Infrared subtraction: a new local analytic method beyond NLO

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LHC is …

- a hadron machine → QCD-based processes
- a high-energy machine → complex processes
- entering a high-precision phase → theory must follow
- searching new physics → must control SM background

High precision computation in QCD needed

- PDFs, resumation, parton shower, hadronization and …
- … fixed order computations

Ambitious goal: Automatic NNLO QCD computations

- Loop computations and …
- … cancellation of soft and collinear singularities → this talk

Well established subtraction schemes at NLO

- Frixione-Kunst-Signer (FKS) subtraction — Frixione, Kunszt, Signer
- Catani-Seymour (CS) Dipole subtraction — Catani, Seymour
- Nagy-Soper subtraction — Nagy, Soper

Many methods available at NNLO

- Antenna subtraction — Gehrmann De Ridder, Gehrmann, Glover, Heinrich, et al.
- Sector-improved residue subtraction — Czakon et al.; Melnikov et al.
- Colourful subtraction — Del Duca, Duhr, Kardos, Somogyi, Troscanyi, et al.
- qT-slicing — Catani, Grazzini, et al.
- N-jettiness slicing — Boughezal, Petriello, et al.
- Projection to Born — Cacciari, Salam, Zanderighi, et al.
- Sector decomposition — Anastasiou, Binoth, et al.
- ε-prescription — Frixione, Grazzini
- Unsubtraction — Rodrigo et al.
- Geometric — Herzog

Why to look for a new method?

- NNLO methods are still not fully general:
  - are they really process-independent?
  - can be automatized?
  - are they efficient?
  - are they local?
  - how they scale with the number of legs?

- More fundamental questions:
  - Is there anything simpler?
  - Are we using all freedom we have in defining subtraction?
  - Can we learn something on subtraction systematically?
  - Can we hope to manage extensions to higher orders?
  - Can we get all-order insights on subtraction from IRC factorisation?

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- Understand the structure of real radiation amplitudes from factorization principles

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- Search for a “minimal” subtraction procedure at NNLO:
  - * We have well established methods at NLO:
    - Frixione-Kunst-Signer (FKS) subtraction — Frixione, Kunszt, Signer 9512328; Frixione 9706545
    - Catani-Seymour (CS) Dipole subtraction — Catani, Seymour 9605323; Catani et al. 0201036
    - Nagy-Soper subtraction — Nagy, Soper, 0308127

Rationale of our approach

- Understand the structure of real radiation amplitudes from factorization principles → not in this talk
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    - Nagy-Soper subtraction — Nagy, Soper, 0308127
  - Understand their basic features
  - Try to find a simpler subtraction at NLO, by merging them
  - Then generalize to NNLO

Structure of subtraction at NLO

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n V\, \delta_{X_n} + \int d\Phi_{n+1} R\, \delta_{X_{n+1}} = \text{finite.}$$

X = IRC safe observable $\qquad \delta_{X_m} = \delta(X - X_m)$

X_m = observable computed with m-body kinematics

V has explicit poles in ϵ, R diverges in phase space integration

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* Introduce counterterms K and their integral I

$$\int d\Phi_{n+1}\,K\,\delta_{X_n} = \int d\Phi_n\,I\,\delta_{X_n}$$

$$\frac{d\sigma_{\rm NLO} - d\sigma_{\rm LO}}{dX} = \int d\Phi_n (V + I)\,\delta_{X_n} + \int d\Phi_{n+1} \Big(R\,\delta_{X_{n+1}} - K\,\delta_{X_n} \Big)$$

V+I is finite in ϵ, R-K converges in phase space integration

Some notations

Center of mass (CM) momentum: $\mathrm{q} = (\sqrt{s}, \vec{0})$

$$s_{\mathrm{q}i} = 2\,\mathrm{q}\cdot k_i \qquad s_{ij} = (k_i + k_j)^2 = 2k_i \cdot k_j$$

$$s_{ijk} = (k_i + k_j + k_k)^2$$

$$s_{ijkl} = (k_i + k_j + k_k + k_l)^2$$

$\mathcal{E}_i = \dfrac{s_{\mathrm{q}i}}{s}$ = rescaled energy of particle i in CM frame

$$w_{ij} = \frac{s\, s_{ij}}{s_{\mathrm{q}i}\, s_{\mathrm{q}j}} = \frac{1 - \cos\theta_{ij}}{2}$$

θ_{ij} = angle between i and j in CM frame

* In the following we consider massless QCD just in final state

Primary IRC limits at NLO

* Soft limit:

$$\mathbf{S}_i \quad \Leftrightarrow \quad k_i^\mu \to 0 \quad \Rightarrow \quad \mathcal{E}_i \to 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{s_{ih}}{s_{kl}} \to 0 & (k, l \neq i) \\ \dfrac{s_{ik}}{s_{il}} \to \text{finite} & (k, l \neq i) \end{cases}$$

Limit on the real matrix element:

$$\mathbf{S}_i \, R(\{k\}) \quad = \quad -\mathcal{N}_1 \sum_{k \neq i,\, l \neq i} \frac{s_{kl}}{s_{ik} s_{il}} B_{kl}\big(\{k\}_{\not i}\big)$$

$$\mathcal{N}_1 = 8\pi\alpha_s \left(\frac{\mu^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon}$$

Primary IRC limits at NLO

* **Collinear limit:** $k^\mu = k_i^\mu + k_j^\mu \qquad a = i, j$

Sudakov parametrization

$$\bar{k}^\mu = k^\mu - \frac{k^2}{2k\cdot r} r^\mu \qquad z_a = \frac{2k_a\cdot r}{2k\cdot r} \qquad \tilde{k}_a^\mu = k_a^\mu - z_a k^\mu - \left(\frac{2k\cdot k_a}{k^2} - 2z_a\right)\frac{k^2}{2k\cdot r} r^\mu$$

$$\bar{k}^2 = 0 \qquad z_i + z_j = 1 \qquad \tilde{k}\cdot\bar{k} = \tilde{k}\cdot r = 0 \qquad \tilde{k}_i^\mu + \tilde{k}_j^\mu = 0$$

$$k_a^\mu = z_a \bar{k}^\mu + \tilde{k}_a^\mu - \frac{1}{z_a}\frac{\tilde{k}_a^2}{2k\cdot r} r^\mu$$

$$\mathbf{C}_{ij} \quad \Leftrightarrow \quad \tilde{k}_i^\mu \to 0 \quad \Rightarrow \quad w_{ij} \to 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{s_{ij}}{s_{kl}} \to 0 & (kl \neq ij,\ k \neq l) \\ \dfrac{s_{ik}}{s_{jk}} \to \text{independent on } k & (k \neq i, j) \end{cases}$$

Limit on the real matrix element:

$$\mathbf{C}_{ij}\, R(\{k\}) = \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij}\, B(\{k\}_{\not{i}\not{j}}, k) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}(\{k\}_{\not{i}\not{j}}, k)\right]$$

$$Q_{ij}^{\mu\nu} = Q_{ij}\left[-g^{\mu\nu} + (d-2)\frac{\tilde{k}_i^\mu \tilde{k}_i^\nu}{\tilde{k}_i^2}\right]$$

Derived IRC limits at NLO

* Soft-collinear limit:

$$z_i = \frac{s_{ir}}{s_{ir}+s_{jr}} \xrightarrow[\mathbf{S}_i]{} 0 \qquad z_j = \frac{s_{jr}}{s_{ir}+s_{jr}} \xrightarrow[\mathbf{S}_i]{} 1$$

$$\frac{z_j}{z_i} = \frac{s_{jr}}{s_{ir}} \xrightarrow[\mathbf{S}_i]{} \frac{s_{jr}}{s_{ir}} = \frac{z_j}{z_i} \qquad \frac{z_i}{z_j} \xrightarrow[\mathbf{S}_i]{} 0$$

$$P_{ij} \xrightarrow[\mathbf{S}_i]{} \frac{2\,C_{f_j}}{s_{ij}} \frac{z_j}{z_i} \delta_{f_i g}$$

$$\frac{s_{kl}}{s_{ik}s_{il}} \xrightarrow[\mathbf{C}_{ij}]{} \mathcal{O}(1) \quad \text{if } k, l \neq j$$

$$\frac{s_{jl}}{s_{ij}s_{il}} \xrightarrow[\mathbf{C}_{ij}]{} \frac{1}{s_{ij}} \frac{z_j}{z_i} \qquad \frac{s_{kj}}{s_{ik}s_{ij}} \xrightarrow[\mathbf{C}_{ij}]{} \frac{1}{s_{ij}} \frac{z_j}{z_i}$$

$$\mathbf{S}_i\,\mathbf{C}_{ij}\,R(\{k\}) = \mathbf{C}_{ij}\,\mathbf{S}_i\,R(\{k\}) = \mathcal{N}_1 \frac{2\,C_{f_j}}{s_{ij}} \frac{z_j}{z_i} B(\{k\}_{\not i})\,\delta_{f_i g}$$

FKS subtraction procedure

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- Divide the phase space through sector functions

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$$\mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sigma} \qquad \sigma_{ij} = \frac{1}{\mathcal{E}_i\, w_{ij}} = \frac{s_{\mathrm{q}j}}{s_{ij}} \qquad \sigma = \sum_{i,j\neq i} \sigma_{ij}$$

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* Basic properties:

$$\boxed{\sum_{i,j\neq i} \mathcal{W}_{ij} = 1}$$

$$\mathbf{S}_i\, \mathcal{W}_{ij} = \frac{\frac{1}{w_{ij}}}{\sum_{j'\neq i} \frac{1}{w_{ij'}}} \qquad \boxed{\sum_{j\neq i} \mathbf{S}_i\, \mathcal{W}_{ij} = 1}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ij} = \frac{\mathcal{E}_j}{\mathcal{E}_i + \mathcal{E}_j} \qquad \boxed{\mathbf{C}_{ij}\, \mathcal{W}_{ij} + \mathbf{C}_{ij}\, \mathcal{W}_{ji} = 1}$$

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Sector $\mathcal{W}_{ij}$

$$d\Phi_{n+1}(\{k\}) = d\Phi_n(\{\bar{k}\}_{\not i \not j}, \bar{k}) d\Phi_1(s, \zeta; \mathcal{E}_i, w_{ij}, \phi)$$

$$\int d\Phi_1(s, \zeta; \mathcal{E}_i, w_{ij}, \phi) \quad = \quad G\, s^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^\zeta d\mathcal{E}_i \int_0^1 dw_{ij} \left[\frac{\mathcal{E}_i^2 (\zeta - \mathcal{E}_i)^2 w_{ij} (1 - w_{ij})}{\zeta^2 (1 - \mathcal{E}_i w_{ij})^2} \right]^{-\epsilon} \frac{\mathcal{E}_i (\zeta - \mathcal{E}_i)}{\zeta (1 - \mathcal{E}_i w_{ij})^2}$$

$$\zeta = \frac{2\bar{k} \cdot q}{s} \qquad G = \frac{(4\pi)^{\epsilon - 2}}{\pi^{1/2} \Gamma(1/2 - \epsilon)}.$$

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$$\mathcal{E}_i^{1-2\epsilon}\, w_{ij}^{-\epsilon}\, R \;=\; \mathcal{E}_i^{-1-2\epsilon}\, w_{ij}^{-1-\epsilon}\, [\mathcal{E}_i^2\, w_{ij} R]$$

$$= \left[-\frac{1}{2\epsilon}\delta(\mathcal{E}_i) + \left(\frac{1}{\mathcal{E}_i} - 2\epsilon\frac{\ln \mathcal{E}_i}{\mathcal{E}_i}\right)_+\right]\left[-\frac{1}{\epsilon}\delta(w_{ij}) + \left(\frac{1}{w_{ij}}\right)_+\right][\mathcal{E}_i^2\, w_{ij} R]$$

- Terms containing **δ**'s $\longrightarrow$ $I\,\delta_{X_n}$
- Remaining term $\longrightarrow$ $R\,\delta_{X_{n+1}} - K\,\delta_{X_n}$

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$$\sum_{i,j\neq i} \delta(\mathcal{E}_i)\, \mathcal{W}_{ij} = \sum_i \delta(\mathcal{E}_i)\, \underbrace{\sum_{j\neq i} \mathbf{S}_i\, \mathcal{W}_{ij}}_{1}$$

$$\begin{aligned}\sum_{i,j\neq i} \delta(w_{ij})\, \mathcal{W}_{ij} &= \sum_{i,j>i} \delta(w_{ij})\, (\mathcal{W}_{ij} + \mathcal{W}_{ji}) \\ &= \sum_{i,j>i} \delta(w_{ij})\, \underbrace{\mathbf{C}_{ij}\, (\mathcal{W}_{ij} + \mathcal{W}_{ji})}_{1}\end{aligned}$$

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* The integration of some counterterms can be non trivial:

$$\int d\Phi_1 \sum_j K_{ij}^{(\mathrm{soft})} \sim \sum_{kl} \int d\bar{\Omega}_i \frac{1 - \cos\bar{\theta}_{kl}}{(1 - \cos\bar{\theta}_{ki})(1 - cos\bar{\theta}_{il})}$$

* Sector parametrization not always optimal

* Can one do something simpler?

CS subtraction procedure

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$$K = \sum_{\text{pairs } ij} \sum_{k \neq i,j} K_{ijk}$$

$$K_{ijk}(\{k\}) = \frac{\mathcal{N}_1}{s_{ij}} \left[V^{[ij]k} B_{[ij]k}(\{k\}_{\not{i}\not{j}\not{k}}, \bar{k}, \bar{r}) + V^{[ij]k}_{\mu\nu} B^{\mu\nu}_{[ij]k}(\{k\}_{\not{i}\not{j}\not{k}}, \bar{k}, \bar{r}) \right]$$

$$\bar{k}^\mu = k_i^\mu + k_j^\mu - \frac{s_{ij}}{s_{ik} + s_{jk}} k_k^\mu \qquad \bar{r}^\mu = \frac{s_{ijk}}{s_{ik} + s_{jk}} k_k^\mu$$

* $V^{[ij]k}$ and $V^{[ij]k}_{\mu\nu}$ need to reproduce **both** soft and collinear limits:

$$\mathbf{S}_i \, V^{[ij]k} = \frac{s_{jk}}{s_{ij} + s_{ik}} \qquad \mathbf{S}_i \, V^{[ij]k}_{\mu\nu} = 0$$

$$\mathbf{C}_{ij} \, V^{[ij]k} B_{[ij]k} = -P_{ij} \, B \qquad \mathbf{C}_{ij} \, V^{[ij]k}_{\mu\nu} B^{\mu\nu}_{[ij]k} = -Q^{\mu\nu}_{ij} \, B_{\mu\nu}$$

CS subtraction procedure

- Counterterms mimic the IRC behaviour in **all** phase space
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- Phase space reparametrized differently for each term of the sum

$$d\Phi_{n+1}(\{k\}) = d\Phi_n(\{k\}_{\not{i}\not{j}\not{k}}, \bar{k}, \bar{r}) d\Phi_1(p^2; y, z, \phi)$$

$$p^2 = (k_i + k_j + k_k)^2 = (\bar{k} + \bar{r})^2$$

$$y = \frac{s_{ij}}{p^2} \qquad\qquad z = \frac{s_{ik}}{s_{ik} + s_{jk}}$$

$$\int d\Phi_1(p^2; y, z, \phi) = G\,(p^2)^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, \left[y\, z(1-y)^2(1-z)\right]^{-\epsilon} (1-y)$$

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* Integration can be non trivial if counterterms are complicated

* Can one introduce simpler counterterms?

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FKS subtraction procedure

- Divide the phase space through sector functions ✓
- Phase space reparametrized differently for each sector ✗
- Identify counterterms through reparametrization ✗
- Integrate analytically after getting rid of sector functions ✓

CS subtraction procedure

- Counterterms mimic the IRC behaviour in **all** phase space ✗
- Counterterms are sums of terms, each with its remapped momenta ✓
- Phase space reparametrized differently for each term of the sum ✓
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A “minimal” subtraction procedure at NLO

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Sector $\mathcal{W}_{ij}$

$\mathbf{S}_i$ and $\mathbf{C}_{ij}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})R\,\mathcal{W}_{ij} \;=\; R\,\mathcal{W}_{ij} - K_{ij} \quad \longrightarrow \text{ finite}$$

Candidate for the counterterm:

$$K_{ij} = \Big[1-(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})\Big]R\,\mathcal{W}_{ij} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]R\,\mathcal{W}_{ij}$$

What is not satisfactory?

Momenta in $\mathbf{S}_i R\,,\mathbf{C}_{ij} R\,,\mathbf{S}_i\mathbf{C}_{ij} R$ do not satisfy

mass-shell condition and momenta conservation

$$B_{\mu\nu}(\{k\}_{\not i \not j}, k) \qquad\qquad B_{kl}(\{k\}_{\not i})$$

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Sector $\mathcal{W}_{ij}$

Counterterm $$\overline{K}_{ij} = \left[\overline{\mathbf{S}}_i + \overline{\mathbf{C}}_{ij}(1 - \overline{\mathbf{S}}_i)\right] R \, \mathcal{W}_{ij}$$

What are $\bar{\mathbf{S}}_i R$, $\bar{\mathbf{C}}_{ij} R$ and $\bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij} R$?

The same as $\mathbf{S}_i R$, $\mathbf{C}_{ij} R$ and $\mathbf{S}_i \mathbf{C}_{ij} R$, but …

… with **remapped momenta in the Born matrix element**

They must satisfy:

$$\mathbf{S}_i \bar{\mathbf{S}}_i R = \mathbf{S}_i R \qquad \mathbf{C}_{ij} \bar{\mathbf{C}}_{ij} R = \mathbf{C}_{ij} R$$

$$\mathbf{S}_i \bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij} R = \mathbf{S}_i \bar{\mathbf{C}}_{ij} R \qquad \mathbf{C}_{ij} \bar{\mathbf{S}}_i \bar{\mathbf{C}}_{ij} R = \mathbf{C}_{ij} \bar{\mathbf{S}}_i R$$

such that:

$$R \, \mathcal{W}_{ij} - \overline{K}_{ij} \longrightarrow \text{finite}$$

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$$\overline{\mathbf{C}}_{ij}\, R \;=\; \frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, B\Big(\{\bar{k}\}^{(ijr)}\Big) + Q_{ij}^{\mu\nu}\, B_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\Big]$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac}+s_{bc}} k_c \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac}+s_{bc}} k_c$$

$\bar{\mathbf{C}}_{ij}R$ and $\bar{\mathbf{S}}_i R$ are the same as $\mathbf{S}_i R$ and $\mathbf{C}_{ij} R$,

with momenta satisfying on-shell condition and momenta conservation

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$$\overline{\mathbf{S}}_i \, R \quad = \quad - \mathcal{N}_1 \sum_{k \neq i, \, l \neq i} \frac{s_{kl}}{s_{ik} s_{il}} B_{kl} \left(\{ \bar{k} \}^{(ikl)} \right)$$

$$\overline{\mathbf{C}}_{ij} \, R \quad = \quad \frac{\mathcal{N}_1}{s_{ij}} \left[P_{ij} \, B \left(\{ \bar{k} \}^{(ijr)} \right) + Q_{ij}^{\mu\nu} \, B_{\mu\nu} \left(\{ \bar{k} \}^{(ijr)} \right) \right]$$

$$\bar{k}_b^{(abc)} = k_a + k_b - \frac{s_{ab}}{s_{ac} + s_{bc}} k_c \qquad \bar{k}_c^{(abc)} = \frac{s_{abc}}{s_{ac} + s_{bc}} k_c$$

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$$d\Phi_{n+1}(\{k\}) = d\Phi_n\left(\{\bar{k}\}^{(abc)}\right) d\Phi_1(s_{abc}; y, z, \phi)$$

$$s_{ab} = y\, s_{abc} \qquad s_{ac} = z(1-y)\, s_{abc} \qquad s_{bc} = (1-z)(1-y) s_{abc}$$

$$\int d\Phi_1(p^2; y, z, \phi) = G\, (p^2)^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, \left[y\, z (1-y)^2 (1-z)\right]^{-\epsilon} (1-y)$$

$\bar{\mathbf{S}}_i R$ B_{kl} term $a, b, c = i, k, l$	$\bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i)R$ $a, b, c = i, j, r$

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$$I = \sum_{i,j\neq i} \int d\Phi_1 \left[\bar{\mathbf{S}}_i + \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i) \right] R\, \mathcal{W}_{ij}$$

$$\sum_{i,j\neq i} \bar{\mathbf{S}}_i\, R\, \mathcal{W}_{ij} = \sum_i \bar{\mathbf{S}}_i\, R \underbrace{\sum_{j\neq i} \mathbf{S}_i\, \mathcal{W}_{ij}}_{1}$$

$$\begin{aligned}
\sum_{i,j\neq i} \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i) R\, \mathcal{W}_{ij} &= \sum_{i,j\neq i} \bar{\mathbf{C}}_{ij}\, R\, \mathbf{C}_{ij} \mathcal{W}_{ij} - \sum_{i,j\neq i} \bar{\mathbf{C}}_{ij} \bar{\mathbf{S}}_i\, R \underbrace{\mathbf{C}_{ij}\mathbf{S}_i \mathcal{W}_{ij}}_{1} \\
&= \sum_{i,j>i} \bar{\mathbf{C}}_{ij}\, R \underbrace{\mathbf{C}_{ij}\left(\mathcal{W}_{ij} + \mathcal{W}_{ji}\right)}_{1} - \sum_{i,j\neq i} \bar{\mathbf{C}}_{ij} \bar{\mathbf{S}}_i\, R \\
&= \sum_{i,j>i} \bar{\mathbf{C}}_{ij}(1-\bar{\mathbf{S}}_i-\bar{\mathbf{S}}_j)\, R
\end{aligned}$$

A “minimal” subtraction procedure at NLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I = \sum_i \int d\Phi_1 \, \bar{\mathbf{S}}_i \, R + \sum_{i,j>i} \int d\Phi_1 \, \bar{\mathbf{C}}_{ij} (1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j) \Big] R$$

$$\int d\Phi_1 \overline{\mathbf{S}}_i R = -\mathcal{N}_1 \sum_{k \neq i,\, l \neq i} B_{kl} \int d\Phi_1 \frac{s_{kl}}{s_{ik} s_{il}}$$

$$\int d\Phi_1 \frac{s_{kl}}{s_{ik} s_{il}} = G\,(p^2)^{1-\epsilon} \int_0^\pi d\phi \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz \left[y\, z (1-y)^2 (1-z) \right]^{-\epsilon} (1-y) \frac{1-z}{y\, z}$$

$$= G\,(p^2)^{1-\epsilon} B(\frac{1}{2}, \frac{1}{2} - \epsilon) B(-\epsilon, 2 - 2\epsilon) B(-\epsilon, -\epsilon)$$

trivial integration

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$$I = \sum_i \int d\Phi_1 \, \bar{\mathbf{S}}_i \, R + \sum_{i,j>i} \int d\Phi_1 \, \bar{\mathbf{C}}_{ij} (1 - \bar{\mathbf{S}}_i - \bar{\mathbf{S}}_j) \Big] R$$

$$\int d\Phi_1 \overline{\mathbf{C}}_{ij} R = \mathcal{N}_1 \left[B \int d\Phi_1 \frac{P_{ij}}{s_{ij}} - B_{\mu\nu} \underbrace{\int d\Phi_1 \frac{Q_{ij}^{\mu\nu}}{s_{ij}}}_{0} \right]$$

trivial integration

$$\int d\Phi_1 \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j) R = \mathcal{N}_1 B \int d\Phi_1 \frac{1}{s_{ij}} \left[P_{ij} - 2C_j \frac{z_j}{z_i} - 2C_j \frac{z_i}{z_j} \right]$$

A “minimal” subtraction procedure at NLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

  - Generate universal local counterterms
  - Exploit the freedom in defining them
  - The counterterms are basically “only” the IRC limits

trivial integration

Hope it can be extended beyond NLO !!

Structure of subtraction at NNLO

$$\frac{d\sigma_{\text{NNLO}} - d\sigma_{\text{NLO}}}{dX} = \int d\Phi_n VV\, \delta_{X_n} + \int d\Phi_{n+1} RV\, \delta_{X_{n+1}} + \int d\Phi_{n+2}\, RR\, \delta_{X_{n+2}}$$

VV and VR have poles in ϵ, VR and RR diverge in phase space

Structure of subtraction at NNLO

$$\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = \int d\Phi_n VV\,\delta_{X_n} + \int d\Phi_{n+1} RV\,\delta_{X_{n+1}} + \int d\Phi_{n+2}\,RR\,\delta_{X_{n+2}}$$

VV and VR have poles in ϵ, VR and RR diverge in phase space

- Counterterms $K^{(1)}$, $K^{(12)}$, $K^{(2)}$, $K^{(RV)}$ and their integrals $I^{(1)}$, $I^{(12)}$, $I^{(2)}$, $I^{(RV)}$

$$\int d\Phi_{n+2}\left[K^{(\mathbf{1})}\,\delta_{X_{n+1}} + \left(K^{(\mathbf{12})} + K^{(\mathbf{2})}\right)\delta_{X_n}\right] = \int d\Phi_{n+1}\,I^{(\mathbf{1})}\,\delta_{X_{n+1}} + \int d\Phi_n \left(I^{(\mathbf{12})} + I^{(\mathbf{2})}\right)\delta_{X_n}$$

$$\int d\Phi_{n+2}\,K^{(\mathbf{RV})}\,\delta_{X_n} = \int d\Phi_{n+1}\,I^{(\mathbf{RV})}\,\delta_{X_n}$$

$$\begin{aligned}\frac{d\sigma_{\rm NNLO} - d\sigma_{\rm NLO}}{dX} = & \int d\Phi_n \left(VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})}\right)\delta_{X_n} \\ & + \int d\Phi_{n+1}\left[\left(RV + I^{(\mathbf{1})}\right)\delta_{X_{n+1}} - \left(K^{(\mathbf{RV})} - I^{(\mathbf{12})}\right)\delta_{X_n}\right] \\ & + \int d\Phi_{n+2}\left[RR\,\delta_{X_{n+2}} - K^{(\mathbf{1})}\delta_{X_{n+1}} - \left(K^{(\mathbf{2})} + K^{(\mathbf{12})}\right)\delta_{X_n}\right]\end{aligned}$$

($V + I^{(2)} + I^{(RV)}$), ($RV + I^{(1)}$) and ($K^{(RV)} - I^{(12)}$) are finite in ϵ

($RR-K^{(1)}-K^{(12)}-K^{(2)}$), ($RV-K^{(RV)}$) and ($I^{(1)}+I^{(12)}$) converge in phase space

Primary IRC limits at NNLO

- Single soft limit
- Single collinear limit
- Double soft limit:

$$\mathbf{S}_{ik} \Leftrightarrow \begin{cases} k_i^\mu = \lambda k_i'^\mu \\ k_k^\mu = \lambda k_k'^\mu \\ \lambda \to 0 \end{cases} \Rightarrow \begin{cases} \mathcal{E}_i, \mathcal{E}_k \to 0 \\ \dfrac{\mathcal{E}_i}{\mathcal{E}_k} \to \text{finite} \end{cases} \Leftrightarrow \begin{cases} \dfrac{s_{ik}}{s_{il}}, \dfrac{s_{ik}}{s_{kl}}, \dfrac{s_{ih}}{s_{lm}}, \dfrac{s_{kh}}{s_{lm}} \to 0 & (h, l, m \neq i, k) \\ \dfrac{s_{il}}{s_{im}}, \dfrac{s_{kl}}{s_{km}}, \dfrac{s_{il}}{s_{km}} \to \text{finite} & (l, m \neq i, k) \end{cases}$$

Limit on the double real matrix element: Catani, Grazzini 9908523

$$\mathbf{S}_{ij} RR(\{k\}) = \frac{\mathcal{N}_1^2}{2} \left[\sum_{\substack{c \neq i,j \\ d \neq i,j}} \sum_{\substack{e \neq i,j \\ f \neq i,j}} \mathcal{I}_{cd}^{(i)} \mathcal{I}_{ef}^{(j)} B_{cdef}\left(\{k\}_{\not{i}\not{j}}\right) + \sum_{\substack{c \neq i,j \\ d \neq i,j}} \mathcal{I}_{cd}^{(ij)} B_{cd}\left(\{k\}_{\not{i}\not{j}}\right) \right]$$

$$\mathcal{I}_{ab}^{(i)} = \begin{cases} 0 & \text{if } i, k \text{ are quarks} \\ \dfrac{s_{ab}}{s_{ia} s_{ib}} & \text{if } i, k \text{ are gluons} \end{cases}$$

Primary IRC limits at NNLO

* Double collinear limit: $k^\mu = k_i^\mu + k_j^\mu + k_k^\mu$ $\qquad a = i, j, k$

$$\bar{k}^\mu = k^\mu - \frac{k^2}{2k\cdot r}\, r^\mu \qquad z_a = \frac{2k_a\cdot r}{2k\cdot r} \qquad \tilde{k}_a^\mu = k_a^\mu - z_a k^\mu - \left(\frac{2k\cdot k_a}{k^2} - 2z_a\right)\frac{k^2}{2k\cdot r}\, r^\mu$$

$$\bar{k}^2 = 0 \qquad z_i + z_j + z_k = 1 \qquad \tilde{k}\cdot\bar{k} = \tilde{k}\cdot r = 0 \qquad \tilde{k}_i^\mu + \tilde{k}_j^\mu + \tilde{k}_k^\mu = 0$$

$$k_a^\mu = z_a \bar{k}^\mu + \tilde{k}_a^\mu - \frac{1}{z_a}\frac{\tilde{k}_a^2}{2k\cdot r}\, r^\mu$$

$$\mathbf{C}_{ijk} \Leftrightarrow \begin{cases} \tilde{k}_i^\mu = \lambda \tilde{k}_i'^\mu \\ \tilde{k}_j^\mu = \lambda \tilde{k}_j'^\mu \\ \tilde{k}_k^\mu = \lambda \tilde{k}_k'^\mu \\ \lambda \to 0 \end{cases} \Rightarrow \begin{cases} w_{ij}, w_{jk}, w_{ik} \to 0 \\ \dfrac{w_{ij}}{w_{jk}}, \dfrac{w_{jk}}{w_{ik}}, \dfrac{w_{ik}}{w_{ij}} \to \text{fin.} \end{cases} \Leftrightarrow \begin{cases} \dfrac{s_{ij}}{s_{lm}}, \dfrac{s_{jk}}{s_{lm}}, \dfrac{s_{ik}}{s_{lm}} \to 0 & (lm \neq ij, jk, ik \;\; l \neq m) \\ \dfrac{s_{ij}}{s_{jk}}, \dfrac{s_{jk}}{s_{ik}}, \dfrac{s_{ik}}{s_{ij}} \to \text{finite} & \\ \dfrac{s_{il}}{s_{jl}}, \dfrac{s_{jl}}{s_{kl}}, \dfrac{s_{il}}{s_{kl}} \to \text{indep. on } l & (l \neq i, j, k) \end{cases}$$

Limit on the double real matrix element: Catani, Grazzini 9908523

$$\mathbf{C}_{ijk}\, RR(\{k\}) = \frac{\mathcal{N}_1^2}{s_{ijk}^2}\left[P_{ijk}\, B(\{k\}_{\not{i}\not{j}\not{k}}, k) + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}(\{k\}_{\not{i}\not{j}\not{k}}, k)\right]$$

$$Q_{ijk}^{\mu\nu} = \sum_{a=i,j,k} Q_{ijk}^{(a)}\left[-g^{\mu\nu} + (d-2)\frac{\tilde{k}_a^\mu \tilde{k}_a^\nu}{\tilde{k}_a^2}\right]$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions

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$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^\alpha (w_{ij})^\beta} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i) w_{kl}} \qquad \alpha > \beta > 1$$

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✱ Primary limits in the sectors:

$$\mathcal{W}_{ijjk} \quad : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ij}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}$$

$$\mathcal{W}_{ijkj} \quad : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijk}\,, \quad \mathbf{SC}_{ijk}\,, \quad \mathbf{CS}_{ijk}$$

$$\mathcal{W}_{ijkl} \quad : \quad \mathbf{S}_i\,, \quad \mathbf{C}_{ij}\,, \quad \mathbf{S}_{ik}\,, \quad \mathbf{C}_{ijkl}\,, \quad \mathbf{SC}_{ikl}\,, \quad \mathbf{CS}_{ijk}$$

$$\mathbf{SC}_{ikl}(f) = \mathbf{C}_{kl}\big(\mathbf{S}_i(f)\big) \qquad \mathbf{CS}_{ijk}(f) = \mathbf{S}_k\big(\mathbf{C}_{ij}(f)\big)$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\, j \neq i \\ k \neq i,\, l \neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\, j \neq i \\ k \neq i,\, l \neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^\alpha (w_{ij})^\beta} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i) w_{kl}} \qquad \alpha > \beta > 1$$

* Single soft and single collinear limits

$$\mathbf{S}_i\, \mathcal{W}_{ijkl} = \left(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{kl}$$

$$\mathbf{S}_i \mathbf{C}_{ij}\, \mathcal{W}_{ijkl} = \left(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{kl}$$

$$\mathcal{W}_{ij}^{(\alpha\beta)} = \frac{\frac{1}{\mathcal{E}_i^\alpha w_{ij}^\beta}}{\sum_{i, j \neq i} \frac{1}{\mathcal{E}_i^\alpha w_{ij}^\beta}}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijjk} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{[ij]k}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijkj} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{k[ij]}$$

$$\mathbf{C}_{ij}\, \mathcal{W}_{ijkl} = \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)}\right) \mathcal{W}_{kl}$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions

$$\mathcal{W}_{ijkl} = \frac{\sigma_{ijkl}}{\sigma} \qquad \sigma = \sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \sigma_{ijkl}$$

$$\sum_{\substack{i,\, j\neq i \\ k\neq i,\, l\neq i,k}} \mathcal{W}_{ijkl} = 1$$

$$\sigma_{ijkl} = \frac{1}{(\mathcal{E}_i)^\alpha (w_{ij})^\beta} \frac{1}{(\mathcal{E}_k + \delta_{kj}\mathcal{E}_i) w_{kl}} \qquad \alpha > \beta > 1$$

* Double soft and double collinear limits

$$\sum_{j\neq i, l\neq i,k} \mathbf{S}_{ik}\, \mathcal{W}_{ijkl} + \sum_{j\neq k, l\neq i,k} \mathbf{S}_{ik}\, \mathcal{W}_{kjil} = 1$$

$$\mathbf{C}_{ijk}\, \left(\mathcal{W}_{ikkj} + \mathcal{W}_{ijkj}\right) + (\text{perm. of } i, j, k) = 1$$

$$\mathbf{S}_{ik}\mathbf{C}_{ijk}(\mathcal{W}_{ikkj} + \mathcal{W}_{ijkj} + \mathcal{W}_{kiij} + \mathcal{W}_{kjij}) = \mathbf{C}_{ijk}\mathbf{S}_{ik}(\mathcal{W}_{ikkj} + \mathcal{W}_{ijkj} + \mathcal{W}_{kiij} + \mathcal{W}_{kjij}) = 1$$

A “minimal” subtraction procedure at NNLO

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A “minimal” subtraction procedure at NNLO

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Sector $\mathcal{W}_{ijjk}$

$\mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{S}_{ij}, \mathbf{C}_{ijk}, \mathbf{SC}_{ijk}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk}) \times \times(1-\mathbf{SC}_{ijk}) RR\, \mathcal{W}_{ijjk} = RR\, \mathcal{W}_{ijjk} - K^{(\mathbf{1})}_{ijjk} - K^{(\mathbf{2})}_{ijjk} - K^{(\mathbf{12})}_{ijjk}$$

→ finite

where the candidates for counterterms are

$$K^{(\mathbf{1})}_{ijjk} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big] RR\, \mathcal{W}_{ijjk}$$

$$K^{(\mathbf{2})}_{ijjk} = \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\Big] RR\, \mathcal{W}_{ijjk}$$

$$K^{(\mathbf{12})}_{ijjk} = -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij})\Big] + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\Big]\Big\} RR\, \mathcal{W}_{ijjk}$$

A "minimal" subtraction procedure at NNLO

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→ finite

where the candidates for counterterms are

cancel in $K^{(\mathbf{2})} + K^{(\mathbf{12})}$

$$K^{(\mathbf{1})}_{ijjk} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]RR\,\mathcal{W}_{ijjk}$$

$$K^{(\mathbf{2})}_{ijjk} = \Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\Big]RR\,\mathcal{W}_{ijjk}$$

$$K^{(\mathbf{12})}_{ijjk} = -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ij} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ij}) + \mathbf{SC}_{ijk}(1-\mathbf{S}_{ij})(1-\mathbf{C}_{ijk})\Big]\Big\}RR\,\mathcal{W}_{ijjk}$$

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits

Sector $\mathcal{W}_{ijkj}$

$\mathbf{S}_i$, $\mathbf{C}_{ij}$, $\mathbf{S}_{ik}$, $\mathbf{C}_{ijk}$, $\mathbf{SC}_{ijk}$, $\mathbf{CS}_{ijk}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk}) \times \times(1-\mathbf{SC}_{ijk})(1-\mathbf{CS}_{ijk}) RR\, \mathcal{W}_{ijkj} = RR\, \mathcal{W}_{ijkj} - K^{(\mathbf{1})}_{ijkj} - K^{(\mathbf{2})}_{ijkj} - K^{(\mathbf{12})}_{ijkj}$$

→ finite

cancel in $K^{(\mathbf{2})} + K^{(\mathbf{12})}$

where the candidates for counterterms are

$$K^{(\mathbf{1})}_{ijkj} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big] RR\, \mathcal{W}_{ijkj}$$

$$K^{(\mathbf{2})}_{ijkj} = \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big] RR\, \mathcal{W}_{ijkj}$$

$$K^{(\mathbf{12})}_{ijkj} = -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijk}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ijk} + \mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijk})\Big]\Big\} RR\, \mathcal{W}_{ijkj}$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
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Sector $\mathcal{W}_{ijkl}$

$\mathbf{S}_i\,, \mathbf{C}_{ij}\,, \mathbf{S}_{ik}\,, \mathbf{C}_{ijkl}, \mathbf{SC}_{ikl}\,, \mathbf{CS}_{ijk}$ commute

$$(1-\mathbf{S}_i)(1-\mathbf{C}_{ij})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl}) \times \times(1-\mathbf{SC}_{ikl})(1-\mathbf{CS}_{ijk})RR\,\mathcal{W}_{ijkl} = RR\,\mathcal{W}_{ijkl} - K^{(\mathbf{1})}_{ijkl} - K^{(\mathbf{2})}_{ijkl} - K^{(\mathbf{12})}_{ijkl}$$

→ finite

where the candidates for counterterms are

cancel in $K^{(\mathbf{2})} + K^{(\mathbf{12})}$

$$K^{(\mathbf{1})}_{ijkl} = \Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]RR\,\mathcal{W}_{ijkl}$$

$$K^{(\mathbf{2})}_{ijkl} = \Big[\mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl}+\mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\Big]RR\,\mathcal{W}_{ijkl}$$

$$K^{(\mathbf{12})}_{ijkl} = -\Big\{\Big[\mathbf{S}_i + \mathbf{C}_{ij}(1-\mathbf{S}_i)\Big]\Big[\mathbf{S}_{ik} + \mathbf{C}_{ijkl}(1-\mathbf{S}_{ik}) + (\mathbf{SC}_{ikl}+\mathbf{CS}_{ijk})(1-\mathbf{S}_{ik})(1-\mathbf{C}_{ijkl})\Big]\Big\}RR\,\mathcal{W}_{ijkl}$$

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits

Momenta in $K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl}$ do not satisfy mass-shell condition and momenta conservation

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
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Momenta in $K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl}$ do not satisfy mass-shell condition and momenta conservation

$$K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl} \longrightarrow \boxed{\overline{K}^{(\mathbf{1})}_{ijkl}, \overline{K}^{(\mathbf{2})}_{ijkl}, \overline{K}^{(\mathbf{12})}_{ijkl}}$$

remapped momenta **in matrix elements** and partially **in IRC kernels**

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Momenta in $K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl}$ do not satisfy mass-shell condition and momenta conservation

$$K^{(\mathbf{1})}_{ijkl}, K^{(\mathbf{2})}_{ijkl}, K^{(\mathbf{12})}_{ijkl} \rightarrow \boxed{\overline{K}^{(\mathbf{1})}_{ijkl}, \overline{K}^{(\mathbf{2})}_{ijkl}, \overline{K}^{(\mathbf{12})}_{ijkl}}$$

remapped momenta **in matrix elements** and partially **in IRC kernels**

They must satisfy:

$$\mathbf{L}_1\, \overline{K}^{(\mathbf{1})}_{ijkl} = K^{(\mathbf{1})}_{ijkl} \qquad \mathbf{L}_1 \in \{\mathbf{S}_i, \mathbf{C}_{ij}\}$$

$$\mathbf{L}_2\, \overline{K}^{(\mathbf{2})}_{ijkl} = K^{(\mathbf{2})}_{ijkl} \qquad \mathbf{L}_2 \in \{\mathbf{S}_{ik}, \mathbf{C}_{ijkl}, \mathbf{SC}_{ikl}, \mathbf{CS}_{ijk}\}$$

$$\mathbf{L}_{12}\, \overline{K}^{(\mathbf{12})}_{ijkl} = K^{(\mathbf{12})}_{ijkl} \qquad \mathbf{L}_{12} \in \{\mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{S}_{ik}.\mathbf{C}_{ijkl}, \mathbf{SC}_{ikl}, \mathbf{CS}_{ijk}\}$$

such that:

$$\boxed{RR\, \mathcal{W}_{ijkl} - \overline{K}^{(\mathbf{1})}_{ijkl} - \overline{K}^{(\mathbf{2})}_{ijkl} - \overline{K}^{(\mathbf{12})}_{ijkl} \longrightarrow \text{finite}}$$

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We use the properties of the sector functions

$$K^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \left[\left(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} RR \right) + \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \right) - \left(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \right) \right] \mathcal{W}_{kl}$$

↓

$$\overline{K}^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \left[\left(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{S}}_i RR \right) \overline{\mathcal{W}}_{kl} + \left(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{C}}_{ij} RR \right) \overline{\mathcal{W}}_{kl} - \left(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \right) \left(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} RR \right) \overline{\mathcal{W}}_{kl} \right]$$

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We use the properties of the sector functions

$$K^{(\mathbf{1})} = \sum_{i,\,j\neq i}\ \sum_{\substack{k\neq i\\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \mathcal{W}^{(\alpha\beta)}_{ij} RR\Big) + \Big(\mathbf{C}_{ij} \mathcal{W}^{(\alpha\beta)}_{ij} RR\Big) - \Big(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}^{(\alpha\beta)}_{ij} RR\Big)\Big] \mathcal{W}_{kl}$$

$$\overline{K}^{(\mathbf{1})} = \sum_{i,\,j\neq i}\ \sum_{\substack{k\neq i\\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \mathcal{W}^{(\alpha\beta)}_{ij}\Big) \big(\overline{\mathbf{S}}_i RR\big)\, \overline{\mathcal{W}}_{kl} + \Big(\mathbf{C}_{ij} \mathcal{W}^{(\alpha\beta)}_{ij}\Big) \big(\overline{\mathbf{C}}_{ij} RR\big)\, \overline{\mathcal{W}}_{kl} - \Big(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}^{(\alpha\beta)}_{ij}\Big) \big(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} RR\big)\, \overline{\mathcal{W}}_{kl}\Big]$$

Z

$$-\mathcal{N}_1 \sum_{\substack{a\neq i\\ b\neq i}} \mathcal{I}^{(i)}_{ab}\, R_{ab}\Big(\{\bar{k}\}^{(iab)}\Big)\, \overline{\mathcal{W}}^{(iab)}_{kl}$$

$$\frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, R\Big(\{\bar{k}\}^{(ijr)}\Big) + Q^{\mu\nu}_{ij}\, R_{\mu\nu}\Big(\{\bar{k}\}^{(ijr)}\Big)\Big] \overline{\mathcal{W}}^{(ijr)}_{kl}$$

NLO sector functions with remapped momenta

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta

We use the properties of the sector functions

$$K^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) + \Big(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) - \Big(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} RR \Big) \Big] \mathcal{W}_{kl}$$

$$\overline{K}^{(\mathbf{1})} = \sum_{i,\, j\neq i} \sum_{\substack{k\neq i \\ l\neq i,k}} \Big[\Big(\mathbf{S}_i \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{S}}_i RR \big)\, \overline{\mathcal{W}}_{kl} + \Big(\mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{C}}_{ij} RR \big)\, \overline{\mathcal{W}}_{kl} - \Big(\mathbf{S}_i \mathbf{C}_{ij} \mathcal{W}_{ij}^{(\alpha\beta)} \Big) \big(\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} RR \big)\, \overline{\mathcal{W}}_{kl} \Big]$$

$$-\mathcal{N}_1 \sum_{\substack{a\neq i \\ b\neq i}} \mathcal{I}_{ab}^{(i)}\, R_{ab} \Big(\{\bar{k}\}^{(iab)} \Big)\, \overline{\mathcal{W}}_{kl}^{(iab)}$$

$$\frac{\mathcal{N}_1}{s_{ij}} \Big[P_{ij}\, R\Big(\{\bar{k}\}^{(ijr)} \Big) + Q_{ij}^{\mu\nu}\, R_{\mu\nu} \Big(\{\bar{k}\}^{(ijr)} \Big) \Big]\, \overline{\mathcal{W}}_{kl}^{(ijr)}$$

single remapping

NLO sector functions with remapped momenta

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta

Examples of double remappings

$$\overline{\mathbf{S}}_{ij}\, RR = \frac{\mathcal{N}_1^{\,2}}{2} \Bigg[\sum_{\substack{c\neq i,j \\ d\neq i,j}} \sum_{\substack{e\neq i,j \\ f\neq i,j}} \mathcal{I}_{cd}^{(i)} \delta_{fjg} \frac{\bar{s}_{ef}^{(icd)}}{\bar{s}_{je}^{(icd)} \bar{s}_{jf}^{(icd)}} B_{cdef}\left(\{\bar{k}\}^{(icd,jef)}\right) + \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \mathcal{I}_{cd}^{(ij)} B_{cd}\left(\{\bar{k}\}^{(ijcd)}\right) + \sum_{c\neq i,j} \mathcal{I}_{cc}^{(ij)} B_{cc}\left(\{\bar{k}\}^{(ijcc')}\right) \Bigg]$$

$$\overline{\mathbf{C}}_{ijk}\, RR = \frac{\mathcal{N}_1^2}{s_{ijk}^2} \left[P_{ijk}\, B\left(\{\bar{k}\}^{(ijkr)}\right) + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}\left(\{\bar{k}\}^{(ijkr)}\right) \right]$$

$$\bar{k}_c^{(abcd)} = k_a + k_b + k_c - \frac{s_{abc}}{s_{ad} + s_{bd} + s_{cd}}\, k_d \qquad \bar{k}_d^{abcd} = \frac{s_{abcd}}{s_{ad} + s_{bd} + s_{cd}}\, k_d$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
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- Phase space reparametrized differently for each term of the sum

$\overline{\mathbf{S}}_{ij} RR$ | B_{cdef} term

$$d\Phi_{n+2}(\{k\}) = d\Phi_n\left(\{\bar{k}\}^{(icd,jef)}\right) d\Phi_1(s_{icd}; y, z, \phi)\, d\Phi_1\left(\bar{s}_{jef}^{(icd)}; y', z', \phi'\right)$$

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum

$\overline{\mathbf{S}}_{ij} RR$ | B_{cdef} term

$$d\Phi_{n+2}(\{k\}) = d\Phi_n\left(\{\bar{k}\}^{(icd,jef)}\right) d\Phi_1(s_{icd}; y, z, \phi)\, d\Phi_1\left(\bar{s}_{jef}^{(icd)}; y', z', \phi'\right)$$

$\overline{\mathbf{S}}_{ij} RR$ | B_{cd} term

$a, b, c, d = i, j, c, d$

$\overline{\mathbf{C}}_{ijk} RR$

$a, b, c, d = i, j, k, r$

$$d\Phi_{n+2}(\{k\}) = d\Phi_n\left(\{\bar{k}\}^{(abcd)}\right) d\Phi_2(s_{abcd}; y, z, \phi, y', z', x')$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum

$$d\Phi_2(s_{abcd}; y, z, \phi, y', z', x')$$

$$\begin{aligned}
2k_a\cdot k_b &= y'\, y\, p^2 \\
2k_a\cdot k_c &= z'(1-y')y\, p^2, \\
2k_b\cdot k_c &= (1-y')(1-z')y\, p^2, \\
2k_c\cdot k_d &= (1-y')(1-y)(1-z)p^2, \\
2k_a\cdot k_d &= (1-y)\left[y'(1-z')(1-z) + z'z - 2\,(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right] p^2 \\
2k_b\cdot k_d &= (1-y)\left[y'z'(1-z) + (1-z')z + 2\,(1-2x')\sqrt{y'z'(1-z')z(1-z)}\right] p^2
\end{aligned}$$

$$\int d\Phi_2(p^2; y, z, \phi, y', z', x') \;=\; G_2\,(p^2)^{2-2\epsilon}\int_0^1 dx' \int_0^1 dy' \int_0^1 dz' \int_0^1 dy \int_0^1 dz\, [x'(1-x')]^{-\epsilon-1/2}$$
$$\left[y'z'(1-y')^2(1-z')y^2 z(1-y)^2(1-z)\right]^{-\epsilon} y(1-y)(1-y')$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

We use the properties of the sector functions

$$I^{(\mathbf{1})} = \sum_{\substack{k\neq i \\ l\neq i,k}} \overline{\mathcal{W}}_{kl} \left[\sum_i \int d\Phi_1 \overline{\mathbf{S}}_i RR + \sum_{i,\, j>i} \int d\Phi_1 \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j) RR \right]$$

trivial integration

- Remapped sector functions sum to 1
- Are kept to combine with sectors of RV

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

We use the properties of the sector functions

$$I^{(\mathbf{1})} = \sum_{\substack{k \neq i \\ l \neq i,k}} \overline{\mathcal{W}}_{kl} \left[\sum_i \int d\Phi_1 \overline{\mathbf{S}}_i RR + \sum_{i,\, j>i} \int d\Phi_1 \overline{\mathbf{C}}_{ij} (1 - \overline{\mathbf{S}}_i - \overline{\mathbf{S}}_j) RR \right]$$

trivial integration

Using the properties of the sector functions one gets for $I^{(\mathbf{12})}$:

$$I^{(\mathbf{12})} = \sum_{\substack{i,j \neq i \\ k \neq i, l \neq i,k}} \int d\Phi_1 \, \overline{K}^{(\mathbf{12})}_{ijkl} = \sum_{\substack{k \neq l \\ l \neq i,k}} \overline{\mathcal{W}}_{kl} \left[\overline{\mathbf{S}}_k + \overline{\mathbf{C}}_{kl} (1 - \overline{\mathbf{S}}_k) \right] I^{(\mathbf{1})}$$

trivial integration

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(2)} = \int d\Phi_2 \left\{ \sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \overline{\mathbf{S}}_{ik} \mathcal{W}_{ijkl} + \sum_{\substack{i,j\neq i \\ k\neq i,j}} \overline{\mathbf{C}}_{ijk} \left[(1-\overline{\mathbf{S}}_{ij}) RR\, \mathcal{W}_{ijjk} + (1-\overline{\mathbf{S}}_{ik}) RR\, \mathcal{W}_{ijkj} \right] + \dots \right\}$$

- "Pure" double-unresolved part
- In all subtraction scheme the more difficult part to be integrated

- Basically products of single unresolved integrals
- Trivial integration

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{2})} = \int d\Phi_2 \Big\{ \sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \overline{\mathbf{S}}_{ik} \mathcal{W}_{ijkl} + \sum_{\substack{i,j\neq i \\ k\neq i,j}} \overline{\mathbf{C}}_{ijk} \Big[(1-\overline{\mathbf{S}}_{ij}) RR\, \mathcal{W}_{ijjk} + (1-\overline{\mathbf{S}}_{ik}) RR\, \mathcal{W}_{ijkj} \Big] + \dots \Big\}$$

We use the properties of the sector functions

$$\sum_{\substack{i,j\neq i \\ k\neq i, l\neq i,k}} \overline{\mathbf{S}}_{ik}\, RR\, \mathcal{W}_{ijkl} = \sum_{i,k>i} \overline{\mathbf{S}}_{ik}\, RR$$

$$\sum_{\substack{i,j\neq i \\ k\neq i}} \overline{\mathbf{C}}_{ijk} \Big[(1-\overline{\mathbf{S}}_{ij}) RR\, \mathcal{W}_{ijjk} + (1-\overline{\mathbf{S}}_{ik}) RR\, \mathcal{W}_{ijkj} \Big] = \sum_{\substack{i,j>i \\ k>j}} \overline{\mathbf{C}}_{ijk} \left(1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}\right) RR$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{2})} = \sum_{i,j>i} \int d\Phi_2\, \overline{\mathbf{S}}_{ij} RR + \sum_{\substack{i,j>i \\ k>j}} \int d\Phi_2\, \overline{\mathbf{C}}_{ijk} (1-\overline{\mathbf{S}}_{ij}-\overline{\mathbf{S}}_{ik}-\overline{\mathbf{S}}_{jk}) RR + \dots$$

trivial integration

$$\int d\Phi_2 \overline{\mathbf{S}}_{ij}\, RR \;=\; \frac{\mathcal{N}_1{}^2}{2} \Bigg[\sum_{\substack{c\neq i,j \\ d\neq i,j}} \sum_{\substack{e\neq i,j \\ f\neq i,j}} B_{cdef}\left(\{\bar{k}\}^{(icd,jef)}\right) \int d\Phi_1\, \mathcal{I}_{cd}^{(i)}\, \delta_{f_j g} \int d\bar{\Phi}_1^{(icd)}\, \frac{\bar{s}_{ef}^{(icd)}}{\bar{s}_{je}^{(icd)}\, \bar{s}_{jf}^{(icd)}}$$

$$+ \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} B_{cd}\left(\{\bar{k}\}^{(ijcd)}\right) \int d\Phi_2\, \mathcal{I}_{cd}^{(ij)} + \sum_{c\neq i,j} B_{cc}\left(\{\bar{k}\}^{(ijcc')}\right) \int d\Phi_2\, \mathcal{I}_{cc}^{(ij)}$$

feasible integration

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$$I^{(\mathbf{2})} = \sum_{i,j>i} \int d\Phi_2\, \overline{\mathbf{S}}_{ij} RR + \sum_{\substack{i,j>i \\ k>j}} \int d\Phi_2\, \overline{\mathbf{C}}_{ijk} (1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}) RR + \dots$$

$$\int d\Phi_2\, \overline{\mathbf{C}}_{ijk}\, RR = \mathcal{N}_1^2 \left[B \int d\Phi_2 \frac{P_{ijk}}{s_{ijk}^2} + B_{\mu\nu} \underbrace{\int d\Phi_2 \frac{Q_{ijk}^{\mu\nu}}{s_{ijk}^2}}_{0} \right]$$

feasible integration

$$\int d\Phi_2\, \overline{\mathbf{C}}_{ijk} (1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}) RR = \mathcal{N}_1^2\, B \int d\Phi_2\, (1 - \overline{\mathbf{S}}_{ij} - \overline{\mathbf{S}}_{ik} - \overline{\mathbf{S}}_{jk}) \frac{P_{ijk}}{s_{ijk}^2}$$

A “minimal” subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

The procedure can be extended beyond NLO !!

- Generate universal local counterterms
- Exploit the freedom in defining them
- The counterterms are basically “only” the IRC limits

→ feasible integration

Proof of concept

- $T_R C_F$ NNLO contribution to the total cross section for $e^+e^- \to q\bar{q}$

  Just contributions from the radiation of a $q'\bar{q}'$ pair

- Known exact NNLO results:

  Hamberg, van Neerven, Matsuura 1991
  Gehrmann De Ridder, Gehrmann, Glover 0403057
  Ellis, Ross, Terrano 1980

$$VV = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left\{ \left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[\frac{1}{3\epsilon^3}+\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{11}{18}\pi^2+\frac{353}{54}\right)+\left(-\frac{26}{9}\zeta_3-\frac{77}{27}\pi^2+\frac{7541}{324}\right)\right] + \left(\frac{\mu^2}{s}\right)^{\epsilon}\left[-\frac{4}{3\epsilon^3}-\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{7}{9}\pi^2-\frac{16}{3}\right)+\left(\frac{28}{9}\zeta_3+\frac{7}{6}\pi^2-\frac{32}{3}\right)\right]\right\}$$

$$\int d\Phi_1\, RV = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3}+\frac{2}{\epsilon^2}+\frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2+\frac{19}{3}\right)+\left(-\frac{100}{9}\zeta_3-\frac{7}{6}\pi^2+\frac{109}{6}\right)\right]$$

$$\int d\Phi_2\, RR = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{18}\pi^2-\frac{407}{54}\right)+\left(\frac{134}{9}\zeta_3+\frac{77}{27}\pi^2-\frac{11753}{324}\right)\right]$$

Proof of concept

- We integrate the known limits $\bar{\mathbf{S}}_{ik}RR$ and $\bar{\mathbf{C}}_{ijk}RR$

$$\int d\Phi_2\,\bar{\mathbf{S}}_{ik}RR = (4\pi\alpha_s^u\mu_0^{2\epsilon})^2 T_R \sum_{l,m=1}^{2} B_{lm}\int d\Phi_2 \frac{4(s_{il}s_{km}+s_{im}s_{kl}-s_{ik}s_{lm})}{s_{ik}^2(s_{il}+s_{kl})(s_{im}+s_{km})}$$

$$= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{17}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{7}{18}\pi^2-\frac{232}{27}\right)+\left(\frac{38}{9}\zeta_3+\frac{131}{54}\pi^2-\frac{2948}{81}\right)\right]$$

$$\int d\Phi_2\,\bar{\mathbf{C}}_{ijk}RR = (8\pi\alpha_s^u\mu_0^{2\epsilon})^2 B\int d\Phi_2 \frac{2T_R C_F}{s_{ijk}s_{ik}}\left[-\frac{t_{ik,j}^2}{s_{ik}s_{ikj}}+\frac{4z_j+(z_i-z_k)^2}{z_i+z_k}+(1-2\epsilon)\left(z_i+z_k-\frac{s_{ik}}{s_{ikj}}\right)\right]$$

$$= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{31}{18\epsilon^2}+\frac{1}{\epsilon}\left(\frac{1}{2}\pi^2-\frac{889}{108}\right)+\left(\frac{80}{9}\zeta_3+\frac{31}{12}\pi^2-\frac{23941}{648}\right)\right]$$

Catani, Grazzini 9908523

- And we get the 2-unresolved integrated counterterm:

$$I^{(2)} = \int d\Phi_2\left[\bar{\mathbf{S}}_{34}+\bar{\mathbf{C}}_{134}(1-\bar{\mathbf{S}}_{34})+\bar{\mathbf{C}}_{234}(1-\bar{\mathbf{S}}_{34})\right]RR$$

$$= B\left(\frac{\alpha_s}{2\pi}\right)^2 T_R C_F\left(\frac{\mu^2}{s}\right)^{2\epsilon}\left[-\frac{1}{3\epsilon^3}-\frac{14}{9\epsilon^2}+\frac{1}{\epsilon}\left(\frac{11}{18}\pi^2-\frac{425}{54}\right)+\left(\frac{122}{9}\zeta_3+\frac{74}{27}\pi^2-\frac{12149}{324}\right)\right]$$

Proof of concept

- From the explicit expression of RV se get for $I^{(\mathbf{RV})}$:

$$I^{(\mathbf{RV})} = \frac{\alpha_S}{2\pi}\frac{2}{3}\frac{T_R}{\epsilon}\left[\int d\Phi_1\, \mathbf{S}_{[34]}\, R + \int d\Phi_1\, \mathbf{C}_{1[34]}(1-\mathbf{S}_{[34]})R + \int d\Phi_1\, \mathbf{C}_{2[34]}(1-\mathbf{S}_{[34]})R\right]$$

$$= \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{\mu^2}{s}\right)^{\epsilon}\left[\frac{4}{3\epsilon^3} + \frac{2}{\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{7}{9}\pi^2 + \frac{20}{3}\right) + \left(-\frac{100}{9}\zeta_3 - \frac{7}{6}\pi^2 + 20\right)\right]$$

- Analytical cancellation of poles in the subtracted VV:

$$VV + I^{(\mathbf{2})} + I^{(\mathbf{RV})} = B\left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(\frac{8}{3}\zeta_3 - \frac{1}{9}\pi^2 - \frac{44}{9} - \frac{4}{3}\ln\frac{\mu^2}{s}\right)$$

- NNLO corrections with the subtraction …

$$\frac{\mu}{\sqrt{s}} = 0.35$$

$$\frac{\sigma_{\mathrm{NNLO}} - \sigma_{\mathrm{NLO}}}{\sigma_{\mathrm{LO}}} = \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(1.40806 \pm 0.00040\right)$$

- … compared with the analytical result

$$\frac{\sigma_{\mathrm{NNLO}} - \sigma_{\mathrm{NLO}}}{\sigma_{\mathrm{LO}}} = \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left[-\frac{11}{2} + 4\zeta_3 - \ln\frac{\mu^2}{s}\right] = \left(\frac{\alpha_S}{2\pi}\right)^2 T_R C_F \left(1.40787186\right)$$

Leading Outlook

- Complete the implementation in a Monte Carlo generator
- Complete the integration of “pure” double-unresolved counterterms

Next-to-Leading Outlook

- Compute counterterms with initial state hadrons

  - Basically implement Catani-Seymour remappings

Next-to-Next-to-Leading Outlook

- Consider the massive case

  - Less singularities, but …
  - … more involved remappings, i.e. integration

Still work in progress …

Back-up slides

A "minimal" subtraction procedure at NNLO

- Divide the phase space through sector functions
- Identify counterterms through IRC limits
- Counterterms are sums of terms, each with its remapped momenta
- Phase space reparametrized differently for each term of the sum
- Integrate analytically each term after getting rid of the sector functions

$\mathcal{I}_{cd}^{(ij)}[q\bar{q}]$	easy	$P_{ijk}[qq'\bar{q}']$	easy
$\mathcal{I}_{cd}^{(ij)}[gg]$	feasible	$P_{ijk}[qq\bar{q}]$	feasible
		$P_{ijk}[q\bar{q}g]$	feasible
		$P_{ijk}[qgg]$	feasible
		$P_{ijk}[ggg]$	feasible

feasible integration

work in progress