The High-Energy Limit of 2-to-2 Partonic Scattering Amplitudes

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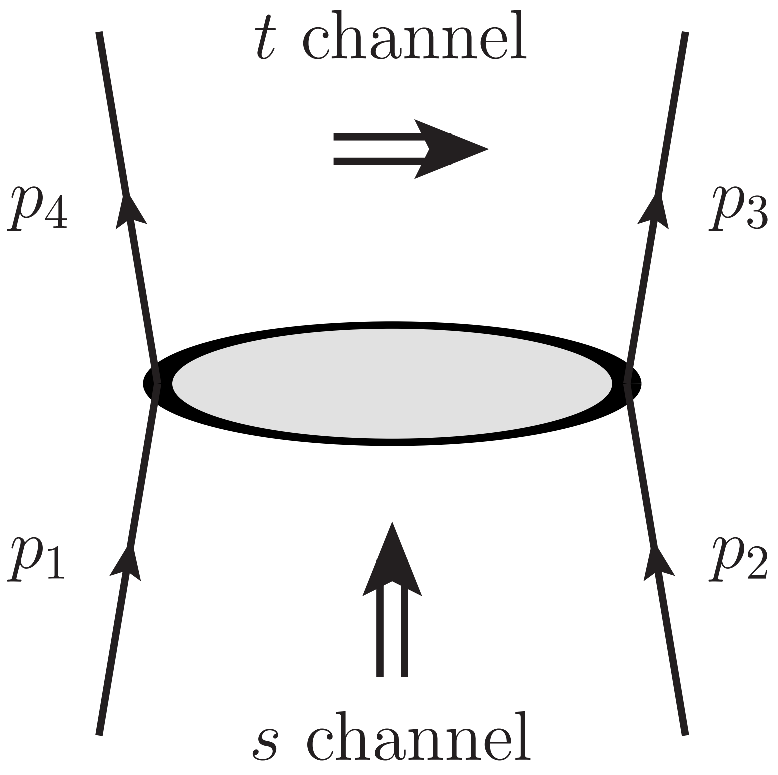
work in collaboration with Simon Caron-Huot, Joscha Reichel and Leonardo Vernazza

Motivation



✓ The High-energy limit is a rich source of experimentally-accessible physical phenomena, e.g.

- total cross section;
- jets at high rapidities;
- high gluon densities

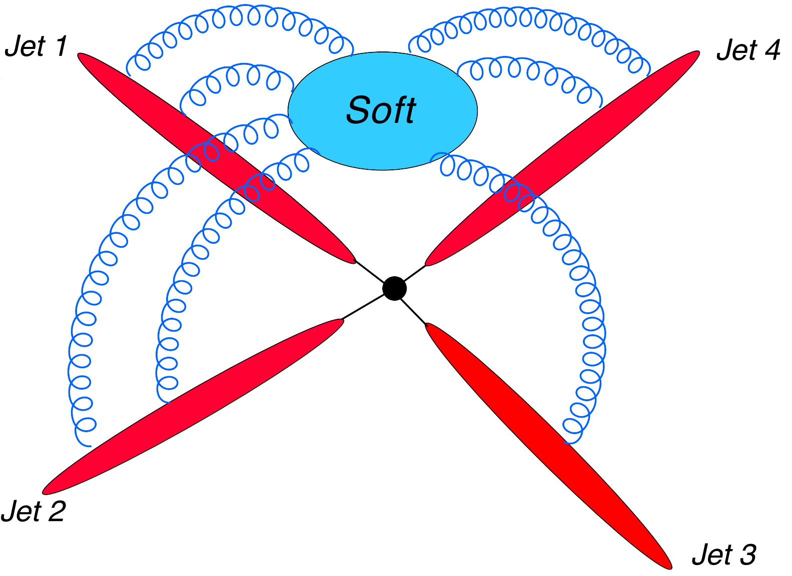
✓ QCD dynamics simplifies, allowing systematic theoretical study using Wilson lines and evolution equations

✓ Unique access to all-order properties of scattering amplitudes — complementing the study of IR divergences and summation of perturbation theory

The Regge limit overlap with infrared singularities

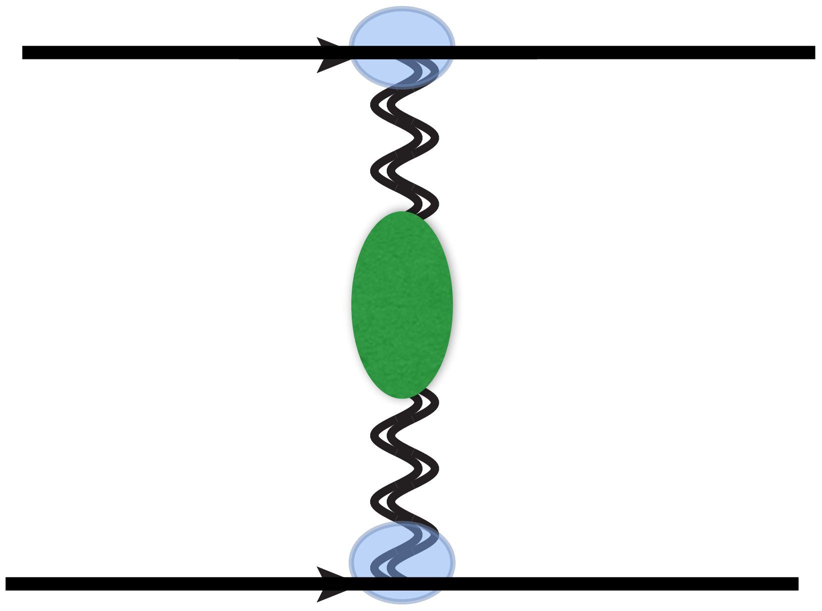
Exponentiation of infrared (soft) singularities $1/\epsilon$
in fixed-angle amplitude using factorisation into
Soft-Collinear-Hard subprocesses

IR



High-energy limit
of the soft
anomalous dimension

= Soft limit of BFKL



High-Energy limit:
BFKL resummation of log(s/t)
Regge factorization

Regge limit

The High-Energy Limit of 2 to 2 Partonic Scattering Amplitudes

Abstract

Recently, there has been significant progress in computing scattering amplitudes in the high-energy limit using rapidity evolution equations. I describe the state-of-the-art and demonstrate the interplay between exponentiation of high-energy logarithms and that of infrared singularities. The focus in this talk is the imaginary part of 2-to-2 partonic amplitudes, which can be determined by solving the BFKL equation. I demonstrate that the wavefunction is infrared finite, and that its evolution closes in the soft approximation. Within this approximation I derive a closed-form solution for the amplitude in dimensional regularization, which fixes the soft anomalous dimension to all orders at NLL accuracy.

I then turn to finite contributions of the amplitude and show that the remaining `hard’ contributions can be determined algorithmically, by iteratively solving the BFKL equation in exactly two dimensions within the class of single-valued harmonic polylogarithms.

To conclude I present numerical results and analyse the large-order behaviour of the amplitude.

The High-Energy Limit of 2 to 2 Scattering Amplitudes

Outline

- Signature and reality properties
- Colour in the high-energy limit
- Gluon Reggization and the odd amplitude (Regge pole)
- Multiple Reggeon exchange (Regge cuts)
- The even amplitude
    - BFKL equation in dimensional regularization
    - Attempting an iterative solution
    - The soft approximation
    - The soft anomalous dimensions to all orders
    - Two-dimensional solution in the space of SVHPLs
    - Numerical results and asymptotic behaviour
- Conclusions

The High-Energy Limit in 2-to-2 Scattering

From the dispersion representation of the amplitude

$$\mathcal{M}(s,t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}-s-i0}\, D_s(\hat{s},t) + \frac{1}{\pi}\int_0^\infty \frac{d\hat{u}}{\hat{u}+s+t-i0}\, D_u(\hat{u},t)$$

with the reality property of the discontinuities, it follows:

Amplitudes of a given signature $\mathcal{M}^{(\pm)}(s,t) = \frac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big)$

are, respectively:

$$\mathcal{M}^{(+)}(s,t) \qquad \text{real}$$

$$\mathcal{M}^{(-)}(s,t) \qquad \text{imaginary}$$

when expressed in terms of the signature-even logarithm:

$$\begin{aligned} L &\equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) \\ &= \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} \end{aligned}$$

The High-Energy Limit in 2-to-2 Scattering: colour and signature

The high-energy limit is dominated by **t-channel exchange, helicity-conserving** configuration.

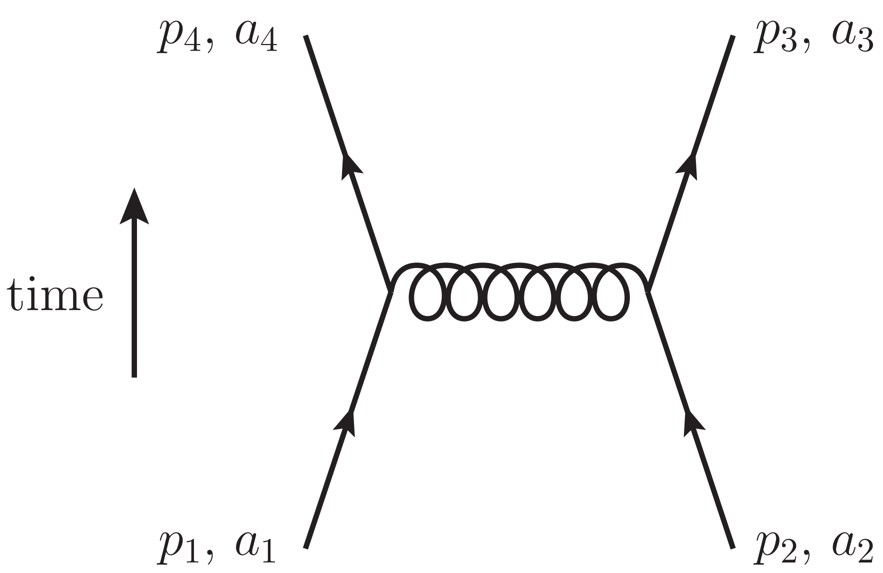
The leading-order amplitude is a **t-channel** gluon exchange, corresponding to an antisymmetric octet representation. It has **odd signature**:

odd under s ⟷ u

helicity is conserved

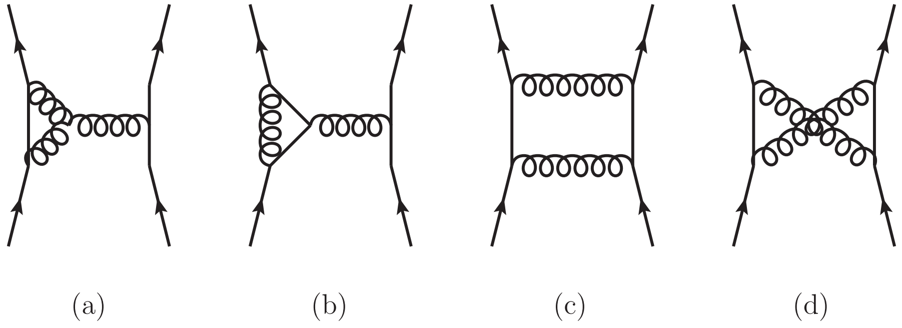
$$\mathcal{M}^{(\text{tree})}_{ij\to ij} = \mathcal{M}^{(\text{tree})\,(-)}_{ij\to ij} = g_s^2\,\frac{2s}{t}\,(\mathbf{T}_i^b)_{a_1a_4}(\mathbf{T}_j^b)_{a_2a_3}\,\delta_{\lambda_1\lambda_4}\delta_{\lambda_2\lambda_3}$$

$$\mathcal{M}^{(\text{tree})\,(+)}_{ij\to ij} = 0$$



At higher orders (and beyond LL accuracy) it's useful to decompose the amplitude using a t-channel colour basis:

$$\mathcal{M}(s,t) = \sum_i c^{[i]}\,\mathcal{M}^{[i]}(s,t)$$



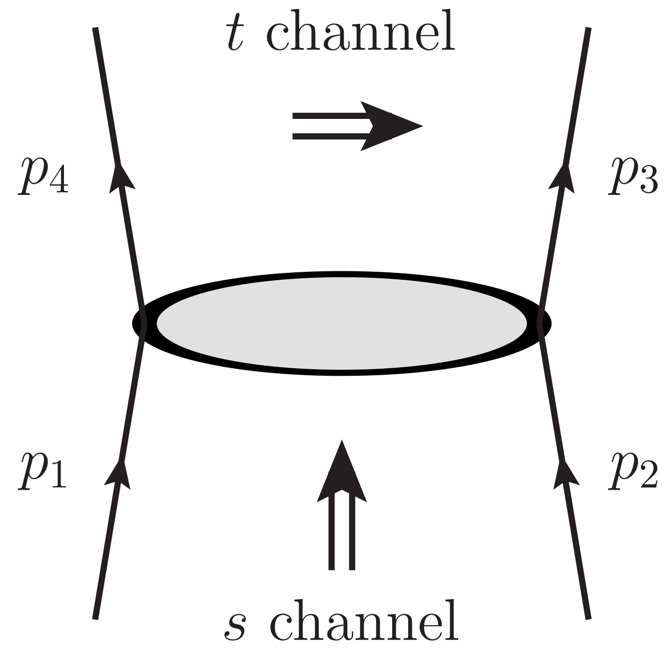
qq, qg scattering: odd: $\mathcal{M}^{[8_a]}$, even: $\mathcal{M}^{[1]}$, $\mathcal{M}^{[8_s]}$

gg scattering: odd: $\mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}$, even: $\mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]}, \mathcal{M}^{[0]}$

The High-Energy Limit in 2-to-2 Scattering: colour and signature

Here, instead of using a particular colour-flow basis,
we use colour operators, acting as generators on a given parton:

$$\begin{cases} \mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{T}_3 - \mathbf{T}_4 \\ \mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_3 = -\mathbf{T}_2 - \mathbf{T}_4 \\ \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_4 = -\mathbf{T}_2 - \mathbf{T}_3 \end{cases}$$



Using the colour conservation: $(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4)\,\mathcal{M} = 0$

One obtains $\mathbf{T}_s^2 + \mathbf{T}_u^2 + \mathbf{T}_t^2 = \sum_{i=1}^{4} C_i$ = sum over the quadratic Casimirs

This leaves just two independent quadratic operators: $\mathbf{T}_t^2$ is even,

$$\mathbf{T}_{s-u}^2 \equiv \frac{\mathbf{T}_s^2 - \mathbf{T}_u^2}{2} \text{ is odd}$$

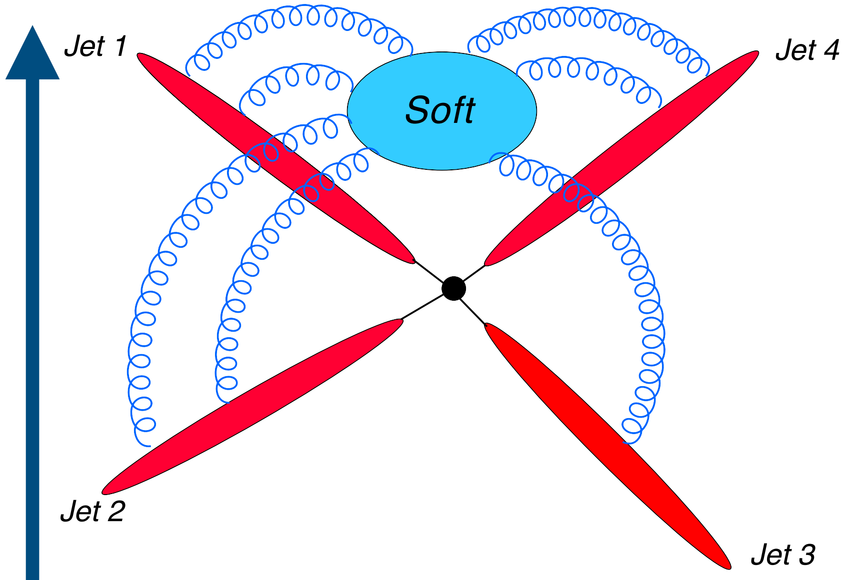
The signature-even amplitude is characterised by
the odd colour operator, acting on the tree amplitude:

$$\mathcal{M}_{\rm NLL}^{(+)} \simeq i\pi \left[\frac{1}{2\epsilon} \frac{\alpha_s}{\pi} + \mathcal{O}\left(\alpha_s^2 L\right) \right] \mathbf{T}_{s-u}^2 \mathcal{M}^{(\rm tree)}$$

The Regge limit overlap with infrared singularities

Exponentiation of infrared (soft) singularities $1/\epsilon$ in fixed-angle amplitude using factorisation into Soft-Collinear-Hard subprocesses

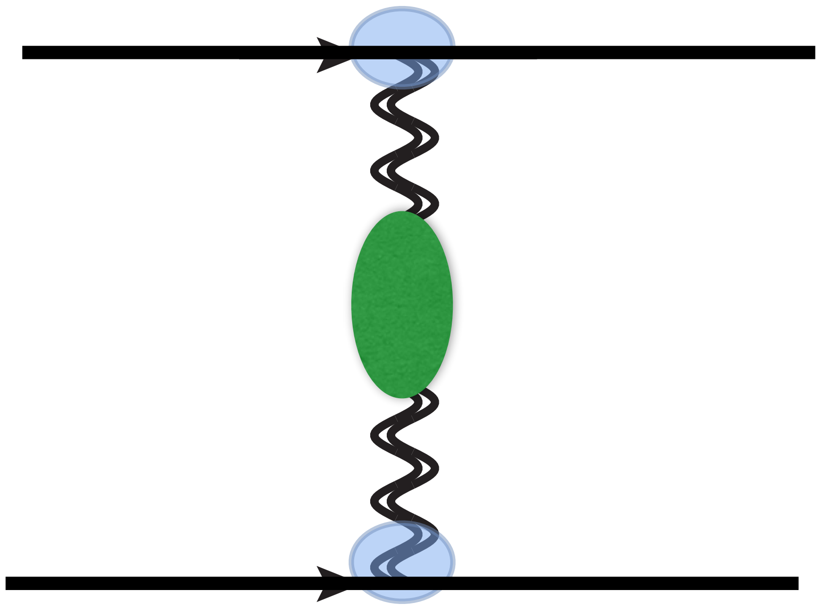
IR



High-energy limit of the soft anomalous dimension

= Soft limit of BFKL

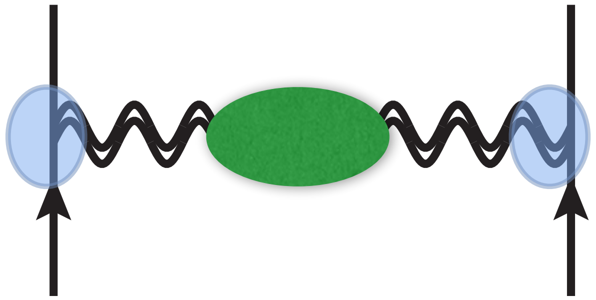
Regge limit



High-Energy limit:
BFKL resummation of log(s/t)
Regge factorization

The High-energy limit: A Reggeized gluon

Leading logs of (-t/s) are summed through gluon Reggeization:



$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

fully consistent with the dipole formula for IR singularities:

$$\alpha(t) = \frac{1}{4}\mathbf{T}_t^2 \int_0^{-t} \frac{d\lambda^2}{\lambda^2}\widehat{\gamma}_K(\alpha_s(\lambda^2,\epsilon))$$

Korchemsky (1993)
Korchemskaya and Korchemsky (1996)
Del Duca, Duhr, EG, Magnea & White (2011)

For the **Real part** of the amplitude, this "Regge pole" factorization can be improved to NLL by introducing impact factors and corrections to the trajectory.
— **but** beyond this the exchange of multiple Reggeized gluons kick in!

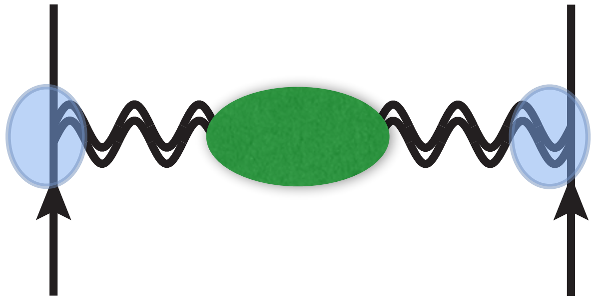
Del Duca, Glover (2001), Del Duca, Falcioni, Magnea, Vernazza (2013)

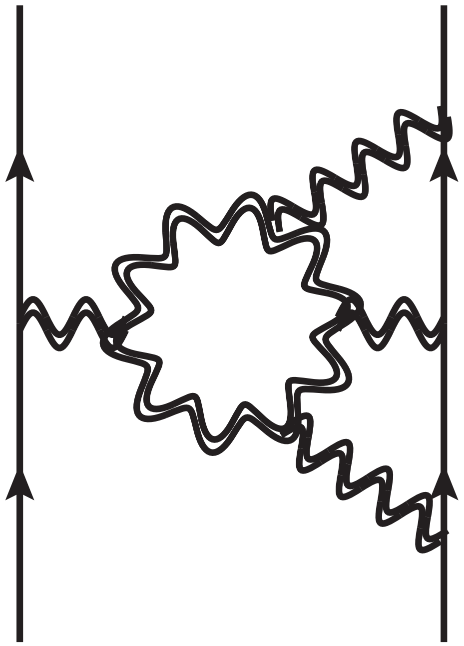
High-energy limit: exchange of multiple Reggeized gluons

Recent progress: we now know to use JIMWLK/BFKL rapidity evolution to **compute** multiple Reggeized gluon contributions to 2-to-2 amplitudes.

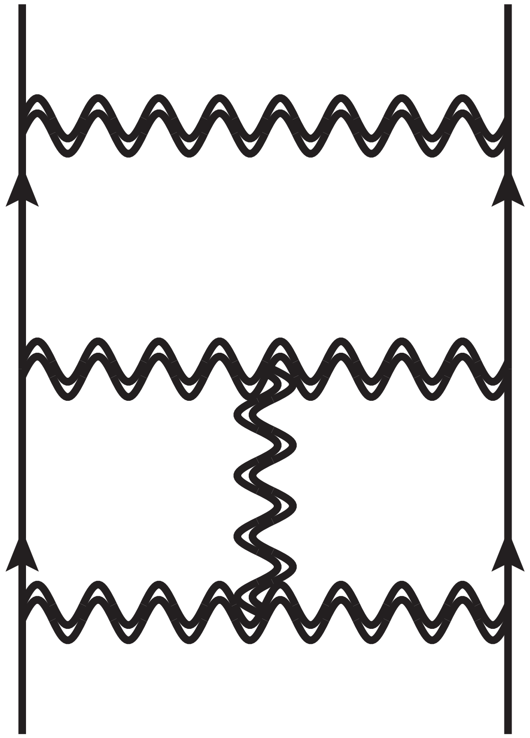
Signature odd (Real) part of the amplitude

LL, NLL





NNLL

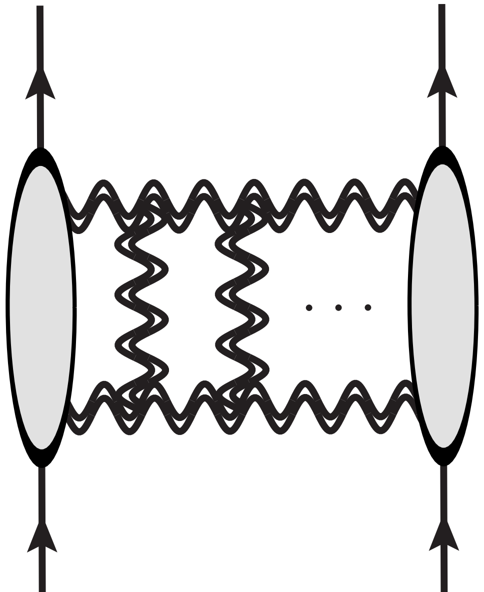


Caron-Huot, EG, Vernazza - JHEP 06 (2017) 016 — checked against Henn & Mistlberger (2017)

Signature even (Imaginary) amplitude

Caron-Huot JHEP 05 (2015) 093

Caron-Huot, EG, Reichel, Vernazza - JHEP 1803 (2018) 098 (and on-going work)



NLL

IR Singularities for amplitudes with massless legs

Exponentiation of IR singularities in fixed-angle scattering:

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\lambda,\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

The Dipole Formula:

$$\Gamma_{\text{Dip.}}(\lambda,\alpha_s) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right)\sum_{(i,j)}\ln\left(\frac{\lambda^2}{-s_{ij}}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s\right)$$

Lightlike Cusp anomalous dimension

Catani (1998)
Dixon, Mert-Aybat and Sterman (2006)
Becher & Neubert, EG & Magnea (2009)

There are two types of **corrections to the dipole formula**:

1. Corrections induced by higher Casimir contributions to the cusp anomalous dimension — starting at 4 loops.
2. Functions of **conformally-invariant cross ratios** — starting at 3-loops:

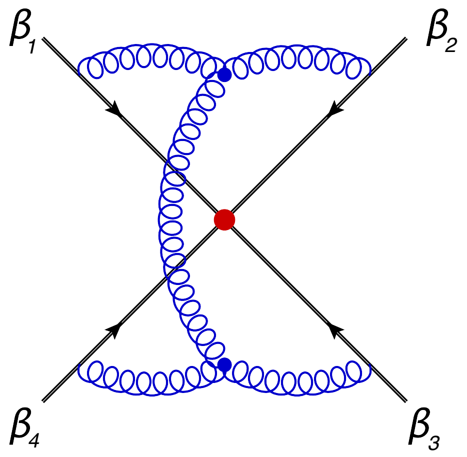
$$\Gamma = \Gamma_{\text{Dip.}} + \Delta(\rho_{ijkl})$$

$$\rho_{ijkl} = \frac{(p_i\cdot p_j)(p_k\cdot p_l)}{(p_i\cdot p_k)(p_j\cdot p_l)}$$

The three-loop correction to the soft anomalous dimension

Ø. Almelid, C. Duhr, EG
Phys. Rev. Lett. **117**, 172002

$$\Delta_n^{(3)}(z,\bar{z}) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 f_{abe}f_{cde}\Bigg\{\sum_{1\leq i<j<k<l\leq n}\Big[\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\left(F\left(1-1/z\right)-F\left(1/z\right)\right)$$
$$+\mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\left(F(1-z)-F(z)\right)$$
$$+\mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\left(F\left(1/(1-z)\right)-F\left(1-1/(1-z)\right)\right)\Big]$$
$$-\sum_{i=1}^{n}\sum_{\substack{1\leq j<k\leq n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\left(\zeta_5+2\zeta_2\zeta_3\right)\Bigg\}$$



$$\boxed{F(z)=\mathcal{L}_{10101}(z)+2\zeta_2\left(\mathcal{L}_{100}(z)+\mathcal{L}_{001}(z)\right)}$$

$$\rho_{1234}=z\bar{z}$$
$$\rho_{1432}=(1-z)(1-\bar{z})$$

$\mathcal{L}_{10\ldots}(z)$ are the single-valued harmonic polylogarithms (SVHPLs) introduced by Francis Brown in 2009. They are single-valued in the region where $\bar{z}=z^*$

The result was recently re-derived it by boostrap!

Ø. Almelid, C. Duhr, EG, A. McLeod, C.D. White, JHEP 09 (2017) 073

The Soft Anomalous dimension in the High-energy limit (NLL)

Results (based on rapidity evolution — see below):

$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\, \mathbf{T}_t^2 + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s, L\right) + \ldots\,.$$

In the high-energy limit the soft anomalous dimension for 2-to-2 scattering is now known **to all orders** at NLL accuracy**:**

Odd Amplitude (Real part)

$$\mathbf{\Gamma}_{\rm NLL}^{(+)} = \left(\frac{\alpha_s(\lambda)}{\pi}\right)^2 \frac{\gamma_K^{(2)}}{2} L\, \mathbf{T}_t^2 + \left(\frac{\alpha_s(\lambda)}{\pi}\right) \sum_{i=1}^{2} \left(\frac{\gamma_K^{(1)}}{2} C_i \log\frac{-t}{\lambda^2} + 2\gamma_i^{(1)}\right)$$

Even Amplitude (Imaginary part)

Caron-Huot, EG, Reichel, Vernazza - JHEP 1803 (2018) 098

$$\mathbf{\Gamma}_{\rm NLL}^{(-)} = i\pi \frac{\alpha_s}{\pi}\, G\left(\frac{\alpha_s}{\pi} L\right) \frac{1}{2}\left(\mathbf{T}_s^2 - \mathbf{T}_u^2\right)$$

$$G^{(l)} = \frac{1}{(l-1)!}\left[\frac{(C_A - \mathbf{T}_t^2)}{2}\right]^{l-1} \left(1 - \frac{C_A}{C_A - \mathbf{T}_t^2} R(\epsilon)\right)^{-1}\Bigg|_{\epsilon^{l-1}}$$

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\, \epsilon^3 - 3\zeta_4\, \epsilon^4 - 6\zeta_5 \epsilon^5 - \left(10\zeta_6 - 2\zeta_3^2\right)\epsilon^6 + \mathcal{O}(\epsilon^7)$$

The Soft Anomalous Dimension in the High-energy limit (beyond NLL)

Results beyond NLL accuracy: $\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\, \mathbf{T}_t^2 + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s, L\right) + \ldots$

Based on rapidity evolution $\boxed{\mathbf{\Gamma}^{(+)}_{\rm NNLL} = \mathcal{O}(\alpha_s^4)}$ Caron-Huot, EG, Vernazza - JHEP 06 (2017) 016

— consistent with the Soft Anomalous Dimension 3-loop result.

The absence of $\alpha_s^3 L^k$ for $k \geq 1$ in the Real part and for $k \geq 2$ in the Imaginary part, is a non-trivial prediction from rapidity evolution, which underpins the structure of corrections to the dipole formula.

Based on the Soft Anomalous Dimension 3-loop result we also know:

$$\mathbf{\Gamma}^{(-)}_{\rm NNLL} = i\pi \left[\frac{\zeta_3}{4}(C_A - \mathbf{T}_t^2)^2 \left(\frac{\alpha_s}{\pi}\right)^3 L + \mathcal{O}(\alpha_s^4)\right] \mathbf{T}^2_{s-u}$$

$$\mathbf{\Gamma}^{(-)}_{\rm N^3LL} = i\pi \left[\frac{11\zeta_4}{4}(C_A - \mathbf{T}_t^2)^2 \left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4)\right] \mathbf{T}^2_{s-u} \qquad \mathbf{\Gamma}^{(+)}_{\rm N^3LL} = \mathcal{O}(\alpha_s^3)$$

State-of-the-art knowledge of the soft anomalous dimension in the high-energy limit

In the high-energy limit the soft anomalous dimension are be arranged in towers of logarithms:

$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\, \mathbf{T}_t^2 + \mathbf{\Gamma}_{\mathrm{NLL}}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\mathrm{NNLL}}\left(\alpha_s, L\right) + \ldots\,.$$

$$\Gamma^{\mathrm{dip.}}\left(\{p_i\}, \lambda, \alpha_s(\lambda^2)\right) \xrightarrow{\mathrm{Regge}} \frac{\gamma_K(\alpha_s)}{2}\left[L\,\mathbf{T}_t^2 + i\pi\,\mathbf{T}_{s-u}^2 + \frac{C_{\mathrm{tot}}}{2}\log\frac{-t}{\lambda^2}\right] + \sum_{i=1}^{4}\gamma_i(\alpha_s) + \mathcal{O}\left(\frac{t}{s}\right),$$

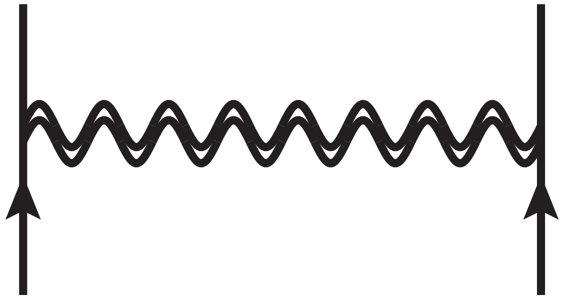
	L^0	L^1	L^2	L^3	L^4	L^5
α_s^1	**Odd(dip) +Even(dip)**	**Odd(dip)**				
α_s^2	**Even(dip)**	**Odd(dip)**	**0**			
α_s^3	**Odd + Even**	**Odd(dip)+ Even**	**0**	**0**		
α_s^4	Odd + Even	Odd + Even	Odd + Even	**Even**	**0**	
α_s^5	Odd + Even	Odd + Even	Odd + Even	Odd + Even	**Even**	**0**

$\mathrm{N^3LL}$ ↘ NNLL ↘ NLL ↘ LL ↘

Leading-logarithmic gluon Reggeization in dimensional regularization

Reggeization can be seen to be a consequence of an evolution equation:

At leading logarithmic accuracy, in dimensional regularization:



$$\frac{d}{dL}\mathcal{M}^{(-)}_{\rm LL} = \alpha(t)\mathcal{M}^{(-)}_{\rm LL} \qquad L = \ln(s/(-t))$$

with
$$\alpha(-p^2) = \alpha_s \mathbf{T}_t^2 \left(\frac{\mu^2}{4\pi e^{-\gamma_E}}\right)^{\epsilon} \int \frac{d^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}} \frac{p^2}{k^2(p-k)^2}$$
$$= \frac{\alpha_s}{\pi}\mathbf{T}_t^2 \left(\frac{\mu^2}{p^2}\right)^{\epsilon} \frac{B_0(\epsilon)}{2\epsilon} + \mathcal{O}(\alpha_s^2)$$

$$B_0(\epsilon) = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \frac{\zeta_2}{2}\epsilon^2 - \frac{7\zeta_3}{3}\epsilon^3 + \ldots$$

$$\mathcal{M}^{(-)}_{\rm LL} = (s/(-t))^{\alpha(t)} \times \mathcal{M}^{\rm tree}$$

From now on we consider the ***reduced amplitude*** $\hat{\mathcal{M}}_{ij\to ij} \equiv e^{-\mathbf{T}_t^2\,\alpha(t)\,L}\,\mathcal{M}_{ij\to ij}$

The BFKL equation in dimensional regularization — an iterative solution

The BFKL equation for the **even amplitude** take the form:

$$\frac{d}{dL}\Omega(p,k) = \frac{\alpha_s B_0(\epsilon)}{\pi}\hat{H}\Omega(p,k)$$

The Hamiltonian is non-trivial, and we do not know to directly diagonalise it, but we can always use an iterative (perturbative) solution:

Substituting:

$$\Omega(p,k) = \sum_{\ell=1}^{\infty}\left(\frac{\alpha_s}{\pi}B_0\right)^{\ell}\frac{L^{\ell-1}}{(\ell-1)!}\,\Omega^{(\ell-1)}(p,k)$$

It follows that the wavefunction is defined by iterating the Hamiltonian:

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\Omega^{(\ell-2)}(p,k)$$

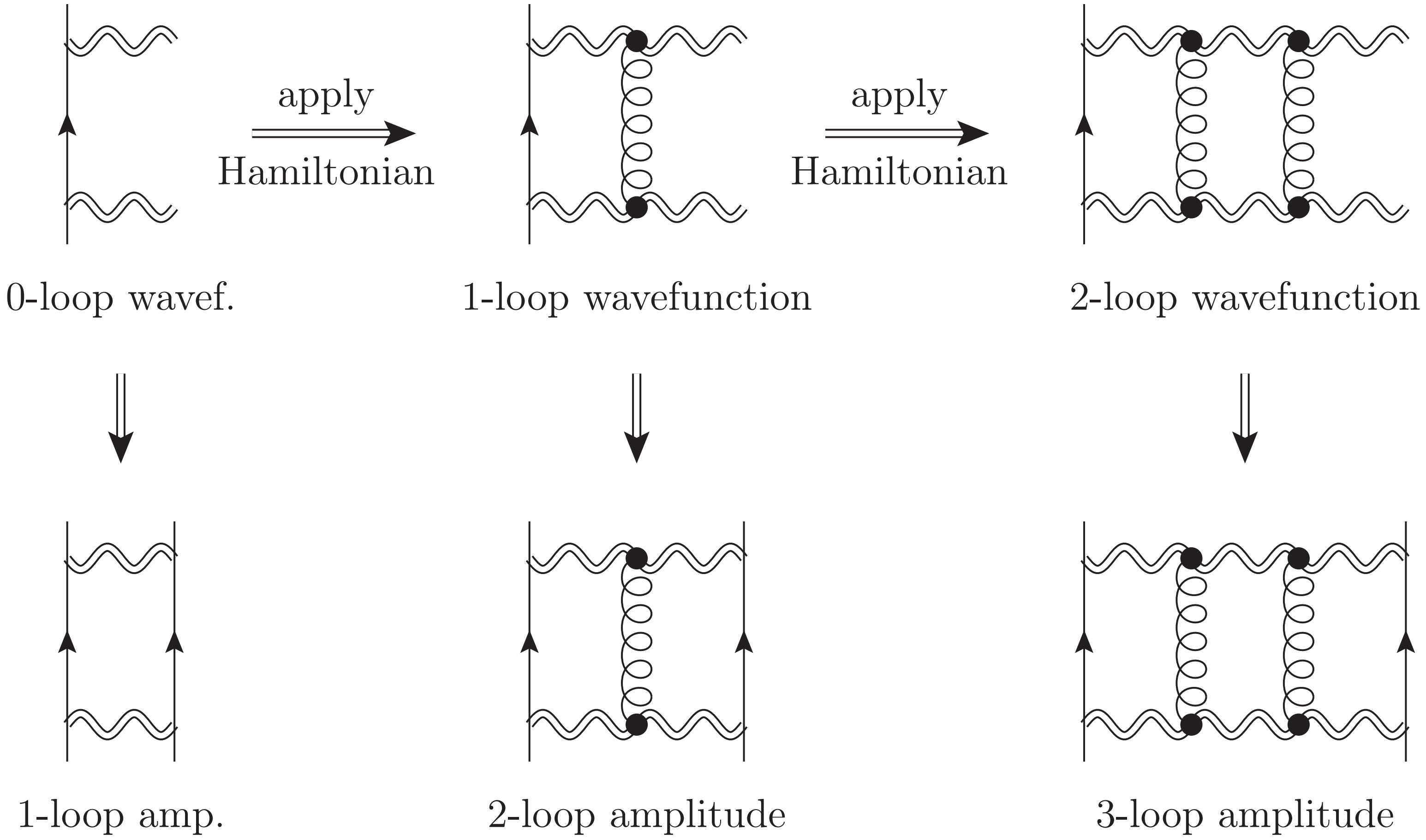
The initial condition: $\Omega^{(0)} = 1$

The BFKL equation in dimensional regularization

The even amplitude is determined by the exchange of a pair of Reggized gluons.

Applying the Hamiltonian is equivalent to adding a rang in the ladder:

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k)$$

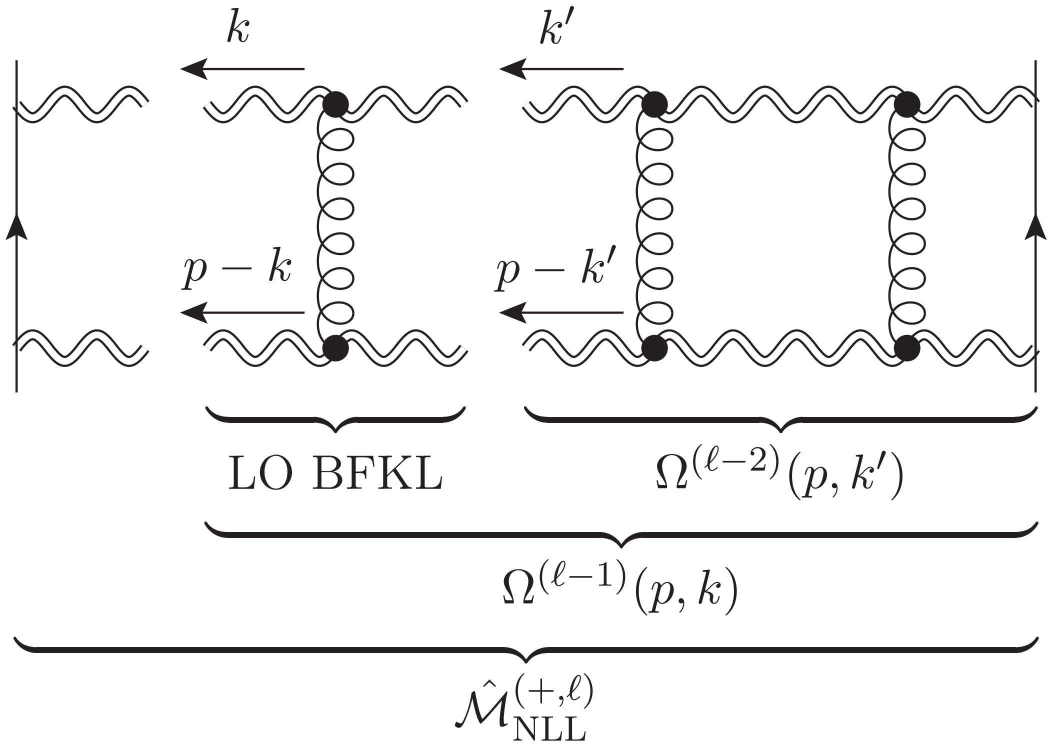


At each order the amplitude is $\hat{\mathcal{M}}^{(+,\ell)}_{\mathrm{NLL}} = -i\pi \frac{B_0^{\ell}}{(\ell-1)!}\int [\mathrm{D}k]\,\frac{p^2}{k^2(p-k)^2}\,\Omega^{(\ell-1)}(p,k)\,\mathbf{T}^2_{s-u}\mathcal{M}^{(\mathrm{tree})}$

$$[\mathrm{D}k] \equiv \frac{\pi}{B_0}\left(\frac{\mu^2}{4\pi e^{-\gamma_E}}\right)^{\epsilon}\frac{d^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}}$$

$$\hat{\mathcal{M}}^{(+)}_{\mathrm{NLL}}\left(\frac{s}{-t}\right) = \sum_{\ell=1}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^{\ell} L^{\ell-1}\,\hat{\mathcal{M}}^{(+,\ell)}_{\mathrm{NLL}}$$

The BFKL equation in more detail



Let us look at the dimensionally-regularised BFKL equation in more detail:

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}_t^2)\,\hat{H}_{\text{i}} + (C_A - \mathbf{T}_t^2)\,\hat{H}_{\text{m}}$$

Integrate

Multiply

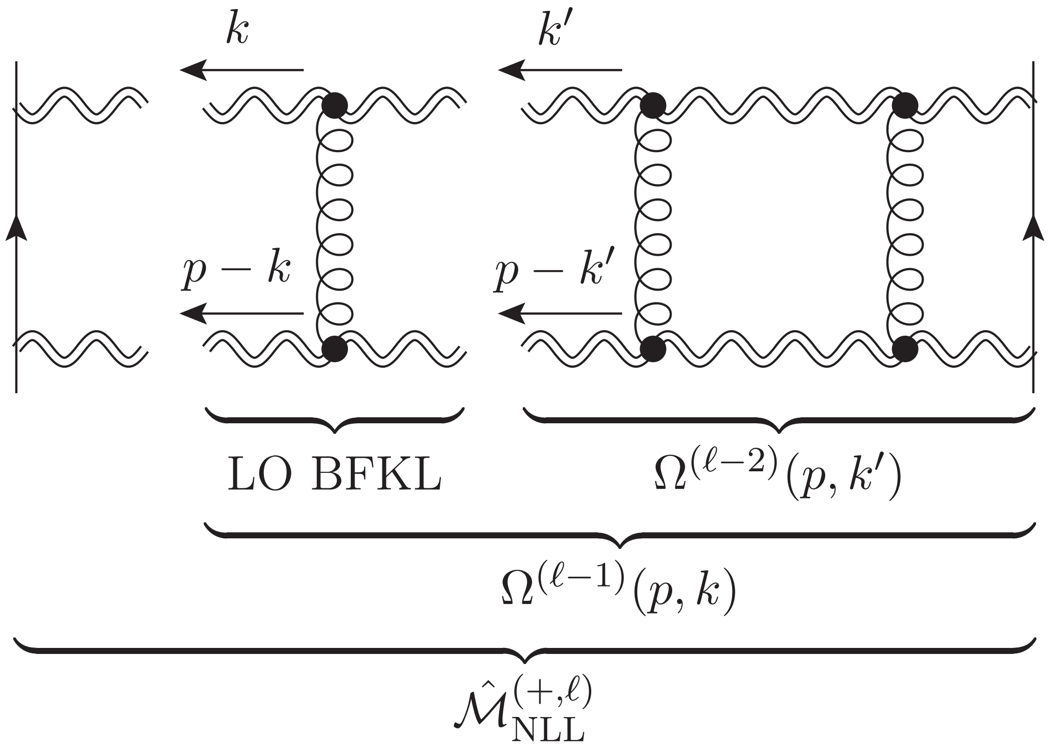
$$\hat{H}_{\text{i}}\,\Psi(p,k) = \int [\mathrm{D}k']\,f(p,k,k')\Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat{H}_{\text{m}}\,\Psi(p,k) = J(p,k)\,\Psi(p,k)$$

$$J(p,k) = \frac{1}{2\epsilon} + \int [\mathrm{D}k']\,f(p,k,k')$$
$$= \frac{1}{2\epsilon}\left[2 - \left(\frac{p^2}{k^2}\right)^{\epsilon} - \left(\frac{p^2}{(p-k)^2}\right)^{\epsilon}\right].$$

$$f(p,k,k') \equiv \frac{k^2}{k'^2(k-k')^2} + \frac{(p-k)^2}{(p-k')^2(k-k')^2} - \frac{p^2}{k'^2(p-k')^2}$$

BFKL iteration through two loops



$$\hat{H}_{\rm i}\,\Psi(p,k) = \int [{\rm D}k']\, f(p,k,k') \Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat{H}_{\rm m}\,\Psi(p,k) = J(p,k)\,\Psi(p,k)$$

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}_t^2)\,\overset{\text{Integrate}}{\overset{\downarrow}{\hat{H}_{\rm i}}} + (C_A - \mathbf{T}_t^2)\,\overset{\text{Multiply}}{\overset{\downarrow}{\hat{H}_{\rm m}}}$$

$$\Omega^{(0)}(p,k) = 1$$

$$\Omega^{(1)}(p,k) = (C_A - \mathbf{T}_t^2) J(p,k)$$

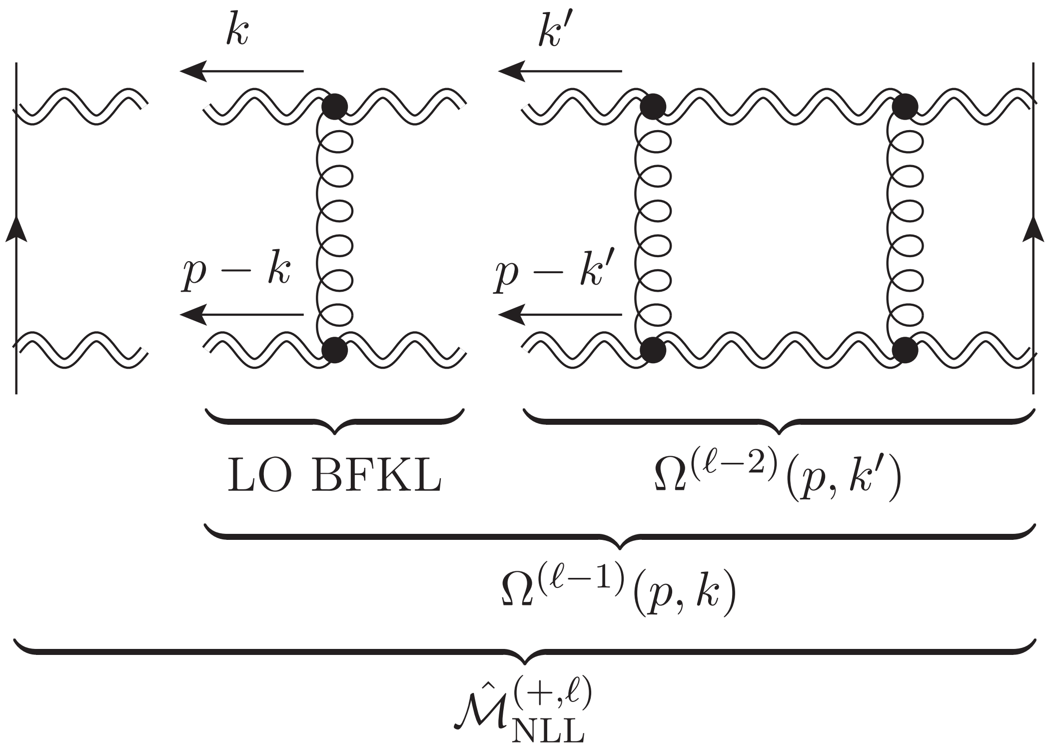
$$\Omega^{(2)}(p,k) = (C_A - \mathbf{T}_t^2)^2 J^2(p,k) + (C_A - 2\mathbf{T}_t^2)(C_A - \mathbf{T}_t^2) \int [{\rm D}k'] f(p,k,k')\,[J(p,k') - J(p,k)]$$

$\vdots$

At higher orders this yields increasingly difficult integrals…

The soft approximation

We observe: the wavefunction, at any loop order, is finite!



$$\hat{H}_{\rm i}\,\Psi(p,k) = \int[{\rm D}k']\,f(p,k,k')\Big[\Psi(p,k')-\Psi(p,k)\Big],$$

$$\hat{H}_{\rm m}\,\Psi(p,k) = J(p,k)\,\Psi(p,k)$$

$$f(p,k,k') \equiv \frac{k^2}{k'^2(k-k')^2} + \frac{(p-k)^2}{(p-k')^2(k-k')^2} - \frac{p^2}{k'^2(p-k')^2}$$

Taking the soft limit $k \ll p$:

$$f(p,k,k')|_{k\ll k'\sim p} \longrightarrow \qquad 0 + \frac{p^2}{(p-k')^2k'^2} - \frac{p^2}{k'^2(p-k')^2} = 0,$$

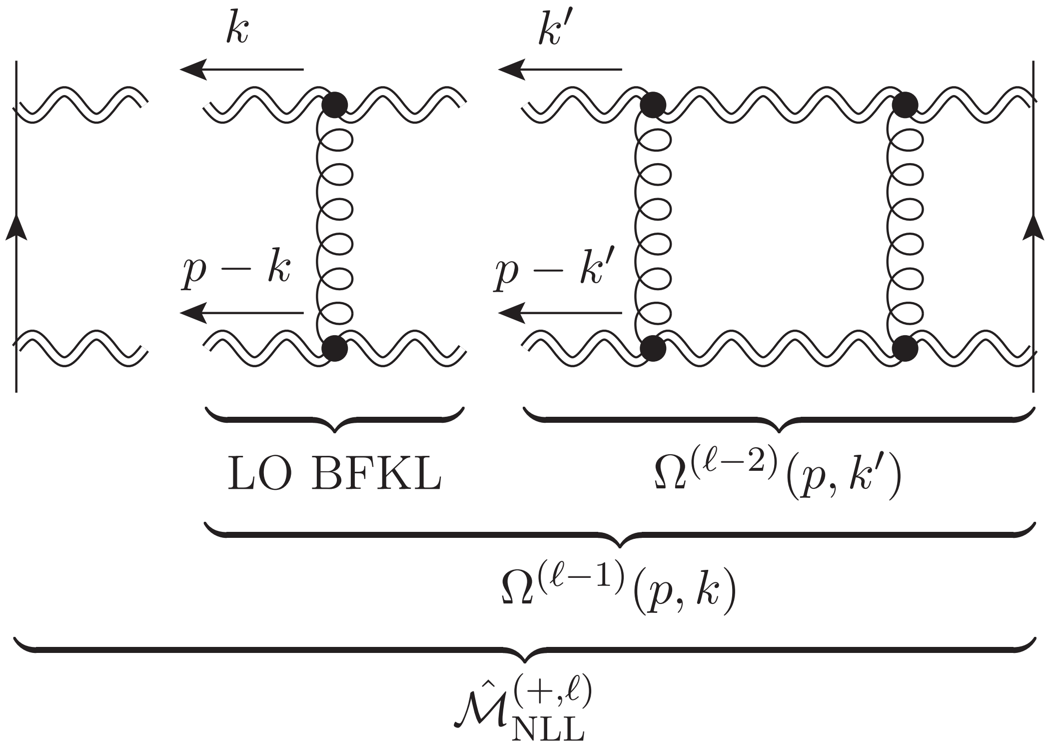
$$f(p,k,k')|_{k\sim k'\ll p} \longrightarrow \qquad \frac{k^2}{k'^2(k-k')^2} + \frac{1}{(k-k')^2} - \frac{1}{k'^2} = \frac{2(k\cdot k')}{k'^2(k-k')^2}.$$

Indeed, for small k the integral over k' is dominated by $k' \simeq k$

Conclusion: The soft limit closes under BFKL evolution! The soft limit corresponds to the entire rail, one of the two Reggeons, being soft.

Iterative solution within the soft approximation

Let us solve for the wavefunction order-by-order within the soft approximation:



$$J_s(p,k) = \frac{1}{2\epsilon}\left[1-\left(\frac{p^2}{k^2}\right)^{\epsilon}\right]$$

$$\Omega_s^{(\ell-1)}(p,k) = \hat{H}_s\,\Omega_s^{(\ell-2)}(p,k)$$

$$\hat{H}_s\Psi(p,k) = (2C_A-\mathbf{T}_t)\int[\mathrm{D}k']\,\frac{2(k\cdot k')}{k'^2(k-k')^2}\Big[\Psi(p,k')-\Psi(p,k)\Big] + (C_A-\mathbf{T})\,J_s(p,k)\,\Psi(p,k)$$

By the action of the Hamiltonian, powers of $\xi \equiv \left(p^2/k^2\right)^{\epsilon}$ transform into such powers:

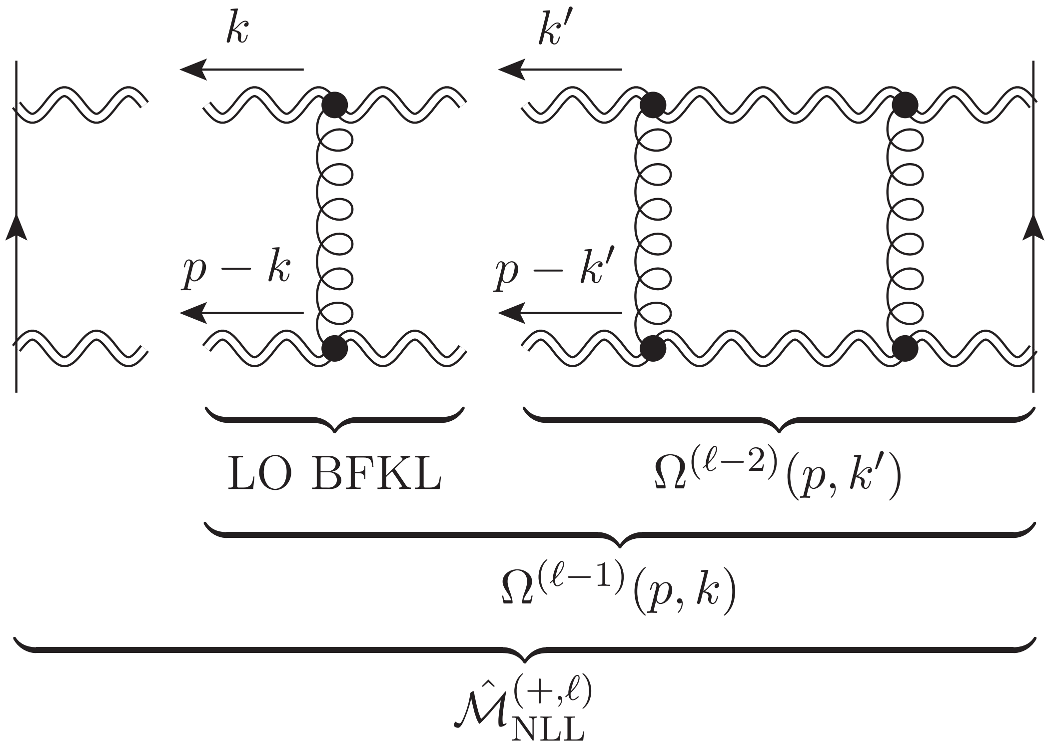
$$\int[\mathrm{D}k']\,\frac{2(k\cdot k')}{k'^2(k-k')^2}\left(\frac{p^2}{k'^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\,\frac{B_n(\epsilon)}{B_0(\epsilon)}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}$$

with $$B_n(\epsilon) = e^{\epsilon\gamma_{\mathrm{E}}}\,\frac{\Gamma(1-\epsilon)}{\Gamma(1+n\epsilon)}\,\frac{\Gamma(1+\epsilon+n\epsilon)\Gamma(1-\epsilon-n\epsilon)}{\Gamma(1-2\epsilon-n\epsilon)}\,.$$

Conclusion: the soft wavefunction is a polynomial in $\xi \equiv \left(p^2/k^2\right)^{\epsilon}$

All orders solution for the soft approximation

Solving for the wavefunction in the soft approximation:



$$J_s(p,k) = \frac{1}{2\epsilon}\left[1-\left(\frac{p^2}{k^2}\right)^{\epsilon}\right] \qquad \xi \equiv \left(p^2/k^2\right)^{\epsilon}$$

$$\begin{aligned}
\Omega^{(0)}(\xi) =& 1, \\
\Omega^{(1)}(\xi) =& \frac{(C_A-\mathbf{T}_t)}{2\epsilon}\Big(1-\xi\Big), \\
\Omega^{(2)}(\xi) =& \frac{(C_A-\mathbf{T}_t)^2}{(2\epsilon)^2}\left\{1-2\xi+\xi^2\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right\}, \\
\Omega^{(3)}(\xi) =& \frac{(C_A-\mathbf{T}_t)^3}{(2\epsilon)^3}\left\{1-3\xi+3\xi^2\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right. \\
& \left. -\,\xi^3\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\left[1-\hat{B}_2(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right\}
\end{aligned}$$

$$\hat{B}_n(\epsilon) = 1-\frac{B_n(\epsilon)}{B_0(\epsilon)} = 2n(2+n)\zeta_3\epsilon^3+3n(2+n)\zeta_4\epsilon^4+\ldots.$$

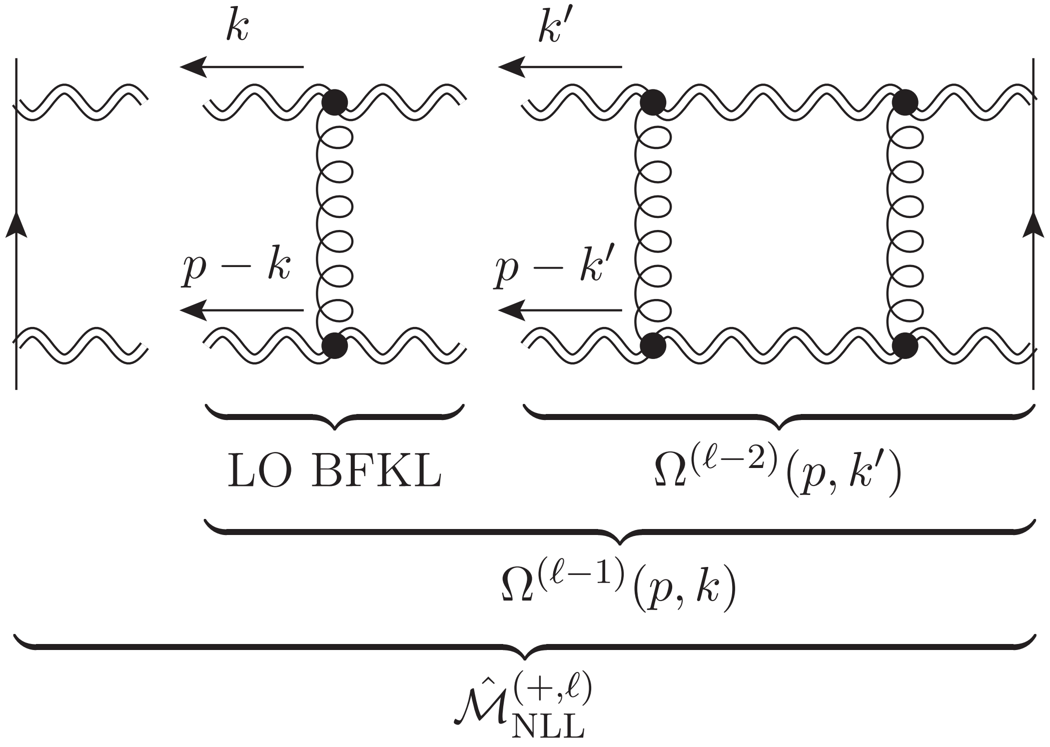
An all-order ansatz:

$$\Omega^{(\ell-1)}(p,k) = \frac{(C_A-\mathbf{T}_t)^{\ell-1}}{(2\epsilon)^{\ell-1}}\sum_{n=0}^{\ell-1}(-1)^n\left(\begin{array}{c}\ell-1\\ n\end{array}\right)\left(\frac{p^2}{k^2}\right)^{n\epsilon}\prod_{m=0}^{n-1}\left\{1-\hat{B}_m(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right\}$$

The amplitude in the soft approximation

Having solved for the wave function we can compute the amplitude.

Summing over the two soft limits, we get (at any given order):



$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi \frac{(B_0)^\ell}{(\ell-1)!}\int [\mathrm{D}k]\, \frac{p^2}{k^2(p-k)^2}\,\Omega^{(\ell-1)}(p,k)\,\mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)}$$

All divergences can be resumed into a closed form expression:

$$\hat{\mathcal{M}}^{(+)}_{\rm NLL}\Big|_s = \frac{i\pi}{L(C_A-\mathbf{T}_t^2)}\left(1-R(\epsilon)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\left[\exp\left\{\frac{B_0(\epsilon)}{2\epsilon}\frac{\alpha_s}{\pi}L(C_A-\mathbf{T}_t)\right\}-1\right]\mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)}+\mathcal{O}(\epsilon^0).$$

$$\begin{aligned}R(\epsilon)\equiv \frac{B_0(\epsilon)}{B_{-1}(\epsilon)}-1 &= \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}-1\\ &= -2\zeta_3\,\epsilon^3-3\zeta_4\,\epsilon^4-6\zeta_5\epsilon^5-\left(10\zeta_6-2\zeta_3^2\right)\epsilon^6+\mathcal{O}(\epsilon^7).\end{aligned}$$

The Soft Anomalous dimension in the high-energy limit (NLL)

Results (based on rapidity evolution):

$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\,\mathbf{T}_t^2 + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s, L\right) + \ldots\,.$$

In the High-Energy Limit the Soft Anomalous Dimension for 2 to 2 scattering is now known **to all orders** at NLL accuracy**:**

Odd Amplitude (Real part)

$$\mathbf{\Gamma}_{\rm NLL}^{(+)} = \left(\frac{\alpha_s(\lambda)}{\pi}\right)^2 \frac{\gamma_K^{(2)}}{2} L\,\mathbf{T}_t^2 + \left(\frac{\alpha_s(\lambda)}{\pi}\right) \sum_{i=1}^{2} \left(\frac{\gamma_K^{(1)}}{2} C_i \log\frac{-t}{\lambda^2} + 2\gamma_i^{(1)}\right)$$

Even Amplitude (Imaginary part)

Caron-Huot, EG, Reichel, Vernazza - JHEP 1803 (2018) 098

$$\mathbf{\Gamma}_{\rm NLL}^{(-)} = i\pi\frac{\alpha_s}{\pi}\, G\left(\frac{\alpha_s}{\pi}L\right) \frac{1}{2}(\mathbf{T}_s^2 - \mathbf{T}_u^2)$$

$$G^{(l)} = \frac{1}{(l-1)!}\left[\frac{(C_A - \mathbf{T}_t^2)}{2}\right]^{l-1} \left(1 - \frac{C_A}{C_A - \mathbf{T}_t^2} R(\epsilon)\right)^{-1}\Bigg|_{\epsilon^{l-1}}$$

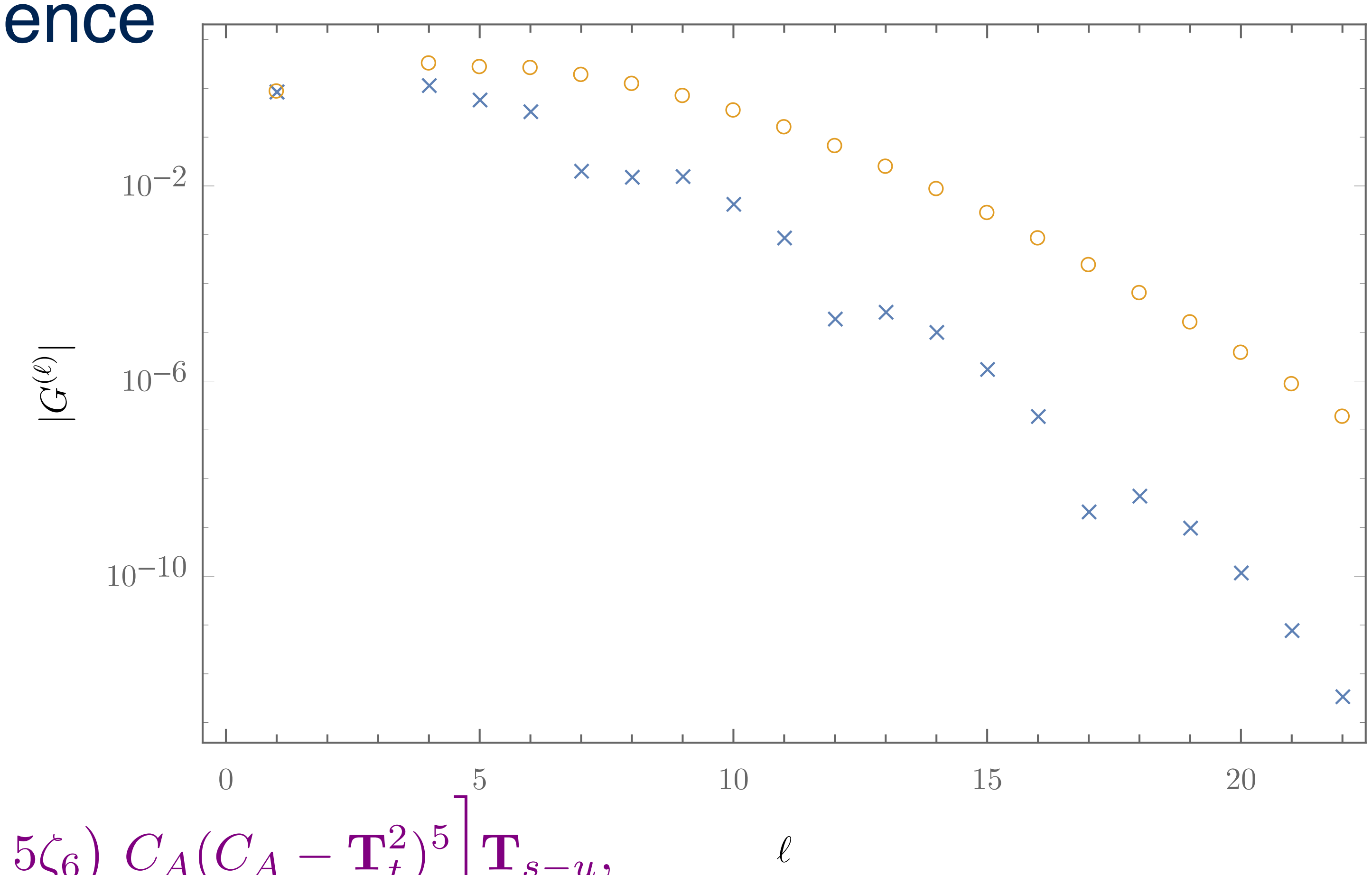
$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(10\zeta_6 - 2\zeta_3^2\right)\epsilon^6 + \mathcal{O}(\epsilon^7)$$

The Soft Anomalous dimension in the High-energy limit (NLL)

$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\,\mathbf{T}_t^2 + \mathbf{\Gamma}_{\mathrm{NLL}}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\mathrm{NNLL}}\left(\alpha_s, L\right) + \ldots .$$

$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-)} = i\pi \frac{\alpha_s}{\pi} G\left(\frac{\alpha_s}{\pi} L\right) \frac{1}{2}(\mathbf{T}_s^2 - \mathbf{T}_u^2)$$

$G(x)$ is an entire function! Its **inverse** Borel transform has a finite radius of convergence



$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-,1)} = i\pi\,\mathbf{T}_{s-u}$$

$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-,2)} = 0$$

$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-,3)} = 0,$$

$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-,4)} = -i\pi\,\frac{\zeta_3}{24}\,C_A(C_A - \mathbf{T}_t^2)^2\,\mathbf{T}_{s-u},$$

$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-,5)} = -i\pi\,\frac{\zeta_4}{128}\,C_A(C_A - \mathbf{T}_t^2)^3\,\mathbf{T}_{s-u},$$

$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-,6)} = -i\pi\,\frac{\zeta_5}{640}\,C_A(C_A - \mathbf{T}_t^2)^4\,\mathbf{T}_{s-u},$$

$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-,7)} = i\pi\frac{1}{720}\left[\frac{\zeta_3^2}{16}\,C_A^2(C_A - \mathbf{T}_t^2)^4 + \frac{1}{32}\left(\zeta_3^2 - 5\zeta_6\right)\,C_A(C_A - \mathbf{T}_t^2)^5\right]\mathbf{T}_{s-u},$$

$$\mathbf{\Gamma}_{\mathrm{NLL}}^{(-,8)} = i\pi\frac{1}{5040}\left[\frac{3\zeta_3\zeta_4}{32}\,C_A^2(C_A - \mathbf{T}_t^2)^5 + \frac{3}{64}\left(\zeta_3\zeta_4 - 3\zeta_7\right)\,C_A(C_A - \mathbf{T}_t^2)^6\right]\mathbf{T}_{s-u}.$$

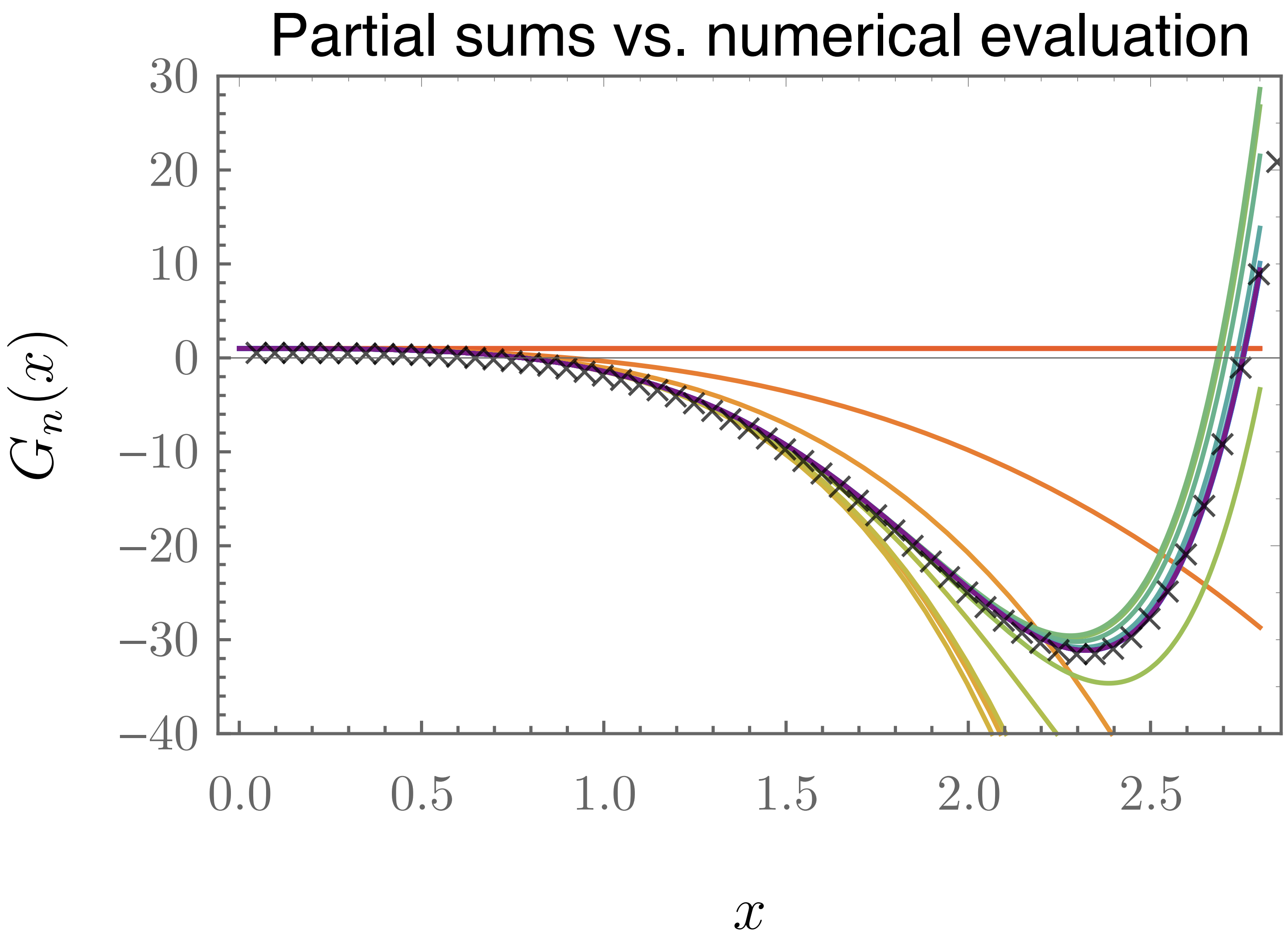
The Soft Anomalous dimension in the High-energy limit (NLL)

$$\mathbf{\Gamma}^{(-)}_{\rm NLL} = i\pi \frac{\alpha_s}{\pi} G\left(\frac{\alpha_s}{\pi} L\right) \frac{1}{2}(\mathbf{T}_s^2 - \mathbf{T}_u^2)$$

$G(x)$ is an entire function! Its **inverse** Borel transform has a finite radius of convergence.

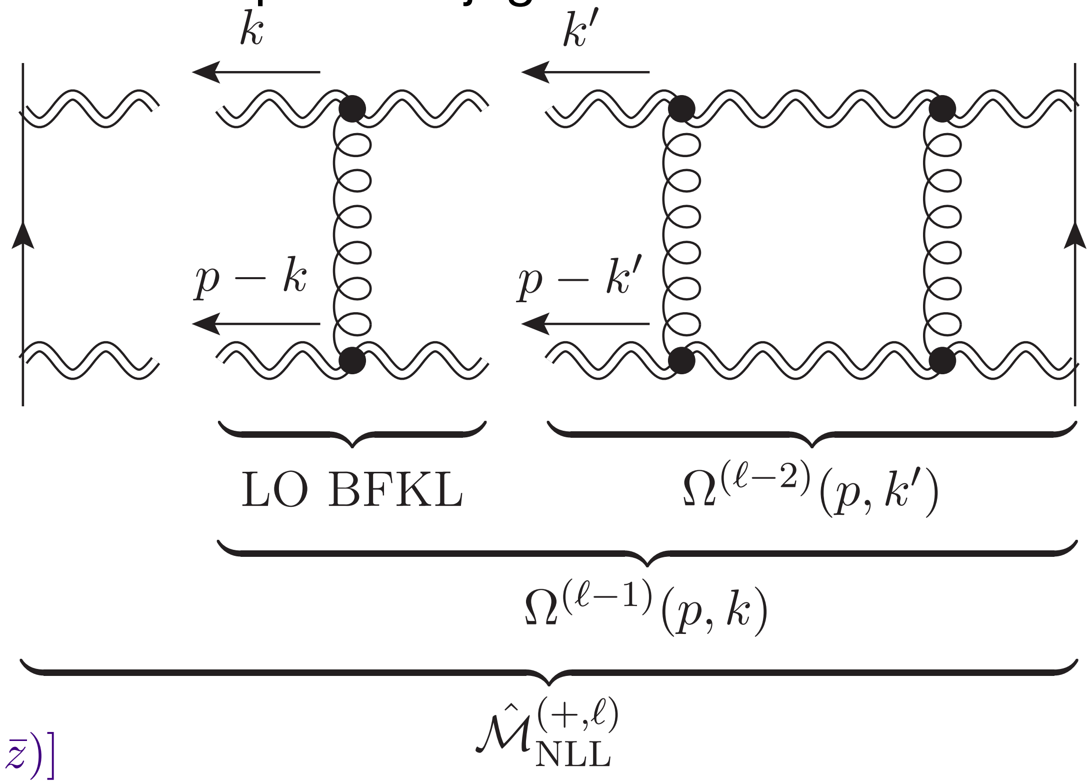
The inverse Borel transform provides a practical way to evaluate G(x) numerically.

The calculation is valid even for $\alpha_s \log(s/(-t)) \gg 1$



The BFKL equation in two dimensions

Taking the two-dimensional limit we can work with a pair of complex-conjugated variables:



$$k = k_x + ik_y, \quad k' = k'_x + ik'_y \quad \text{and} \quad p = p_x + ip_y$$

$$\frac{k_x + ik_y}{p_x + ip_y} = \frac{z}{z-1} \qquad \text{and} \qquad \frac{k'_x + ik'_y}{p_x + ip_y} = \frac{w}{w-1}$$

$$\Omega^{(\ell-1)}_{2d}(z,\bar{z}) = \hat{H}_{2d}\Omega^{(\ell-2)}_{2d}(z,\bar{z})$$

$$\hat{H}_{2d}\psi(z,\bar{z}) = C_1\hat{H}_{2d,\,i}\psi(z,\bar{z}) + C_2\hat{H}_{2d,\,m}\psi(z,\bar{z})$$

Integrate $\hat{H}_{2d,\,i}\psi(z,\bar{z}) = \frac{1}{4\pi}\int d^2w K(w,\bar{w},z,\bar{z})\left[\psi(w,\bar{w}) - \psi(z,\bar{z})\right]$

Multiply $\hat{H}_{2d,\,m}\psi(z,\bar{z}) = j(z,\bar{z})\psi(z,\bar{z})$

With the kernel: $K(w,\bar{w},z,\bar{z}) = \frac{z\bar{w}+w\bar{z}}{w\bar{w}(z-w)(\bar{z}-\bar{w})} = \frac{1}{\bar{w}(z-w)} + \frac{2}{(z-w)(\bar{z}-\bar{w})} + \frac{1}{w(\bar{z}-\bar{w})}$

$$j(z,\bar{z}) = \frac{1}{2}\log\left[\frac{z}{(1-z)^2}\frac{\bar{z}}{(1-\bar{z})^2}\right] = \frac{1}{2}\mathcal{L}_0(z,\bar{z}) + \mathcal{L}_1(z,\bar{z})$$

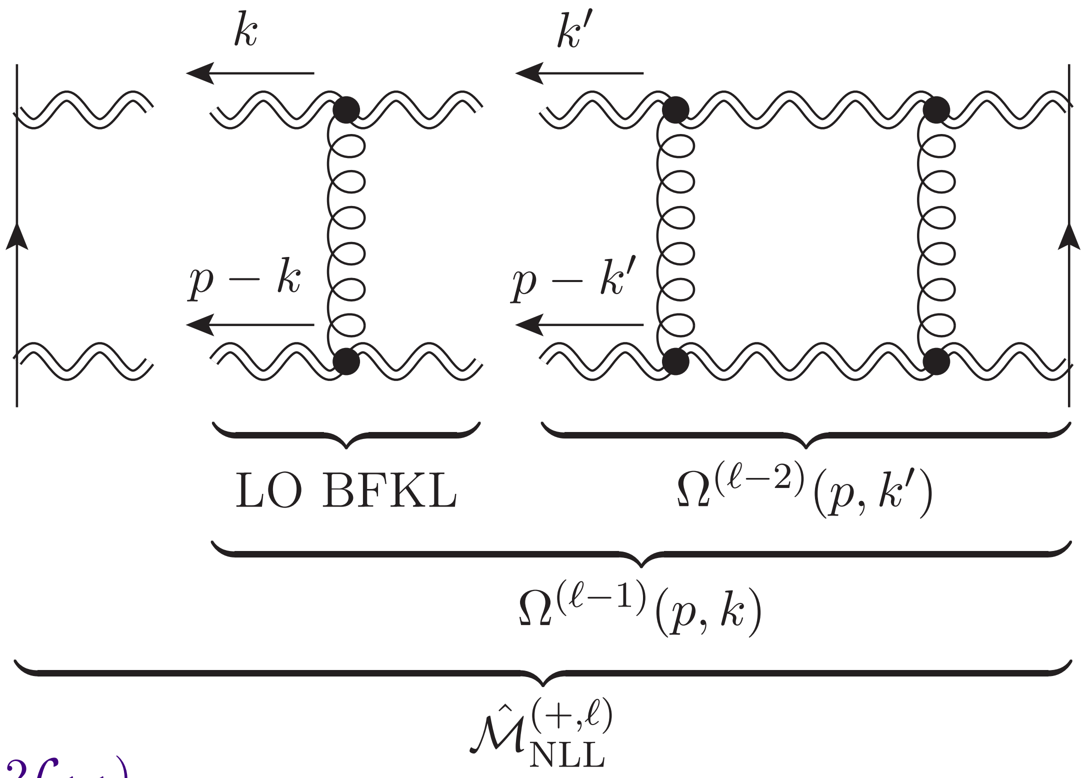
We observe:

- Two symmetries: $z \longleftrightarrow 1/z$ and $z \longleftrightarrow \bar{z}$
- The 2d wavefunction (at any order) can be expressed in terms of pure **Single-Valued Harmonic Polylogarithms (SVHPLs)** of uniform weight.

Iterating the BFKL Hamiltonian in two dimensions

The 2d wavefunction computed in terms of pure **Single-Valued Harmonic Polylogarithms (SVHPLs)**

$$\Omega_{2d}^{(\ell-1)}(z,\bar{z}) = \hat{H}_{2d}\Omega_{2d}^{(\ell-2)}(z,\bar{z})$$



$$\Omega_{2d}^{(1)} = \frac{1}{2}C_2\left(\mathcal{L}_0 + 2\mathcal{L}_1\right)$$

$$\Omega_{2d}^{(2)} = \frac{1}{2}C_2^2\left(\mathcal{L}_{0,0} + 2\mathcal{L}_{0,1} + 2\mathcal{L}_{1,0} + 4\mathcal{L}_{1,1}\right) + \frac{1}{4}C_1C_2\left(-\mathcal{L}_{0,1} - \mathcal{L}_{1,0} - 2\mathcal{L}_{1,1}\right)$$

$$\begin{aligned}\Omega_{2d}^{(3)} = &\frac{1}{4}C_1C_2^2\left(-2\mathcal{L}_{0,0,1} - 3\mathcal{L}_{0,1,0} - 7\mathcal{L}_{0,1,1} - 2\mathcal{L}_{1,0,0} - 7\mathcal{L}_{1,0,1} - 7\mathcal{L}_{1,1,0}\right.\\ &\left.-14\mathcal{L}_{1,1,1} + 2\zeta_3\right) + \frac{3}{4}C_2^3\left(\mathcal{L}_{0,0,0} + 2\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1} + 2\mathcal{L}_{1,0,0}\right.\\ &\left.+4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right) + \frac{1}{16}C_1^2C_2\left(\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1}\right.\\ &\left.+\mathcal{L}_{1,0,0} + 4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right)\end{aligned}$$

- An algorithm is set up to iteratively determine the wavefunction to any loop order.
- A closed-form expression is yet unknown.

The full amplitude: combining the soft and 2d calculations

The soft wavefunction generates all IR singularities in the amplitude.

We can therefore
split the full wavefunction into soft and hard:
and use dim. reg. only for the soft:

$$\Omega(p,k) = \Omega_{\text{hard}}(p,k) + \Omega_{\text{soft}}(p,k)$$

$$\Omega^{(\text{2d})}_{\text{hard}}(z,\bar{z}) \equiv \lim_{\epsilon\to 0} \Omega_{\text{hard}} = \Omega^{(\text{2d})}(z,\bar{z}) - \Omega^{(\text{2d})}_{\text{soft}}(z,\bar{z})$$

The full amplitude is therefore recovered by summing two integrals:

$$\hat{\mathcal{M}}^{(+,\,\text{NLL})}_{ij\to ij}\left(\frac{s}{-t}\right) = -i\pi\left[\int [\text{D}k]\frac{p^2}{k^2(p-k)^2}\Omega_{\text{soft}}(p,k) + \frac{1}{4\pi}\int\frac{d^2z}{z\bar{z}}\Omega^{(\text{2d})}_{\text{hard}}(z,\bar{z})\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\text{tree})}_{ij\to ij}$$

In principle the algorithm can be run to any oder. In practice we stopped at 12 loops.
The first few orders are:

$$\hat{\mathcal{M}}^{(1)}_{ij\to ij} = i\pi\frac{1}{2\epsilon}\mathbf{T}^2_{s-u}\mathcal{M}^{(\text{tree})}_{ij\to ij}$$

$$\hat{\mathcal{M}}^{(2)}_{ij\to ij} = i\pi C_2\left[\frac{1}{8\epsilon^2} - \frac{\zeta(2)}{8}\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\text{tree})}_{ij\to ij}$$

$$\hat{\mathcal{M}}^{(3)}_{ij\to ij} = i\pi C_2^2\left[\frac{1}{48\epsilon^3} - \frac{\zeta(2)}{32\epsilon} - \frac{29}{48}\zeta(3)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\text{tree})}_{ij\to ij}$$

$$\hat{\mathcal{M}}^{(4)}_{ij\to ij} = i\pi\left[\frac{C_2^3}{384\epsilon^4} - \frac{C_2^3\zeta(2)}{192\epsilon^2} - \left(\frac{7}{288}C_2^3\zeta(3) + \frac{1}{192}C_2^2C_A\zeta(3)\right)\frac{1}{\epsilon} - \frac{C_2^3\zeta(4)}{48} - \frac{C_2^2C_A\zeta(4)}{128}\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\text{tree})}_{ij\to ij}$$

$$\begin{aligned}\hat{\mathcal{M}}^{(5)}_{ij\to ij} = i\pi\Bigg[&\frac{C_2^4}{3840\epsilon^5} - \frac{C_2^4\zeta(2)}{1536\epsilon^3} + \left(-\frac{7C_2^4\zeta(3)}{2304} - \frac{C_2^3C_A\zeta(3)}{1920}\right)\frac{1}{\epsilon^2} + \left(-\frac{9C_2^4\zeta(4)}{4096} - \frac{C_2^3C_A\zeta(4)}{1280}\right)\frac{1}{\epsilon} \\ &+ C_2^4\left(\frac{35\zeta(2)\zeta(3)}{4608} - \frac{293\zeta(5)}{1280}\right) + C_2^3C_A\left(\frac{1}{768}\zeta(2)\zeta(3) + \frac{253\zeta(5)}{1920}\right) - \frac{1}{48}C_2^2C_A^2\zeta(5)\Bigg]\mathbf{T}^2_{s-u}\mathcal{M}^{(\text{tree})}_{ij\to ij}\end{aligned}$$

To this order (five loops) all integrals have also been computed directly in dim. reg.

The BFKL equation in two dimensions - radius of convergence

Considering the exchange of specific colour representations in the t-channel, we get the following numerical coefficients:

$$\mathcal{M}^{(-)}_{\mathrm{NLL}\,27} = \frac{-i\pi}{L}\Big(0.8225x^2 - 11.62x^3 + 1.037x^4 - 86.76x^5 + 142.8x^6 - 880.3x^7 + \\ + 2555.x^8 - 10536.x^9 + 35577.x^{10} - 133005.x^{11} + 467988.x^{12}\Big)\mathbf{T}^2_{s-u}\mathcal{M}^{(-,\,\mathrm{tree})}$$

$$\mathcal{M}^{(-)}_{\mathrm{NLL}\,\mathrm{singlet}} = \frac{-i\pi}{L}\Big(- 0.6169x^2 - 6.536x^3 - 0.8371x^4 - 8.483x^5 - 1.529x^6 - 12.67x^7 + \\ + 1.609x^8 - 20.62x^9 + 16.48x^{10} - 35.98x^{11} + 46.07x^{12}\Big)\mathbf{T}^2_{s-u}\mathcal{M}^{(-,\,\mathrm{tree})}$$

Applying Padé Approximants on these partial sums we extract the position of the nearest singularity:

singularity at	$L\alpha_s/\pi \equiv x$		$(C_A - \mathbf{T}_t^2)\alpha_s L/\pi = (C_A - \mathbf{T}_t^2)x$	
Representation	27	singlet	27	singlet
soft amplitude	-0.2	0.333	1	1
hard amplitude	~ -0.19	0.333	~ 0.95	1
full amplitude	-0.237	0.666	1.185	-2

The soft amplitude:

- Confirms Padé analysis
- Singularity at $C_2 x = 1$ cancels in the full amplitude

$$\hat{\mathcal{M}}_{\mathrm{NLL,\,soft}} = \frac{i\pi}{LC_2}\left\{\left(e^{\frac{B_0}{2\epsilon}C_2 x} - 1\right)\frac{B_{-1}(\epsilon)}{B_0(\epsilon)}\left(1 - B_{-1}(\epsilon)\frac{C_1}{C_2}\right)^{-1} \right.$$
$$\left. + \left(1 - e^{\gamma_E C_1\,x}\frac{\Gamma(1-C_2x)}{\Gamma(1+C_2x)}\frac{\Gamma^{2-\frac{C_1}{C_2}}\left(1+C_2\frac{x}{2}\right)}{\Gamma^{2-\frac{C_1}{C_2}}\left(1-C_2\frac{x}{2}\right)}\right)\right\}\mathbf{T}_{s-u}\,\mathcal{M}^{(\mathrm{tree})}$$

$C_1 = 2C_A - \mathbf{T}_t^2$ $\qquad$ $C_2 = C_A - \mathbf{T}_t^2$

Conclusions

- Rapidity evolution equations can be efficiently used to compute partonic scattering amplitudes to high loop orders.
- The high-energy limit and infrared factorization are complementary avenues in studying these amplitudes.
- Number-theory findings:
  - Soft amplitude can be resummed using Gamma functions.
  - Hard amplitude is expressed in terms of SV Zeta values (e.g. no even Zetas; first multi-Zeta occurs at 11 loops). It cannot be resummed into Gamma functions.
- Large order behaviour aspects:
  - The soft anomalous dimension (at NLL in the Regge limit) is an entire function. Its calculation extends to $\alpha_s \log(s/(-t)) \gg 1$
  - The finite part of the NLL amplitude has a finite radius of convergence, with asymptotically sign-oscillating coefficients.