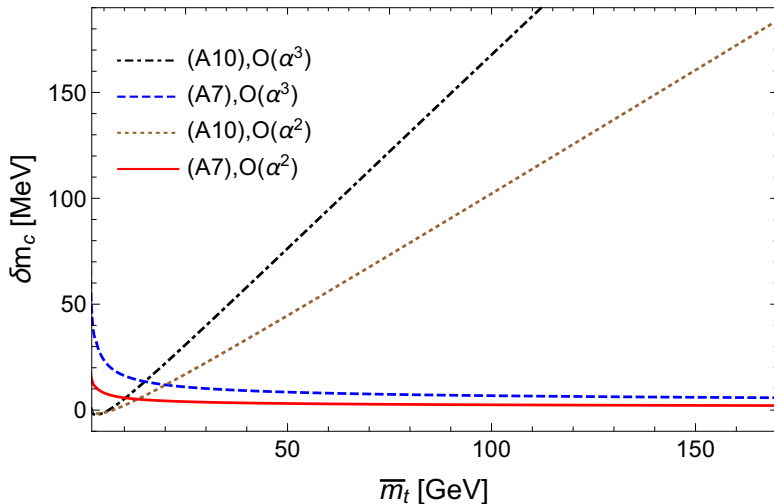


Figure: Plot of the correction to the PV mass of a top mass with varying  $\bar{m}_t$  mass due to a heavy quark with  $\overline{\text{MS}}$  mass equal to 1.223 GeV (charm).

$\mu_b = 20 \text{ GeV}$  and  $\mu_c = 5 \text{ GeV}$ .

$$\begin{aligned}
 & \int_{\mu_b}^{\bar{m}_t} d\bar{m} \frac{d}{d\bar{m}} (\delta m_b^{(5)}(\bar{m}) + \delta m_c^{(5)}(\bar{m}) + \delta m_{(bc)}^{(5)}(\bar{m})) \\
 + & \int_{\mu_c}^{\mu_b} d\bar{m} \frac{d}{d\bar{m}} (\delta m_b^{(4)}(\bar{m}) + \delta m_c^{(4)}(\bar{m}) + \delta m_{(bc)}^{(4)}(\bar{m})) \\
 + & \delta m_b^{(3)}(\mu_c) + \delta m_c^{(3)}(\mu_c) + \delta m_{(bc)}^{(3)}(\mu_c) = -2.5 \Big|_{\mathcal{O}(\alpha^2)} + 0.8 \Big|_{\mathcal{O}(\alpha^3)} = -1.7 \text{ MeV} .
 \end{aligned}$$

$$\int_{\mu_b}^{\bar{m}_t} d\bar{m} \mathcal{F}(\bar{m}, 5) + \int_{\mu_c}^{\mu_b} d\bar{m} \mathcal{F}(\bar{m}, 4) = 8445 + 837 + 53 - 43 = 9291(22) \text{ MeV} .$$

$$(m_P(\mu_c) + \mu_c \Omega_m) \Big|_{\mu_c=5 \text{ GeV}} = 5744(\mu)_{-15}^{+7} (Z_m)_{-9}^{+9} \text{ MeV} .$$

Finally, we also include the error associated to  $\alpha$ . Combining all errors we obtain

$$m_{t,\text{PV}}(163\text{MeV}) = 173033(\text{h.o.})_{-22}^{+22}(\mu)_{-15}^{+7}(Z_m)_{-9}^{+9}(\alpha)_{-123}^{+119} \text{ MeV} .$$

$$\left[ \frac{m_{t,\text{PV}}}{\bar{m}_t} - 1 \right] \times 10^5 = 6155(\text{h.o.})_{-13}^{+13}(\mu)_{-9}^{+4}(Z_m)_{-6}^{+6}(\alpha)_{-75}^{+73} .$$

$$m_{t,\text{PV}}(163\text{MeV}) = 173033(\text{th})_{-28}^{+25}(\alpha)_{-123}^{+119} \text{ MeV} ,$$

$$\left[ \frac{m_{t,\text{PV}}}{\bar{m}_t} - 1 \right] \times 10^5 = 6155 (\text{th})_{-17}^{+15} (\alpha)_{-75}^{+73} ,$$

## Conclusions

- ▶ We have organized the OPE along an hyperasymptotic expansion. We use the PV prescription of the Borel integral for reference (scheme/scale independence). Analytic control of the error.
- ▶ We have checked the method in the case of the static potential in the large  $\beta_0$  approximation for two terms of the hyperasymptotic expansion.
- ▶ Applied to the top and bottom mass (very competitive error determination)

### Not discussed:

- ▶ Preliminary discussion of the gluon condensate
- ▶ It is possible to connect truncated sums with  $RS^{(n)}$  schemes
- ▶ It is possible to connect truncated sums with the MRS scheme
- ▶ Alternative?  $\mu \rightarrow \infty$
- ▶ Renormalization group analysis of RS masses.
- ▶ Any summation method preserves the structure of the OPE?

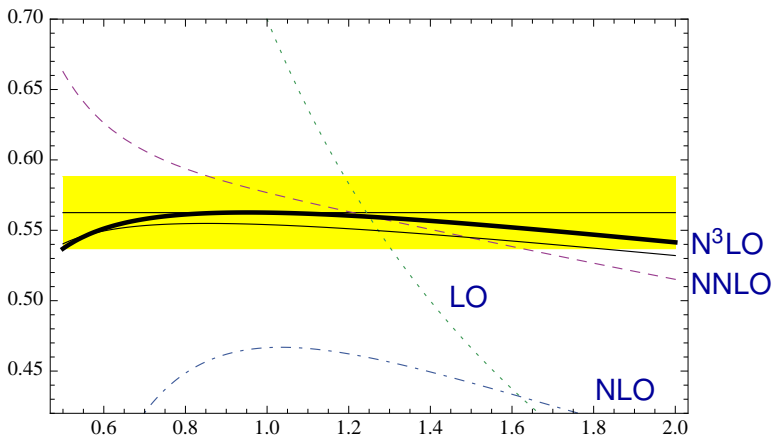


Figure:  $2m + V$  renormalon free.  $-N_V/2 = N_m$  for  $n_l = 3$ , as a function of  $x \equiv \nu r$ , obtained from  $-(N_V/2)v_n/v_n^{asym}$ .  $v_n^{asym}$  is truncated at  $\mathcal{O}(1/n^3)$ .

$$N_m(n_l = 0) = 0.600(29) ,$$

$$N_m(n_l = 3) = 0.563(26) .$$