Topics in the computation of gauge theory amplitudes

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Motivation

Outline:

- Some results in pure Yang-Mills theory (no quarks) and in N=4 SYM
- Tree level: Parke-Taylor formulas, BCFW recursion, a 6-point amplitude in twistor language, NMHV superamplitude
- One-loop: computation of box and triangle coefficients

Motivation:

- Understand the symmetries of pure Yang-Mills theory
- Find an economical language in which to express amplitudes, avoiding redundancy
- Gluon amplitudes are necessary input for eg. jet distributions

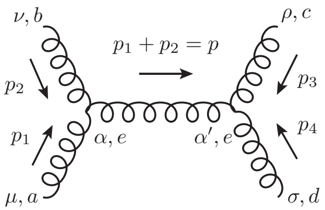
Difficulties in the traditional perturbative approach

- Pure Yang-Mills theory (no quarks)
- Large number of diagrams even for small number n of particles
- Large number of terms in each diagram

n	4	5	6	7	8	9	10
diagrams	4	25	120	2485	34300	559405	10525900

- Big simplifications in final results (symmetries?)

Example $n = 4$



$$= ig^2 (if^{abe}) (if_e{}^{cd}) \frac{1}{(p_1+p_2)^2} R_1$$

$$\begin{aligned} R_1 = {} & (\varepsilon_1 \cdot \varepsilon_2)[(\varepsilon_3 \cdot \varepsilon_4)(p_1 - p_2) \cdot (p_3 - p_4) \\ & - 2(\varepsilon_4 \cdot p_3)\varepsilon_3 \cdot (p_1 - p_2) + 2(\varepsilon_3 \cdot p_4)\varepsilon_4 \cdot (p_1 - p_2)] \\ & + (\varepsilon_1 \cdot \varepsilon_3)[4(\varepsilon_4 \cdot p_3)(\varepsilon_2 \cdot p_1)] + (\varepsilon_1 \cdot \varepsilon_4)[-4(\varepsilon_2 \cdot p_1)(\varepsilon_3 \cdot p_4)] \\ & + (\varepsilon_2 \cdot \varepsilon_3)[-4(\varepsilon_1 \cdot p_2)(\varepsilon_4 \cdot p_3)] + (\varepsilon_2 \cdot \varepsilon_4)[4(\varepsilon_1 \cdot p_2)(\varepsilon_3 \cdot p_4)] \\ & + (\varepsilon_3 \cdot \varepsilon_4)[2(\varepsilon_1 \cdot p_2)\varepsilon_2 \cdot (p_3 - p_4) - 2(\varepsilon_2 \cdot p_1)\varepsilon_1 \cdot (p_3 - p_4)] \end{aligned}$$

Color decomposition

- Use Gell-Mann matrices instead of structure constants
- Factor the color part, focus on their coefficients

$$if^{abc} = 2\mathrm{Tr}\left(\left[T^a, T^b\right] T^c\right) = 2\left[\mathrm{Tr}\left(T^a T^b T^c\right) - \mathrm{Tr}\left(T^b T^a T^c\right)\right]$$

$$\mathcal{M}_n^{(0)}(p_1^{\lambda_1}, \ldots, p_n^{\lambda_n}) = g^{n-2} \sum_{\sigma \in S_{n-1}} \mathrm{Tr}\left(T^{a_1} T^{a_{\sigma(2)}} T^{a_{\sigma(3)}} \ldots T^{a_{\sigma(n)}}\right) \cdot$$
$$\cdot A_n^{(0)}(p_1^{\lambda_1}, p_{\sigma(2)}^{\lambda_{\sigma(2)}} \ldots, p_{\sigma(n)}^{\lambda_{\sigma(n)}})$$

- The A_n (partial amplitudes) can be computed using only planar cyclically ordered diagrams [1]
- The A_n are gauge invariant, and are not all independent

[1] F. A. Berends and W. Giele, *The six-gluon process as an example of Weyl-van der Waerden spinor calculus*, Nucl. Phys. B 294 (1987) 700.

Spinor decomposition

Useful to use the $(1/2,0)$ e $(0,1/2)$ representation of the Lorentz group

- External p^μ are light-like and on-shell:

$$p^\mu \longrightarrow p_{\alpha\dot{\alpha}} = p_\mu(\sigma^\mu)_{\alpha\dot{\alpha}} = |p\rangle_\alpha [p|_{\dot{\alpha}}$$

- Can rewrite all Lorentz invariants into spinor notation

$$\langle p_1 p_2\rangle = \langle p_1|^\alpha |p_2\rangle_\alpha = \varepsilon^{\alpha\beta}|p_1\rangle_\beta |p_2\rangle_\alpha$$
$$[p_1 p_2] = [p_1|_{\dot{\alpha}}|p_2]^{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}|p_1]^{\dot{\beta}}|p_2]^{\dot{\alpha}}$$

Example

$(p+q)^2 = 2p\cdot q = \langle qp\rangle[pq]$

Spinor helicity formalism

- Polarizations can be represented by[2,3]

$$\varepsilon^-_{\alpha\dot{\alpha}} = \sqrt{2}\frac{|p\rangle_\alpha [\mu|_{\dot{\alpha}}}{[p\mu]}, \quad \varepsilon^+_{\alpha\dot{\alpha}} = \sqrt{2}\frac{|\mu\rangle_\alpha [p|_{\dot{\alpha}}}{\langle p\mu\rangle}$$

- They have the desired properties

$$\varepsilon^\pm \cdot p = 0, \quad \varepsilon^+ \cdot \varepsilon^- = 1$$

- The spinor μ can be chosen arbitrarily. $\mu \to \mu + yp \Longleftrightarrow$ gauge invariance

Easier to compute amplitudes for polarized gluons, es. $A_n^{(0)}(+,+,+,+)$

[2] P. De Causmaecker, R. Gastmans, W. Troost, and T. T. Wu, *Helicity Amplitudes for Massless QED*, Phys. Lett. 105B (1981) 215.

[3] P. De Causmaecker, R. Gastmans, W. Troost, and T. T. Wu, *Multiple Bremsstrahlung in Gauge Theories at High-Energies. 1. General Formalism for Quantum Electrodynamics*, Nucl. Phys. B206 (1982) 53-60.

Parke-Taylor formulas

$$A_n^{(0)}(++\cdots+) = 0 \quad (1)$$

$$A_n^{(0)}(-+\cdots+) = 0 \quad (2)$$

$$A_n^{(0)}(1^+,2^+,\ldots,j^-,\ldots,k^-,\ldots,n^+) = i(\sqrt{2})^n \frac{\langle jk\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \quad (3)$$

- Convention: all momenta incoming
- Crossing:

$$(1) \Longleftrightarrow ++ \to --\cdots- \text{ is forbidden}$$

$$(2) \Longleftrightarrow ++ \to +-\cdots- \text{ is forbidden}$$

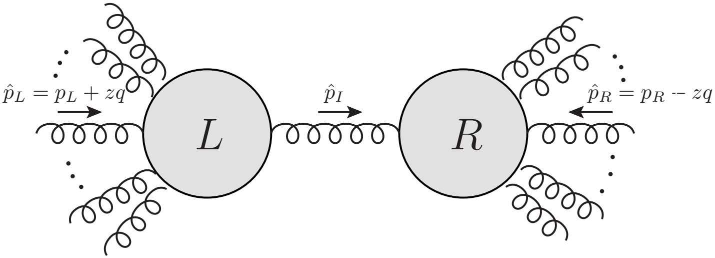
$$(3) \Longleftrightarrow ++ \to ++-\cdots- \text{ is allowed}$$

- (3) is «maximally helicity violating»[4] (MHV)

[4] S. J. Parke and T. R. Taylor, *An Amplitude for n Gluon Scattering*, Phys. Rev. Lett. 56 (1986) 2459.

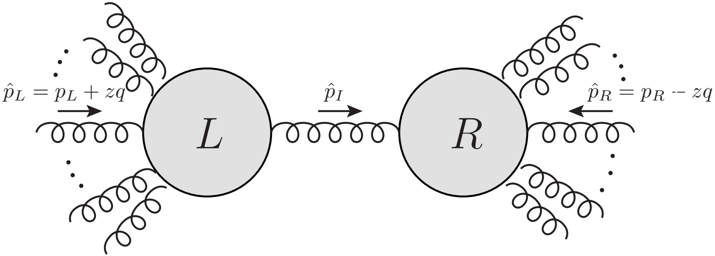
BCFW method

- Recursive method, allows to compute an amplitude starting from on-shell amplitudes with fewer external legs[5]
- Can be derived from the study of the singularities of the S matrix with complex momenta

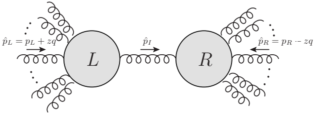


- Limitations: not easy to prove equivalence of different «shifts»; method can give rise to «spurious» poles

[5] R. Britto, F. Cachazo, B. Feng, and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, Phys. Rev. Lett. 94 (2005) 181602



- Add a complex momentum zq^μ, $z \in \mathbb{C}$ which traverses the diagram
- Impose that external momenta stay on-shell $\Longrightarrow$ q^μ must be complex, and $q^2 = 0$
- Intereted in $z \to 0$ limit of the modified amplitude
- Where are the poles in the z plane?



- One simple pole for each possible partitioning:

$$\frac{1}{\hat{p}_I^2} = \frac{1}{p_I^2 + 2zq \cdot p_I}$$

- Pole $\longleftrightarrow$ propagator is on shell, at

$$z_s = -\frac{p_I^2}{2q \cdot p_I}$$

- Consider $\int_C \frac{\hat{A}(z)}{z} dz$ over large contour, if $\hat{A}(z) \to 0$ as $z \to \infty$, then

$$A = \operatorname*{Res}_{z \to 0} \frac{\hat{A}(z)}{z} = -\sum_{\text{partitions}\lambda_I} \left[\hat{A}_L(z_s) \frac{1}{p_I^2} \hat{A}_R(z_s) \right]$$

Behaviour of $\hat{A}$ for $z \to \infty$

Background field method

- BCFW formula requires $\hat{A}(z) \to 0$ as $z \to \infty$. Not true in every theory, eg. in ϕ^4 have $\hat{A}(z) \to$const.
- $z \to \infty$ corresponds to one hard gluon in a sea of soft gluons: approach with background field method[6]
- Divide gauge field $\mathbf{A}^\mu = A^\mu + a^\mu$, where only a has high-frequency modes, A is background field

$$D_\mu = \partial_\mu - igA^a_\mu t_a, \qquad F^a_{\mu\nu} = \partial_{[\mu} A^a_{\nu]} + gf^a{}_{bc} A^b_\mu A^c_\nu$$

$$\mathbf{F}^a_{\mu\nu} = F^a_{\mu\nu} + gf^a{}_{bc} a^b_\mu a^c_\nu + D_{[\mu} a^a_{\nu]}$$

$$\begin{aligned}\mathscr{L} &= -\frac{1}{4}\left[F^a_{\mu\nu} F^{\mu\nu}_a + D_{[\mu} a^a_{\nu]} D^{[\mu} a^{\nu]}_a + 2gF^{\mu\nu}_a f_{abc} a^b_\mu a^c_\nu + +2F^{\mu\nu}_a D_{[\mu} a^a_{\nu]}\right] + \mathscr{O}(a^3) \\ &= -\frac{1}{2}\left[\eta^{\rho\sigma} D_\mu a^a_\rho D^\mu a_{\sigma a} + 2gF^{\rho\sigma}_a f^a{}_{bc} a^b_\rho a^c_\sigma\right]\end{aligned}$$

[6] N. Arkani-Hamed and J. Kaplan, "On Tree Amplitudes in Gauge Theory and Gravity," JHEP 04 (2008) 076, arXiv:0801.2385 [hep-th]

$$\mathcal{M}^{\rho\sigma} = (cz + \cdots)\eta^{\rho\sigma} + D^{\rho\sigma} + \frac{1}{z}E^{\rho\sigma} + \cdots$$

- Apply Ward identity

$$\begin{cases} \hat{p}_{L\rho}\mathcal{M}^{\rho\sigma}\varepsilon_{R\sigma} = 0 \to (p_L + zq)_\rho \mathcal{M}^{\rho\sigma}\varepsilon_{R\sigma} = 0, \\ \varepsilon_{L\rho}\mathcal{M}^{\rho\sigma}\hat{p}_{R\sigma} = 0. \end{cases}$$

to prove

$$\mathcal{M}(\hat{p}_L^-, \hat{p}_R^+)(z) = \varepsilon_{L\rho}^- \varepsilon_{R\sigma}^+ \mathcal{M}^{\rho\sigma} \to \frac{1}{z}$$
$$\mathcal{M}(\hat{p}_L^-, \hat{p}_R^-)(z) = \varepsilon_{L\rho}^- \varepsilon_{R\sigma}^- \mathcal{M}^{\rho\sigma} \to \frac{1}{z}$$

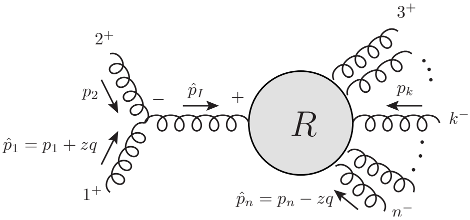
- Every nonzero tree amplitude has a negative-helicity gluon, so method is always applicable

Proof of Parke-Taylor formulas

- Translate shift into spinor notation

$$|\hat{1}\rangle = |1\rangle + z|n\rangle, \quad |\hat{n}] = |n] - z|1]$$

- Only one diagram contributes[7]



$$\begin{aligned}
&A_n^{(0)}(1^+ 2^+ \ldots k^- \ldots n^-) = \\
&= -A_3^{(0)}(\hat{1}^+ 2^+ \hat{I}^-)\,\frac{1}{P_I^2}\,A_{n-1}^{(0)}(-I^+ \ldots k^- \ldots n^-) \\
&= \frac{[\hat{1}2]^3}{[2\hat{I}][\hat{I}\hat{1}]}\,\frac{1}{\langle 21\rangle[12]}\,\frac{\langle k\hat{n}\rangle^4}{\langle \hat{I}3\rangle\langle 34\rangle\cdots\langle k\,k{+}1\rangle\cdots\langle \hat{n}\hat{I}\rangle} \\
&= \frac{\langle kn\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}
\end{aligned}$$

- On-shell 3-point vertex can be defined with complex momenta

[7] R. Britto, F. Cachazo, and B. Feng, *New recursion relations for tree amplitudes of gluons*, Nucl. Phys. B715 (2005) 499-522

$N = 4$ supersymmetry

- At tree level, amplitudes are the same as in regular Yang-Mills theory

	field	creation operator
gluon	g^+	$a^\dagger$
4 gluinos	ψ^A	$a^{\dagger A}$
6 scalars	ϕ^{AB}	$a^{\dagger AB}$
4 gluinos	$\psi^{ABC} \sim \bar{\psi}^D$	$a^{\dagger ABC}$
gluon	g^-	$a^{\dagger ABCD}$

$$\delta_\varepsilon \varphi = [\varepsilon Q, \varphi] = \varepsilon \psi$$

$$\{Q, Q^\dagger\} \propto P$$

- Ward identities[8] relate different amplitudes

$$0 = \langle 0| \left[Q_A, a_1^\dagger a_2^\dagger \cdots a_n^\dagger \right] |0\rangle = \text{sum of amplitudes}$$

Example

$$0 = |1\rangle \langle 0| a_1^\dagger a_2^{\dagger BCDE} a_3^\dagger a_4^\dagger \cdots a_n^\dagger |0\rangle - |2\rangle 4! \delta_A^{[B} \langle 0| a_1^{\dagger A} a^{\dagger CDE]} a_3^\dagger a_4^\dagger \cdots a_n^\dagger |0\rangle$$

[8] M. L. Mangano and S. J. Parke, "Multiparton amplitudes in gauge theories," Phys. Rept. 200 (1991) 301–367

$N = 4$ supersymmetry

Superfield and superamplitude

- Superfield: used to represent any field present in the theory

$$\Omega = g^+ + \eta_A \psi^A - \frac{1}{2!}\eta_A\eta_B\phi^{AB} - \frac{1}{3!}\eta_A\eta_B\eta_C\psi^{ABC} + \eta_A\eta_B\eta_C\eta_D g^-$$

- η are Grassmann variables

To obtain...	g^+	ψ^A	ϕ^{AB}	ψ^{ABC}	g^-
apply... then set $\eta_i = 0$	1	$\frac{\partial}{\partial\eta_A}$	$\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\eta_B}$	$\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\eta_B}\frac{\partial}{\partial\eta_C}$	$\frac{\partial}{\partial\eta_A}\frac{\partial}{\partial\eta_B}\frac{\partial}{\partial\eta_C}\frac{\partial}{\partial\eta_D}$

- Superamplitude: $\mathscr{A}(\Omega_1(p_1),\ldots,\Omega_n(p_n))$, from which partial amplitudes can be extracted

$N = 4$ supersymmetry
Superfield and superamplitude

Example

$$A_n(g_1^+,\dots,g_i^-,\dots,g_j^-,\dots,g_n^+) = \left(\prod_{A=1}^{4}\frac{\partial}{\partial\eta_{iA}}\right)\left(\prod_{B=1}^{4}\frac{\partial}{\partial\eta_{jB}}\right) \mathscr{A}_n(\Omega_1,\dots,\Omega_n)\big|_{\eta_{kC}=0}$$

- Expand the superamplitude in a series in the Grassmann variables: the coefficients are the partial amplitudes

Example

$$\mathscr{A}_n(\Omega_1,\dots,\Omega_n) = \cdots + A_n(g_1^-,g_2^+,g_3^+\dots,g_n^+)\eta_{11}\eta_{12}\eta_{13}\eta_{14} + \cdots$$

$N = 4$ supersymmetry

Superfield and superamplitude

- The series contains terms of degree 8, 12, 16, ... (due to R symmetry) in the Grassmann variables

$$\mathscr{A} = \mathscr{A}^{\mathrm{MHV}} + \mathscr{A}^{\mathrm{NMHV}} + \cdots + \mathscr{A}^{\mathrm{N}^K\mathrm{MHV}} + \cdots + \mathscr{A}^{\mathrm{anti-MHV}}$$

- Charges can be written $(Q_{iA})_{\dot\alpha} = [i|_{\dot\alpha} \frac{\partial}{\partial \eta_{iA}}, \quad \left(Q_i^{\dagger A}\right)_\alpha = |i\rangle_\alpha \eta_A$
- Conservation of supermomentum $\sum_i |i\rangle \eta_{iA}$ corresponds to the Ward identities. Can be factored

$$\frac{1}{2^4} \prod_A \sum_{i,j=1}^{n} \langle ij\rangle \eta_{iA}\eta_{jA} = \delta^{(8)}(Q^\dagger)$$

$$\mathscr{A} = \delta^{(8)}(Q^\dagger) P_n$$

Example

$$\mathscr{A}_n^{\mathrm{MHV}} = \frac{\delta^{(8)}(Q^\dagger)}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}$$

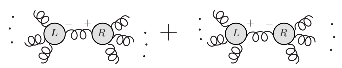
$N = 4$ supersymmetry

Supersymmetric BCFW method

- The shift must conserve supermomentum: the Grassmann variables must also be shifted
- Example: $|\hat{1}] = |1] + z|2], \quad |\hat{2}\rangle = |2\rangle - z|1\rangle, \quad \hat{\eta}_{1A} = \eta_{1A} + z\eta_{2A}$
- Sum over all possible helicities and fields in the internal propagator
- Result:

$$\mathscr{A}_n = \int d^4\eta_{\hat{I}} \hat{\mathscr{A}}_L \frac{1}{p_I^2} \hat{\mathscr{A}}_R$$

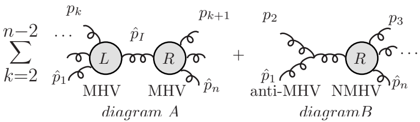
Example: gluon in the internal propagator



$$\left[\left(\prod_{A=1}^{4} \frac{\partial}{\partial \eta_{\hat{I}A}}\right) \hat{\mathscr{A}}_L\right] \frac{1}{p_I^2} \hat{\mathscr{A}}_R + \hat{\mathscr{A}}_L \frac{1}{p_I^2} \left[\left(\prod_{A=1}^{4} \frac{\partial}{\partial \eta_{\hat{I}A}}\right) \hat{\mathscr{A}}_R\right]\Bigg|_{\eta_{\hat{I}A}=0}$$

$N = 4$ supersymmetry

NMHV superamplitude



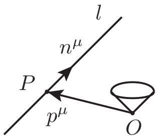
- BCFW method is efficient to compute $N^K MHV$ amplitudes
- NMHV superamplitude is:[9] $\mathscr{A}_n^{\text{NMHV}} = \mathscr{A}_n^{\text{MHV}} \sum_{j=2}^{n-3} \sum_{k=j+2}^{n-1} R_{njk}$, with

$$R_{n,j,k} = \frac{\langle j-1, j\rangle \langle k-1, k\rangle \delta^{(4)}(\Xi)}{x_{j,k}^2 \langle n|x_{nk} \cdot x_{kj}|j\rangle \langle n|x_{nk} \cdot x_{kj}|j-1\rangle \langle n|x_{nj} \cdot x_{jk}|k\rangle \langle n|x_{nj} \cdot x_{jk}|k-1\rangle}$$

$$\Xi_{njk} = \langle n|x_{nj} \cdot x_{jk}|\theta_{kn}\rangle + \langle n|x_{nk} \cdot x_{kj}|\theta_{jn}\rangle, \quad \theta_{ij} = p_i \eta_i + \cdots + p_{j-1}\eta_{j-1}, \quad x_{ij} = p_i + \cdots + p_j$$

[9] J. M. Drummond and J. M. Henn, *All tree-level amplitudes in N=4 SYM*, JHEP 04 (2009) 018

Twistors



$$L^a = \begin{pmatrix} \lambda_\alpha \\ \mu^{\dot\alpha} \end{pmatrix}$$

- Representation of the conformal group[10]
- (projective) twistor $\Longleftrightarrow$ null line in Minkowski space

$$n^\mu \to \lambda_{\alpha\dot\alpha} \to \lambda_\alpha \tilde\lambda_{\dot\alpha}, \quad \mu^{\dot\alpha} = -ip^{\dot\alpha\alpha}\lambda_\alpha$$

- Motivation: remove spurious poles resulting from BCFW, prove equivalence of different shifts

[10] Roger Penrose. *Twistor algebra*. Journal of Mathematical Physics, 8(2):345, 1967.

Twistors

space $M*$	T	C
null line	ray zL^a	point L^a
point	$c_1 L_1 + c_2 L_2$	line $M^{ab} = L_1^a L_2^b - L_1^b L_2^a$

- Conformal invariants, eg.

$$\langle L_1 L_2 L_3 L_4 \rangle = \varepsilon_{abcd} L_1^a L_2^b L_3^c L_4^d = \det \begin{pmatrix} | & | & | & | \\ L_1 & L_2 & L_3 & L_4 \\ | & | & | & | \end{pmatrix}$$

- Lorentz invariants, eg.

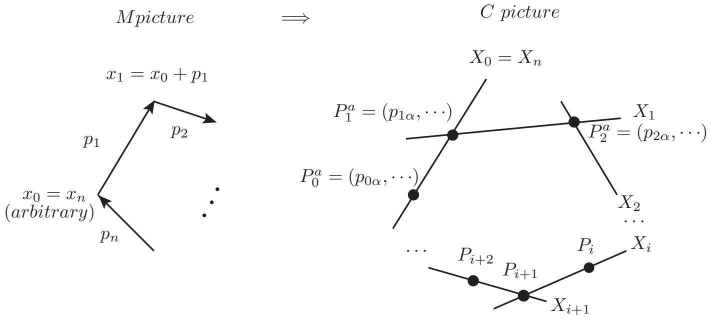
$$\langle L_1 L_2 \rangle = I_{ab} L_1^a L_2^b = \langle \lambda_1, \lambda_2 \rangle$$

- I_{ab} is the «metric twistor» representing the point at infinity
- Distance between two points in M:

$$X_1^{ab} \leftrightarrow P^{[a} Q^{b]}, \quad X_2^{ab} \leftrightarrow R^{[a} S^{b]}, \quad (x_1 - x_2)^2 = -2 \frac{\varepsilon_{abcd} P^a Q^b R^c S^d}{(I_{ab} P^a Q^b)(I_{cd} R^c S^d)}$$

Hodges construction

- Suitable[11] for theories with a cyclic ordering of the external particles, such as Yang-Mills in the planar limit



$$p_i = x_i - x_{i-1}$$

[11] Andrew Hodges. *Eliminating spurious poles from gauge-theoretic amplitudes*. JHEP, 05:135, 2013

Application

- The NMHV amplitude $A(---+++)$ is

$$\frac{[4|5+6|1\rangle^3}{[34][23]\langle 56\rangle\langle 61\rangle[2|3+4|5\rangle S_{234}} + \frac{[6|1+2|3\rangle^3}{[61][12]\langle 34\rangle\langle 45\rangle[2|3+4|5\rangle S_{612}}$$

$$S_{ijk} = \frac{1}{2}(p_i + p_j + p_k)^2$$

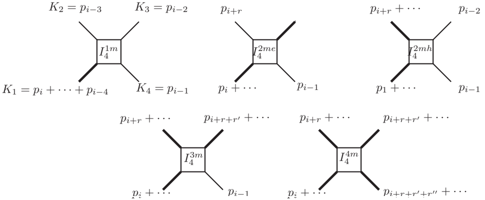
- However, $[2|3+4|5\rangle = [23]\langle 35\rangle + [24]\langle 45\rangle = 0$ is not really a pole. In twistor language the amplitude becomes

$$\frac{\langle 12\rangle^4\langle 23\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle\langle 1235\rangle}\left(\frac{\langle 1345\rangle^3}{\langle 2345\rangle\langle 1234\rangle\langle 1245\rangle} - 4 \leftrightarrow 6\right)$$

- One can prove that $\langle 1235\rangle = 0 \Longleftrightarrow$ parenthesis is 0. No pole is present

Basis of scalar integrals

- Any diagram/amplitude can be expanded in a basis of known scalar integrals with 4 external momenta at most[12]
- The computation of the amplitude boils down to the computation of the coefficients



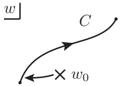
+ triangles, bubbles...

$$A_n^{(1)} = \sum_{\text{partitions } \chi} b + c_\chi (I_1^{(1)})_\chi + d_\chi (I_2^{(1)})_\chi + e_\chi (I_3^{(1)})_\chi + f_\chi (I_4^{(1)})_\chi$$

[12] W. L. van Neerven and J. A. M. Vermaseren, *Large loop integrals*, Phys. Lett. 137B (1984) 241-244.

Unitarity cuts

- Discontinuities of a function of complex variables



$$f(z) = \int_C g(z,w)dw \Longrightarrow \mathrm{disc}(f) = 2\pi i \operatorname*{Res}_{w\to w_0} g(z,w)$$

- A loop diagram has the same structure, over more variables

$$I^{(L)}(p_1,\ldots,p_n) = \int d^D\ell_1 \cdots d^D\ell_L \frac{n}{(q_1^2-m_1^2)\cdots(q_k^2-m_k^2)}$$

$$= \int dq_1^2 \frac{R_1(q_i^2,\xi,\ldots)}{(q_1^2-m_1^2)} = \int dq_1^2 g(q_1^2,\ldots)$$

$$\operatorname*{disc}_{q_1^2}\left(I^{(L)}\right) = 2\pi i \operatorname*{Res}_{q_1^2\to m_1^2} g(z,w) = 2\pi i R_1|_{q_1^2=m_1^2}$$

$$= \int dq_1^2 \left[2\pi i\delta(q_1^2-m_1^2)\right] R_1(q_i^2,\xi,\ldots)$$

Unitarity cuts

- Cutkosky rules: delete one propagator and set the related particle on shell. Diagrammatically, a «cut»
- Key idea: apply cuts to both sides of

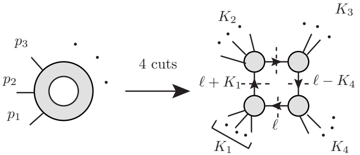
$$A_n^{(1)} = \sum_{\text{partitions } \chi} b + c_\chi (I_1^{(1)})_\chi + d_\chi (I_2^{(1)})_\chi + e_\chi (I_3^{(1)})_\chi + f_\chi (I_4^{(1)})_\chi$$

The «rational part» b cannot be computed with this method

«Box» coefficient $\longleftrightarrow$ 4 cuts

$$A_n^{(1)} = \sum b + c_\chi (I_1^{(1)})_\chi + d_\chi (I_2^{(1)})_\chi + e_\chi (I_3^{(1)})_\chi + f_\chi (I_4^{(1)})_\chi$$

- Apply cuts to $A_n^{(1)}$:



$$\ell^2 = 0, \quad (\ell + K_1)^2 = 0, \quad (\ell + K_1 + K_2)^2 = 0, \quad (\ell - K_4)^2 = 0$$

- Two solutions ℓ_i. The coefficient of I_4 is[13]

$$f = \frac{1}{2} \sum_i \left(A_1^{\text{tree}} A_2^{\text{tree}} A_3^{\text{tree}} A_4^{\text{tree}} \right) |_{\ell_i}$$

[13] R. Britto, F. Cachazo, and B. Feng, *Generalized unitarity and one-loop amplitudes in N=4 super-Yang-Mills*, Nucl. Phys. B725 (2005) 275-305.

«Triangle» coefficient ⟷ 3 cuts

$$A_n^{(1)} = \sum b + c_\chi (I_1^{(1)})_\chi + d_\chi (I_2^{(1)})_\chi + e_\chi (I_3^{(1)})_\chi + f_\chi (I_4^{(1)})_\chi$$

- Apply cuts to $A_n^{(1)}$: $\ell^2 = 0, \quad (\ell + K_1)^2 = 0, \quad (\ell + K_2)^2 = 0$
- Solutions $\ell(t)$ depend on one parameter t. [14]

$$\int dt\, J_t A_1^{\text{tree}}(t) A_2^{\text{tree}}(t) A_3^{\text{tree}}(t) = \int dt\, J_t \left(\sum_\sigma \frac{B_\sigma}{t - t^\sigma} + e \right)$$

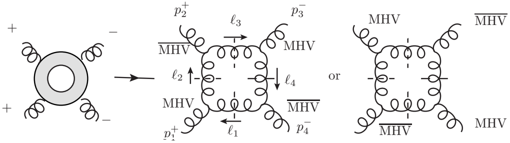
$$A_1^{\text{tree}}(t) A_2^{\text{tree}}(t) A_3^{\text{tree}}(t) = \sum_\sigma \frac{B_\sigma}{t - t^\sigma} + c_0 + c_1 t + c_2 t^2 + \cdots + \frac{c_{-1}}{t} + \frac{c_{-2}}{t^2} + \cdots$$

- Integrals over t give zero.

$$e = c_0 = \left(A_1^{\text{tree}}(t) A_2^{\text{tree}}(t) A_3^{\text{tree}}(t) - \sum_\sigma \frac{B_\sigma}{t - t^\sigma} \right)_{t=0}$$

[14] D. Forde, *Direct extraction of one-loop integral coefficients*, Phys. Rev. D75 (2007) 125019.

Example: $A(++--)$



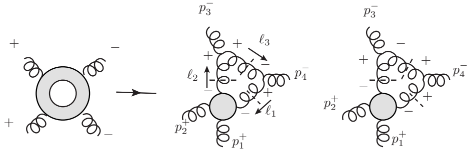
- Two solutions; one is

$$\ell_1 = \frac{\langle 12\rangle}{\langle 24\rangle}|4\rangle[1|, \quad \ell_2 = \frac{\langle 41\rangle}{\langle 42\rangle}|2\rangle[1|, \quad \ell_3 = \frac{[21]}{[31]}|2\rangle[3|, \quad \ell_4 = \frac{\langle 23\rangle}{\langle 24\rangle}|4\rangle[3|$$

- Multiply the 4 amplitudes

$$f = \frac{1}{2}\sum_i \left(A_1^{\rm tree} A_2^{\rm tree} A_3^{\rm tree} A_4^{\rm tree}\right)|_{\ell_i} = -suA^{\rm MHV}$$

Example: $A(++--)$



- Multiply the 3 amplitudes

$$\frac{\langle \ell_2 \ell_1\rangle^3}{\langle \ell_1 1\rangle\langle 12\rangle\langle 2\ell_2\rangle}\frac{[\ell_3\ell_2]^3}{[\ell_2 3][3\ell_3]}\frac{\langle \ell_3 4\rangle^3}{\langle 4\ell_1\rangle\langle \ell_1\ell_3\rangle}$$

$$=\frac{\langle 34\rangle^4[43]}{\langle 12\rangle\langle 13\rangle\langle 23\rangle\left(t+\frac{\langle 14\rangle}{\langle 13\rangle}\right)}$$

- Coefficient of this «triangle» is zero

Other coefficients

The coefficient of the «bubble» can be computed with two cuts after computing the contributions of the integrals with more legs. Not the rational part: different methods are needed, such as

- Cuts in $4+\varepsilon$ dimensions[15]
- Shifts of BCFW kind[16]
- Special «Feynman rules»[17]

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[15] C. Anastasiou, R. Britto, B. Feng, Z. Kunst, P. Mastrolia, *D-dimensional unitarity cut method*, Phys.Lett. B645 (2007) 213-216

[16] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde, and D. A. Kosower, *Bootstrapping One-Loop QCD Amplitudes with General Helicities*, Phys. Rev. D74 (2006) 036009

[17] P. Draggiotis, M. V. Garzelli, C. G. Papadopoulos and R. Pittau, *Feynman Rules for the Rational Part of the QCD 1-loop amplitudes*, JHEP 0904 (2009) 072

Conclusion

- Some of these methods (spinors, twistors) are useful for massless theories. Some can be generalized to theories with massive particles
- How can unitarity methods be extended to many loops?
- Problem: no explicit analytic basis of integrals