Higgs pair production at NLO

Combine numerical evaluation and analytic high energy approximation

Go Mishima
Karlsruhe Institute of Technology (KIT), TTP
in collaboration with **Joshua Davies, Gudrun Heinrich, Stephen P. Jones, Matthias Kerner, Matthias Steinhauser, David Wellmann**

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Karlsruhe Institute of Technology

Institute for Theoretical Particle Physics

λ_{HHH} in the Standard Model

Higgs potential $$V(H) = \frac{1}{2} m_H^2 H^2 + \lambda_{HHH} v H^3 + \frac{1}{4} \lambda_{HHHH} H^4$$

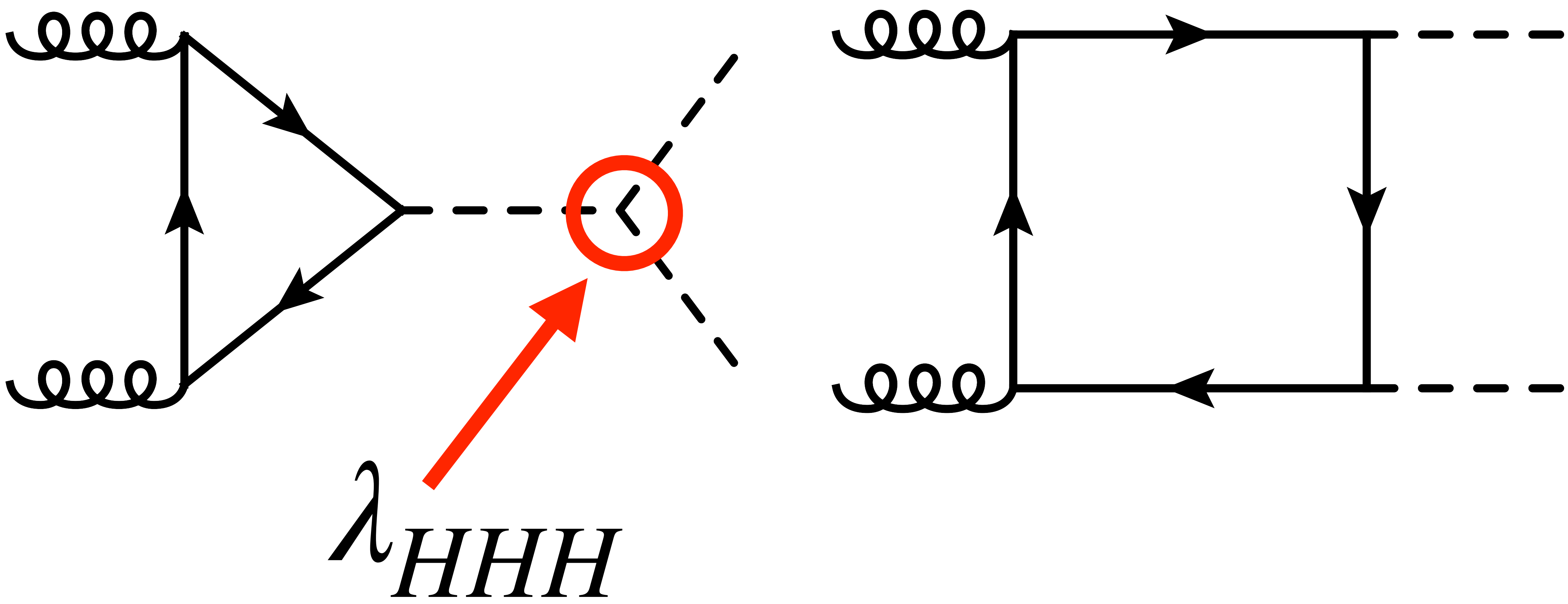
in SM: $$\lambda_{HHH} = \frac{m_H^2}{2v^2} = 0.13\ldots$$ (not directly measured)

[CMS: arXiv:1811.09689]: $-11.8 < \lambda/\lambda_{\text{SM}} < 18.8$

[ATL-PHYS-PUB-2019-009]: $-3.2 < \lambda/\lambda_{\text{SM}} < 11.9$

λ_{HHH} in the Standard Model

The simplest process is Higgs pair production.



λ_{HHH} in the Standard Model

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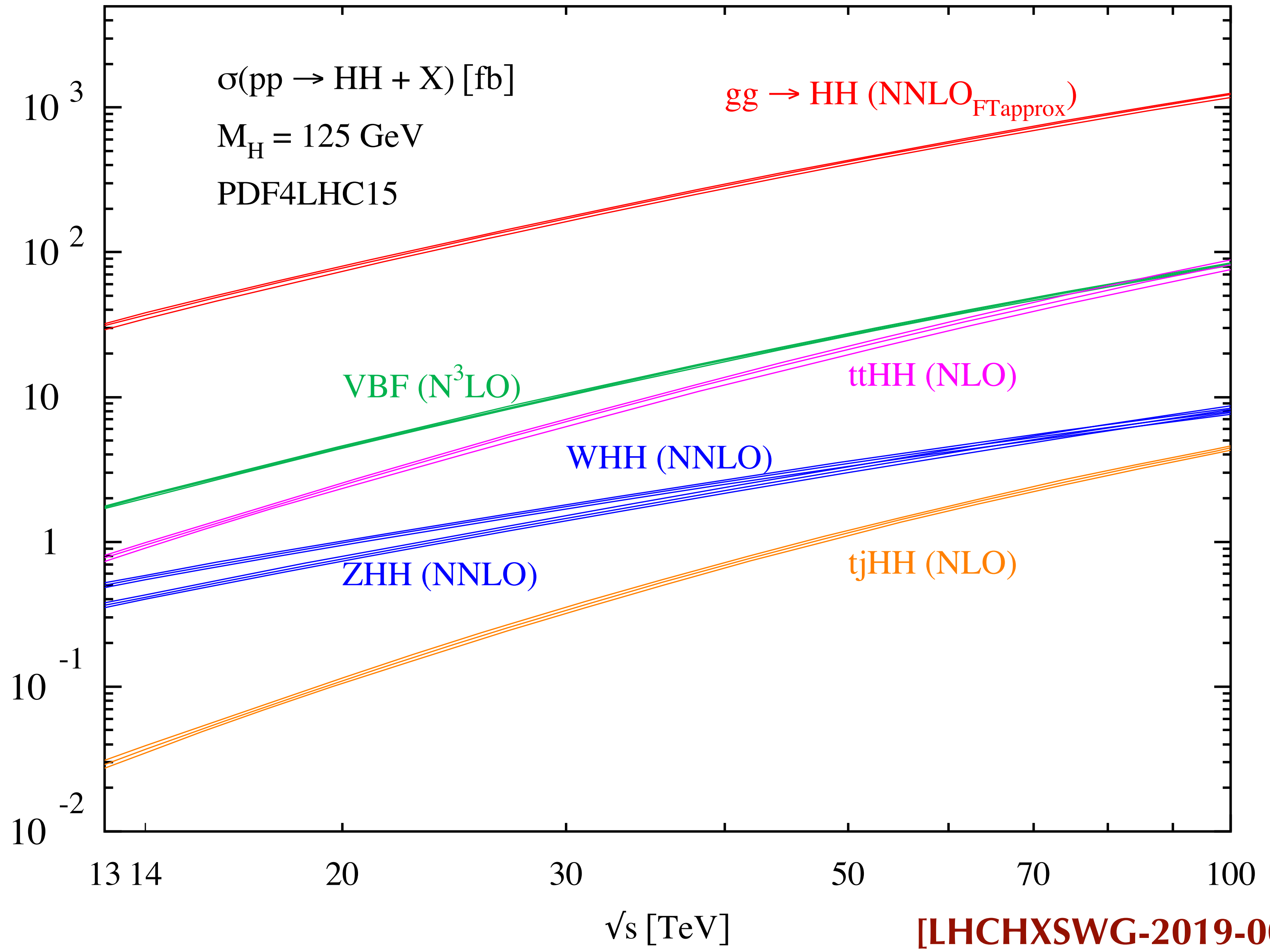
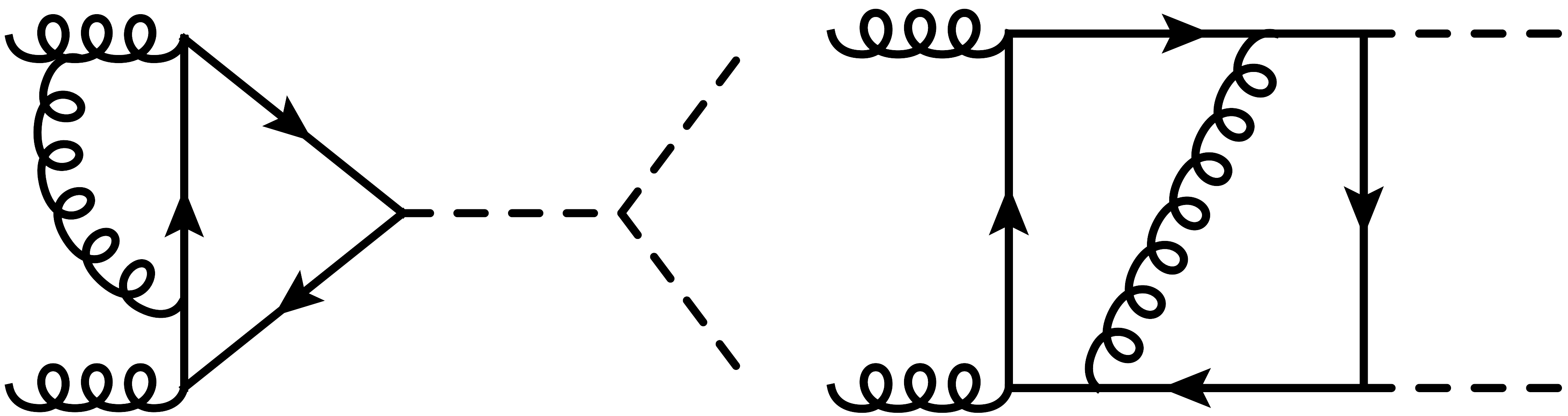


[LHCHXSWG-2019-005, arXiv:1901.00012]

Main topic



two-loop integrals: high-energy approximation
and
numerical evaluation

Previous works (concerning up to NLO)

exact analytic@LO

[Eboli, Marques, Novaes, Natale, ’87, Glover, van der Bij ’88, Plehn, Spira, Zerwas, ‘96]

Born-improved HEFT@NLO

[Dawson, Dittmaier Spira, ’98]

$\mathrm{FT}_{\mathrm{approx}}$, $\mathrm{FT}'_{\mathrm{approx}}$

[Maltoni, Vryonidou, Zaro, ’14]

HEFT@NLO with 1/mt corr.

[Grigo, Hoff, Melnikov, Steinhauser, ’13,
Grigo, Melnikov, Steinhauser, ’14,
Grigo, Hoff, Steinhauser, ’15, Degrassi, Giardino, Gröber, ‘16]

exact numerical@NLO

[Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Zicke, ’16,
Baglio, Campanario, Glaus, Mühlleitner, Spria, Streicher, ’18]

Padé approximation using the large top-mass and the threshold expansion@NLO

[Gröber, Maier Rauh, ’17]

small p_T expansion@NLO

[Bonciani, Degrassi, Giardino, Gröber, ’18]

large-mt expansion vs high-energy expansion

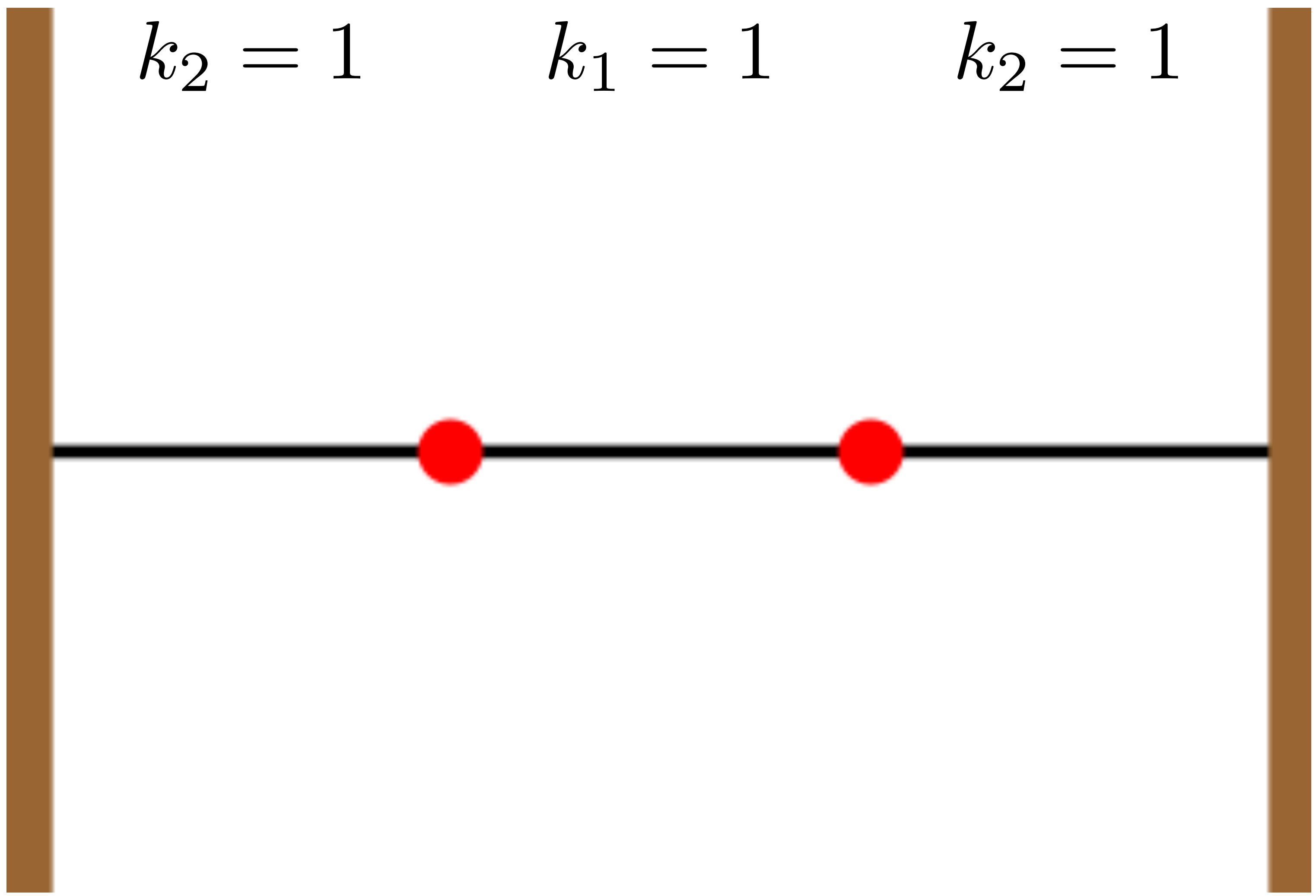
	large-mt expansion	high-energy expansion
NLO was done in	1998 for $m_t \to \infty$ (HEFT) 2013 for $1/m_t$ corrections	2018
Effective field theory	local	non-local (if exists)
expansion-by-subgraphs	applicable	not applicable
expansion in the momentum representation	possible	not possible (to my knowledge)
Padé approximation outside the radius of convergence	bad	better

An analogy for large-mt vs high-energy

free particle ⇔ harmonic oscillator

mass ⇔ spring constant

$$\partial^2\phi(x) + m^2\phi(x) = 0 \quad \Leftrightarrow \quad \ddot{x} + kx = 0$$



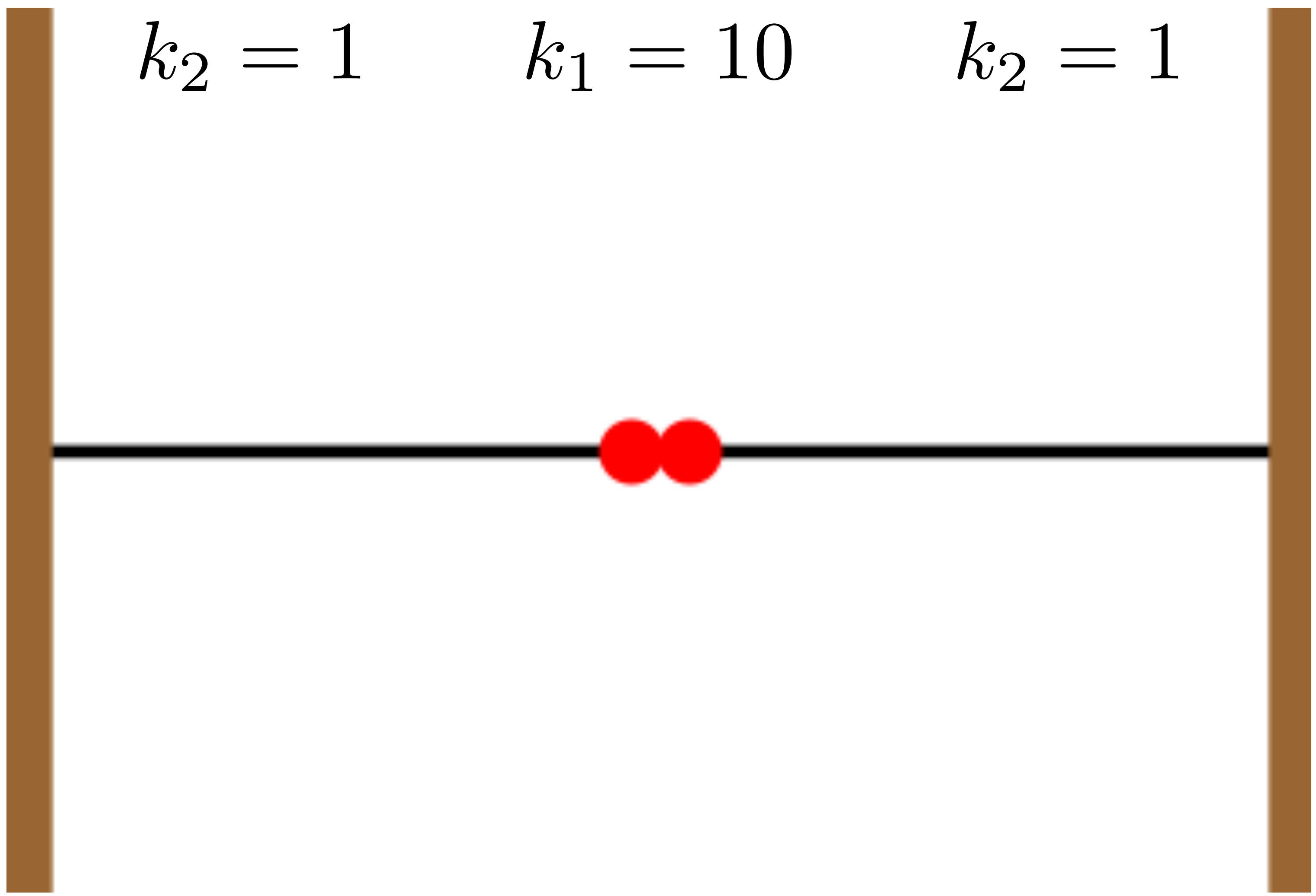
Analogy for the large-mt approximation

$$\text{free particle} \Leftrightarrow \text{harmonic oscillator}$$

$$\text{mass} \Leftrightarrow \text{spring constant}$$

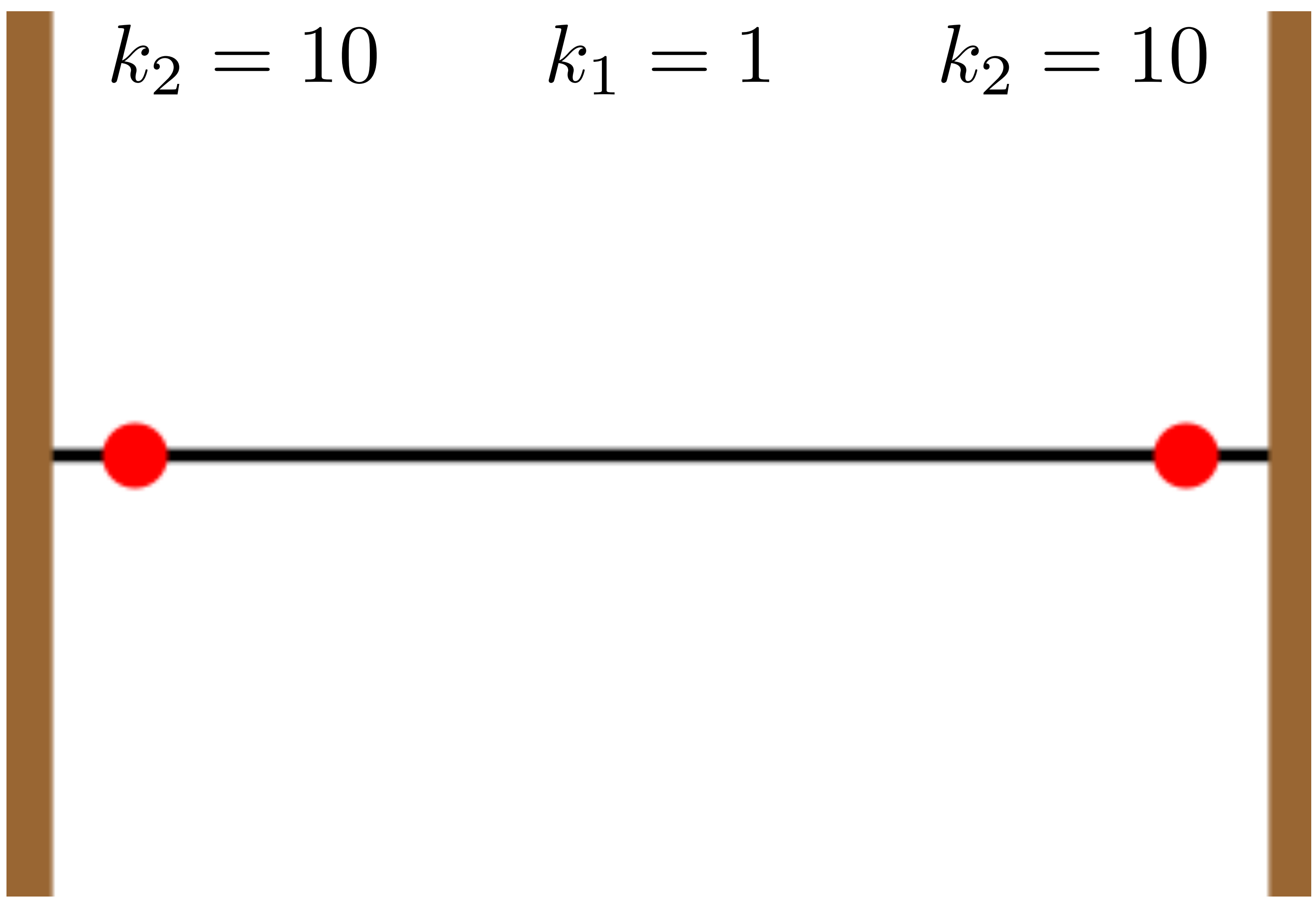
$$\partial^2\phi(x) + m^2\phi(x) = 0 \Leftrightarrow \ddot{x} + kx = 0$$

$k_2 = 1$ $k_1 = 10$ $k_2 = 1$

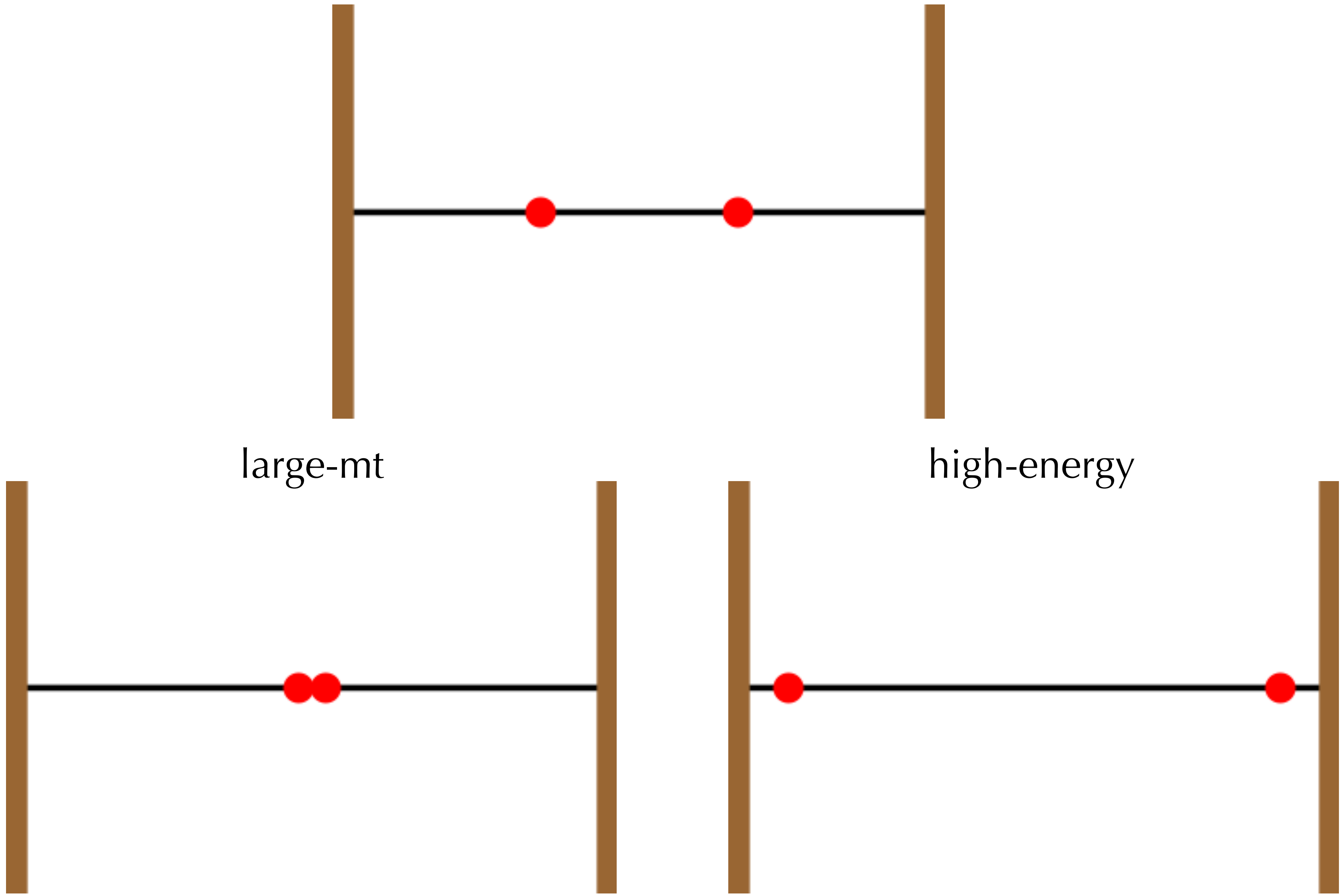


Analogy for the high-energy approximation

free particle	$\Leftrightarrow$	harmonic oscillator
mass	$\Leftrightarrow$	spring constant
$\partial^2\phi(x) + m^2\phi(x) = 0$	$\Leftrightarrow$	$\ddot{x} + kx = 0$



An analogy for large-mt vs high-energy



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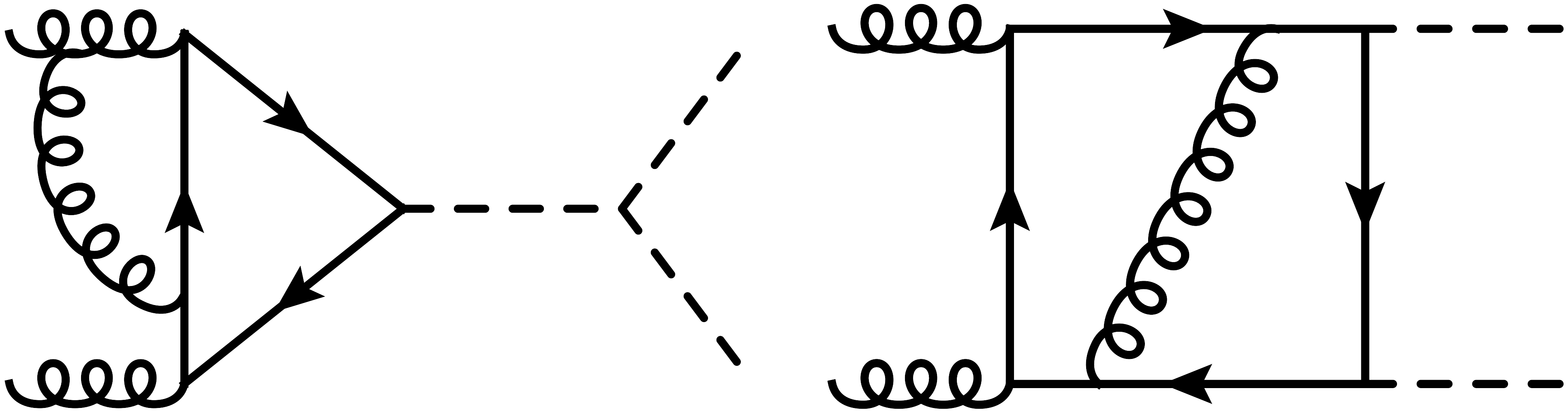
Main topic



two-loop integrals: high-energy approximation
and
numerical evaluation

Our setup to calculate the two-loop amplitude

qgraf [Nogueira, '93] : generate amplitudes

q2e/exp [Harlander, Seidensticker, Steinhauser, '98, Seidensticker, '99] : rewrite output to FORM notation

TFORM 4.2 [Rujil, Ueda, Vermaseren '17] : projection to the form factors

LiteRed [Lee, '13] : mH expansion

FIRE [Smirnov, '14] (with LiteRed rules [Lee, '13]) : IBP reduction to master integrals

tsort [Smirnov, Pak] : minimization of master integrals

Up to this point, we retain the full top mass dependence.

Feynman Diagrams: 8 (LO) + 118 (NLO)

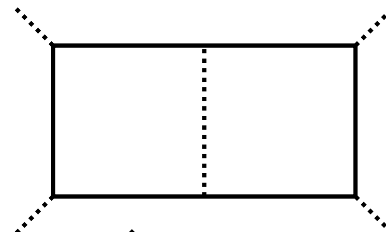
Scalar Integrals: 26K (+120K mH expansion)

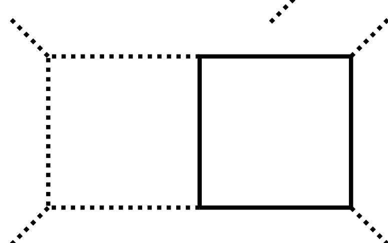
Master Integrals: 10 (LO) + 221 (NLO)

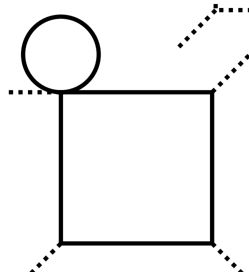
Minimal Master Integrals: 10 (LO) + 161 (NLO)

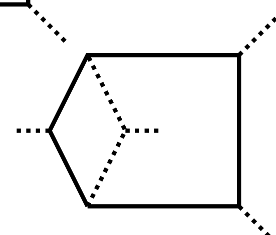
Master integrals at 2 loop

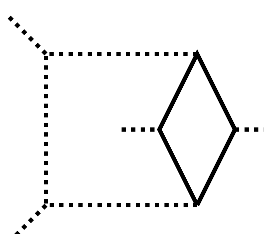
161 = 131 (planar&crossing) + 30 (nonplanar&crossing)

=28 +(20+15+19+11+9)  +crossing

+15 +(8+3)  +crossing

+1 +(2)  +crossing

+11 +(6+6)  +crossing

+7 

JHEP 1803 (2018) 048 [arXiv:1801.09696] (planar&crossing: lines 1–3)

JHEP 1901 (2019) 176 [arXiv:1811.05489] (nonplanar&crossing: lines 4–5)

Asymptotic expansion of Feynman integrals

Expansion by subgraphs

Mathematical proof based on graph-theoretical language exists. [Smirnov '90, Smirnov '02]
Valid only for the cases of large-mass, or Euclidian kinematics.
The procedure is implemented and can be performed automatically.

Method of regions (expansion by regions) [Beneke, Smirnov '97]

Mathematical proof is not yet given.
More broader application is assumed to be possible.

Step 1. Assign a hierarchy to the dimensionful parameters.

Step 2. Reveal the relevant scaling of the integration variable.

Step 3. For each region, expand the integrand according to its scaling.

Step 4. Integrate. Scaleless integrals like $\int_0^\infty dx\ x^a$ are set to zero.

Step 5. Sum over the contributions from all the relevant regions.

Asymptotic expansion of Feynman integrals

Expansion by subgraphs

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Valid only for the cases of large-mass, or Euclidian kinematics.
The procedure is implemented and can be performed automatically.

$$p = \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix}$$

Method of regions (expansion by regions) [Beneke, Smirnov '97]

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Method of regions: toy model

$$I = \int_0^\infty \frac{x^\epsilon \, \mathrm{d}x}{(x+m)(x+M)}$$

$m \ll M$

$$= \int_0^\Lambda dx \, \frac{x^\epsilon}{(x+m)(x+M)} + \int_\Lambda^\infty dx \, \frac{x^\epsilon}{(x+m)(x+M)}$$

$m < \Lambda < M$

$$= \int_0^\Lambda dx \, \frac{x^\epsilon}{(x+m)} \left(\sum_{n=0}^\infty \frac{(-x)^n}{M^{n+1}} \right) + \int_\Lambda^\infty dx \, \frac{x^\epsilon}{(x+M)} \left(\sum_{n=0}^\infty \frac{(-m)^n}{x^{n+1}} \right)$$

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$$= \frac{m^\epsilon}{M} \sum_{n=0}^\infty \left(\frac{-m}{M} \right)^n \Gamma(1+n+\epsilon)\Gamma(-\epsilon-n) + \frac{M^\epsilon}{M} \sum_{n=0}^\infty \left(\frac{-m}{M} \right)^n \Gamma(1+n-\epsilon)\Gamma(\epsilon-n)$$

$$= \left(-\frac{1}{\epsilon M} - \frac{\log m}{M} \right) \sum_{n=0}^\infty \left(\frac{m}{M} \right)^n + \left(\frac{1}{\epsilon M} + \frac{\log M}{M} \right) \sum_{n=0}^\infty \left(\frac{m}{M} \right)^n + \mathcal{O}(\epsilon) .$$

$$= \frac{\log M - \log m}{M - m} + \mathcal{O}(\epsilon)$$

Method of regions: toy model

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Method of regions: geometric approach

[Pak, Smirnov '10, Jantzen, Smirnov, Smirnov '12]

	mathematical derivation	geometric approach
toy model	possible	possible
actual Feynman integrals	not known	possible

$$I = \int_0^\infty \frac{x^\epsilon\,\mathrm{d}x}{(x+m)}\left(\frac{1}{M}+\frac{-x}{M^2}+\frac{x^2}{M^3}+\cdots\right)+\int_0^\infty \frac{x^\epsilon\,\mathrm{d}x}{(x+M)}\left(\frac{1}{x}+\frac{-m}{x^2}+\frac{m^2}{x^3}+\cdots\right)$$

Method of regions: geometric approach

[Pak, Smirnov '10, Jantzen, Smirnov, Smirnov '12]

most non-trivial

Step 1. Assign a hierarchy to the dimensionful parameters.

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Step 5. Sum over the contributions from all the relevant regions.

$$I = \int_0^\infty \frac{x^\epsilon\,\mathrm{d}x}{(x+m)}\left(\frac{1}{M} + \frac{-x}{M^2} + \frac{x^2}{M^3} + \cdots\right) + \int_0^\infty \frac{x^\epsilon\,\mathrm{d}x}{(x+M)}\left(\frac{1}{x} + \frac{-m}{x^2} + \frac{m^2}{x^3} + \cdots\right)$$

$x \sim m$ $\qquad$ $x \sim M$

Method of regions: geometric approach

[Pak, Smirnov '10, Jantzen, Smirnov, Smirnov '12]

$$I = \int_0^\infty \mathrm{d}x\, x^\epsilon \int_0^\infty \mathrm{d}\alpha_1 e^{-\alpha_1(x+m)} \int_0^\infty \mathrm{d}\alpha_2 e^{-\alpha_2(x+M)}$$

$$= \Gamma(1+\epsilon) \int_0^\infty \mathrm{d}\alpha_1 \int_0^\infty \mathrm{d}\alpha_2\ (\alpha_1+\alpha_2)^{-1-\epsilon} e^{-m\alpha_1 - M\alpha_2}$$

$$= \Gamma(1+\epsilon) \int \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \tilde{\mathcal{U}}^{-d/2} e^{-\tilde{\mathcal{F}}/\tilde{\mathcal{U}}}$$

$$\tilde{\mathcal{U}} = \alpha_1 + \alpha_2$$

$$\tilde{\mathcal{F}} = (m\alpha_1 + M\alpha_2)(\alpha_1 + \alpha_2)$$

$$\tilde{\mathcal{U}}\tilde{\mathcal{F}} = m\alpha_1^3 + m\alpha_1^2\alpha_2 + m\alpha_1\alpha_2^2 + M\alpha_1^2\alpha_2 + M\alpha_1\alpha_2^2 + M\alpha_2^3\,,$$

Method of regions: geometric approach

[Pak, Smirnov '10, Jantzen, Smirnov, Smirnov '12]

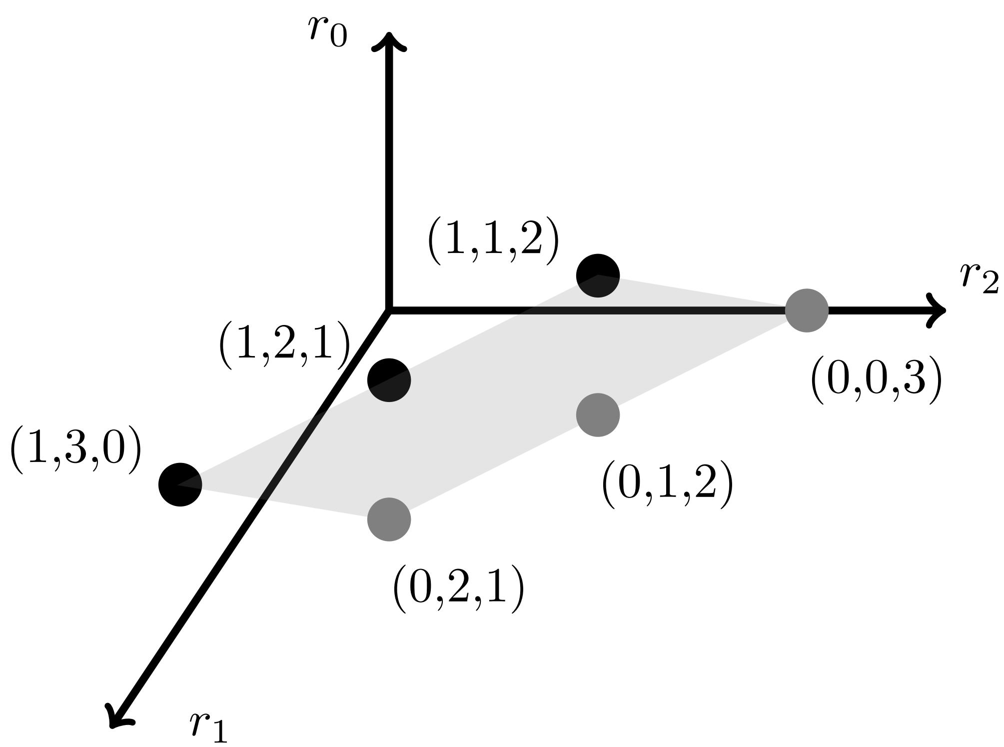
$$\tilde{\mathcal{U}}\tilde{\mathcal{F}} = m\alpha_1^3 + m\alpha_1^2\alpha_2 + m\alpha_1\alpha_2^2 + M\alpha_1^2\alpha_2 + M\alpha_1\alpha_2^2 + M\alpha_2^3\,,$$

Map each term to a point (r_0, r_1, r_2) where $m^{r_0}\alpha_1^{r_1}\alpha_2^{r_2}$

Assume a certain scaling $(m, \alpha_1, \alpha_2) \sim (m, m^{s_1}, m^{s_2})$

The scaling of each term can be expressed as $m^{\vec{r}\cdot\vec{s}}$
where $\vec{s} = (1, s_1, s_2)$

Consider the leading order terms,
which corresponds to the points in the bottom side



Non-vanishing contributions consist of the points having the same scaling $m^{\vec{r}_A\cdot\vec{s}} = m^{\vec{r}_B\cdot\vec{s}}$

$$\Rightarrow \vec{s}\cdot(\vec{r}_A - \vec{r}_B) = 0$$

Method of regions: geometric approach

[Pak, Smirnov '10, Jantzen, Smirnov, Smirnov '12]

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Algorithm is implemented in the Mathematica package "asy.m"
[Pak, Smirnov '10, Jantzen, Smirnov, Smirnov '12]

$$I = \int_0^\infty \frac{x^\epsilon\,\mathrm{d}x}{(x+m)}\left(\frac{1}{M}+\frac{-x}{M^2}+\frac{x^2}{M^3}+\cdots\right)+\int_0^\infty \frac{x^\epsilon\,\mathrm{d}x}{(x+M)}\left(\frac{1}{x}+\frac{-m}{x^2}+\frac{m^2}{x^3}+\cdots\right)$$

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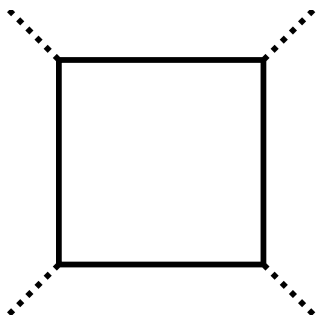
Expansion in m_t



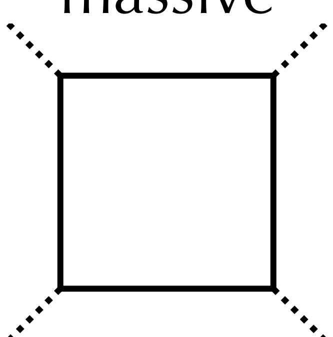
$$= \int Dk \frac{1}{k^2 - m_t^2} \frac{1}{(k+p_1)^2 - m_t^2} \frac{1}{(k+p_1+p_2)^2 - m_t^2} \frac{1}{(k+p_3)^2 - m_t^2}$$

$$= \sum_{n=0}^{\infty} (m_t^2)^n f_n(S, T, \log m_t)$$

Naive expansion of the integrand like

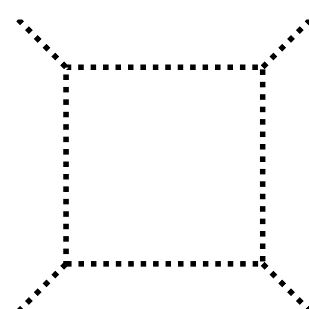
$$\frac{1}{k^2 - m_t^2} = \frac{1}{k^2} + \frac{m_t^2}{(k^2)^2} + \cdots$$ gives wrong result.

massive



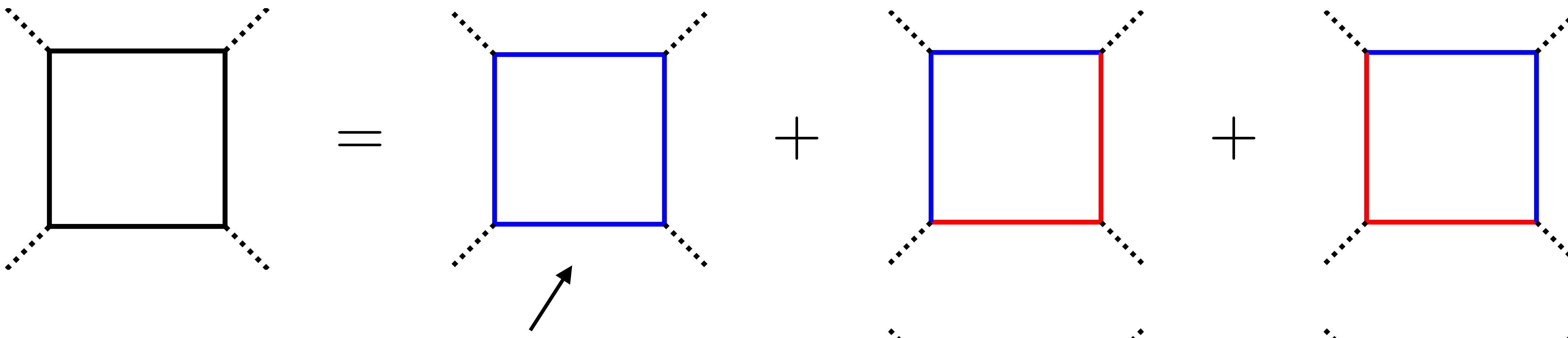
is finite.

massless

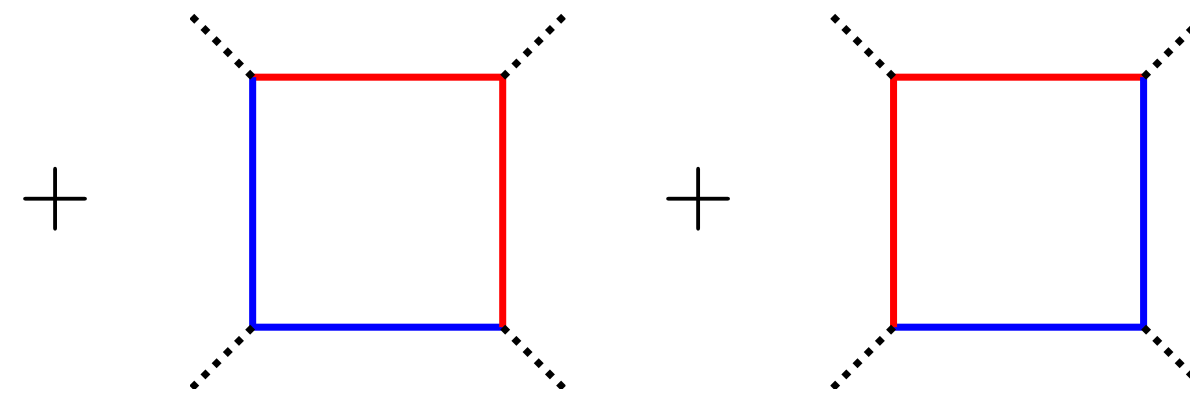


$$= \frac{1}{st}\left(\frac{4}{\varepsilon^2} - \frac{2\log st}{\varepsilon} + 2\log s \log t - \frac{4\pi^2}{3}\right) + \mathcal{O}(\epsilon)$$

Method of region [Beneke, Smirnov '97, Smirnov `02, Jantzen `11]



``all-hard'' region = massless integral



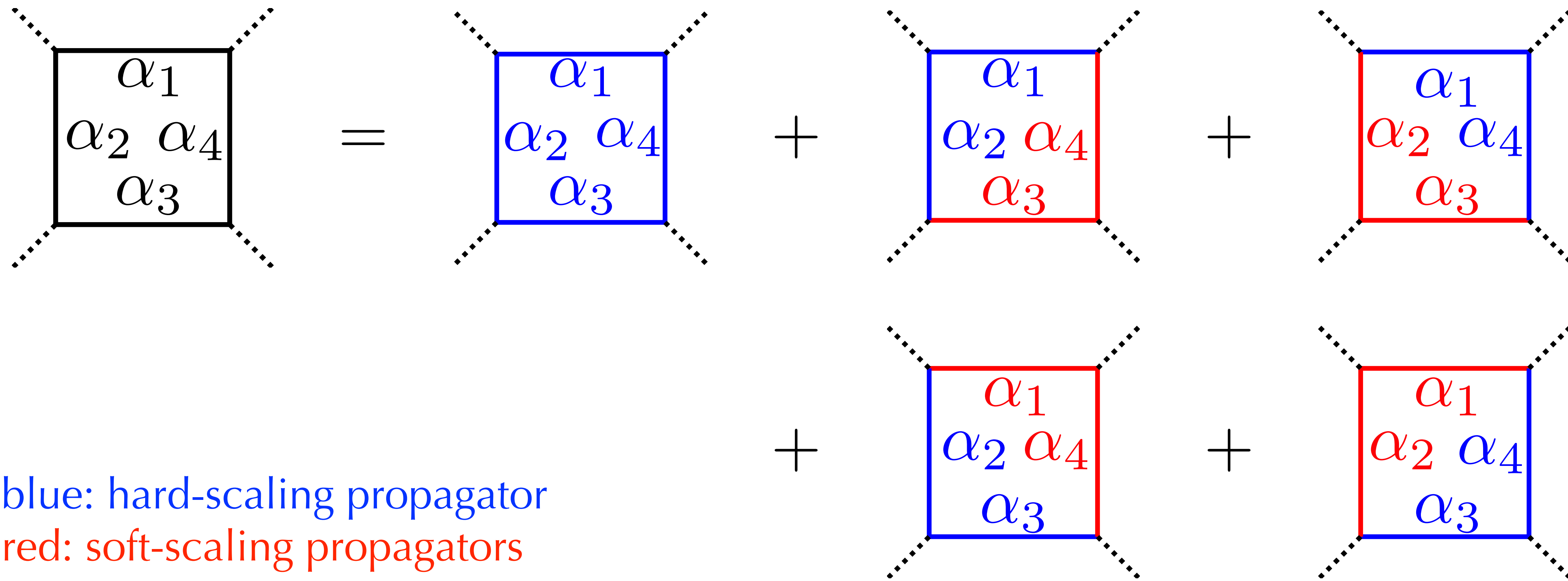
blue: hard-scaling propagator
red: soft-scaling propagators

the scaling of propagators in terms of alpha-parameter representation

$$\int_0^\infty \left(\prod_{n=1}^{4} d\alpha_n \right) \alpha_{1234}^{-d/2}\ \mathrm{e}^{-m^2 \alpha_{1234} - (s\alpha_1\alpha_3 + t\alpha_2\alpha_4)/\alpha_{1234}}$$

$$\alpha_{1234} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

Method of region [Beneke, Smirnov '97, Smirnov `02, Jantzen `11]



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the scaling of propagators in terms of alpha-parameter representation

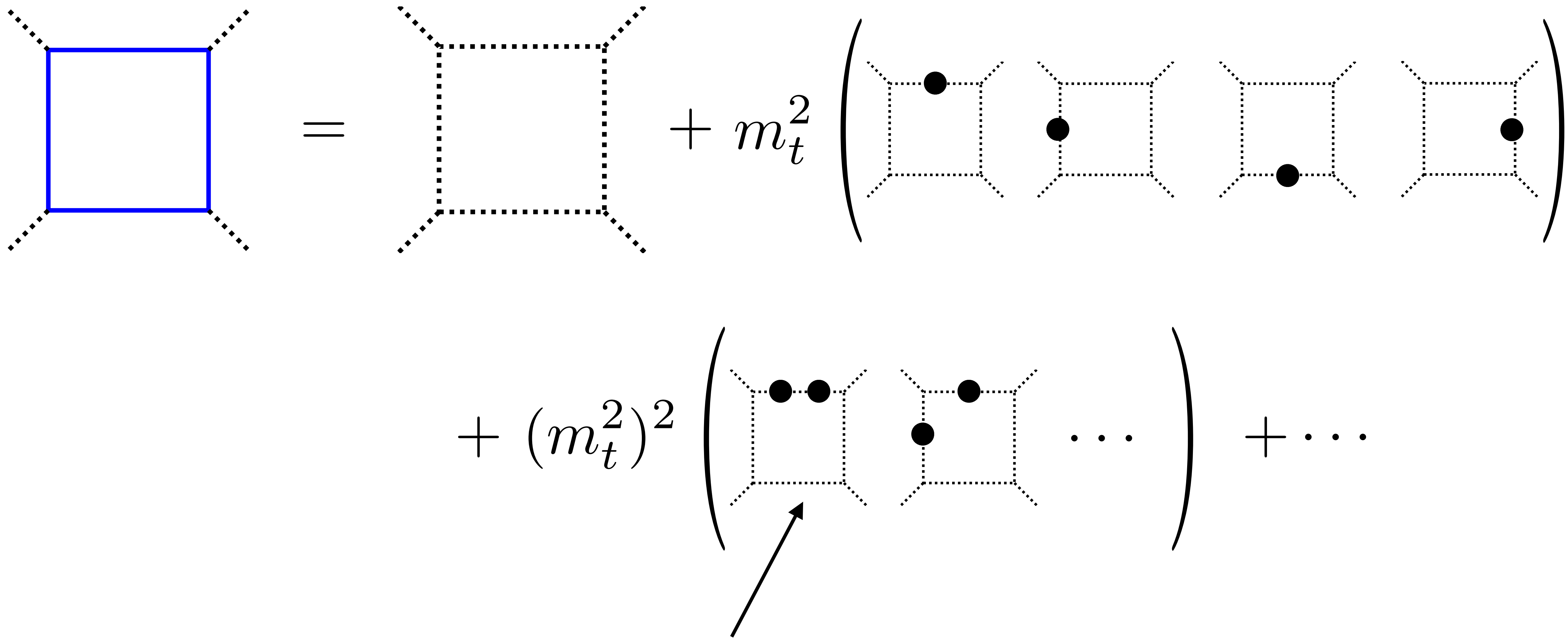
$$\int_0^\infty \left(\prod_{n=1}^{4} d\alpha_n\right) \alpha_{1234}^{-d/2}\, \mathrm{e}^{-m^2\alpha_{1234}-(s\alpha_1\alpha_3+t\alpha_2\alpha_4)/\alpha_{1234}}$$

$$\alpha_{i_1\ldots i_n} \equiv \alpha_{i_1} + \cdots + \alpha_{i_n}$$

$$\alpha_{1234} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$$

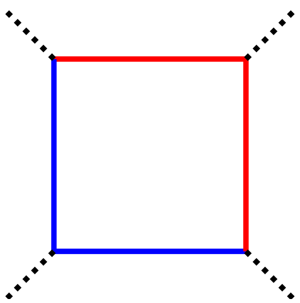
Method of region: ``all-hard" region

In our case, the expansion in this region corresponds to the **naive Taylor expansion**.
The right hand side consists of massless diagrams with dots.



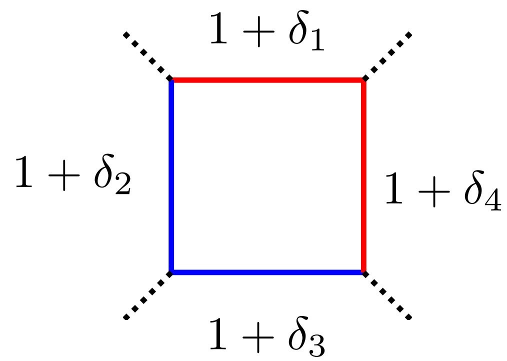
We can apply the integration by parts (IBP) reduction.

Method of region: soft-collinear regions



$$= \int_0^\infty \left(\prod_{n=1}^{4} d\alpha_n\right) \begin{array}{l} \alpha_{12}^{-d/2}\ \mathrm{e}^{-m^2\alpha_{12}-(s\alpha_1\alpha_3+t\alpha_2\alpha_4)/\alpha_{12}} \\ -\,\alpha_{12}^{-d/2-2}(\alpha_3+\alpha_4)((d/2)\alpha_{12}+m^2(\alpha_{12})^2-s\alpha_1\alpha_3-t\alpha_2\alpha_4) \\ \times\ \mathrm{e}^{-m^2\alpha_{12}-(s\alpha_1\alpha_3+t\alpha_2\alpha_4)/\alpha_{12}} \\ +\cdots \end{array}$$

Usual momentum representation is not always possible…



The integrals are ill-defined,
so we have to introduce
analytic regularization
of the exponent of propagators.

$$f_0^{(2)} = \frac{(m^2)^{-\varepsilon}}{st}\left[\frac{1}{\varepsilon}\left(-\frac{1}{\delta_3}-\frac{1}{\delta_4}+\log st\right)\right]$$

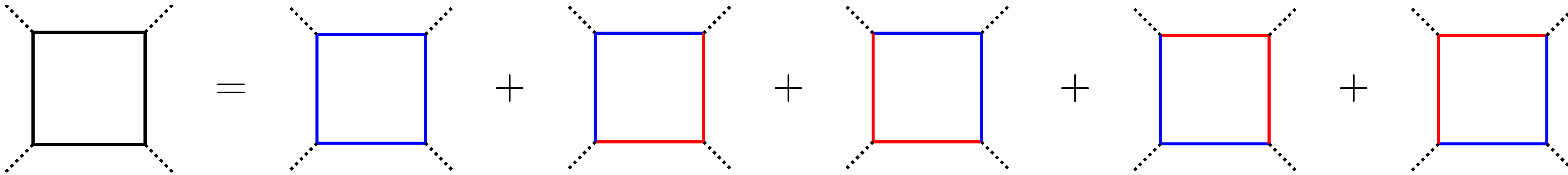
$$f_0^{(3)} = \frac{(m^2)^{-\varepsilon}}{st}\left[-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{1}{\delta_3}-\frac{1}{\delta_4}+\log t/m^2\right)+\frac{\pi^2}{12}\right]$$

$$f_0^{(4)} = \frac{(m^2)^{-\varepsilon}}{st}\left[-\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left(-\frac{1}{\delta_3}+\frac{1}{\delta_4}+\log s/m^2\right)+\frac{\pi^2}{12}\right]$$

$$f_0^{(5)} = \frac{(m^2)^{-\varepsilon}}{st}\left[-\frac{2}{\varepsilon^2}+\frac{1}{\varepsilon}\left(\frac{1}{\delta_3}+\frac{1}{\delta_4}-2\log m^2\right)+\frac{\pi^2}{6}\right]$$

Cancellation of auxiliary parameters between soft regions occurs.

Method of region: total



$$f_0^{(1)} = \frac{1}{st}\left(\frac{4}{\varepsilon^2} - \frac{2\log st}{\varepsilon} + 2\log s \log t - \frac{4\pi^2}{3}\right)$$

$$f_0^{(2)} = \frac{(m^2)^{-\varepsilon}}{st}\left[\frac{1}{\varepsilon}\left(-\frac{1}{\delta_3} - \frac{1}{\delta_4} + \log st\right)\right]$$

$$f_0^{(3)} = \frac{(m^2)^{-\varepsilon}}{st}\left[-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{1}{\delta_3} - \frac{1}{\delta_4} + \log t/m^2\right) + \frac{\pi^2}{12}\right]$$

$$f_0^{(4)} = \frac{(m^2)^{-\varepsilon}}{st}\left[-\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\left(-\frac{1}{\delta_3} + \frac{1}{\delta_4} + \log s/m^2\right) + \frac{\pi^2}{12}\right]$$

$$f_0^{(5)} = \frac{(m^2)^{-\varepsilon}}{st}\left[-\frac{2}{\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{1}{\delta_3} + \frac{1}{\delta_4} - 2\log m^2\right) + \frac{\pi^2}{6}\right]$$

$$I = \sum_{n=0}^{\infty} (m^2)^n f_n$$

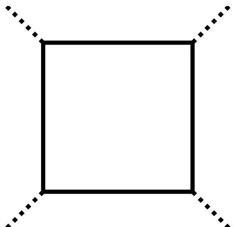
$$f_0 = \frac{1}{st}\left(2\log\frac{s}{m^2}\log\frac{t}{m^2} - \pi^2\right)$$

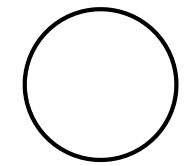
Cancellation of auxiliary parameters between soft regions occurs.

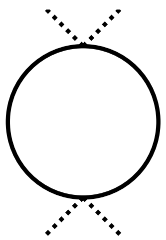
Expansion in m_t : using differential equation

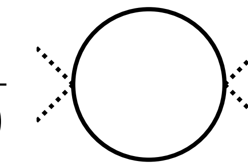
[Kotikov '91]

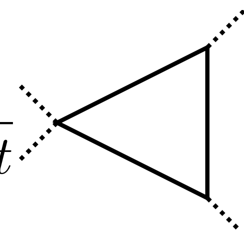
We used LiteRed **[Lee '13]** for obtaining the diff.-eq.

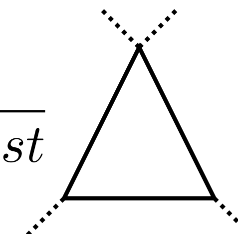
$$\frac{\partial}{\partial(m_t^2)}$$


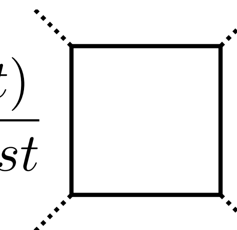
$$= -\frac{2(d-2)\left(2m^2(s+t)+st\right)}{m^4\left(4m^2+s\right)\left(4m^2+t\right)\left(4m^2(s+t)+st\right)}$$


$$-\frac{2(d-3)t}{m^2\left(4m^2+t\right)\left(4m^2(s+t)+st\right)}$$


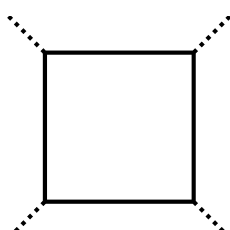
$$-\frac{2(d-3)s}{m^2\left(4m^2+s\right)\left(4m^2(s+t)+st\right)}$$


$$-\frac{(d-4)s}{4m^4(s+t)+m^2st}$$


$$-\frac{(d-4)t}{4m^4(s+t)+m^2st}$$


$$-\frac{2(d-5)(s+t)}{4m^2(s+t)+st}$$


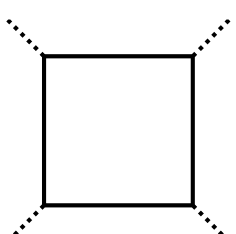
Substituting the form,



$$= \sum_{n_1,n_2} c_{n_1,n_2}(m_t^2)^{n_1}(\log m_t)^{n_2}$$

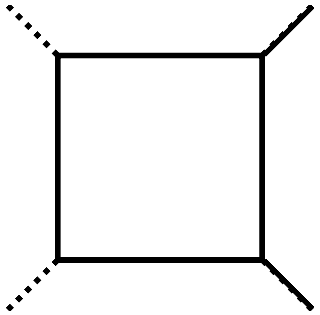
we obtain recursive relations of C_n's.

See also **[Melnikov, Tancredi, Wever '16]**

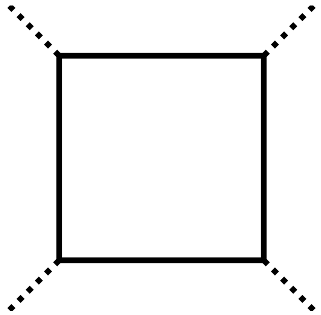


$$= (m_t^2)^0 f_0 + (m_t^2)^1 f_1 + (m_t^2)^2 f_2 + \cdots$$

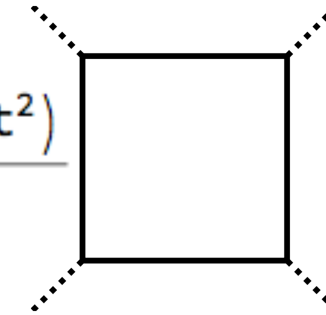
Expansion in m_h: only the all-hard region



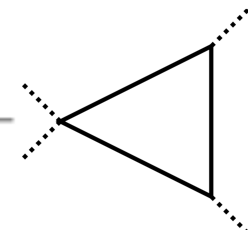
$$=$$



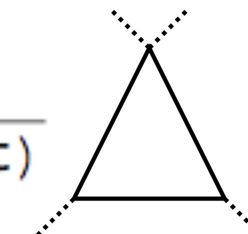
$$+m_h^2 \Bigg(\frac{s\,(4\,mts\,s + 4\,mts\,t - 6\,s\,t + d\,s\,t - 10\,t^2 + 2\,d\,t^2)}{t\,(s+t)\,(-4\,mts\,s - 4\,mts\,t + s\,t)}$$



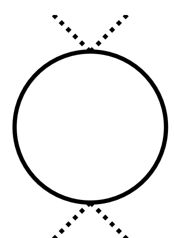
$$+ \frac{2\,(-4+d)\,s\,(s+2\,t)}{t\,(s+t)\,(-4\,mts\,s - 4\,mts\,t + s\,t)}$$



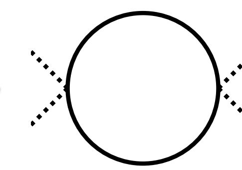
$$+ \frac{2\,(-4+d)\,(s+2\,t)}{(s+t)\,(-4\,mts\,s - 4\,mts\,t + s\,t)}$$



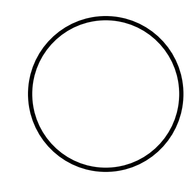
$$- \frac{4\,(-3+d)\,(s+2\,t)}{s\,t\,(-4\,mts\,s - 4\,mts\,t + s\,t)}$$



$$+ \frac{8\,(-3+d)\,(2\,mts - t)}{(4\,mts - t)\,t\,(4\,mts\,s + 4\,mts\,t - s\,t)}$$



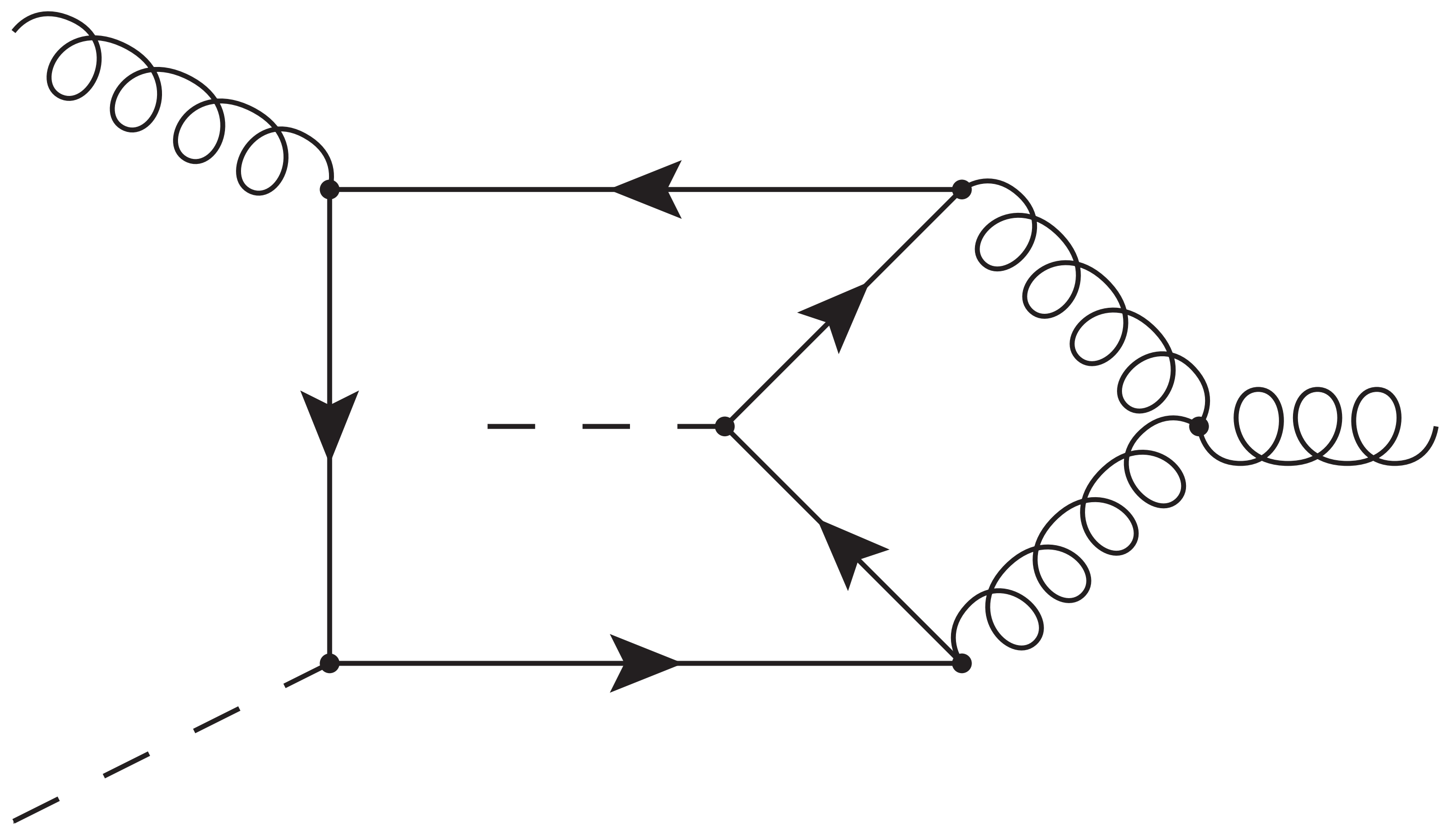
$$\frac{(-2+d)\,(-48\,mts^2\,s + 16\,d\,mts^2\,s - 48\,mts^2\,t + 16\,d\,mts^2\,t + 26\,mts\,s\,t - 8\,d\,mts\,s\,t + 12\,mts\,t^2 - 4\,d\,mts\,t^2 - 4\,s\,t^2 + d\,s\,t^2)}{mts^2\,s\,(4\,mts - t)\,t\,(4\,mts\,s + 4\,mts\,t - s\,t)}$$



$$+\mathcal{O}(m_h^4) \Bigg)$$

The massive-Higgs diagram can be expressed as an infinite sum of the massless-Higgs diagrams.

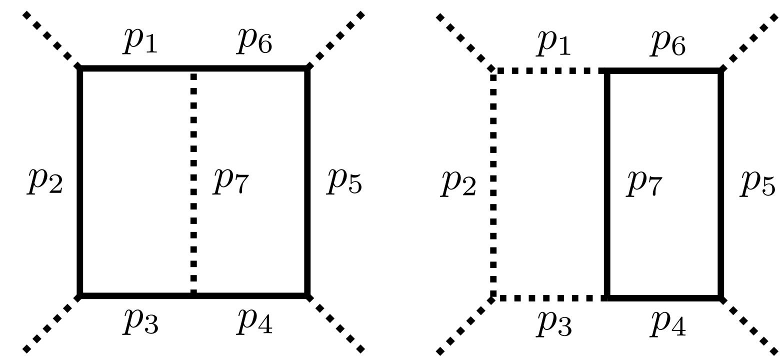
Expansion of non-planar two-loop integrals



The geometric approach rely on the positivity of $\mathcal{F}$

1-loop box

$$\mathcal{U} = \alpha_{1234}, \qquad \mathcal{F} = m_t^2 \alpha_{1234} \mathcal{U} + S\alpha_1\alpha_3 + T\alpha_2\alpha_4$$



PL1 PL2

2-loop planar double box

$$\mathcal{U}^{\rm PL1} = \mathcal{U}^{\rm PL2} = \alpha_{123}\alpha_{456} + \alpha_{123456}\alpha_7$$

$$\mathcal{F}^{\rm PL1} = m_t^2 \alpha_{123456} \mathcal{U}^{\rm PL1} + S\left[\alpha_1\left(\alpha_4\alpha_{67} + \alpha_3\alpha_{4567}\right) + \alpha_6\left(\alpha_{23}\alpha_4 + \alpha_{34}\alpha_7\right)\right] + T\alpha_2\alpha_5\alpha_7$$

$$\mathcal{F}^{\rm PL2} = m_t^2 \alpha_{1237} \mathcal{U}^{\rm PL2} + S\left[\alpha_1\left(\alpha_4\alpha_{67} + \alpha_3\alpha_{4567}\right) + \alpha_6\left(\alpha_{23}\alpha_4 + \alpha_{34}\alpha_7\right)\right] + T\alpha_2\alpha_5\alpha_7\,.$$

2-loop non-planar

$$\mathcal{U} = \alpha_{12}\alpha_{34567} + \alpha_{34}\alpha_{567}$$

$$\mathcal{F} = m_t^2 \alpha_{34567} \mathcal{U} + S\left(\alpha_1\alpha_7\alpha_{45} + \alpha_2\alpha_5\alpha_{37} + \alpha_5\alpha_7\alpha_{34}\right) + T\alpha_1\alpha_3\alpha_6 + U\alpha_2\alpha_4\alpha_6$$

Negative terms appear if you apply $S + T + U = 0$

Our solution:

Assume S>0, T>0, U>0 putting aside S+T+U=0 and proceed the computation anyway.

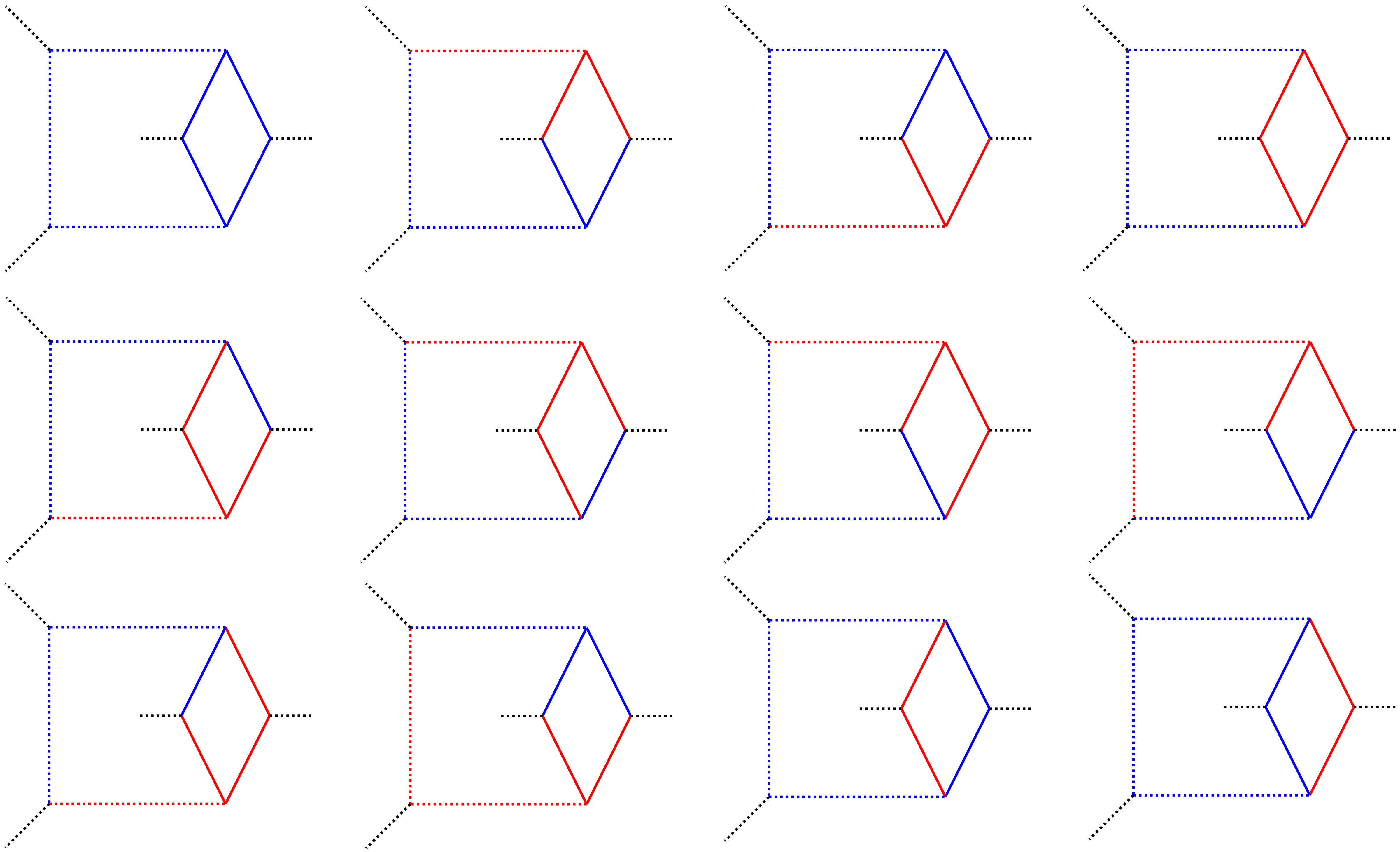
At the end of the calculation, restore the physical kinematics by an analytic continuation.

Our solution:

Assume S>0, T>0, U>0 putting aside S+T+U=0 and proceed the computation anyway. At the end of the calculation, restore the physical kinematics by an analytic continuation.

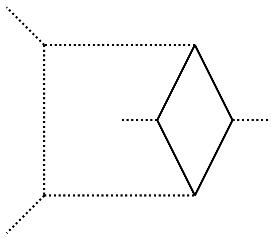
It works!!

Method of region: non-planar diagrams



blue: hard-scaling propagator, red: soft-scaling propagators

Method of region: non-planar diagrams



$$= \frac{e^{2i\pi\epsilon}e^{-2\epsilon\gamma_E}}{s^{3+2\epsilon}}\left[\frac{i\pi^3\sqrt{s}}{m_t\sqrt{v(1-v)}}\left[\frac{1}{\epsilon}-2\log\left(\frac{m_t^2}{32s}\right)\right]+\sum_{i_1=-1}^{0}\sum_{i_2=0}^{4+i_1}\frac{d_{i_1,i_2}}{v}\epsilon^{i_1}\log^{i_2}(m_t)\right]+\mathcal{O}(m_t,\epsilon)$$

$$d_{-1,3} = -\frac{4}{3} \qquad d_{-1,2} = h_0(4v+2)+h_1(4v-6)+6i\pi$$

$$d_{-1,1} = -h_0^2+2h_0h_1+8h_0v+2i\pi h_0(2v-1)-h_1^2+8h_1(v-1)+2i\pi h_1(2v-1)-\frac{10\pi^2}{3}+8i\pi$$

$$\begin{aligned} d_{-1,0} = {} & -4h_0h_1\frac{1}{2}i\pi h_0^2+i\pi h_0h_1-\frac{1}{2}i\pi h_1^2-\frac{4i\pi^3}{3}-8i\pi+h_0h_1^2\left(\frac{1}{2}-v\right)+4i\pi h_0(v-1)-8h_0v+4i\pi h_1 v \\ & +\frac{1}{6}h_0^3(1-2v)+h_0^2h_1\left(\frac{1}{2}-v\right)+\frac{1}{3}\pi^2h_0(5-8v)+\frac{1}{6}h_1^3(1-2v)+h_1(8-8v)+\pi^2h_1\left(1-\frac{8v}{3}\right)\end{aligned}$$

$$d_{0,4} = -\frac{10}{3} \qquad d_{0,3} = h_0(4-8v)+h_1(4-8v)-\frac{20i\pi}{3}$$

$$d_{0,2} = -h_0^2+6h_0h_1-2i\pi h_0(6v+1)-h_1^2-2i\pi h_1(6v-7)+\frac{47\pi^2}{3}$$

$$\begin{aligned} d_{0,1} = {} & -i\pi h_0^2+16h_0h_1-2i\pi h_0h_1-16i\pi h_0-i\pi h_1^2+16i\pi h_1-32i\pi \\ & +\frac{1}{3}h_0^3(-2v-1)+h_0^2h_1(-2v-1)+h_0h_1^2(3-2v)+h_1^3\left(1-\frac{2v}{3}\right)-32h_1(v-1)-i\pi^3+16\pi^2 \\ & -32h_0v+\pi^2h_0\left(2v-\frac{7}{3}\right)+\pi^2h_1\left(2v+\frac{1}{3}\right)+6\zeta_3\end{aligned}$$

$$\begin{aligned} d_{0,0} = {} & 8h_0h_1^2-32h_0h_1-12i\pi h_1h_2-16\pi^2+64i\pi+16v\zeta_3+i\pi(8v-61)\zeta_3 \\ & +h_0^2h_1(8-8v)+h_0^2h_2(10-8v)+\frac{11}{12}\pi^2h_1^2(1-4v)+h_1h_3(8-16v)+h_4(74-76v) \\ & +h_0^2h_1^2\left(-v-\frac{1}{4}\right)+h_0h_1^3\left(\frac{1}{2}-2v\right)+\frac{1}{8}h_1^4(5-4v)-\frac{2}{3}\pi^2h_1(v-13)+\frac{22}{3}\pi^2h_2(1-2v)\end{aligned}$$

$$
\begin{aligned}
&- 8i\pi h_0^2(v-1) - 16i\pi h_0(v-2) - \frac{8}{3}h_1^3(v-1) + 64h_1(v-1) - 16h_3(v-2) \\
&- \frac{8h_0^3 v}{3} + 64h_0 v + 8i\pi h_1^2 v - 16i\pi h_1(v+1) - 16h_2(v+1) \\
&- \frac{1}{6}i\pi h_0^3(2v-11) + 8i\pi h_0 h_2(v+1) - \frac{2}{3}\pi^2 h_0(v+12) + \frac{1}{6}i\pi^3 h_1(2v-7) + \pi^4\left(\frac{151v}{90} - \frac{19}{9}\right) \\
&+ h_0^3 h_1\left(2v - \frac{3}{2}\right) - \frac{1}{2}i\pi h_0^2 h_1(2v+1) - 8i\pi h_0 h_1(2v-1) - 12h_0 h_2(2v+1) + 12i\pi h_2(2v+1) \\
&+ \frac{11}{12}\pi^2 h_0^2(4v-3) + 8h_0 h_3(4v-5) + \frac{1}{6}i\pi^3 h_0(2v+17) - \frac{1}{6}i\pi h_1^3(2v+9) - 4i\pi h_3(2v+5) \\
&+ \frac{1}{8}h_0^4(4v+1) + \frac{1}{2}i\pi h_0 h_1^2(6v-1) + h_0 h_1 h_2(16v-8) + h_2^2(14v-7) - \frac{2}{3}i\pi^3(4v+1) \\
&+ \frac{1}{6}\pi^2 h_0 h_1(44v-31) + h_0(-30v-3)\zeta_3 + h_1 h_2(16v+4) + h_1(21-30v)\zeta_3 - 12(4h_{211}v + h_{211})
\end{aligned}
$$

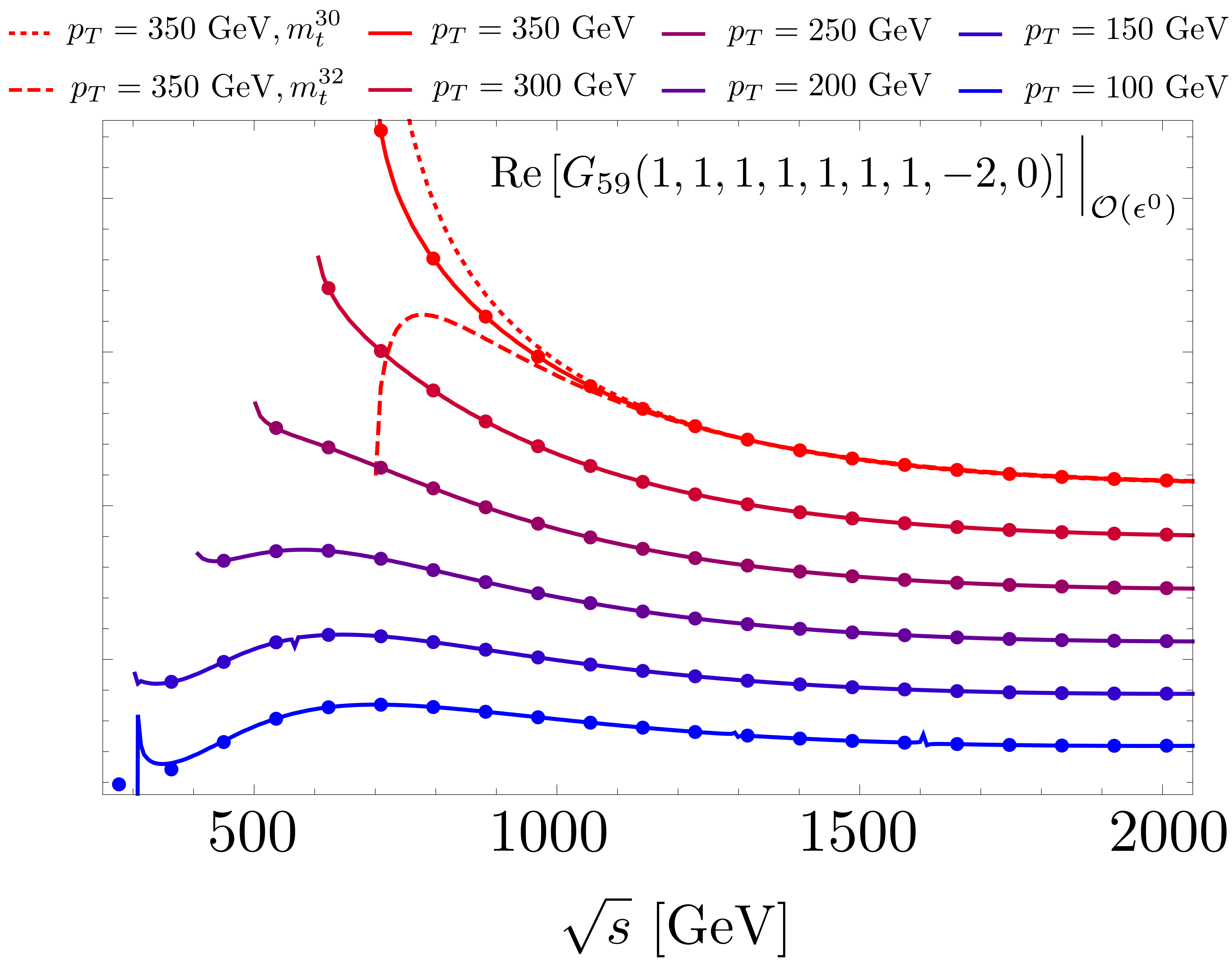
$h_0 = \mathrm{HPL}(\{0\}, v), \quad h_1 = \mathrm{HPL}(\{1\}, v), \quad h_2 = \mathrm{HPL}(\{2\}, v), \quad h_{2,1} = \mathrm{HPL}(\{2,1\}, v), \ \ldots$ $\qquad v = -t/s$

- used as the boundary condition of m_t-differential equation (solved up to mt^32)
- expressed in terms of HPL's
- suited for the evaluation at the physical kinematical configuration (s>0, t<0)
- satisfy the t-differential equation (independent check)
- consistent with the Higgs+jet result **[Kudashkin, Melnikov, Wever, '17]**

Improve the series expansion using Padé approximant

$$f_0 + f_1 x + \cdots + f_{n+m} x^{n+m}$$
$$\rightarrow \frac{a_0 + a_1 x + \cdots + a_n x^n}{1 + b_1 x + \cdots + b_m x^m}$$

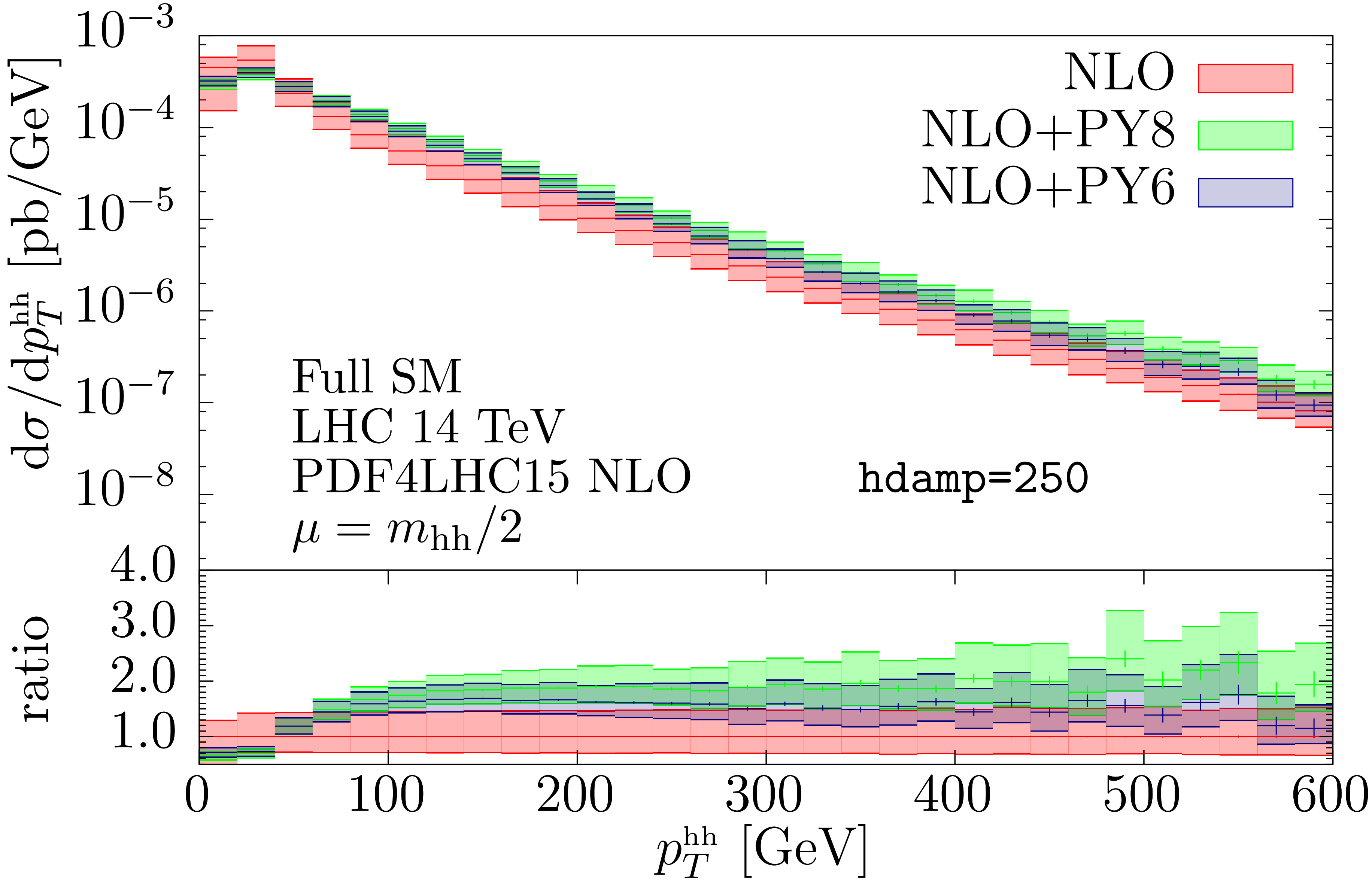
Applying Padé approximant for master integrals



numerical evaluation

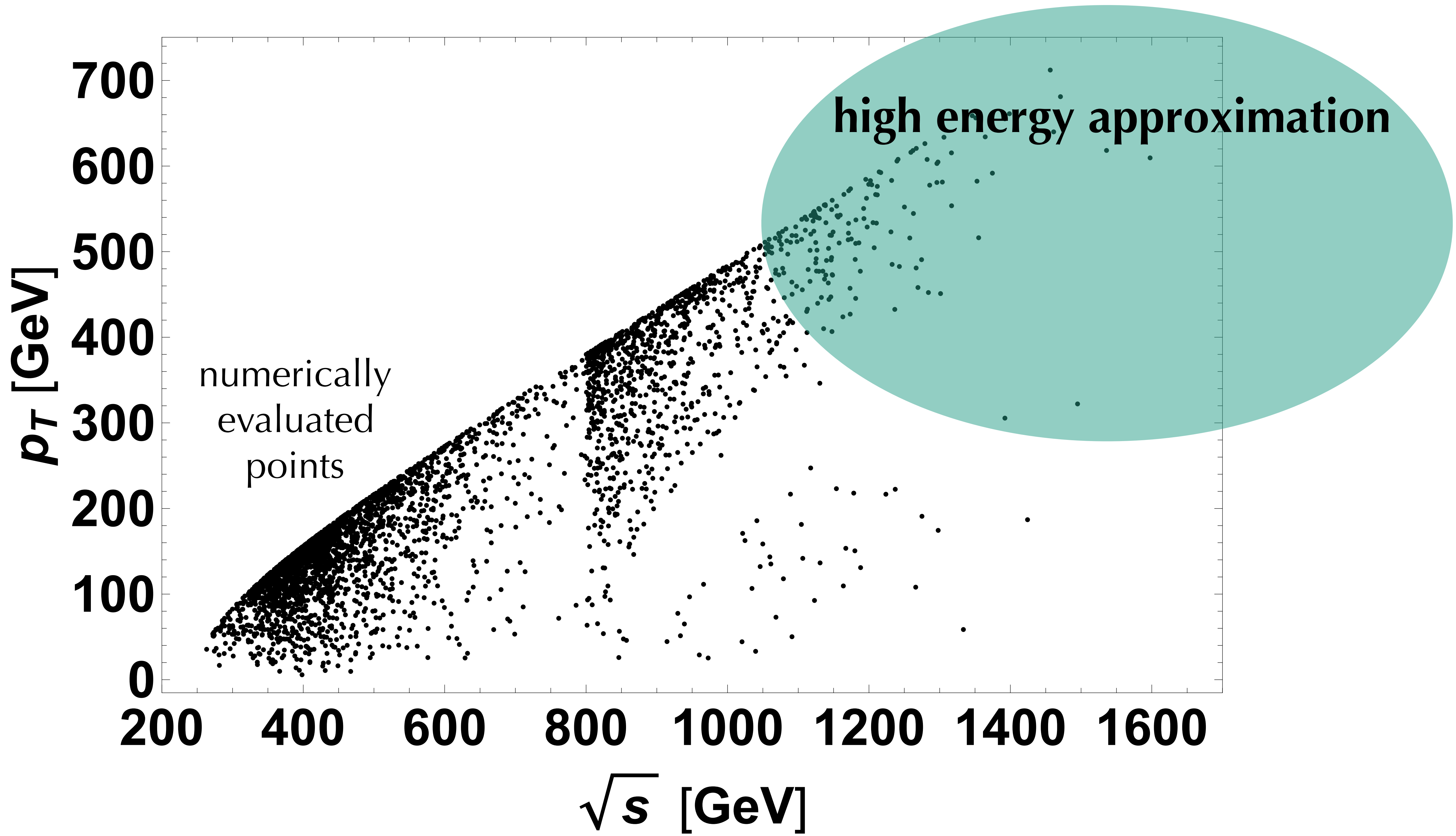
based on PRL 117 (2016) 012001, JHEP 1610 (2016) 107, JHEP 1708 (2017) 088

Numerically evaluated two-loop integrals (virtual correction) combined with parton showers within the POWHEG-BOX-V2 and MG5_aMC@NLO frameworks.

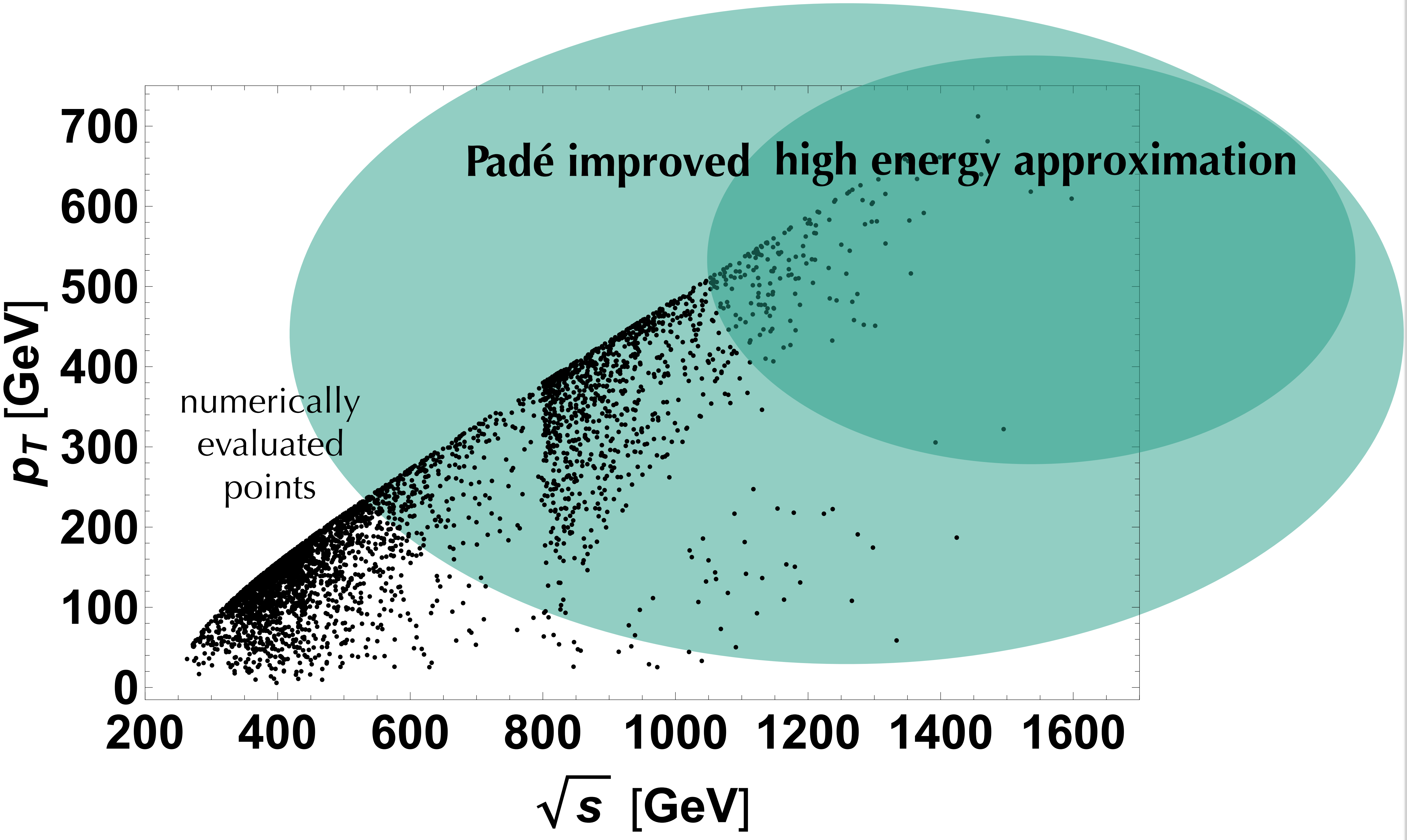


Two-loop integrals: evaluated points are increased: 3398 → 6320

Complementarity of HE approximation and numerics

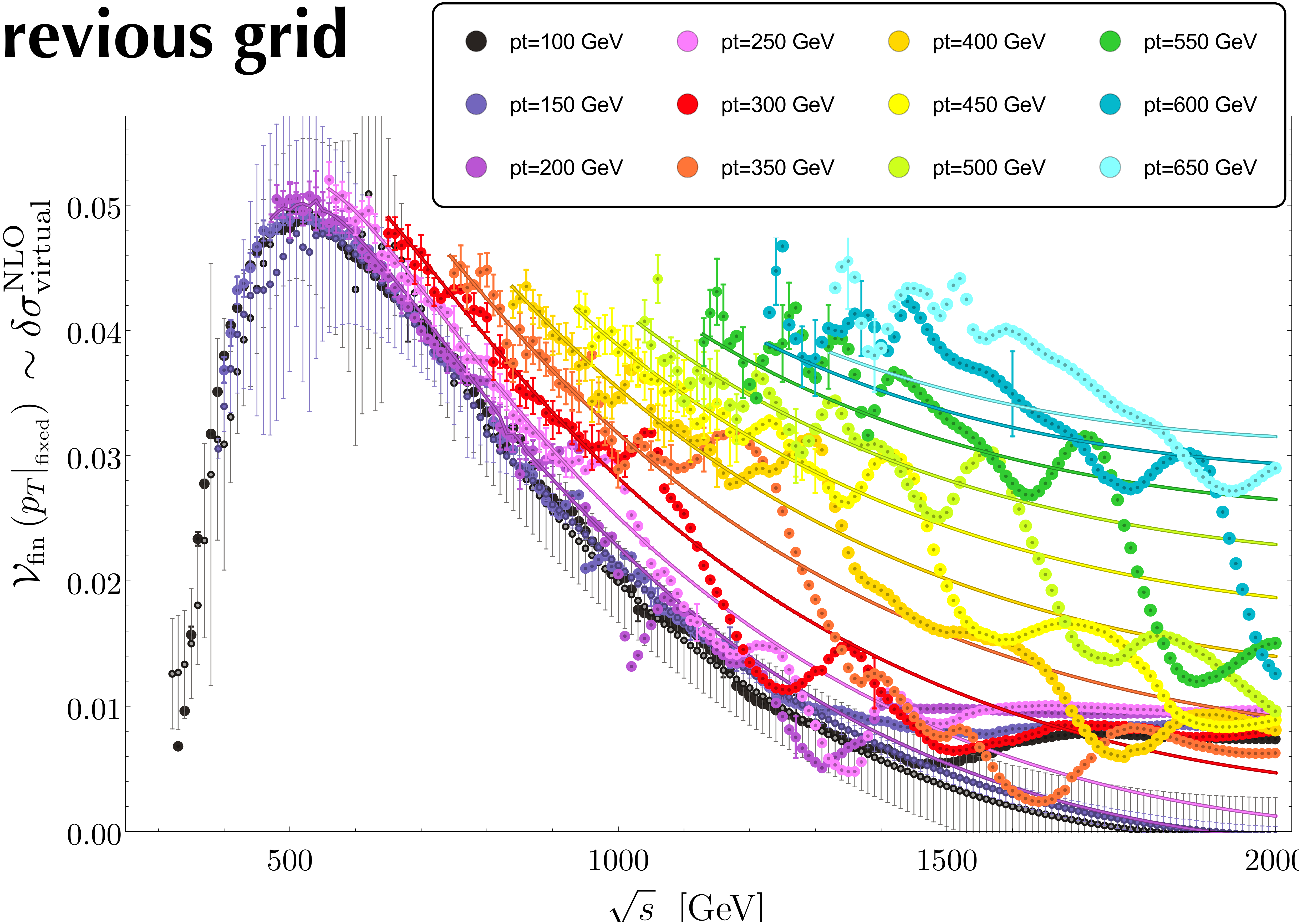


Complementarity of HE approximation and numerics

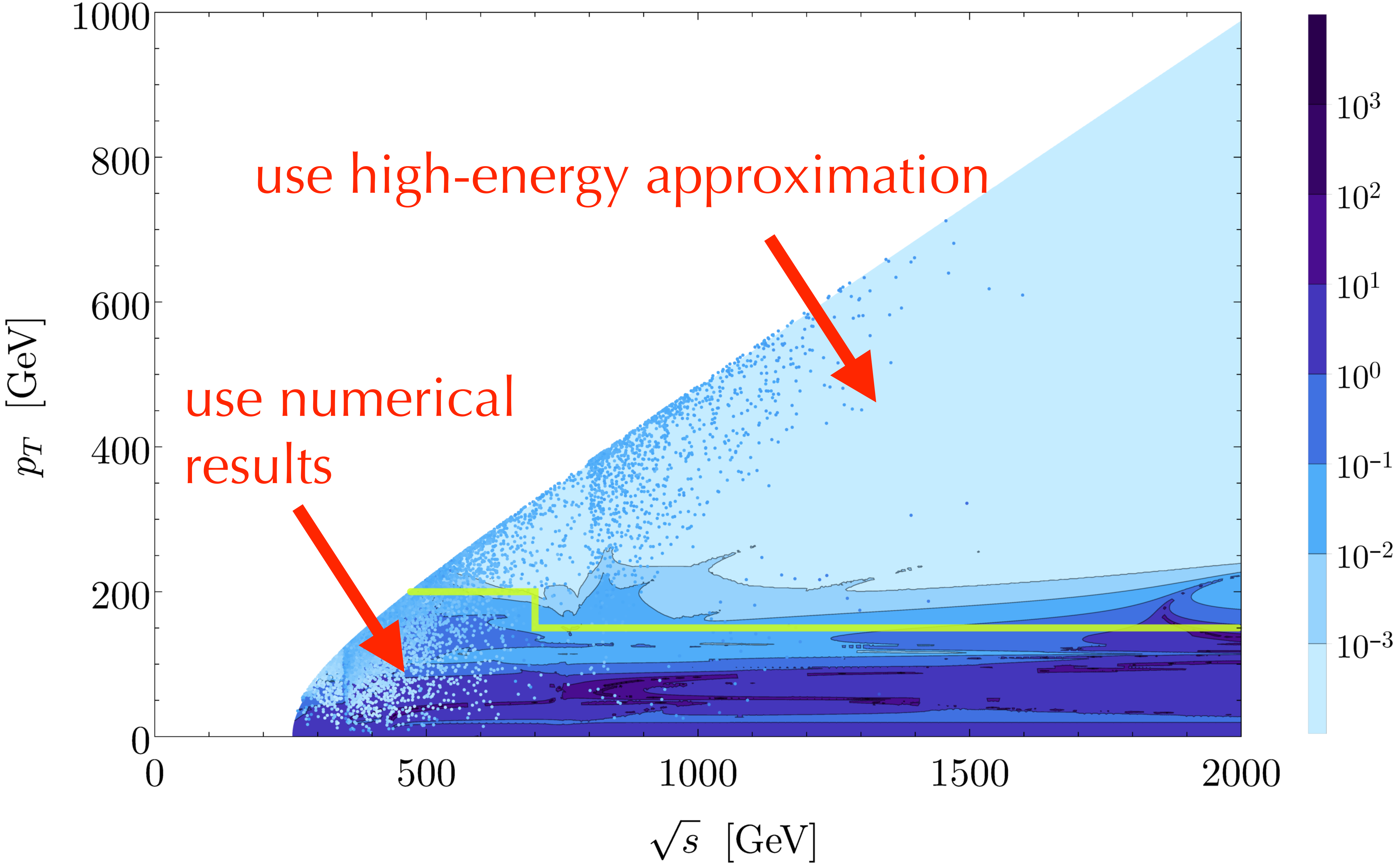


Higgs pair production at NLO

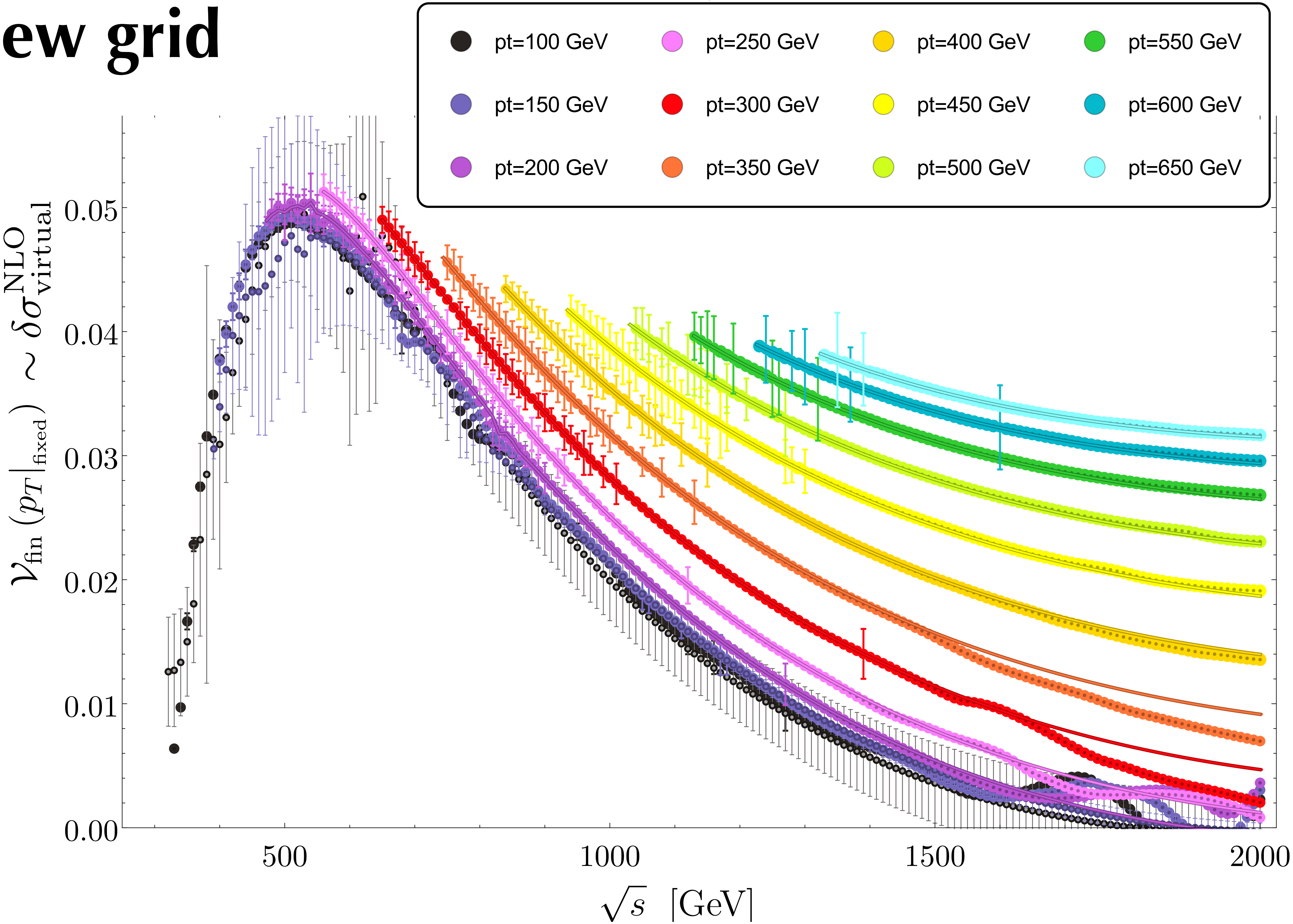
Previous grid



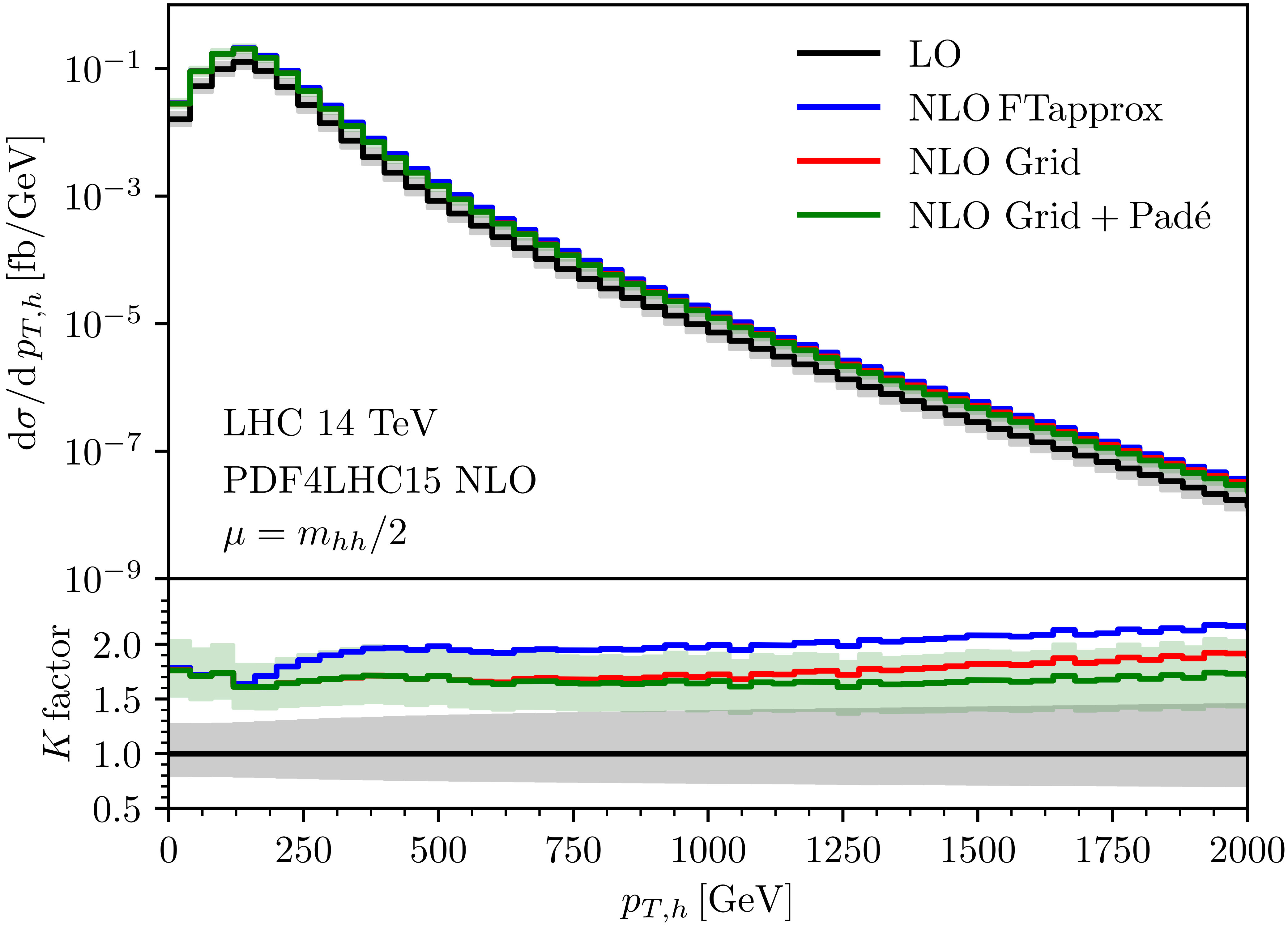
Combine HE approximation and numerics



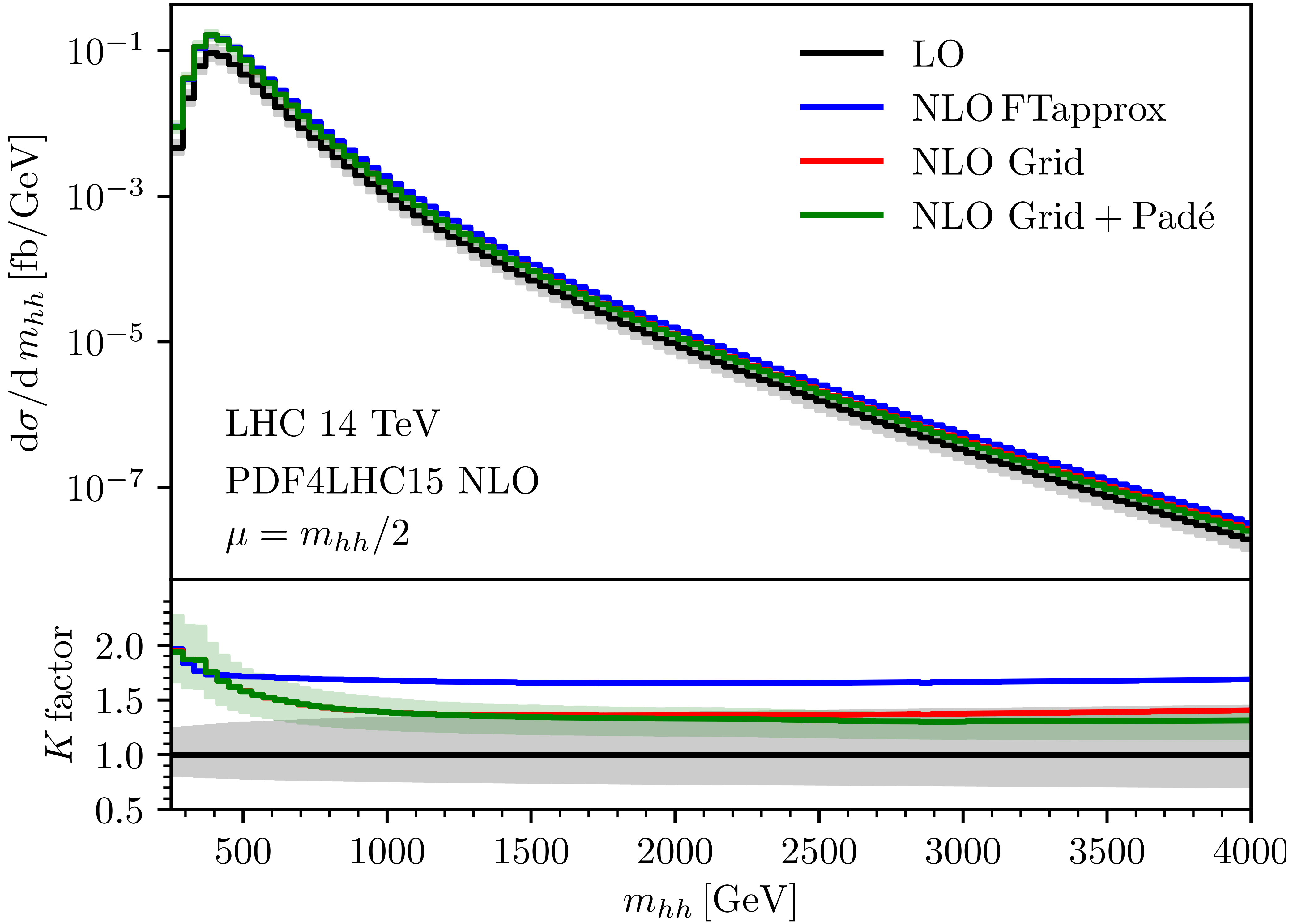
New grid


pt=100 GeV
pt=150 GeV
pt=200 GeV
pt=250 GeV
pt=300 GeV
pt=350 GeV
pt=400 GeV
pt=450 GeV
pt=500 GeV
pt=550 GeV
pt=600 GeV
pt=650 GeV
$\mathcal{V}_{\text{fin}}\left(p_T|_{\text{fixed}}\right) \sim \delta\sigma^{\text{NLO}}_{\text{virtual}}$
0.00
0.01
0.02
0.03
0.04
0.05
500
1000
1500
2000
$\sqrt{s}$ [GeV]

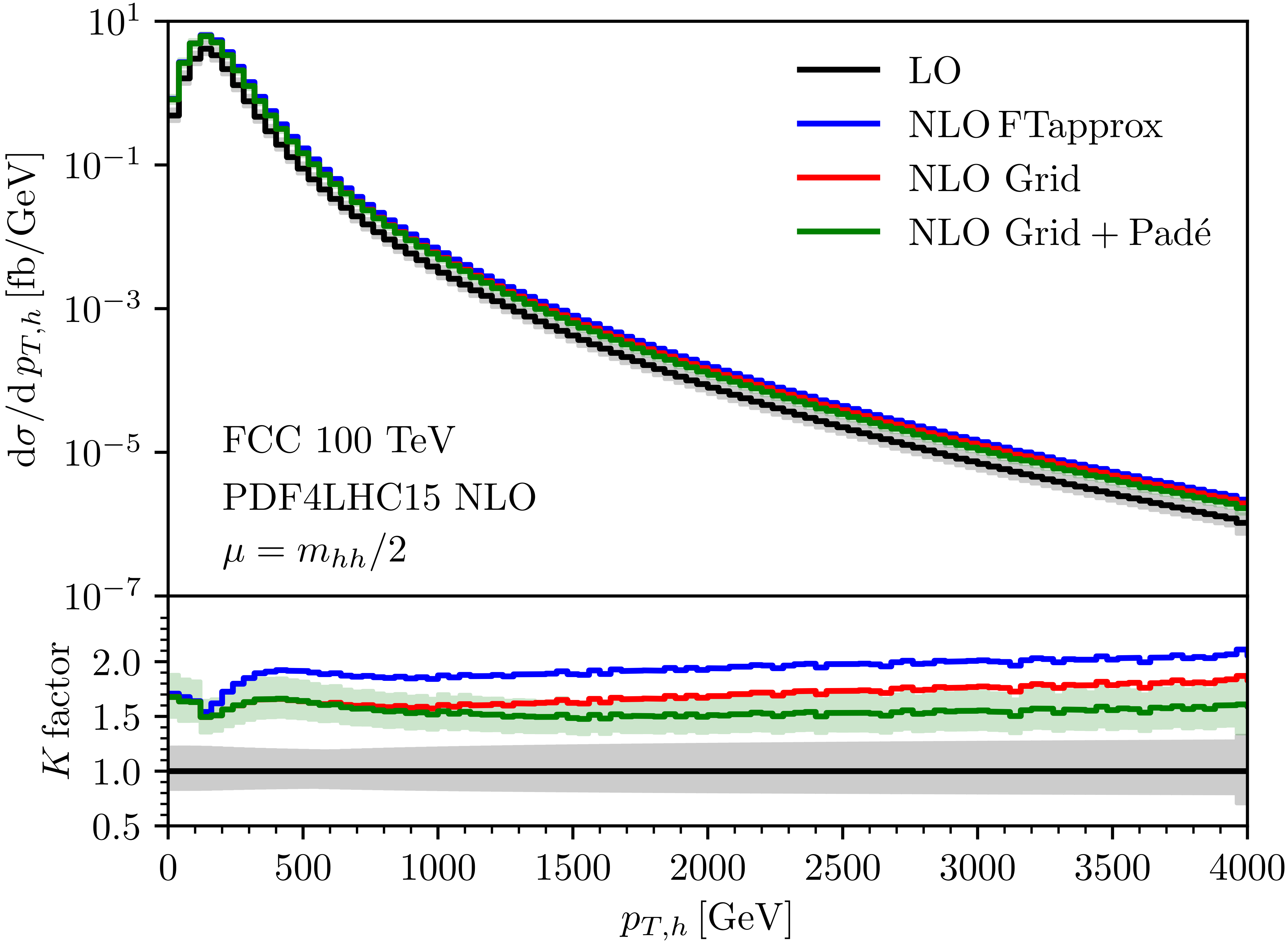
Result: p_T distribution at 14 TeV



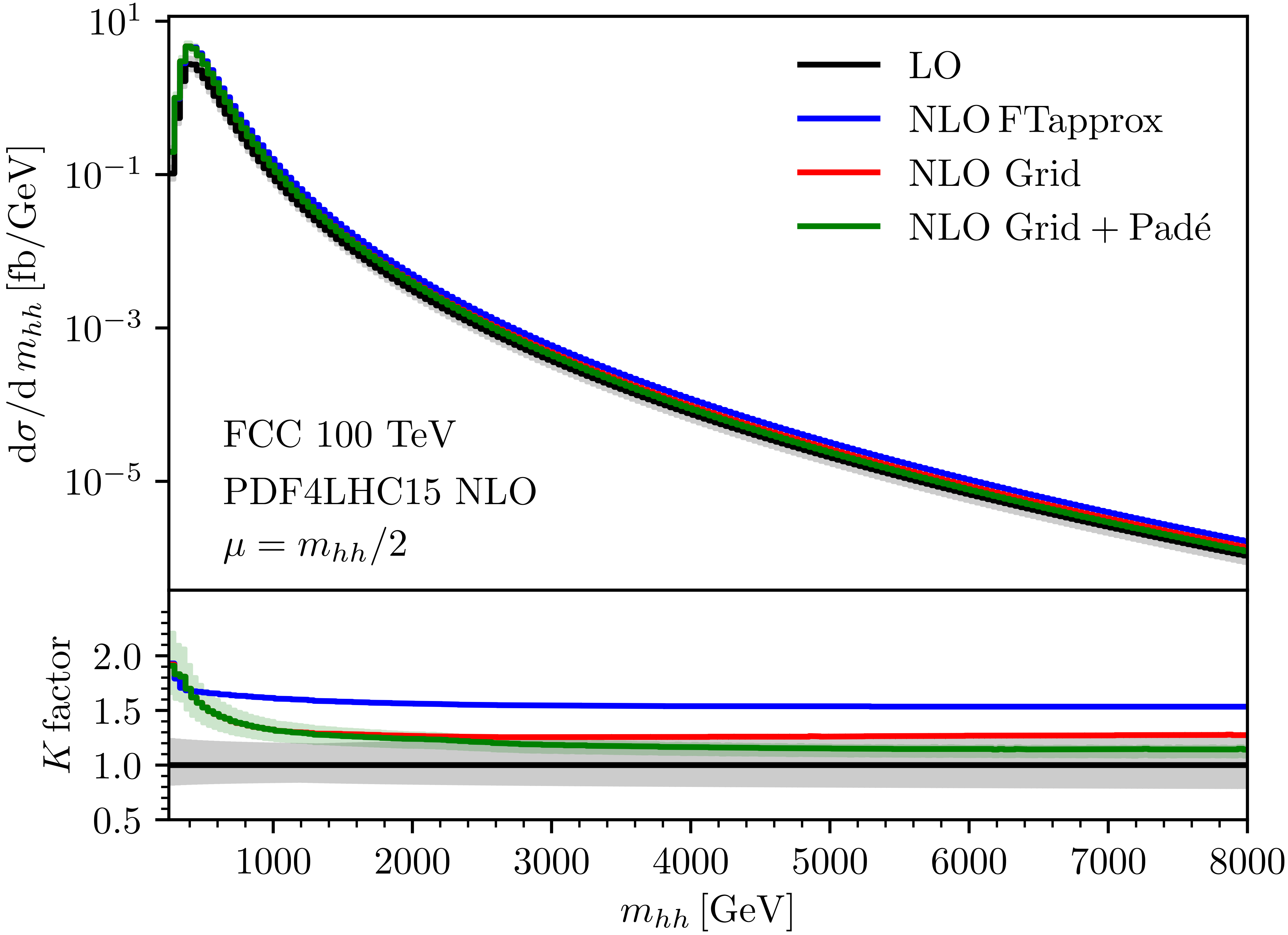
Result: m_{hh} distribution at 14 TeV



Result: p_T distribution at 100 TeV



Result: m_{hh} distribution at 100 TeV



Summary

- We have improved the NLO virtual corrections to the Higgs pair production cross section via gluon fusion by combining **numerical evaluation** and the **high-energy approximation**.

- We apply the method of regions to two-loop four-point functions for the first time.

- For the non-planar integrals, we solve the problem of the indefinite sign of F-function using an analytic continuation.

- **Numerical evaluation** and the **high-energy approximation** agree when $200\ \mathrm{GeV} < p_T < 400\ \mathrm{GeV},\ \sqrt{s} < 800\ \mathrm{GeV}$.

- **Padé improved high-energy approximation** provides reasonable results even down to $p_T \simeq 150\ \mathrm{GeV}$.

- The updated gird is available at https://github.com/mppmu/hhgrid