In Search of Explicitly Represented Solutions

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Outline

1. P-recursive sequences
2. Convolutions of Liouvillian sequences
3. Definite-sum solutions
4. Sieved polynomial bases

P-recursive sequences

Notation:

$\mathbb{K}$ ... algebraically closed field of characteristic 0

$\mathbb{K}^{\mathbb{N}}$ ... the set of *all sequences* over $\mathbb{K}$

Let $E: \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ be the *shift operator* w.r.t. n, acting on $a = \langle a_n \rangle_{n=0}^{\infty} \in \mathbb{K}^{\mathbb{N}}$ by

$$Ea_n = a_{n+1} \quad \text{for all } n \geq 0, \quad \text{or}$$

$$(a_0, a_1, a_2, \ldots) \mapsto (a_1, a_2, a_3, \ldots)$$

P-recursive sequences

Let $d \in \mathbb{N}$ and $p_0, p_1, \ldots, p_d \in \mathbb{K}[n]$, $p_d \neq 0$. The operator $L : \mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$ defined by

$$L = \sum_{j=0}^{d} p_j E^j$$

is a *linear recurrence operator* with polynomial coefficients, acting on $a \in \mathbb{K}^{\mathbb{N}}$ by

$$(La)_n = \sum_{j=0}^{d} p_j(n)\, a_{n+j}.$$

P-recursive sequences

Notation:

$\mathbb{K}[n]\langle E\rangle$ ... the *algebra of linear recurrence operators* with polynomial coefficients in n

$\mathcal{P}(\mathbb{K})$ ... the set of *P-recursive sequences* over $\mathbb{K}$

Definition

A sequence $a \in \mathbb{K}^{\mathbb{N}}$ is *P-recursive* if there is $L \in \mathbb{K}[n]\langle E\rangle$, $L \neq 0$, such that

$$(La)_n = 0$$

for all $n \geq 0$.

P-recursive sequences

Definition

A sequence $a \in \mathbb{K}^{\mathbb{N}}$ is *hypergeometric* if:

1. $\exists N \in \mathbb{N}: \; a_n \neq 0$ for all $n \geq N$,
2. $\exists p, q \in \mathbb{K}[n] \setminus \{0\}$:

$$p(n)\, a_{n+1} + q(n)\, a_n \; = \; 0$$

for all $n \geq 0$.

Notation: $\mathcal{H}(\mathbb{K})$ ... hypergeometric sequences in $K^{\mathbb{N}}$

P-recursive sequences

Theorem

$\mathcal{P}(\mathbb{K})$ is closed under the following unary operations $a \mapsto c$:

1. *shift:* $c_n = a_{n+1}$
2. *inverse shift:* $c_n = \begin{cases} a_{n-1}, & n \geq 1, \\ \lambda, & n = 0 \end{cases} \quad (\lambda \in \mathbb{K})$
3. *indefinite summation:* $c_n = \sum_{k=0}^{n} a_k$
4. *multisection:* $c_n = a_{kn+r} \quad (k \in \mathbb{N},\ 0 \leq r < k)$

P-recursive sequences

Theorem

$\mathcal{P}(\mathbb{K})$ is closed under the following binary operations $(a, b) \mapsto c$:

5. *addition:* $c_n = a_n + b_n$
6. *multiplication:* $c_n = a_n b_n$
7. *convolution:* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

P-recursive sequences

Theorem

$\mathcal{P}(\mathbb{K})$ is closed under the following variadic operation $(a^{(0)}, a^{(1)}, \ldots, a^{(m-1)}) \mapsto c$:

8. *interlacing:* $c_n = a_{n \operatorname{div} m}^{(n \bmod m)} \quad (m \in \mathbb{N})$

Example

The interlacing of $a, b \in \mathbb{K}^{\mathbb{N}}$ is the sequence

$$c = \langle a_0, b_0, a_1, b_1, a_2, b_2, \ldots \rangle$$

P-recursive sequences

Definition

$\mathcal{A}(\mathbb{K})$ is the least subring of $\mathcal{P}(\mathbb{K})$ containing $\mathcal{H}(\mathbb{K})$, closed under

- σ, σ^{-1},
- Σ.

The elements of $\mathcal{A}(\mathbb{K})$ are d'Alembertian sequences.

Example

Some d'Alembertian sequences:

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$
- Derangement numbers $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$
- $a_n = \frac{(n+1)!}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1}$

P-recursive sequences

Definition

$\mathcal{L}(\mathbb{K})$ is the least subring of $\mathcal{P}(\mathbb{K})$ containing $\mathcal{H}(\mathbb{K})$, closed under

- σ, σ^{-1},
- Σ,
- interlacing.

The elements of $\mathcal{L}(\mathbb{K})$ are Liouvillian sequences.

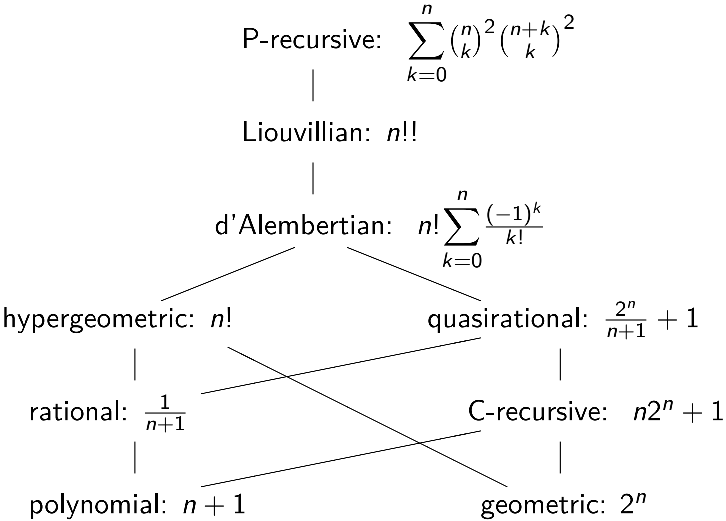
Example

The sequence

$$n!! = \begin{cases} 2^k k!, & n = 2k, \\ \frac{(2k+1)!}{2^k k!}, & n = 2k+1 \end{cases}$$

is Liouvillian (as an interlacing of two hypergeometric sequences).

P-recursive sequences



P-recursive sequences

Theorem

$\mathcal{L}(\mathbb{K})$ is closed under the following operations:

1. *shift,*
2. *inverse shift,*
3. *indefinite summation,*
4. *multisection,*
5. *addition,*
6. *multiplication,*
7. *interlacing.*

Convolutions of Liouvillian sequences

Question: What about *convolution* $(a * b)_n = \sum_{k=0}^{n} a_k b_{n-k}$?

Example

The convolution of $1/n!$ with itself

$$\frac{1}{n!} * \frac{1}{n!} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} = \frac{2^n}{n!}$$

is hypergeometric.

Convolutions of Liouvillian sequences

Example

The convolution of $n!$ with itself

$$c_n \;:=\; n! * n! \;=\; \sum_{k=0}^{n} k!(n-k)!$$

satisfies

$$2c_{n+1} - (n+2)c_n \;=\; 2(n+1)!, \qquad c_0 = 1$$

with solution

$$c_n \;=\; \frac{(n+1)!}{2^n} \sum_{k=0}^{n} \frac{2^k}{k+1}.$$

Note: c_n is d'Alembertian, not hypergeometric.

Convolutions of Liouvillian sequences

Example

The convolution of $n!$ with $1/n!$

$$c_n \;:=\; n! * \frac{1}{n!} \;=\; \sum_{k=0}^{n} \frac{k!}{(n-k)!}$$

satisfies

$$c_{n+2} - (n+2)c_{n+1} + c_n = \frac{1}{(n+2)!}.$$

Note: This equation has no nonzero Liouvillian solution, so $\mathcal{L}(\mathbb{K})$ is *not closed* under convolution!

Convolutions of Liouvillian sequences

Definition

$\mathcal{A}_{rat}(\mathbb{K})$ is the least subring of $\mathcal{P}(\mathbb{K})$ containing $\mathbb{K}(n)$, closed under

- σ, σ^{-1},
- Σ.

The elements of $\mathcal{A}_{rat}(\mathbb{K})$ are rationally d'Alembertian sequences.

Convolutions of Liouvillian sequences

Definition

$\mathcal{L}_{rat}(\mathbb{K})$ is the least subring of $\mathcal{P}(\mathbb{K})$ containing $\mathbb{K}(n)$, closed under

- σ, σ^{-1},
- Σ,
- interlacing.

The elements of $\mathcal{L}_{rat}(\mathbb{K})$ are rationally Liouvillian sequences.

Convolutions of Liouvillian sequences

Example

- Harmonic numbers $H_n = \sum_{k=1}^{n} \frac{1}{k}$ are rationally d'Alembertian.
- The interlacing of H_n with H_n is rationally Liouvillian.

Convolutions of Liouvillian sequences

Theorem

The convolution of a d'Alembertian sequence with a rationally d'Alembertian sequence is d'Alembertian.

Theorem

The convolution of a Liouvillian sequence with a rationally Liouvillian sequence is Liouvillian.

Convolutions of Liouvillian sequences

Definition

$CL(\mathbb{K})$ is the least subring of $\mathcal{P}(\mathbb{K})$ containing $\mathcal{H}(\mathbb{K})$, closed under

- σ, σ^{-1},
- interlacing,
- convolution.

Call its elements *Cauchy-Liouvillian (CL) sequences*.

Open problems:

- Construct an algorithm for finding solutions of LRE with polynomial coefficients which are *convolutions of hypergeometric sequences*.
- Construct an algorithm for finding solutions of LRE with polynomial coefficients which are *convolutions of Liouvillian sequences*.
- Construct an algorithm for finding *CL solutions* of LRE with polynomial coefficients.
- Is $CL(\mathbb{K}) \subset \mathcal{P}(\mathbb{K})$ or $CL(\mathbb{K}) = \mathcal{P}(\mathbb{K})$?

Inverse Zeilberger's problem

GIVEN $L \in \mathbb{K}[n]\langle E \rangle$,

FIND a sequence $F(n,k) \neq 0$ such that

$$\frac{F(n+1,k)}{F(n,k)}, \frac{F(n,k+1)}{F(n,k)} \in \mathbb{K}(n,k),$$

$$L\left(\sum_{k=0}^{n} F(n,k)\right) = 0.$$

Definite-sum solutions

A simpler version

Given $L \in \mathbb{K}[n]\langle E\rangle$ and a kernel $K(n,k)$,
find $h \in \mathbb{K}^{\mathbb{N}}$ such that

$$L\left(\sum_{k=0}^{n} h_k K(n,k)\right) = 0.$$

Solved only for some kernels $K(n,k)$.

Definite-sum solutions

Example

Take $K(n,k) = \binom{n}{k} \in \mathbb{K}[n]$. Then

$$\sum_{k=0}^{n} h_k K(n,k) = \sum_{k=0}^{\infty} h_k \binom{n}{k}.$$

Idea: Interpret $\sum_{k=0}^{\infty} h_k \binom{x}{k}$ as a *formal polynomial series*.

Definition

A sequence of polynomials $\mathcal{B} = \langle P_k(x) \rangle_{k=0}^{\infty}$ from $\mathbb{K}[x]$ is a factorial basis for $\mathbb{K}[x]$ if

(1) $\deg P_k = k$,

(2) $P_k \mid P_m$ for $k < m$.

Definite-sum solutions

Proposition

$\mathcal{B} = \langle P_k(x) \rangle_{k=0}^{\infty}$ *is a factorial basis* $\iff$

$$\exists c_0, c_1, \ldots \in \mathbb{K}^* \ \ \exists \rho_1, \rho_2, \rho_3, \ldots \in \mathbb{K} :$$

$$P_k(x) = c_k (x - \rho_1)(x - \rho_2) \cdots (x - \rho_k) \ \ \textit{for all } k \in \mathbb{N}.$$

Also known as a *sequence of polynomials with persistent roots* $\rho = (\rho_1, \rho_2, \rho_3, \ldots)$ in *umbral calculus*.

Notation:

$$\mathcal{L}(\mathbb{K}[x]) = \{L : \mathbb{K}[x] \to \mathbb{K}[x];\ L \text{ linear operator}\}$$

Definite-sum solutions

Example

Some factorial bases:

- $\mathcal{P} := \langle x^k \rangle_{k=0}^{\infty}$ *power basis*
- $\mathcal{F} := \langle x^{\underline{k}} \rangle_{k=0}^{\infty}$ *falling-factorial basis*
- $\mathcal{C} := \left\langle \binom{x}{k} \right\rangle_{k=0}^{\infty}$ *binomial-coefficient basis*
- $\mathcal{R} := \langle x^{\overline{k}} \rangle_{k=0}^{\infty}$ *rising-factorial basis*
- $\mathcal{A} := \left\langle \binom{x+k}{k} \right\rangle_{k=0}^{\infty}$ *Apéry basis*

Note:

$$\mathcal{C} = \left\langle \frac{x^{\underline{k}}}{k!} \right\rangle_{k=0}^{\infty}, \qquad \mathcal{A} = \left\langle \frac{(x+1)^{\overline{k}}}{k!} \right\rangle_{k=0}^{\infty}$$

Definite-sum solutions

Definition (ABP 1995)

An operator $L \in \mathcal{L}(\mathbb{K}[x])$ is compatible with a factorial basis $\mathcal{B}$ iff there are $A, B \in \mathbb{N}$ and $\alpha_{i,k} \in \mathbb{K}$ for $-A \leq i \leq B$, $k \geq 0$ s.t.

$$LP_k = \sum_{i=-A}^{B} \alpha_{i,k} P_{k+i} \tag{1}$$

where $P_j = 0$ if $j < 0$.

If (1) holds, L is (A, B)-compatible with $\mathcal{B}$.

Definite-sum solutions

Definition

Define $D, E, Q, X \in \mathcal{L}(\mathbb{K}[x])$ for $p \in \mathbb{K}[x]$ and $q \in \mathbb{K}^*$ by

$$\begin{array}{lcll} Dp(x) & := & p'(x) & \text{(differentiation)}, \\ Ep(x) & := & p(x+1) & \text{(shift)}, \\ Qp(x) & := & p(qx) & \text{(q-shift)}, \\ Xp(x) & := & xp(x) & \text{(multiplication by x)}. \end{array}$$

Example

- $(x^k)' = kx^{k-1} \quad \Longrightarrow \quad D$ is (1,0)-compatible with $\mathcal{P}$
- $(qx)^k = q^k x^k \quad \Longrightarrow \quad Q$ is (0,0)-compatible with $\mathcal{P}$
- $\binom{x+1}{k} = \binom{x}{k-1} + \binom{x}{k} \quad \Longrightarrow \quad E$ is (1,0)-compatible with $\mathcal{C}$

Definite-sum solutions

Example

- $(x+1)^k = \sum_{j=0}^{k} \binom{k}{j} x^j \implies E$ is not compatible with $\mathcal{P}$
- $(x+1)^{\overline{k}} = \sum_{j=0}^{k} \frac{k!}{j!} x^{\overline{j}} \implies E$ is not compatible with $\mathcal{R}$
- $\binom{x+1+k}{k} = \sum_{j=0}^{k} \binom{x+j}{j} \implies E$ is not compatible with $\mathcal{A}$
- $\binom{x}{k}' = \sum_{j=0}^{k-1} \frac{(-1)^{j+k}}{j-k} \binom{x}{j} \implies D$ is not compatible with $\mathcal{C}$

Example

- $xP_k(x) = u_k P_k(x) + v_k P_{k+1}(x)$

  $\implies X$ is (0,1)-compatible with every factorial basis

Definite-sum solutions

Lemma

E compatible with $\langle P_k(x)\rangle_{k=0}^{\infty} \iff$

$$\exists A \in \mathbb{N}\ \forall n \in \mathbb{N}\colon \{\rho_1{+}1, \rho_2{+}1, \ldots, \rho_n{+}1\} \subseteq \{\rho_1, \rho_2, \ldots, \rho_{n+A}\}$$

as multisets.

Example

- $P_k(x) = x^k$: $\rho = (0, 0, 0, \ldots)$, $\rho + 1 = (1, 1, 1, \ldots)$ X
- $P_k(x) = x^{\underline{k}}$: $\rho = (0, 1, 2, \ldots)$, $\rho + 1 = (1, 2, 3, \ldots)$ $\surd$
- $P_k(x) = x^{\overline{k}}$: $\rho = (0, -1, -2, \ldots)$, $\rho + 1 = (1, 0, -1, \ldots)$ X

Definite-sum solutions

Proposition

The algebra $\mathbb{K}[x]$ *naturally embeds into the algebra* $\mathbb{K}[[\mathcal{B}]]$ *of formal polynomial series in basis* $\mathcal{B}$ *of the form*

$$y = \sum_{k=0}^{\infty} c_k P_k(x) \qquad (c_k \in \mathbb{K}),$$

with multiplication defined by

$$\left(\sum_{k=0}^{\infty} c_k P_k(x)\right)\left(\sum_{k=0}^{\infty} d_k P_k(x)\right) = \sum_{k=0}^{\infty} e_k P_k(x),$$

$$e_k = \sum_{\max\{i,j\} \le k \le i+j} c_i d_j \, [P_k](P_i P_j).$$

Definite-sum solutions

Let $L \in \mathcal{L}(\mathbb{K}[x])$ be (A, B)-compatible with $\mathcal{B}$.

Extend $L : \mathbb{K}[x] \to \mathbb{K}[x]$ to $L : \mathbb{K}[[\mathcal{B}]] \to \mathbb{K}[[\mathcal{B}]]$ by defining

$$\begin{aligned} L\left(\sum_{k=0}^{\infty} c_k P_k(x)\right) &:= \sum_{k=0}^{\infty} c_k L P_k(x) \\ &= \sum_{k=0}^{\infty} c_k \sum_{i=-A}^{B} \alpha_{i,k} P_{k+i}(x) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=-B}^{A} \alpha_{-i,k+i} c_{k+i}\right) P_k(x) \end{aligned}$$

where $P_k(x) = 0$ for $k < 0$ and $c_{k-i} = 0$ for $i > k$.

Definite-sum solutions

Theorem

For any polynomial series $y = \sum_{k=0}^{\infty} c_k P_k(x) \in \mathbb{K}[[\mathcal{B}]]$ *we have*

$$L(y) = 0 \iff L'(c) = 0$$

where $L' = \mathcal{R}_{\mathcal{B}} L$ *is the operator induced by L in basis* $\mathcal{B}$*:*

$$\mathcal{R}_{\mathcal{B}} L := \sum_{i=-B}^{A} \alpha_{-i,k+i} E_k^i, \qquad (2)$$

$$E_k^i(c)_k = c_{k+i} \quad \text{for all } i, k \in \mathbb{Z},$$
$$c_j = 0 \quad \text{for } j < 0.$$

Definite-sum solutions

Definition

Let $\mathcal{B}$ be a factorial basis of $\mathbb{K}[x]$.

- $\mathcal{L}_\mathcal{B} := \{L \in \mathcal{L}(\mathbb{K}[x]);\ L \text{ compatible with } \mathcal{B}\}$,
- $\mathcal{E} := \{ \sum_{i=-S}^{R} a_k^{(i)} E_k^i;\ R, S \in \mathbb{N}, a^{(i)} \in \mathbb{K}^\mathbb{Z} \text{ for } -S \leq i \leq R\}$.

Proposition

$\mathcal{L}_\mathcal{B}$ and $\mathcal{E}$ are $\mathbb{K}$-algebras, and the transformation

$$\mathcal{R}_\mathcal{B} : \mathcal{L}_\mathcal{B} \to \mathcal{E}$$

is an isomorphism of $\mathbb{K}$-algebras.

Definite-sum solutions

Example

Differential operators:

$$\begin{aligned}\mathcal{R}_{\mathcal{P}} D &= (k+1)E_k \\ \mathcal{R}_{\mathcal{P}} X &= E_k^{-1}\end{aligned}$$

q-Difference operators:

$$\begin{aligned}\mathcal{R}_{\mathcal{P}} Q &= q^k \\ \mathcal{R}_{\mathcal{P}} X &= E_k^{-1}\end{aligned}$$

Recurrence operators:

$$\begin{aligned}\mathcal{R}_{\mathcal{C}} E &= E_k + 1 \\ \mathcal{R}_{\mathcal{C}} X &= k(E_k^{-1} + 1)\end{aligned}$$

Definite-sum solutions

Example

$$L = E^3 - (n^2 + 6n + 10)E^2 + (n+2)(2n+5)E - (n+1)(n+2)$$

$$\mathcal{R}_{\mathcal{C}} L = E_k^3 - (k^2 + 6k + 7)E_k^2 - (2k^2 + 8k + 7)E_k - (k+1)^2$$

Solution : $(\mathcal{R}_{\mathcal{C}} L)(k!^2) = 0 \Rightarrow L\left(\sum_{k=0}^{n} \binom{n}{k} k!^2\right) = 0.$

Sieved polynomial bases

Problem: We need more bases compatible with E.

Idea: Use *products* of compatible bases.

Example

$$\mathcal{B}^{(1)} = \mathcal{B}^{(2)} = \mathcal{C}, \quad P_k^{(1)}(x) = P_k^{(2)}(x) = \binom{x}{k} :$$

$$\begin{aligned} P_{2k}^{(\pi)}(x) &= P_k^{(1)}(x) P_k^{(2)}(x) = \binom{x}{k}^2 \\ P_{2k+1}^{(\pi)}(x) &= P_{k+1}^{(1)}(x) P_k^{(2)}(x) = \binom{x}{k+1}\binom{x}{k} \end{aligned}$$

Sieved polynomial bases

Definition

For $a \in \mathbb{N} \setminus \{0\}$, $b \in \mathbb{K}$, the *generalized binomial-coefficient basis* is defined by

$$\mathcal{C}_{a,b} := \left\langle \binom{ax+b}{k} \right\rangle_{k=0}^{\infty}$$

Proposition

E is $(a,0)$-compatible with $\mathcal{C}_{a,b}$.

Proof: By the Chu-Vandermonde identity,

$$\binom{a(x+1)+b}{k} = \sum_{i=0}^{a} \binom{a}{i}\binom{ax+b}{k-i} = \sum_{i=-a}^{0} \binom{a}{-i}\binom{ax+b}{k+i}.$$

Sieved polynomial bases

Definition

For $i = 1, 2, \ldots, m$, let $\mathcal{B}_i = \langle P_k^{(i)}(x) \rangle_{k=0}^{\infty}$ be a basis of $\mathbb{K}[x]$.
For all $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, m-1\}$, let

$$P_{mk+j}^{(\pi)}(x) \ := \ \prod_{i=1}^{j} P_{k+1}^{(i)}(x) \cdot \prod_{i=j+1}^{m} P_k^{(i)}(x).$$

Then $\prod_{i=1}^{m} \mathcal{B}_i := \langle P_n^{(\pi)}(x) \rangle_{n=0}^{\infty}$ is the *product* of $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$.

Sieved polynomial bases

Theorem

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m$ be factorial bases of $\mathbb{K}[x]$, and $L \in \mathcal{L}(\mathbb{K}[x])$.

1. $\prod_{i=1}^{m} \mathcal{B}_i$ *is a factorial basis of* $\mathbb{K}[x]$.
2. *Assume that E is $(A_i, 0)$-compatible with each $\mathcal{B}_i$. Let*

$$A = \max_{1 \le i \le m} A_i.$$

Then E is $(mA, 0)$-compatible with $\prod_{i=1}^{m} \mathcal{B}_i$.

Sieved polynomial bases

Example

$$\mathcal{B} \;=\; \mathcal{C}^2 \;=\; \left\langle \binom{x}{k} \right\rangle_{k=0}^{\infty} \cdot \left\langle \binom{x}{k} \right\rangle_{k=0}^{\infty} :$$

$$P_{2k}(x) = \binom{x}{k}^2, \quad P_{2k+1}(x) = \binom{x}{k+1}\binom{x}{k}$$

$$m = 2, \quad A = 1 \quad \Longrightarrow \quad E \text{ is (2,0)-compatible with } \mathcal{B}$$

$$\begin{aligned} P_{2k}(x+1) &= P_{2k}(x) + 2P_{2k-1}(x) + P_{2k-2}(x) \\ P_{2k+1}(x+1) &= P_{2k+1}(x) + \frac{2k+1}{k+1} P_{2k}(x) + \frac{k}{k+1} P_{2k-1}(x) \end{aligned}$$

Sieved polynomial bases

Example (cont'd)

X is $(0,1)-$compatible with $\mathcal{B}$:

$$\begin{aligned} x \cdot P_{2k}(x) &= (k+1)P_{2k+1}(x) + kP_{2k}(x), \\ x \cdot P_{2k+1}(x) &= (k+1)P_{2k+2}(x) + kP_{2k+1}(x). \end{aligned}$$

Problems:

1. The coefficients of $\mathcal{R}_{\mathcal{B}}L$ need not belong to $\mathbb{K}(k)$.
2. $\operatorname{ord} \mathcal{R}_{\mathcal{B}}L$ may exceed $\operatorname{ord} L$ by a factor of mA.
3. We need only those $h \in \ker \mathcal{R}_{\mathcal{B}}L$ satisfying

$$k \not\equiv 0 \pmod{m} \implies h_k = 0.$$

Sieved polynomial bases

Definition

Call the sequence $b \in \mathbb{K}^{\mathbb{N}}$ defined by

$$b_k = a_{mk+j} \text{ for all } k \in \mathbb{N}$$

the *j-th m-section* of $a \in \mathbb{K}^{\mathbb{N}}$, and denote it by $s_j^m a$.

Notation

For

$$y(x) = \sum_{k=0}^{\infty} c_k P_k(x) \in \mathbb{K}[[\mathcal{B}]]$$

let $\sigma_{\mathcal{B}} y = c' \in \mathbb{K}^{\mathbb{Z}}$ *where*

$$c'_k = \begin{cases} c_k, & \textit{if } k \geq 0, \\ 0, & \textit{if } k < 0. \end{cases}$$

Sieved polynomial bases

Theorem

For $L \in \mathcal{L}_{\mathbb{K}[x]}$, $k \in \mathbb{N}$, $m \in \mathbb{N} \setminus \{0\}$, $r, j \in \{0, \ldots, m-1\}$, *let*

$$L_{r,j} \; := \sum_{\substack{-A \le i \le B \\ i+j \equiv r \pmod m}} \alpha_{k+\frac{r-i-j}{m},j,i} E_k^{\frac{r-i-j}{m}} \; \in \; \mathcal{E}$$

where

$$LP_{mk+j}(x) \; = \; \sum_{i=-A}^{B} \alpha_{k,j,i} P_{mk+j+i}(x).$$

Then for every $y \in \mathbb{K}[[\mathcal{B}]]$ *and* $r \in \{0, 1, \ldots, m-1\}$,

$$s_r^m \sigma_{\mathcal{B}}(Ly) \; = \; \sum_{j=0}^{m-1} L_{r,j} \left(s_j^m \sigma_{\mathcal{B}} y \right).$$

Sieved polynomial bases

Corollary

$Ly = 0 \iff \forall r \in \{0, 1, \ldots, m-1\}$:

$$\sum_{j=0}^{m-1} L_{r,j} \left(s_j^m \sigma_{\mathcal{B}} y \right) = 0.$$

Notation

$$[\mathcal{R}_{\mathcal{B}} L] \ := \ [L_{r,j}]_{r,j=0}^{m-1}$$

Proposition

$$\left[\mathcal{R}_{\mathcal{B}} \left(L^{(1)} L^{(2)} \right)\right] \ = \ \left[\mathcal{R}_{\mathcal{B}} L^{(1)}\right] \left[\mathcal{R}_{\mathcal{B}} L^{(2)}\right]$$

Sieved polynomial bases

- To construct $[\mathcal{R}_{\mathcal{B}} L]$ for some $L \in \mathbb{K}[x]\langle E \rangle$:
  1. compute $[\mathcal{R}_{\mathcal{B}} E]$ and $[\mathcal{R}_{\mathcal{B}} X]$;
  2. everywhere in L substitute
     - $E \mapsto [\mathcal{R}_{\mathcal{B}} E]$,
     - $x \mapsto [\mathcal{R}_{\mathcal{B}} X]$,
     - $1 \mapsto I_m$.

- We are looking for $y \in \ker L$ of the form

$$y(x) \;=\; \sum_{k=0}^{\infty} h_k P_{mk}(x),$$

so we have $s_0^m \sigma y = h$ and $s_j^m \sigma y = 0$ for all $j \neq 0$.

- For such y, the last Corollary implies

$$
\begin{aligned}
Ly = 0 \iff & \forall r \in \{0,1,\ldots,m-1\}: L_{r,0}\, h &=& \ 0 \\
\iff & \operatorname{gcrd}(L_{0,0}, L_{1,0}, \ldots, L_{m-1,0})\, h &=& \ 0.
\end{aligned}
$$

- So we only need column 0 of $[\mathcal{R}_{\mathcal{B}} L]$ to construct the desired annihilator

$$L' \ = \ \operatorname{gcrd}(L_{0,0}, L_{1,0}, \ldots, L_{m-1,0})$$

of the unknown h.

Sieved polynomial bases

This approach sometimes generalizes to kernels of the form

$$F(n,k) = \prod_{i=1}^{m} \binom{a_i n + b_i k + c_i}{e_i k + f_i}$$

where $a_i, e_i \in \mathbb{N} \setminus \{0\}$, $b_i, f_i \in \mathbb{N}$, $c_i \in \mathbb{K}$.

Example

Let $F(n,k) = \binom{n}{k}\binom{n+k}{2k}$. Take

$$P_{3k}(x) = \binom{x}{k}\binom{x+k}{2k},$$

Sieved polynomial bases

Example (cont'd)

$$P_{3k+1}(x) = \binom{x}{k}\binom{x+k}{2k+1},$$

$$P_{3k+2}(x) = \binom{x}{k+1}\binom{x+k}{2k+1}.$$

More concisely,

$$P_k(x) = \binom{x}{\lfloor \frac{k+1}{3} \rfloor}\binom{x+\lfloor \frac{k}{3} \rfloor}{\lfloor \frac{2k+1}{3} \rfloor}.$$

Sieved polynomial bases

Example (cont'd)

As $\mathcal{B} = \langle P_k(x) \rangle_{k=0}^{\infty}$ *is factorial,* X *is* $(0,1)$*-compatible with* $\mathcal{B}$*:*

$$\begin{aligned} xP_{3k}(x) &= (2k+1)P_{3k+1}(x) + kP_{3k}(x) \\ xP_{3k+1}(x) &= (k+1)P_{3k+2}(x) + kP_{3k+1}(x) \\ xP_{3k+2}(x) &= 2(k+1)P_{3k+3}(x) - (k+1)P_{3k+2}(x) \end{aligned}$$

Generalizations 4

Example (cont'd)

Also E *is* $(3,0)$*-compatible with* $\mathcal{B}$*:*

$$\begin{aligned} P_{3k}(x+1) &= P_{3k}(x) \\ &+ \frac{3}{2}P_{3k-1}(x) + \frac{8k-3}{2k}P_{3k-2}(x) + P_{3k-3}(x) \\ P_{3k+1}(x+1) &= P_{3k+1}(x) \\ &+ \frac{3k+1}{2k+1}P_{3k}(x) + \frac{k}{2k+1}P_{3k-1}(x) + \frac{2k-1}{2k+1}P_{3k-2}(x) \\ P_{3k+2}(x+1) &= P_{3k+2}(x) + \frac{3k+2}{k+1}P_{3k+1}(x) + P_{3k}(x) \end{aligned}$$

Sieved polynomial bases

Example (cont'd)

The associated operator matrices are:

$$[\mathcal{R}_{\mathcal{B}}E] = \begin{bmatrix} E_k + 1 & \frac{3k+1}{2k+1} & 1 \\ \frac{8k+5}{2(k+1)}E_k & \frac{2k+1}{2k+3}E_k + 1 & \frac{3k+2}{k+1} \\ \frac{3}{2}E_k & \frac{k+1}{2k+3}E_k & 1 \end{bmatrix},$$

$$[\mathcal{R}_{\mathcal{B}}X] = \begin{bmatrix} k & 0 & 2k\,E_k^{-1} \\ 2k+1 & k & 0 \\ 0 & k+1 & -(k+1) \end{bmatrix}$$

Take

$$L := (n+2)^2E^2 - (11n^2 + 33n + 25)E - (n+1)^2.$$

Then:

Sieved polynomial bases

Example (cont'd)

$$L_{0,0} = (k+2)^2 E_k^2 + \frac{29k^3 + 46k^2 + 14k - 1}{2k+1} E_k - 2(37k^2 + 41k + 11),$$

$$L_{1,0} = \frac{(k+2)(4k+5)(12k^2+26k+11)}{2(k+1)(2k+3)} E_k^2 - \frac{79 + 237k + 199k^2 + 47k^3}{2(1+k)} E_k - (2k+1)(49k+31),$$

$$L_{2,0} = \frac{(k+2)(22k^2+62k+43)}{2(2k+3)} E_k^2 - \frac{3}{2}(11k^2 + 34k + 25) E_k - 11(k+1)(2k+1),$$

Example (cont'd)

and

$$\mathrm{gcrd}(L_{0,0}, L_{1,0}, L_{2,0}) = E_k - 2\,\frac{2k+1}{k+1}.$$

So $h_k = \binom{2k}{k}$ *satisfies* $L_{0,0}h_k = L_{1,0}h_k = L_{2,0}h_k = 0$. *From*

$$h_k\binom{n+k}{2k} = \binom{2k}{k}\binom{n+k}{2k} = \binom{n}{k}\binom{n+k}{k}$$

it follows that

$$h_k P_{3k}(n) = \binom{2k}{k}\binom{n}{k}\binom{n+k}{2k} = \binom{n}{k}^2\binom{n+k}{k}.$$

Example (cont'd)

Therefore Apéry's $\zeta(2)$-sequence

$$y_n = \sum_{k=0}^{\infty} h_k P_{3k}(n) = \sum_{k=0}^{\infty} \binom{n}{k}^2 \binom{n+k}{k}$$

satisfies

$$(n+2)^2 y_{n+2} - (11n^2 + 33n + 25) y_{n+1} - (n+1)^2 y_n = 0.$$