A NEW METHOD TO COMPUTE PHYSICAL OBSERVABLES

THE FDU FRAMEWORK

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- “LTD en acción”
  - I)- Applications of LTD
  - II)- FDU approach
  - III)- Singular structures
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Supported by
STARTING CONCEPT:

Inside a “0” there COULD BE many hidden things
Basic introduction

Theoretical motivation

- Theoretical framework: **Standard Model + factorization theorem**

\[
\frac{d\sigma}{d^2 q_T \, dM^2 \, d\Omega \, dy} = \sum_{a,b} \int dx_1 dx_2 \, f^h_a(x_1) f^h_b(x_2) \frac{d\hat{\sigma}_{ab \to V+X}}{d^2 q_T \, dM^2 \, d\Omega \, dy}
\]

- **Deal with ill-defined expressions** in intermediate steps **DREG!!!**
  - Proposed by {Giambiagi&Bollini, t’Hooft&Veltman, Cicuta&Montaldi, Ashmore,…}, it becomes a standard in HEP since it preserves gauge invariance
  - **Abstract idea:** «Change the dimension of the loop-momentum space»
  - **Reality:** «Introduce a parameter \(\varepsilon\) in order to make everything integrable»

PDFs (non-perturbative)
Partonic cross-section (perturbative)

\[ O_d[F] = \int d^d x \, F(x) \]
\[ d = 4 - 2\varepsilon \]

1. linearity \[ \int d^d x \, (a F(x) + b G(x)) = a \int d^d x \, F(x) + b \int d^d x \, G(x) \]
2. scaling \[ \int d^d x \, F(s x) = s^{-d} \int d^d x \, F(x) \]
3. translational invariance \[ \int d^d x \, F(x + y) = \int d^d x \, F(x) \]

Nice mathematical properties!!
(“Todo bonito”)

What we want to compute!
Basic introduction

- Singularities in perturbative theories with DREG

- Two kinds of physical singularities:
  - **Ultraviolet poles** coming from the high-energy region in loop integrals
    
    **SOLUTION:** Add proper counterterms obtained from RENORMALIZATION procedure. These counterterms have EXPLICIT $\varepsilon$-poles and are proportional to lower-order amplitudes.
  
  - **Infrared poles** associated with degenerate configurations: extra-particle radiation in soft (i.e. low energy) or collinear (i.e. parallel) configurations
    
    **SOLUTION:** Kinoshita-Lee-Nauenberg theorem states that adding real-emission processes and computing IR-safe observables guarantees the cancellation of all the IR poles present in renormalized virtual amplitudes and INTEGRATED real-radiation contributions.
  
- **Loop integrals** could contain **UV and IR** singularities
Basic introduction

**Theoretical motivation**

- **Summary scheme:** obtaining finite physical results at higher-orders

- **Vacuum quantum fluctuations (no experimental signature)**

- **Virtual corrections (loop integrals)**
  - IR/UV divergences

- **Renormalization counter-terms**
  - UV divergences

- **Renormalization procedure**

- **KLN theorem + IR safe observables**

- **Finite physical observable**

- **Renormalized virtual corrections**
  - IR divergences

- **Real corrections (PS integrals)**
  - IR divergences

- **Contributions with extra-radiation (included in the definition of the observable)**

**WE WANT INTEGRAND LEVEL CANCELLATION!!!**
Towards Loop-Tree Duality

Residue theorem (from Wikipedia)

\[ \oint_{\gamma} f(z) \, dz = 2\pi i \sum \text{Res}(f, a_k) \]

«If \( f \) is a holomorphic function in \( U/\{a_i\} \), and \( g \) a simple positively oriented curve, then the integral is given by the sum of the residues at each singular point \( a_i \)»

**Feynman propagator**

\[ [G(q)]^{-1} = 0 \implies q_0 = \pm \sqrt{q^2 - i0} \]

**Advanced propagator**

\[ [G_A(q)]^{-1} = 0 \implies q_0 \simeq \pm \sqrt{q^2 + i0} \]

NO POLES CLOSED BY \( C_L \)!

Residue theorem can be used to compute integrals involving propagators: the prescription and the contour that we choose determine the result!
Towards Loop-Tree Duality

**Dual representation of one-loop integrals**

\[ L^{(1)}(p_1, \ldots, p_N) = \int_\ell \prod_{i=1}^{N} G_F(q_i) = \int_\ell \prod_{i=1}^{N} \frac{1}{q_i^2 - m_i^2 + i0} \]

Even at higher-orders, the number of cuts is equal the number of loops

\[ G_D(q_i, q_j) = \frac{1}{q_j^2 - m_j^2 - i0\eta(q_j - q_i)} \]

\[ \tilde{\delta}(q_i) = i2\pi \theta(q_i,0) \delta(q_i^2 - m_i^2) \]
Part I: Applications of LTD

• I)- Deal with massless Feynman integrals
• II)- Analysis of IR-divergent integrals
• III)- Study of UV-divergent integrals and local UV counter-terms
LTD for Feynman integrals

Motivation and introduction

- Two different kinds of physical singularities: **UV** and **IR**
  - IR divergences: *massless triangle*
    \[ L^{(1)}(p_1, p_2, -p_3) = \int \prod_{i=1}^{3} G_F(q_i) = -\frac{c_T}{\epsilon^2 s_{12}} \left( \frac{-s_{12} - i0}{\mu^2} \right)^{-\epsilon} \]

  **IR pole**

  **IDEA:** Define a proper **MOMENTUM MAPPING** to generate **REAL EMISSION KINEMATICS**, and use **REAL TERMS** as fully local IR counter-terms!

- UV divergences: *bubble with massless propagators*
  \[ L^{(1)}(p, -p) = \int \prod_{i=1}^{2} G_F(q_i) = c_T \frac{\mu^{2\epsilon}}{\epsilon (1 - 2\epsilon)} \left( -p^2 - i0 \right)^{-\epsilon} \]

  **UV pole**

  **IDEA:** Define an **INTEGRAND LEVEL REPRESENTATION** of standard UV counter-terms, and combine it with the **DUAL REPRESENTATION** of virtual terms!

LTD for Feynman integrals: IR case

Reference example: Massless scalar three-point function in the time-like region

\[ L^{(1)}(p_1, p_2, -p_3) = \int \prod_{i=1}^{3} G_F(q_i) = -\frac{c\Gamma}{\epsilon^2} \left(-\frac{s_{12}}{\mu^2} - i0\right)^{-1-\epsilon} = \sum_{i=1}^{3} I_i \]

\[ I_1 = \frac{1}{s_{12}} \int d[\xi_{1,0}] d[v_1] \xi_{1,0}^{-1} (v_1(1-v_1))^{-1} \]
\[ I_2 = \frac{1}{s_{12}} \int d[\xi_{2,0}] d[v_2] \frac{(1-v_2)^{-1}}{1-\xi_{2,0} + i0} \]
\[ I_3 = \frac{1}{s_{12}} \int d[\xi_{3,0}] d[v_3] \frac{v_3^{-1}}{1+\xi_{3,0} - i0} \]

- This integral is UV-finite (power counting); there are only IR-singularities, associated to soft and collinear regions

- **OBJECTIVE:** Define a IR-regularized loop integral by adding real corrections at integrand level (i.e. no epsilon should appear, 4D representation)

LTD for Feynman integrals: IR case

Location of IR singularities in the dual-space

- Analyze the dual integration region. It is obtained as the positive energy solution of the on-shell condition:

\[ G^{-1}_F(q_i) = q_i^2 - m_i^2 + i\epsilon = 0 \]

\[ q_{i,0}^{(\pm)} = \pm \sqrt{q_i^2 + m_i^2 - i\epsilon} \]

- **Forward** (backward) on-shell hyperboloids associated with **positive** (negative) energy solutions.
- Degenerate to light-cones for massless propagators.
- Dual integrands become **singular at intersections** (two or more on-shell propagators)

Massive case: hyperboloids

Massless case: light-cones

The application of LTD converts loop-integrals into PS ones: integration over forward light-cones.

- Only forward-backward interferences originate threshold or IR poles (other propagators become singular in the integration domain)
- Forward-forward singularities cancel among dual contributions
- Threshold and IR singularities associated with finite regions (i.e. contained in a compact region)
- No threshold or IR singularity at large loop momentum

This structure suggests how to perform real-virtual combination! Also, how to overcome threshold singularities (integrable but numerically unstable)
LTD for Feynman integrals: UV case

- **Reference example:** two-point function with massless propagators

\[
L^{(1)}(p, -p) = \int_\ell \prod_{i=1}^{2} G_F(q_i) = \frac{c_T}{\epsilon(1 - 2\epsilon)} \left( -\frac{p^2}{\mu^2} - i0 \right)^{-\epsilon} = \sum_{i=1}^{2} I_i
\]

- **OBJETIVE:** Define a UV-regularized loop integral by adding unintegrated UV counter-terms, and find a purely 4-dimensional representation of the loop integral

- In this case, the integration regions of dual integrals are two energy-displaced forward light-cones. This integral contains UV poles only

- **OBJETIVE:** Define a UV-regularized loop integral by adding unintegrated UV counter-terms, and find a purely 4-dimensional representation of the loop integral

LTD for Feynman integrals: UV case

**UV counter-term**

- Divergences arise from the high-energy region (UV poles) and can be cancelled with a suitable renormalization counter-term. For the scalar case, we use

\[
I_{\text{cnt}}^{\text{UV}} = \int_{\ell} \frac{1}{(q_{\text{UV}}^2 - \mu_{\text{UV}}^2 + i0)^2}
\]


- Dual representation (new: double poles in the loop energy)

\[
I_{\text{cnt}}^{\text{UV}} = \int_{\ell} \frac{\tilde{\delta}(q_{\text{UV}})}{2 (q_{\text{UV},0}^{(+)} )^2}
\]

\[
q_{\text{UV},0}^{(+)} = \sqrt{q_{\text{UV}}^2 + \mu_{\text{UV}}^2 - i0}
\]

Bierenbaum et al. JHEP 03 (2013) 025

- Loop integration for loop energies larger than \( \mu_{\text{UV}} \)

Part II: FDU formalism

I)- Adding real contributions to locally cancel IR singularities: Universal kinematical mappings

II)- Local four-dimensional representation of renormalization counter-terms

OBJETIVE: Avoid using DREG (or any other regularization) through a purely 4D representation of physical observables
Towards local IR regularization

According to KLN theorem real contributions. Suppose one-loop scalar scattering amplitude given by the triangle.

\[ |\mathcal{M}^{(0)}(p_1, p_2; p_3)\rangle = ig \]
\[ |\mathcal{M}^{(1)}(p_1, p_2; p_3)\rangle = -ig^3 L^{(1)}(p_1, p_2, -p_3) \Rightarrow \text{Re} \langle \mathcal{M}^{(0)} | \mathcal{M}^{(1)} \rangle \]

- 1->2 one-loop process \[\rightarrow\] 1->3 with unresolved extra-parton
- Add scalar tree-level contributions with one extra-particle; consider interference terms:

\[ |\mathcal{M}_{ir}^{(0)} (p'_1, p'_2, p'_r; p_3)\rangle = -ig^2 / s'_{ir} \Rightarrow \text{Re} \langle \mathcal{M}_{ir}^{(0)} | \mathcal{M}_{jr}^{(0)} \rangle = \frac{g^4}{s'_{ir} s'_{jr}} \]

- Generate 1->3 kinematics starting from 1->2 configuration plus the loop three-momentum \[l\] !!!

Generalization of mappings

Real-virtual momentum mapping with massive particles:

- Consider 1 the emitter, r the radiated particle and 2 the spectator.
- Apply the PS partition and restrict to the only region where 1/r is allowed (i.e. $R_1 = \{ y_{1r} < \min y_{kj} \}$).
- Propose the following mapping:

$$
\begin{align*}
\hat{p}_r^\mu &= q_1^\mu \\
\hat{p}_1^\mu &= (1 - \alpha_1) \hat{p}_1^\mu + (1 - \gamma_1) \hat{p}_2^\mu - q_1^\mu \\
\hat{p}_2^\mu &= \alpha_1 \hat{p}_1^\mu + \gamma_1 \hat{p}_2^\mu
\end{align*}
$$

with $\hat{p}_i$ massless four-vectors build using $p_i$ (simplify the expressions).

- Express the loop three-momentum with the same parameterization used for describing the dual contributions!

Repeat in each region of the partition...

Generalization of mappings

- We combine the dual contributions with the real terms (after applying the proper mapping) to get the total decay rate in the scalar toy-model.
  - The result agrees perfectly with standard DREG.
  - Massless limit is smoothly approached due to proper treatment of quasi-collinear configurations in the RV mapping.

Example: massive scalar three-point function (DREG vs LTD)

Local renormalization within LTD

- LTD can also deal with **UV singularities** by building local versions of the usual UV counterterms.

  **1: Expand** internal propagators around the “UV propagator”

  \[
  \frac{1}{q_i^2 - m_i^2 + i0} = \frac{1}{q_{UV}^2 - \mu_{UV}^2 + i0} \\
  \times \left[ 1 - \frac{2q_{UV} \cdot k_{i,UV} + k_{i,UV}^2 - m_i^2 + \mu_{UV}^2}{q_{UV}^2 - \mu_{UV}^2 + i0} + \frac{(2q_{UV} \cdot k_{i,UV})^2}{(q_{UV}^2 - \mu_{UV}^2 + i0)^2} \right] + \mathcal{O} \left( (q_{UV}^2)^{-5/2} \right)
  \]

  - **2: Apply LTD** to get the **dual representation** for the expanded UV expression, and **subtract** it from the **dual+real** combined integrand.

    \[
    I_{UV}^{\text{cnt}} = \int \frac{\tilde{\delta}(q_{UV})}{2 (q_{UV,0}^{(+)})^2} \\
    q_{UV,0}^{(+)} = \sqrt{q_{UV}^2 + \mu_{UV}^2 - i0}
    \]

    **LTD extended to deal with multiple poles**
    (use residue formula to obtain the dual representation)

    Bierenbaum et al. JHEP 03 (2013) 025

In the massless case, the renormalization factors are usually ignored because they are “0”: but they hide a cancelation between UV and IR singularities...
Local renormalization within LTD

- Requires **unintegrated** wave-function, mass and vertex renormalization constants

- Self-energy corrections with **on-shell renormalization** conditions

\[
\Sigma_R(p_1 = M) = 0 \quad \frac{d\Sigma_R(p_1)}{dp_1}\bigg|_{p_1=M} = 0
\]

- Wave function renormalization constant, **both IR and UV poles**

\[
\Delta Z_2(p_1) = -g_5^2 C_F \int G_F(q_1) G_F(q_3) \left( (d-2) \frac{q_1 \cdot p_2}{p_1 \cdot p_2} + 4M^2 \left( 1 - \frac{q_1 \cdot p_2}{p_1 \cdot p_2} \right) G_F(q_3) \right)
\]

- **Remove UV poles** by expanding around the UV-propagator (same for the vertex counterterm)

- Integrated form of local counterterms agrees with standard UV counterterms

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Part III: Physical examples

- Application of the FDU/LTD formalism to express amplitudes and observables in four space-time dimensions.
  
  I)- Vector boson decays at NLO 
  II)- Higgs amplitudes at one and two loop

- REMARK: First application of two-loop local renormalization in 4D!!!
Physical example: $A^* \rightarrow q\bar{q}(g)$ @NLO

Results and comparison with DREG

- Total decay rate for Higgs into a pair of massive quarks:
  - Agreement with the standard DREG result
  - Smoothly achieves the massless limit
  - Local version of UV counterterms successfully reproduces the expected behaviour
  - Efficient numerical implementation

Rodrigo et al, JHEP10(2016)162
Physical example: $A^* \rightarrow q\bar{q}(g)@$NLO

Results and comparison with DREG

- Total decay rate for a vector particle into a pair of massive quarks:
  - Agreement with the standard DREG result
  - Smoothly achieves the massless limit
  - Efficient numerical implementation

Rodrigo et al, JHEP10(2016)162
Physical example: $A^* \rightarrow q\bar{q}(g)@NLO$

- The total decay-rate can be expressed using purely **four-dimensional integrands** (which are **integrable** functions!!)

- We recover the total NLO correction, **avoiding to deal with** DREG (ONLY used for comparison with known results)

- **Main advantages:**
  - Direct **numerical** implementation (integrable functions for $\epsilon=0$)
  - **Smooth transition** to the massless limit (due to the efficient treatment of quasi-collinear configurations)
  - **Mapped real-contribution used as a fully local IR counter-term for the dual contribution!**

**Important remarks**

- Finite integral for $\epsilon=0$  
  - Integrability with $\epsilon=0$  
  - With FDU is true!
**Physical example: Higgs@((N)NLO)**

Using LTD to regularize finite amplitudes

- Application of LTD to compute one-loop Higgs amplitudes:

  ![Diagram of gg → H and H → γγ](image)

- They are IR/UV finite BUT still not well-defined in 4D!!! Hidden cancellation of singularities leads to potentially undefined results (scheme dependence!!!)

- We start by defining a tensor basis and projecting (amplitude level!):

  \[
  A_{\mu\nu}^{(1,f)} = \sum_{i=1}^{5} A_{i}^{(1,f)} T_{i}^{\mu\nu}
  \]

  with

  \[
  T_{i}^{\mu\nu} = \left\{ g^{\mu\nu}, \frac{2 p_{1}^{\mu} p_{2}^{\nu}}{s_{12}}, \frac{2 p_{1}^{\mu} p_{1}^{\nu}}{s_{12}}, \frac{2 p_{2}^{\mu} p_{2}^{\nu}}{s_{12}} \right\}
  \]

  \[
  P_{1}^{\mu\nu} = \frac{1}{d-2} \left( g^{\mu\nu} - \frac{2 p_{1}^{\mu} p_{2}^{\nu}}{s_{12}} - (d-1) P_{2}^{\mu\nu} \right)
  \]

  \[
  P_{2}^{\mu\nu} = \frac{2 p_{1}^{\mu} p_{2}^{\nu}}{s_{12}}
  \]

- Then, scalar coefficients \( P_{i}^{\mu\nu} A_{i}^{(1,f)} = A_{i}^{(1,f)} \) are dualized.

- **IMPORTANT:** Take into account 1-2 exchange symmetry (different cuts and non-trivial cancellations!!!)
**Physical example: Higgs@NLO**

**Using LTD to regularize finite amplitudes**

- Combine expressions (use “zero integrals” in DREG associated with Ward identities):

\[
A^{(1,f)}_1 = g_f \int_\ell \delta(\ell) \left[ \left( \frac{\ell_0^{(+)}}{q_{1,0}^{(+)}} + \frac{\ell_0^{(+)}}{q_{4,0}^{(+)}} + \frac{2 (2\ell \cdot p_{12})^2}{s_{12} - (2\ell \cdot p_{12} - i0)^2} \right) \frac{s_{12} M_f^2}{(2\ell \cdot p_1)(2\ell \cdot p_2)} \right] c_1^{(f)} + \frac{2 s_{12}^2}{s_{12} - (2\ell \cdot p_{12} - i0)^2} c_{23}^{(f)} \nonumber
\]

- Use local renormalization (equivalent to Dyson’s prescription…)

\[
A^{(1,f)}_{1,R} \bigg|_{d=4} = \left( A^{(1,f)}_1 - A^{(1,f)}_{1,UV} \right)_{d=4},
A^{(1,f)}_{1,UV} = -g_f \int_\ell \delta(\ell) \frac{\ell_0^{(+)}}{2(q_{UV,0}^{(+)})^3} \left( 1 + \frac{1}{(q_{UV,0}^{(+)})^2} \frac{3 \mu_{UV}^2}{d-4} \right) c_{23}^{(f)}
\]

- Counter-term mimics UV behaviour at integrand level.
- Term proportional to $\mu_{UV}^2$ used to fix DREG scheme (vanishing counter-term in d-dim!!)
- Valid also for W amplitudes in unitary-gauge (naive Dyson’s prescription fails to subtract subleading terms due to enhanced UV divergences)

Physical example: Higgs@(N)NLO

- Infinite-mass limit used to define effective vertices. Equivalent to explore asymptotic expansions!
- Expansions at **integrand level** are non-trivial in **Minkowski** space (i.e. within Feynman integrals) and additional factors are necessary
- **Dual amplitudes** are expressed as **phase-space integrals** → **Euclidean space!!**

\[ \delta(q_3) G_D(q_3; q_2) = \frac{\tilde{\delta}(q_3)}{s_{12} + 2q_3 \cdot p_{12} - i0} M_f^2 \gg s_{12} \quad \tilde{\delta}(q_3) G_D(q_3; q_2) = \frac{\tilde{\delta}(q_3)}{2q_3 \cdot p_{12}} \sum_{n=0}^{\infty} \left( \frac{-s_{12}}{2q_3 \cdot p_{12}} \right)^n \]

Expansion of the dual propagator (q$_3$ on-shell)

- **Example**: Higgs amplitudes with heavy-particles within the loop

\[ A_{1,R}^{(1,f)}(s_{12} < 4M_f^2) \bigg|_{d=4} = \frac{M_f^2}{\langle v \rangle} \int \tilde{\delta}(\ell) \left[ \frac{3 \mu_{UV}^2 \ell_0^{(+)}}{(q_{UV,0}^{(+)})^5} \tilde{c}_2(f) + \frac{M_f^2}{(\ell_0^{(+)})^4} \left( \sum_{n=0}^{\infty} Q_n(z) \left( \frac{s_{12}}{(2\ell_0^{(+)})^2} \right)^n \right) c_1^{(f)} \right] \]

\[ z = (2\ell \cdot p_1)/(\ell_0^{(+)})^2 \quad \text{and} \quad Q_n(z) = \frac{1}{1 - z^2} (P_{2n}(z) - 1) \]

Reproduces all the known-results!!

Introducing the notation

- Dual amplitudes can be defined at higher-orders (even with multiple poles)
  Bierenbaum, Catani, Draggiotis, Rodrigo; JHEP 10 (2010) 073

- Standard example: two-loop N-point scalar amplitude

\[
L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{\ell_1} \int_{\ell_2} G_F(\alpha_1 \cup \alpha_2 \cup \alpha_3) \\
\int_{\ell_i} \cdot = -i \int \frac{d^d \ell_i}{(2\pi)^d} \cdot , \quad G_F(\alpha_k) = \prod_{i \in \alpha_k} G_F(q_i)
\]

- Three possible sets of momenta, according to their dependence on \( l_1, l_2 \) or \( l_1 + l_2 \) (integration variables)

\[
\alpha_1 \equiv \{0, 1, \ldots, r\}, \quad \alpha_2 \equiv \{r + 1, r + 2, \ldots, l\}, \quad \alpha_3 \equiv \{l + 1, l + 2, \ldots, N\}
\]

\[
q_i = \begin{cases} 
\ell_1 + p_{1,i}, & i \in \alpha_1 \\
\ell_2 + p_{i,l-1}, & i \in \alpha_2 \\
\ell_1 + \ell_2 + p_{i,l-1}, & i \in \alpha_3
\end{cases} \quad \text{with} \quad \ell_1 \text{ anti-clockwise} \\
\ell_2 \text{ clockwise}
\]

Generic two-loop diagram

Driencourt-Mangin, Rodrigo, G.S., Torres Bobadilla, JHEP 02 (2019) 143
About Higgs@NNLO...

- **“The number of cuts equals the number of loops”**
- **Derivation:** “Iterate” the one-loop formula and use propagator properties
- **Standard example:** two-loop N-point scalar amplitude

\[
L^{(2)}(p_1, p_2, \ldots, p_N) = \int_{\ell_1} \int_{\ell_2} \{G_D(\alpha_2) G_D(\alpha_1 \cup \alpha_3) \\
+ G_D(-\alpha_2 \cup \alpha_1) G_D(\alpha_3) - G_F(\alpha_1) G_D(\alpha_2) G_D(\alpha_3)\}
\]

where we used 
\[
G_D(\alpha_k) = \sum_{i \in \alpha_k} \tilde{\delta}(q_i) \prod_{j \in \alpha_k \setminus j \neq i} G_D(q_i; q_j)
\]

- **Remarks and subtleties:**
  - Modified prescription depends on loop momenta.
  - Not a “trivial” iteration: connection with Cauchy’s theorem and multivariable residues.
  - **Thesis:** “Virtual-real amplitudes mapped with one-loop formulae” (partial cancellations), but a new mapping required for double-real emission.

*Driencourt-Mangin, Rodrigo, G.S., Torres Bobadilla, JHEP 02 (2019) 143*
Part IV: Singular structures

- LTD formalism recasts Minkowski integrals into Euclidean ones, thus leading to some “nice” mathematical properties.

- 1)- Characterization of threshold singularities within the LTD approach
In general, the location of the singularities is given by the solutions of

\[ \lambda_{ij}^{\pm} = \pm q_{i,0}^{(+)} \pm q_{j,0}^{(+)} + k_{ji,0} = 0 \]

with \( q_i \) on-shell and \( k_{ji} = q_j - q_i \).

We consider the following test functions

\[ S_{ij}^{(1)} = (2\pi\nu)^{-1} G_D(q_i; q_j) \tilde{\delta}(q_i) + (i \leftrightarrow j) \]

Up to 2 on-shell states
(statistical thresholds)

\[ S_{ijk}^{(1)} = (2\pi\nu)^{-1} G_D(q_i; q_k) G_D(q_i; q_j) \tilde{\delta}(q_i) + \text{perm.} \]

Up to 3 on-shell states
(anomalous thresholds)

**IMPORTANT:** The singular structure of scattering amplitudes is dictated by their propagators. So, the proposed test functions are general enough to do a proper analysis of threshold singularities.

The singular structure depends on the separation among momenta:

- **Time-like separation (causal connection):**

\[
 k_{ji}^2 - (m_j + m_i)^2 \geq 0
\]

Physical threshold singularities are originated.

\[
 \lim_{\lambda_{ij}^{++} \to 0} S_{ij}^{(1)} = \frac{\theta(-k_{ji,0}) \theta(k_{ji}^2 - (m_i + m_j)^2)}{x_{ij}(-\lambda_{ij}^{++} - i0k_{ji,0})} + \mathcal{O} \left( (\lambda_{ij}^{++})^0 \right)
\]

Always +i0 !!!

The prescription is crucial to determine the imaginary part: it is always +i0 and corresponds to the usual Feynman prescription! For this configuration, LTD and FTT give equivalent descriptions!
Characterization of singularities with LTD

The singular structure depends on the separation among momenta:

- **Space-like separation:**
  \[ k_{ji}^2 - (m_j - m_i)^2 \leq 0 \]
  The dual-prescription changes sign within the different contributions, which allows a perfect cancellation of any singular behaviour.

  \[ q_{j,0}^{(+) \, G_D(q_i; q_j)}|_{\lambda_{ij}^{+-} \to 0} = -q_{i,0}^{(+) \, G_D(q_j; q_i)}|_{\lambda_{ij}^{+-} \to 0} \]

  \[ \lim_{\lambda_{ij}^{+-} \to 0} S_{ij}^{(1)} = \mathcal{O} \left( (\lambda_{ij}^{+-})^0 \right) \]

- **Light-like separation:**
  It originates IR and threshold singularities that remain in a compact region of the integration domain. There is a partial cancellation among dual contributions, but IR might remain!

Characterization of singularities with LTD

- **Anomalous thresholds:** causal (i.e. time-like separated) singularities originated by multiple propagators going on-shell.

\[
\lim_{\lambda_{ij}^{++}, \lambda_{ik}^{++} \to 0} S_{ijk}^{(1)} = \frac{1}{x_{ijk}} \prod_{r=j,k} \theta(-k_{ri,0}) \theta(k_{ri}^2 - (m_i + m_r)^2) \left( -\lambda_{ir}^{++} - \nu 0 k_{ri,0} \right)
+ \mathcal{O} \left( (\lambda_{ij}^{++})^{-1}, (\lambda_{ik}^{++})^{-1} \right)
\]

\[x_{ijk} = 8 q_{i,0}^{(+)} q_{j,0}^{(+)} q_{k,0}^{(+)}\]

- Intersections of two hyperboloids lead to the standard IR and threshold singularities.

- Anomalous thresholds are originated from the intersection of two forward (backward) and one backward (forward) hyperboloids.

- **There are not singularities for** \(\lambda_{jk}^{-+} = \lambda_{ik}^{++} - \lambda_{ij}^{++} \to 0\) !!!
LTD allows to transform loop into phase-space integrals: we integrate on space components!

I)- Combination with spinor-helicity formalism in four-dimensions (NEW!)

II)- Novel multiloop formulae based on iterative LTD!!! (ONGOING, RESULTS TO APPEAR SOON!)
Towards multiloop & multileg numerical LTD

- We consider three processes which are UV and IR finite, at one-loop.

- Purely 4D numerical implementation for fixed helicity configurations!

- Complete agreement with well-known analytical results (solid blue lines)

- Smooth dependence on the external parameters (kinematical invariants & masses)

- Competitive computational performance (vs. SecDec and standard tools)

Rodrigo et al., arXiv:1911.11125 [hep-ph]
Conclusions and perspectives

- Loop-tree duality allows to treat virtual and real contributions in the same way (implementation simplified)
- Physical interpretation of IR/UV singularities in loop integrals (light-cone diagrams) and proper disentanglement
- Combined virtual-real terms are integrable in 4D!!
- Local 4-dimensional 2-loop results (Higgs/heavy quarks)
- Efficient numerical implementation for multiloop multileg processes!! ongoing!!! (“paso a paso”)

- Perspectives:
  - Automation of multileg processes at NLO and beyond!!!
  - Carefull comparison with other schemes

“WorkStop-ThinkStart”
NEW! Firenze Meeting 2019
Inside a “0” there ARE many hidden things

GRACIAS!!!