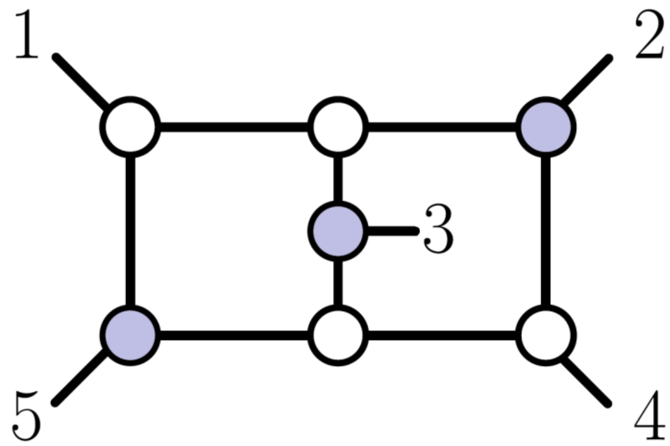
The Two-Loop Five-Point Amplitude in N=4 Super-Yang-Mills Theory and N=8 Supergravity



Lance Dixon (SLAC)

S. Abreu, LD, E. Herrmann, B. Page and M. Zeng

1812.08941, 1901.nnnnn

DESY Zeuthen, 24 January 2019

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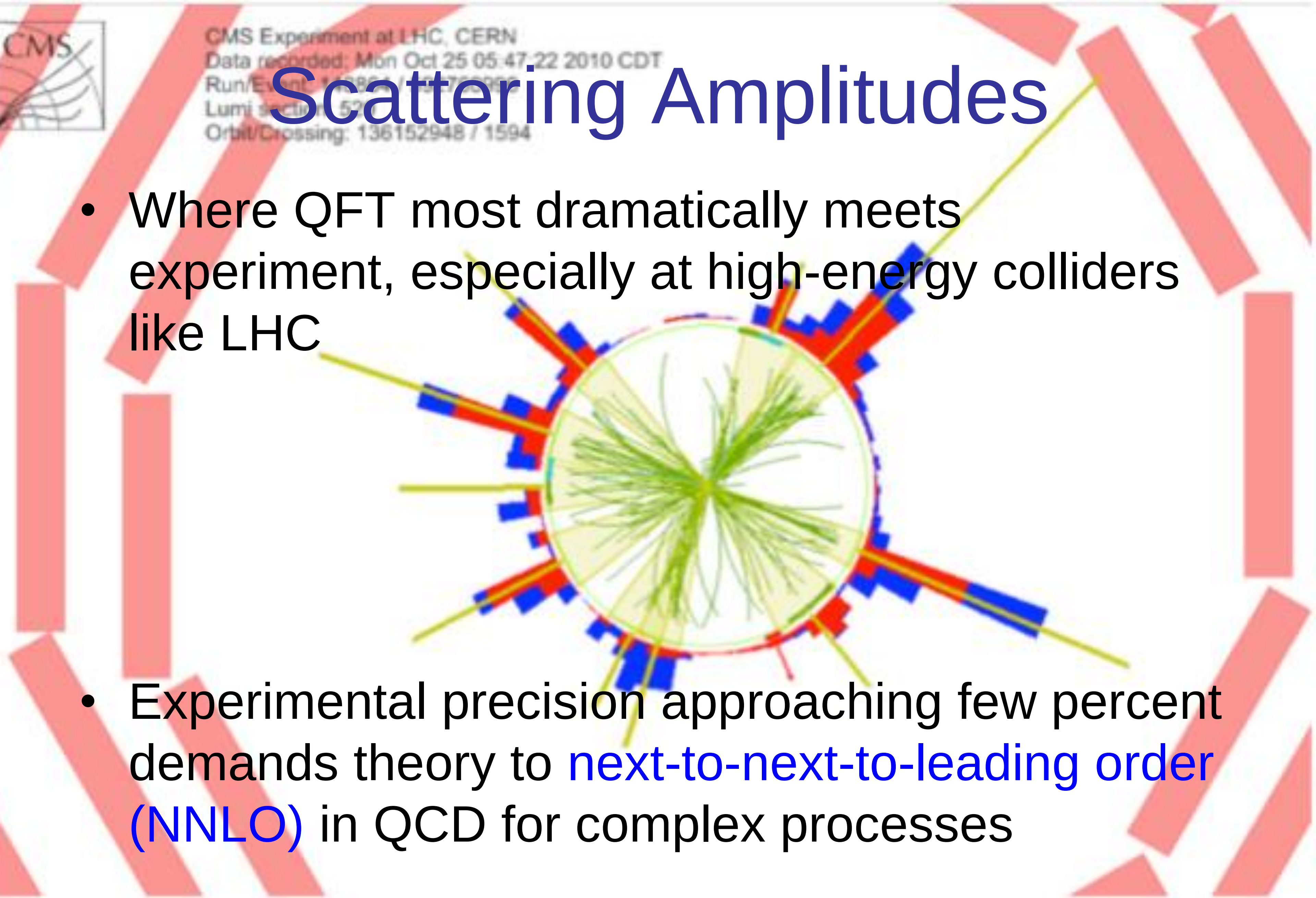
Scattering Amplitudes

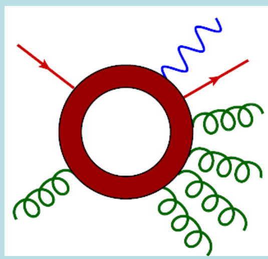


- Where QFT most dramatically meets experiment, especially at high-energy colliders like LHC

- Experimental precision approaching few percent demands theory to next-to-next-to-leading order (NNLO) in QCD for complex processes

QCD Loop Amplitude Bottleneck

- NLO: Have efficient, unitarity-based methods for computing one-loop amplitudes at high multiplicity, e.g. the 2 → 6 process $pp \to W + 5$ jets



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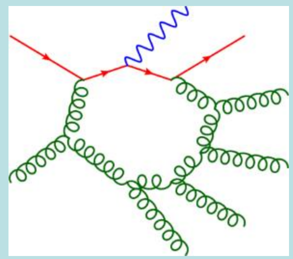
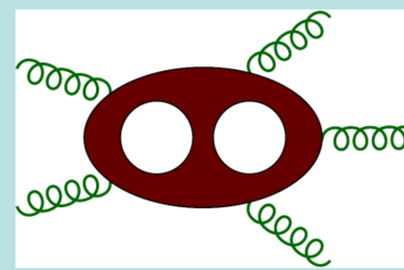


+ 256,264 more

Bern, LD, et al., 1304.1253

- NNLO: Two-loop QCD amplitudes unknown beyond 2 → 2 processes, except for one very recent 2 → 3 case ($gg \to ggg$) in large N_c (planar) limit



Badger et al., 1712.02229; Abreu et al., 1712.03946, 1812.04586

Why is two loops so hard?

- Primarily because two-loop integrals are intricate, transcendental, multi-variate functions
- In contrast, at one loop all integrals are reducible to scalar box integrals + simpler

→ combinations of dilogarithms

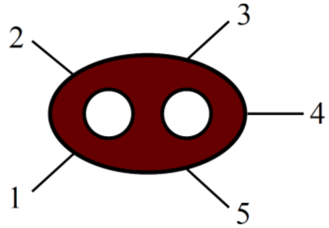
$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t)$$

+ logarithms and rational terms

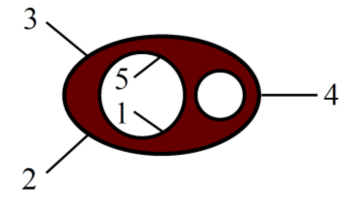
't Hooft, Veltman (1974)

A "toy" model

- Today, explore some of the complexity of multi-loop, multi-leg QCD amplitudes in a controlled setting: QCD's maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, but **NOT** in the large N_c (planar) limit.
- First two-loop amplitude for 2 → 3 processes with full color dependence – albeit still at the level of the symbol
- We expect that the same space of functions we encounter here will also be relevant for two-loop 5-point amplitudes in full-color QCD.



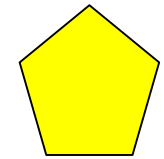
Two-loop color decomposition



Bern, Rozowsky, Yan, hep-ph/9702424

$$\mathcal{A}_5^{(2)} = \left[\frac{N_c g^2 e^{-\epsilon\gamma_E}}{(4\pi)^{2-\epsilon}}\right]^2 \Big\{ \sum_{S_5/D_5} (\mathrm{Tr}[12345] - \mathrm{Tr}[54321])\Big(A^{\mathrm{ST}}[12345] + \frac{A^{\mathrm{SLST}}[12345]}{N_c^2}\Big)$$

$$+ \sum_{S_5/(S_3\times Z_2)} \frac{\mathrm{Tr}[15](\mathrm{Tr}[234] - \mathrm{Tr}[432])}{N_c} A^{\mathrm{DT}}[15|234] \Big\}$$



$$\mathrm{Tr}[12345] \equiv \mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}T^{a_5}]$$

- Leading color coefficient A^{ST} obeys ABDK/BDS ansatz,
  Anastasiou, Bern, LD, Kosower, hep-th/0309040,
  Bern, LD, Smirnov, hep-th/0505205,
- Verified numerically long ago
  Cachazo, Spradlin, Volovich, hep-th/0602228;
  Bern, Czakon, Kosower, Roiban, Smirnov, hep-th/0604074
- Given by exponential of one-loop amplitude (need $O(\varepsilon^2)$ terms) Bern, LD, Dunbar, Kosower, hep-th/9611127

Color trace relations

Kleiss, Kuijf (1989); Bern, Kosower, (1991); Del Duca, LD, Maltoni, hep-ph/9910563; Edison, Naculich, 1111.3821

- Tree-level: $A_n[1,\ldots,n,\ldots]$ given in terms of permutations of $(n\text{-}2)!$ independent $A_n[1,\ldots,n]$ by Kleiss-Kuijf relations
- One loop: subleading-color A^{DT} completely determined by permutations of A^{ST}
- Both follow from applying Jacobi relations to all-adjoint color structures.
- Two loops: Same method → Edison-Naculich relations, which we solve as:

$$A^{\mathrm{SLST}}[12345] = 5A^{\mathrm{ST}}[13254] + \sum_{\mathrm{cyclic}} \Big[A^{\mathrm{ST}}[12435] - 2A^{\mathrm{ST}}[12453] + \frac{1}{2}\Big(A^{\mathrm{DT}}[12|345] + A^{\mathrm{DT}}[13|245] \Big) \Big]$$

Integrands

- First obtained Carrasco, Johansson, 1106.4711
  in a "BCJ" form Bern, Carrasco, Johansson, 1004.0476
  which simultaneously gives the integrand for N=8 supergravity as a "square" of the N=4 SYM integrand.
  This integrand is manifestly D-dimensional
- Integrand also given in a four-dimensional form
  Bern, Herrmann, Litsey, Stankowicz, Trnka, 1512.08591
  which exposes the expected rational prefactors for pure transcendental functions g^{DT} as 6 "KK" independent Parke-Taylor factors,

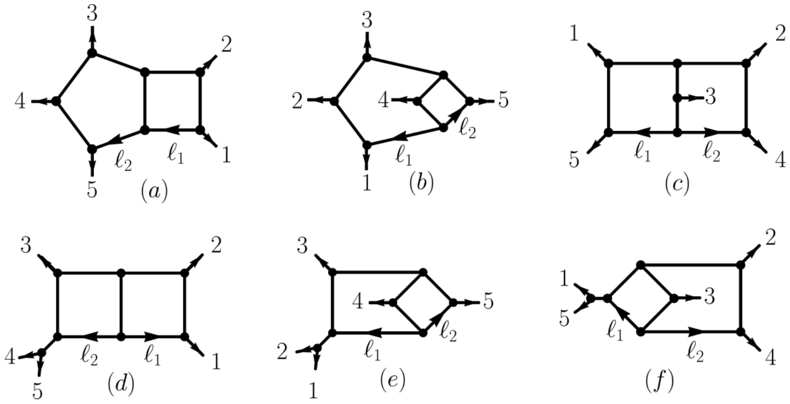
$$\mathrm{PT}[\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5] \equiv \frac{\delta^8(Q)}{\langle\sigma_1\,\sigma_2\rangle\,\langle\sigma_2\,\sigma_3\rangle\,\langle\sigma_3\,\sigma_4\rangle\,\langle\sigma_4\,\sigma_5\rangle\,\langle\sigma_5\,\sigma_1\rangle}$$

$$A^{\mathrm{DT}}[15|234] = \sum_{\sigma(234)\in S_3} \mathrm{PT}[1\sigma_2\sigma_3\sigma_4 5]\; g^{\mathrm{DT}}_{\sigma_2\sigma_3\sigma_4}$$

← pure

BCJ Integrand

Carrasco, Johansson, 1106.4711



$$\gamma_{12} \equiv \gamma_{12345} \equiv i \frac{[12][23][34][45][51]}{[12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]} = i \frac{[12][23][34][45][51]}{\mathrm{tr}_5}$$

Linear in loop momentum for N=4 SYM, quadratic for N=8 SUBRA

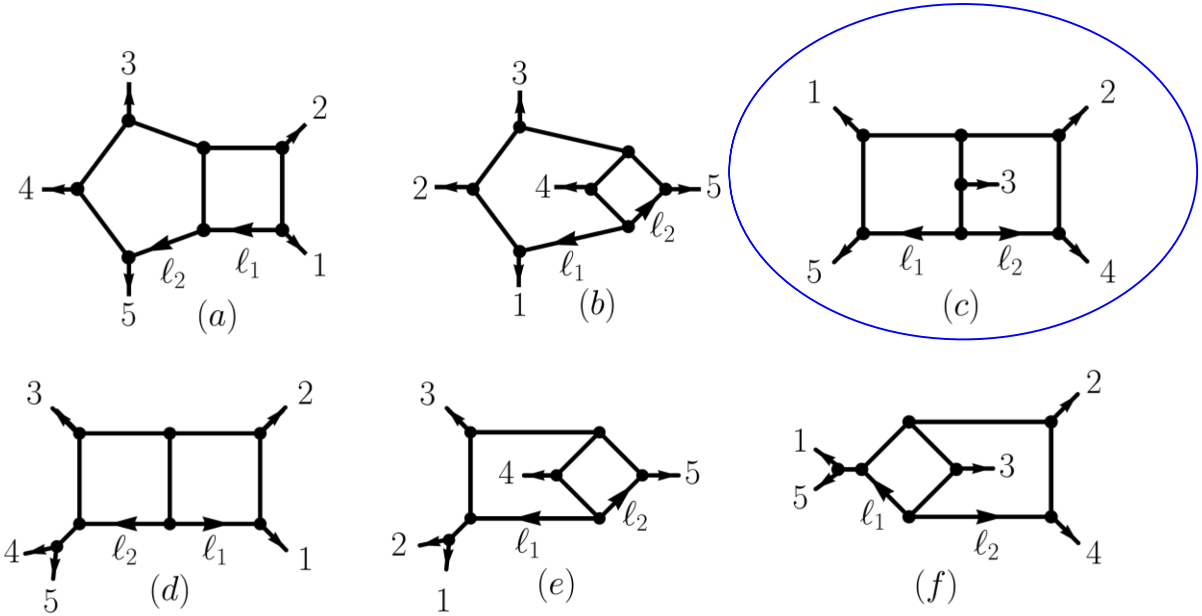
$$N^{(a,b)} = \frac{1}{4}\Big(\gamma_{12}(2s_{45} - s_{12} + \tau_{2\ell_1} - \tau_{1\ell_1}) + \gamma_{23}(s_{45} + 2s_{12} - \tau_{2\ell_1} + \tau_{3\ell_1})$$
$$+ 2\gamma_{45}(\tau_{5\ell_1} - \tau_{4\ell_1}) + \gamma_{13}(s_{12} + s_{45} - \tau_{1\ell_1} + \tau_{3\ell_1})\Big)\,,$$

$$N^{(c)} = \frac{1}{4}\Big(\gamma_{15}(\tau_{5\ell_1} - \tau_{1\ell_1}) + \gamma_{25}(s_{12} - \tau_{2\ell_1} + \tau_{5\ell_1}) + \gamma_{12}(s_{34} + \tau_{2\ell_1} - \tau_{1\ell_1} + 2[s_{15} + \tau_{1\ell_2} - \tau_{2\ell_2}])$$
$$+ \gamma_{45}(\tau_{4\ell_2} - \tau_{5\ell_2}) - \gamma_{35}(s_{34} - \tau_{3\ell_2} + \tau_{5\ell_2}) + \gamma_{34}(s_{12} + \tau_{3\ell_2} - \tau_{4\ell_2} + 2[s_{45} + \tau_{4\ell_1} - \tau_{3\ell_1}])\Big)\,,$$

$$N^{(d-f)} = \gamma_{12}s_{45} - \frac{1}{4}\Big(2\gamma_{12} + \gamma_{13} - \gamma_{23}\Big)s_{12}\,,$$

$$s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j\,, \quad \tau_{i\ell_j} = 2k_i \cdot \ell_j$$

Integrals



non-planar
double
pentagon
the crux

Most topologies were known previously, e.g.
planar (a) Papadopoulous, Tommasini, Wever, 1511.09404;
Gehrmann, Henn, Lo Presti, 1511.05409, 1807.09812;
hexabox (b) Chicherin, Henn, Mitev, 1712.09610
planar (d) Gehrmann, Remiddi, hep-ph/000827
nonplanar (e,f) Gehrmann, Remiddi, hep-ph/0101124

Integrals (cont.)

- Use IBP reduction method of Abreu, Page, Zeng, 1807.11522 building off earlier work based on generalized unitarity and computational algebraic geometry Gluza, Kajda, Kosower, 1009.0472; Ita, 1510.05626; Larsen, Zhang, 1511.01071; Abreu, Febres Cordero, Ita, Page, Zeng, 1712.03946
- Reduction performed numerically, at numerous rational phase space points, over a prime field to avoid intermediate expression swell
- Quite sufficient for full analytic reconstruction when structure of the rational function prefactors is so heavily constrained, as in N=4 SYM
- Even works for planar QCD
  Abreu, Dormans, Febres Cordero, Ita, Page, 1812.04586
- Subsequently, our results were reproduced by
  Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1812.11057, 1812.11160

Iterated integrals

Chen; Goncharov; Brown

- Generalized polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives:

$$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d \ln s_k$$

S = finite set of rational expressions, "symbol letters", and

$$f^{s_k} \equiv \{n-1,1\} \text{ component of a "coproduct" } \Delta$$

f^{s_k} are also pure functions, weight n-1

- Iterate: $d\,f^{s_k} \;\Rightarrow\; f^{s_j,s_k} \equiv \{n-2,1,1\} \text{ component}$

- Symbol = {1,1,…,1} component (maximally iterated)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example: Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalization of classical polylogs:

$$\mathrm{Li}_n(u) \;=\; \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t), \quad \mathrm{Li}_1(t) = -\ln(1-t)$$

- Define HPLs by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Just two symbol letters: $\mathcal{S} \;=\; \{u, 1-u\}$
- Weight n = length of binary string $\vec{w}$

$$\mathcal{S}[\mathrm{Li}_n(u)] \;=\; -\ (1-u) \otimes \underbrace{u \otimes u \otimes \cdots \otimes u}_{n-1}$$

Symbol alphabet for planar 5-point

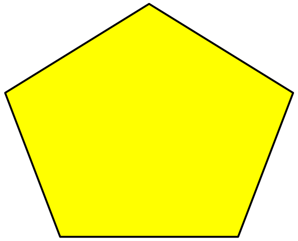
5 x 5 + 1 = 26 letters

Gehrmann, Henn, Lo Presti, 1511.05409

$$\mathcal{S} = \{s_{i,i+1}, s_{i-1,i} + s_{i,i+1}, s_{i,i+1} - s_{i+2,i+3}, s_{i+3,i+4} - s_{i,i+1} - s_{i+1,i+2}, o_i, \Delta\}$$

$$s_{i,i+1} \equiv (k_i + k_{i+1})^2 \qquad o_1 = \frac{[12]\langle 23\rangle[34]\langle 41\rangle}{\langle 12\rangle[23]\langle 34\rangle[41]}$$

$$\Delta = \mathrm{tr}[\gamma_5 1234] = [12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]$$



Closed under dihedral permutations, $\mathbf{D_5}$, subset of $\mathbf{S_5}$

O_i are odd under parity, $\langle ab\rangle \Leftrightarrow [ab]$

- Most letters seen already in one-mass four-point integrals
- But not O_i or Δ

Symbol alphabet for nonplanar 5-point

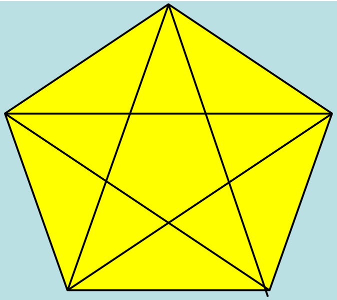
10 + 15 + 5 + 1 = 31 letters

Chicherin, Henn, Mitev, 1712.09610

$$\mathcal{S} = \{ s_{i,j},\ s_{i,j} - s_{k,l},\ o_i,\ \Delta \}$$

$$o_1 = \frac{[12]\langle 23\rangle[34]\langle 41\rangle}{\langle 12\rangle[23]\langle 34\rangle[41]} \qquad \Delta = \mathrm{tr}[\gamma_5 1234] = [12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]$$

- Obtained by applying full $\mathbf{S_5}$ to planar alphabet; only generates 5 new letters
- However, function space is much bigger because branch-cut condition now allows 10 first entries, $s_{i,j} \equiv (k_i + k_j)^2$



- In planar case there were only 5, $s_{i,i+1} \equiv (k_i + k_{i+1})^2$

Nonplanar 5-point function space

Chicherin, Henn, Mitev, 1712.09610

- Also empirical constraint on first 2 entries of the symbol.
- Imposing this and integrability, dimension of even | odd part of function space is:

Weight	1	2	3	4
# of integrable symbols for $\mathbb{A}_{\mathrm{P}}$	$5 \| 0$	$25 \| 0$	$125 \| 1$	$635 \| 16$
after 2nd entry condition	$5 \| 0$	$20 \| 0$	$80 \| 1$	$335 \| 11$
# of integrable symbols for $\mathbb{A}_{\mathrm{NP}}$	$10 \| 0$	$100 \| 9$	$1000 \| 180$	$9946 \| 2730$
after 2nd entry condition	$10 \| 0$	$70 \| 9$	$505 \| 111$	$3736 \| 1191$

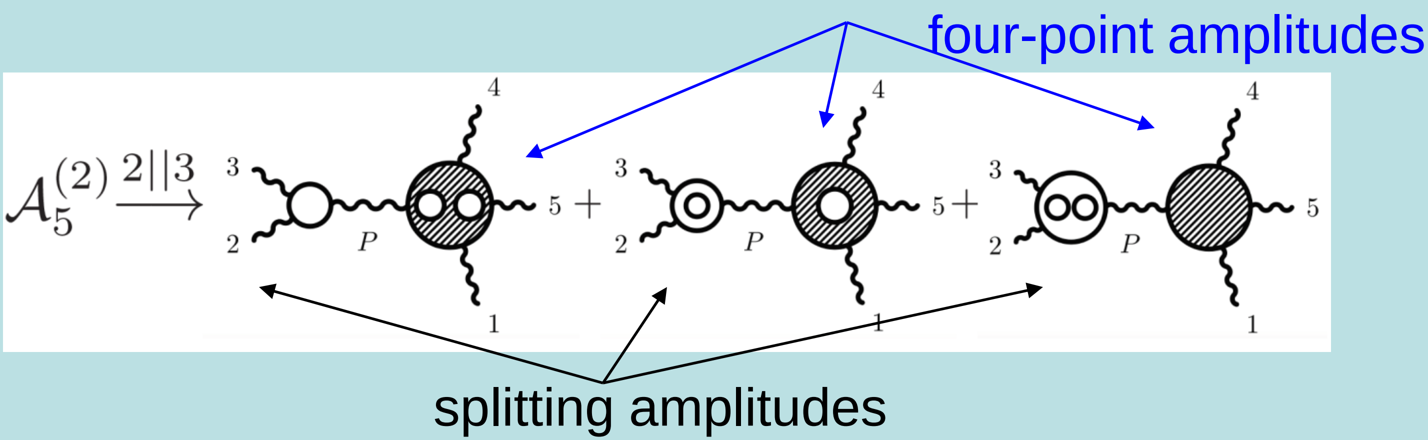
- The SYM and SUGRA amplitudes both lie in this space

Numerical reduction and assembly

- Given decomposition into 6 PT factors, suffices to perform reduction to master integrals at 6 numerical kinematic points
- Use mod p arithmetic with p a 10-digit prime; reconstruct rational numbers using Wang's algorithm Wang (1981); von Manteuffel, Schabinger, 1406.4513; Peraro, 1608.01902
- Inserting symbols of all master integrals, we obtain symbols of all the pure functions
- Basic result is for g_{234}^{DT} , but also recover M^{BDS}, where $A^{ST}[12345] = \mathrm{PT}[12345]\, M^{BDS}$
- Also computed $A^{SLST}[12345]$, so color algebra could be checked via Edison-Naculich relations

Validation

- Five-point gauge theory amplitudes have a stringent set of limiting behavior as a gluon becomes soft or two partons become collinear.
- E.g. as legs 2 and 3 become collinear:



- We checked the collinear limit, as well as the soft limit, and the IR poles in ε which are predicted by Catani, hep-ph/9802439; Bern, LD, Kosower, hep-ph/0404293

Structure

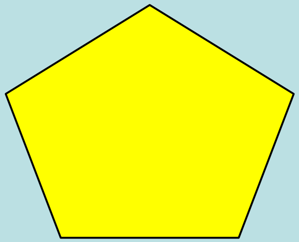
- Symbols are large: M^{BDS} (planar) has "only" 2,365 terms, while g_{234}^{DT} (nonplanar) has 24,653 terms.
- How many functions are there in the full amplitude?
- Take linear span of all 120 permutations of g_{234}^{DT} and M^{BDS}
- At order ε^0, there are 52 weight 4 functions.
  Naively there should be 72 = 12 (planar) + 6 ·10 (nonplanar)
- So there are 20 relations among the permutations, e.g.

$$g[12345] + g[12453] + g[12534] + g[21345] + g[21453] + g[21534]$$
$$-g[12435] - g[12543] - g[12354] - g[21435] - g[21543] - g[21354] = 0$$

- The 20 relations are also obeyed by the lower-weight $1/\varepsilon$ pole terms.
- What do they mean? Do they reflect a nonplanar version of dual conformal invariance or integrated BCJ relations?

Structure (cont.)

- Take first derivatives, i.e. {3,1} coproducts.
- How many functions are there?
- Weight 3 even: 362 (out of a possible 505).
- But only 40 of them have (two) odd letters. Rest simple.
- Weight 3 odd is even more restricted: only 12 (out of a possible 111)
- They are just the 12 $\mathbf{S_5/D_5}$ permutations of the D=6 one-loop pentagon integral!!



- Weight 2 is not restricted at all; the {2,1,1} coproducts include all 70 even and 9 odd functions obeying the second entry condition.

Structure (cont.)

- Simplicity of weight 3 odd space lets us present the odd part of the derivative of the odd part of the basic double trace function:

$$\frac{\partial}{\partial x_i}\left[g_{234}^{\mathrm{DT,odd}}\right]\Big|_{\mathrm{odd}} = \sum_{j=1}^{12} \mathcal{I}_5^{d=6}(\Sigma_j) \sum_{\gamma} m_{j\gamma} \frac{\partial \log W_\gamma}{\partial x_i}$$

- $\gamma \in \{1, \ldots, 5, 16, \ldots, 20, 31\} = \{s_{ij}, \Delta\}$ only, $\Sigma_j \in S_5/D_5$
- {3,1} coproduct matrix $m_{j\gamma}$ on next page

{3,1} coproduct matrix $m_{j\gamma}$

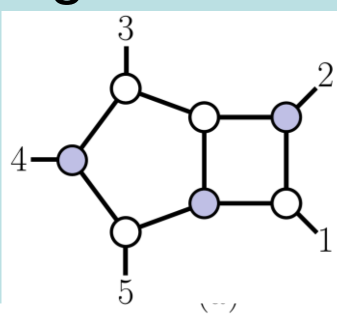
$$m_{j\gamma} = \begin{pmatrix}
-\frac{17}{4} & -\frac{5}{4} & -6 & -\frac{17}{4} & -\frac{7}{2} & -\frac{17}{4} & -\frac{7}{4} & \frac{1}{2} & -1 & -\frac{17}{4} & 10 \\
\frac{17}{4} & \frac{5}{4} & \frac{5}{4} & \frac{17}{4} & 4 & \frac{17}{4} & \frac{11}{2} & \frac{17}{4} & \frac{1}{2} & \frac{1}{2} & -10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 \\
0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\
-\frac{17}{4} & -6 & -\frac{5}{4} & -\frac{17}{4} & -\frac{7}{2} & \frac{1}{2} & -\frac{7}{4} & -\frac{17}{4} & -\frac{17}{4} & -1 & 10 \\
-\frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\
\frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} & 1 & 0 & 0 \\
0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{4} & \frac{1}{4} & 0 & 1 & 0 \\
-\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\
0 & \frac{1}{4} & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} & -\frac{1}{2} & 0 \\
\frac{17}{4} & 6 & 6 & \frac{17}{4} & 9 & -\frac{1}{2} & 4 & -\frac{1}{2} & -\frac{5}{4} & -\frac{5}{4} & -10
\end{pmatrix}$$

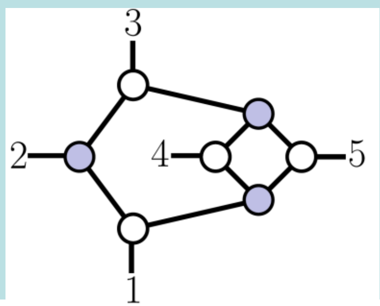
rank 8 → only 8 independent linear combinations of final entries appear, not 11

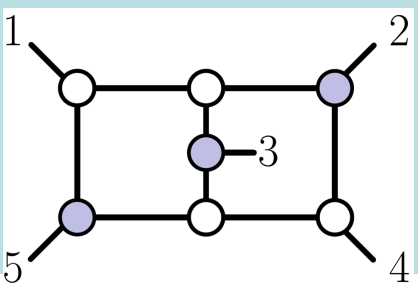
N=8 supergravity

Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia, 1901.05932;
Abreu, LD, Herrmann, Page, Zeng, 1901.nnnnn

- Same integration methods can be applied to the double-copy N=8 supergravity integrand
  Carrasco, Johansson, 1106.4711
- Loop-momentum numerator is quadratic instead of linear in the loop momentum
- Richer set of rational function prefactors.
- 40 can be inferred from four-dimensional leading singularities computed from on-shell diagrams, e.g.





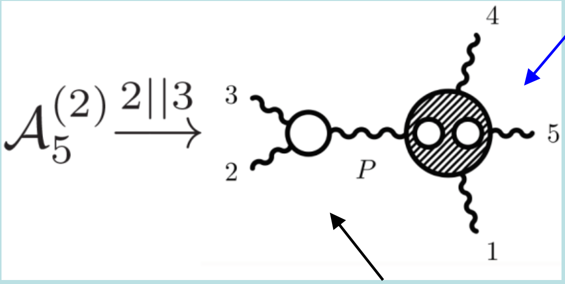


N=8 supergravity (cont.)

- After reductions for > 45 phase space points, discover 5 additional rational structures (*D*-dim'l leading sing's)
- Result has uniform transcendentality
- Because there is no color, there are exactly 45 pure function components to the amplitude
- 5 of the 45 are removed by a natural IR subtraction.
- We can compare the 45 functions to the 52 for N=4 SYM: They overlap a lot; their span has dimension 62
- The weight 3 odd part of the {3,1} coproduct space has dimension 11, not 12, because the S_5 symmetric pentagon sum $\sum_{j=1}^{12} \mathcal{I}_5^{d=6}(\Sigma_j)$ drops out

Validation

- Five-point gravity amplitudes have a stringent set of limiting behavior as a graviton becomes soft
  Weinberg (1965); Berends, Giele, Kuijf (1988);
  Bern, LD, Perelstein, Rozowsky, hep-th/9811140
  or two gravitons become collinear
  Bern, LD, Perelstein, Rozowsky, hep-th/9811140
- E.g. as legs 2 and 3 become collinear:



four-point two-loop amplitude only

tree splitting amplitude only

- We checked the collinear limit, as well as the soft limit, and the IR poles in ε which are predicted by
  Weinberg (1965); Naculich, Nastase, Schnitzer, 0805.2347

Structure

- Give same odd, odd {3,1} coproduct matrix as in N=4 SYM, but now for a component of the N=8 finite remainder:

$$m_{j\alpha_1} = \frac{1}{12}\begin{pmatrix} -3 & -2 & 2 & 2 & -2 & 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & -1 & -3 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & -2 & 1 & 0 & 5 & 0 & 1 & 0 \\ 3 & 0 & -3 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & -2 & 4 & -3 & 0 & -3 & 0 & 1 & 0 \\ -3 & -1 & 1 & 3 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -3 & 2 & -1 & 3 & -3 & 2 & 0 & 3 & 0 & -3 & 0 \\ 3 & 4 & -2 & 0 & 0 & 1 & 0 & -3 & 0 & -3 & 0 \\ 3 & -1 & 0 & 0 & -1 & 2 & 0 & -5 & 0 & 2 & 0 \\ -3 & 0 & 3 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ -3 & -3 & 3 & -1 & 2 & -3 & 0 & 3 & 0 & 2 & 0 \\ 3 & 2 & -2 & -2 & 2 & -1 & 0 & -1 & 0 & -1 & 0 \end{pmatrix}$$

rank 5 → only 5 independent linear combinations of final entries!

Conclusions

- Two-loop five-point nonplanar amplitudes now available at symbol level in maximally supersymmetric theories
- Also, all required master integrals needed for QCD have now been computed at symbol level
- Results expressed in terms of a symbol alphabet with 26 parity even and 5 parity odd letters
  Chicherin, Henn, Mitev, 1712.09610
- Still more to explore about how the amplitudes' pure functions sit inside this function space.
- Also, need to promote symbols → functions
- Opens the door to full-color 2 → 3 massless QCD amplitudes for e.g. NNLO 3 jet production at hadron colliders