Hatted representation of massless propagators and its implications for 2-point correlators and anomalous dimensions

Konstantin Chetyrkin (UHH)

in collaboration with **Pavel Baikov** (MSU)

based on P. Baikov, K.Ch.: arXiv:1804.10088, arXiv:1808.00237 and "in preparation"

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Starting point: 1991

The seminal calculation /Gorishnii, Kataev, Larin/ of the $\mathcal{O}(\alpha_s^3)$ Adler function demonstrated for the first time a mysterious complete cancellation of **all** contributions proportional to ζ_4 (abounding in separate diagrams) while odd zetas ζ_3 and ζ_5 survive! The result is π-**free** ($\zeta_4 = \frac{\pi^4}{90}$ and $\zeta_6 = \frac{\pi^6}{945}$)

$$d_2 = -\frac{3}{32}C_F^2 + C_F T_f\left[\zeta_3 - \frac{11}{8}\right] + C_F C_A\left[\frac{123}{32} - \frac{11\zeta_3}{4}\right],$$

$$d_3 = -\frac{69}{128}C_F^3 + C_F^2 T_f\left[-\frac{29}{64} + \frac{19}{4}\zeta_3 - 5\zeta_5\right] + C_F T_f^2\left[\frac{151}{54} - \frac{19}{9}\zeta_3\right] + C_F^2 C_A\left[-\frac{127}{64} - \frac{143}{16}\zeta_3 + \frac{55}{4}\zeta_5\right]$$
$$+ C_F T_f C_A\left[-\frac{485}{27} + \frac{112}{9}\zeta_3 + \frac{5}{6}\zeta_5\right] + C_F C_A^2\left[\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right],$$

the authors wrote: **“We would like to stress the cancellations of ζ_4 in the final results for $R(s)$. It is very interesting to find the origin of the cancellation of ζ_4 in the physical quantity.”**

The situation got even more interesting about 20 years later: the $\mathcal{O}({\alpha_s}^4)$ contributions to the Adler function and to the coefficient function (CF) of C_{Bjp} the Bjorken sum rule /Baikov, Kühn, K. Ch. (2009-2010)/ were found to be

completely π-free*

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* **we do not consider any powers of π which are routinely generated during the procedure of analytical continuation to the Minkowskian (negative) values of the momentum transfer Q^2)**

	d_4	$(1/C^{Bjp})_4$
C_F^4	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$	$\frac{4157}{2048} + \frac{3}{8}\,\zeta_3$
$n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R}$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\,\zeta_5$	$-\frac{13}{16} - \zeta_3 + \frac{5}{2}\,\zeta_5$
$\frac{d_F^{abcd} d_A^{abcd}}{d_R}$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$	$\frac{3}{16} - \frac{1}{4}\,\zeta_3 - \frac{5}{4}\,\zeta_5$
$C_F T_f^3$	$-\frac{6131}{972} + \frac{203}{54}\,\zeta_3 + \frac{5}{3}\,\zeta_5$	$-\frac{605}{972}$
$C_F^2 T_f^2$	$\frac{5713}{1728} - \frac{581}{24}\,\zeta_3 + \frac{125}{6}\,\zeta_5 + 3\,\zeta_3^2$	$\frac{869}{576} - \frac{29}{24}\,\zeta_3$
$C_F T_f^2 C_A$	$\frac{340843}{5184} - \frac{10453}{288}\,\zeta_3 - \frac{170}{9}\,\zeta_5 - \frac{1}{2}\,\zeta_3^2$	$\frac{165283}{20736} + \frac{43}{144}\,\zeta_3 - \frac{5}{12}\,\zeta_5 + \frac{1}{6}\,\zeta_3^2 EQN$
$C_F^3 T_f$	$\frac{1001}{384} + \frac{99}{32}\,\zeta_3 - \frac{125}{4}\,\zeta_5 + \frac{105}{4}\,\zeta_7$	$-\frac{473}{2304} - \frac{391}{96}\,\zeta_3 + \frac{145}{24}\,\zeta_5$
$C_F^2 T_f C_A$	$\frac{32357}{13824} + \frac{10661}{96}\,\zeta_3 - \frac{5155}{48}\,\zeta_5 - \frac{33}{4}\,\zeta_3^2 - \frac{105}{8}\,\zeta_7$	$-\frac{17309}{13824} + \frac{1127}{144}\,\zeta_3 - \frac{95}{144}\,\zeta_5 - \frac{35}{4}\,\zeta_7$
$C_F T_f C_A^2$	$-\frac{(\cdots)}{(\cdots)} + \frac{8609}{72}\,\zeta_3 + \frac{18805}{288}\,\zeta_5 - \frac{11}{2}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$	$-\frac{(\cdots)}{(\cdots)} - \frac{59}{64}\,\zeta_3 + \frac{1855}{288}\,\zeta_5 - \frac{11}{12}\,\zeta_3^2 + \frac{35}{16}\,\zeta_7$
$C_F^3 C_A$	$-\frac{253}{32} - \frac{139}{128}\,\zeta_3 + \frac{2255}{32}\,\zeta_5 - \frac{1155}{16}\,\zeta_7$	$-\frac{8701}{4608} + \frac{1103}{96}\,\zeta_3 - \frac{1045}{48}\,\zeta_5$
$C_F^2 C_A^2$	$-\frac{592141}{18432} - \frac{43925}{384}\,\zeta_3 + \frac{6505}{48}\,\zeta_5 + \frac{1155}{32}\,\zeta_7$	$-\frac{435425}{55296} - \frac{1591}{144}\,\zeta_3 + \frac{55}{9}\,\zeta_5 + \frac{385}{16}\,\zeta_7$
$C_F C_A^3$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{77995}{1152}\,\zeta_5 + \frac{605}{32}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$	$\frac{(\cdots)}{(\cdots)} - \frac{(\cdots)}{(\cdots)}\,\zeta_3 - \frac{12545}{1152}\,\zeta_5 + \frac{121}{96}\,\zeta_3^2 - \frac{385}{64}\,\zeta_7$

Transcedentals: odd zetas: $\zeta_3, \zeta_5, \zeta_7$ BUT NOT even one ζ_4 or ζ_6 (both appear eventually in *every* separate input diagram /from about 20 thousand!/)

What is common between the Adler function and C_{Bjp}? They both are "physical" (no anomalous dimension, depend only on the *bare* cc α_s).
The Adler function D^{SS} for the scalar correlator is π-dependent already at $\mathcal{O}({\alpha_s}^3)$★
and even more at the next loop (expilict ζ_4 and ζ_6 terms)★★

In fact, one can construct a physical (read: scale-independent) object from $\mathcal{O}(\alpha_s^L)$ D^{SS} and the (L+1)-loop quark mass anomalous dimension γ_m.

For $\mathcal{O}(\alpha_s^3)$ D^{SS} it was done with expected result: all π dependence indeed disappeared! /Vermaseren, Larin van Ritbergen (1997)/

BUT for $\mathcal{O}(\alpha_s^4)$ correlators this stopped to work:

It was found /Baikov, K. Ch. Kühn (2017)/ that ζ_4 does not dissappear from a scale-independent (SI) object constructed from $\mathcal{O}(\alpha_s^4)$ D^{SS} and 5-loop AD γ_m.

ζ_4 also does not dissappear from the 5-loop gluon correlator (enters the hadronic decays of the Higgs boson) computed in
/ Herzog, Ruijl, Ueda, Vermaseren and Vogt (2017)/.

★ K. K. Ch. (1997).

★★ Baikov, Kühn, K. Ch. (2006)

2017: 2 new important developments

- 5-loop QCD β-function and quark AD γ_m were computed /Baikov, K. Ch. Kühn; Herzog, Ruijl, Ueda, Vermaseren and Vogt; Luthe, Maier, Marquard and Schroder/.

  First appearence of π in β_5 (in form of ζ_4)

- Jamin and Miravitllas have discovered that after a transition to a new so-called C-scheme all terms proportional to even zetas (ζ_4 and ζ_6) do disappear from (SI versions of) the 5-loop scalar correlator and the 5-loop gluon correlator (both enter the hadronic decays of the Higgs boson /Baikov, K. Ch. Kühn (2005); Herzog, Ruijl, Ueda, Vermaseren and Vogt (2017)/.

  They also suggested that the absence of even zetas after transition to the C-scheme is an universal feature of *all* $\mathcal{O}(\alpha_s^5)$ physical quantities $\equiv$ **no π-conjecture**

  Many more confirmations of the conjecture in /J. Davies and A. Vogt (2017)/
  /K. Ch, G. Falcioni, Herzog and Vermaseren (2017)/
  /A. Vogt, F. Herzog, S. Moch, B. Ruijl, T. Ueda, J. Vermaseren, (2018)/
  /S. Moch, B. Ruijl, T. Ueda, J. Vermaseren, A. Vogt (2018) /

Main points

- The aim of the talk is to prove the no π-conjecture and completely elucidate the origin of the π-dependent terms in massless correlators as well as in AD's and β-function

- The resulting formulas are completely general and do not depend on a specific field model

- The main tool is so-called "hatted" representation of transcendental numbers appearing in massless props The representation in a sense "hides" all π-dependent terms inside, roughly speaking, odd zetas ...

A word about notations and conventions (goodbye β_0 and γ_0)

we use

1. $$\gamma(a) = \sum_{i\geq 1} \gamma_i\, a^i, \quad a = \frac{\alpha_s}{4\pi}$$

2. $$\beta(a) = \sum_{i\geq 1} \beta_i\, a^i$$

3. Landau gauge for QCD (for simplicity, could be relaxed)

4. G-scheme instead of $\overline{\text{MS}}$ one: all ADs and betas are *not* different from their $\overline{\text{MS}}$ versions but the simplest 1-loop p-integral is tuned to be maximally simple:

$$\frac{1}{i(2\pi)^D}\int \frac{d^D l}{(-l^2)(-(q-l)^2)} = \frac{1}{(4\pi)^2\,(-q^2)^\epsilon}\,\frac{1}{\epsilon}$$

for finite renormalized quantities: $\left(\ln\frac{\mu^2}{Q^2}\right)_G \to \left(\ln\frac{\mu^2}{Q^2}\right)_{\overline{\text{MS}}} + 2$

2017: BIG PUZZLE

What is special in the C-scheme★?

$$a = \bar{a}\,(1 + c_1\,\bar{a} + c_2\,\bar{a}^2 + c_3\,\bar{a}^3 + c_4\bar{a}^4)$$

with c_1, c_2 and c_3 are made from $\beta_1 - \beta_4$ (all free from even zetas) and with

$$c_4 = \frac{1}{3}\frac{\beta_5}{\beta_1}$$

any SI $\mathcal{O}(\alpha_s^5)$ correlator $F(\bar{a})$ as well as the very β-function $\bar{\boldsymbol{\beta}}(\bar{\boldsymbol{a}})$ loose any dependence on even zetas. We will call the class of renormalization schemes for which

$$\bar{\boldsymbol{\beta}}(\bar{\boldsymbol{a}}) \stackrel{\boldsymbol{\pi}}{=} \mathbf{0}$$

as $\boldsymbol{\pi}$-*independent* schemes

★C-scheme has some interesting features and applications, not relevant in our context of π-hunting; see /Boito, Jamin and Miravitllas, [1606.06175]/

To really appreciate the mystery behind these cancellations induced by the C-scheme, please, look on the following simple facts:

1. **a bare physical (massless!) quantity depends on the bare coupling constant, say, α_s^B;**

2. **its renormalization is done with the replacement $\alpha_s^B = Z_a \alpha_s$;**

3. **the charge renormalization constant Z_a depends on the five-loop coefficient in the β-function—β_5—starting from the *fifth* order, α_s^5;**

4. **as a result the renormalized physical quantity starts to "feel" β_5 only at astonishingly large *sixth* order in α_s;**

5. **for the case of the scalar correlator the contribution of order α_s^6 corresponds to the fabulously large 7-loop level**

Explanation of the mystery (comes from the hatted representation to be discussed later): the ζ_4 term in the β_5 is, in fact, not independent and not genuinely 5-loop but meets a simple factorization formula ($F^{\zeta_i} = \lim_{\zeta_i \to 0} \frac{\partial}{\partial \zeta_i} F$):

$$\beta_5^{\zeta_4} = \frac{9}{8} \beta_1 \, \beta_4^{\zeta_3}$$

The factorization is not trivial at all:

$$\beta_1 \qquad\qquad\qquad (\partial/\partial\zeta_3)\beta_4$$

$$\begin{aligned}
\frac{\partial}{\partial \zeta_4} \beta_5 \quad = \quad & \frac{9}{8} \Big(\frac{4}{3} n_f \, T_F - \frac{11}{3} C_A \Big) \Big(\frac{44}{9} C_A^4 - \frac{136}{3} C_A^3 \, n_f \, T_F \\
& + \frac{656}{9} C_A^2 \, C_F \, n_f \, T_F - \frac{224}{9} C_A^2 \, n_f^2 \, T_F^2 - \frac{352}{9} C_A \, C_F^2 \, n_f \, T_F \\
& - \frac{448}{9} C_A \, C_F \, n_f^2 \, T_F^2 + \frac{704}{9} C_F^2 \, n_f^2 \, T_F^2 - \frac{704}{3} \frac{d_A^{abcd} \, d_A^{abcd}}{N_A} \\
& + \frac{1664}{3} \frac{d_F^{abcd} \, d_A^{abcd}}{N_A} n_f - \frac{512}{3} \frac{d_F^{abcd} \, d_F^{abcd}}{N_A} n_f^2 \Big)
\end{aligned}$$

π-structure of p-integrals

We will call a (bare) L-loop p-integral $F(Q^2, \epsilon)$ π-*safe* if the π-dependence of its pole in ϵ *and* constant part can be completely absorbed into the properly defined "hatted" odd zetas.

The first observation of a non-trivial class of π-safe p-integrals — all 3-loop ones — was made in /Broadhurst (1999)/ An extension of the observation on the class of all 4-loop p-integrals was performed in /Baikov, K.Ch. (2010)/ Here it was shown that, given an arbitrary 4-loop p-integral, its pole in ϵ *and* constant part depend on even zetas *only* via the following combinations:

$$\hat{\zeta}_3 := \zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6, \ \hat{\zeta}_5 := \zeta_5 + \frac{5\epsilon}{2}\zeta_6 \quad \text{and} \quad \hat{\zeta}_7 := \zeta_7.$$

Exact meaning: for any 4-loop p-integral F_4:

$$F_4(\zeta_3, \zeta_4, \zeta_5, \zeta_6, \zeta_7) = F_4(\hat{\zeta}_3, 0, \hat{\zeta}_5, 0, \hat{\zeta}_7) + \mathcal{O}(\epsilon) \qquad \star$$

A generalization of the $\star$ for L=5 has been recently constructed in /Georgoudis, Goncalves, Panzer, Pereira, [1802.00803]/ (and confirmed independently by us, see later)

Remainder on connection between L-loop p-integrals and (L+1) loop Z-factors (a Minimal scheme is assumed!)

The connection is given by the following (35 years old!) Theorem (V. Smirnov, K. Ch, /1983/)

Theorem *Any (L+1)-loop UV counterterm for any Feynman integral may be expressed in terms of pole and finite parts of some appropriately constructed L-loop p-integrals.*

Corollary *Any (L+1)-loop anomalous dimension or a beta-function in any theory may be expressed in terms of pole and finite parts of some appropriately constructed L-loop p-integrals.*

$\hat{G}$-scheme

Let us define the $\widehat{\mathrm{G}}$-scheme by pretending that hatted zetas do not depend on ϵ. This means that all p-integrals are assumed to be expressed in term of the hatted zetas and that the extraction of the pole part of a p-integral is defined as:

$$\hat{K}\Big(\mathcal{P}(\epsilon)\prod_j \hat{\zeta}_j\Big) := \left(\sum_{i<0} \mathcal{P}_i\, \epsilon^j\right) \prod_j \hat{\zeta}_j,$$

with $\mathcal{P}(\epsilon) = \sum_i \epsilon^i \mathcal{P}_i$ being a polynomial in ϵ with rational coefficients. The corresponding coupling constant will be denoted as $\hat{a}$.

The $\hat{G}$-scheme has some remarkable features. Indeed, one can see just from its definition that the corresponding "hatted" Green function, ADs and Z-factors can be obtained from the normal (that is computed with the G-scheme) by very simple rules.

- As a first step we make a formal replacement of the coupling constant a by $\hat{a}$ in every G-renormalized Green function, AD and Z-factor we want to transform to the $\hat{G}$-scheme.
- Renormalized Green function $\hat{F}(\hat{a})$ is obtained from $F(\hat{a})$ by setting to zero *all* even zetas in the latter (both are assumed as taken at $\epsilon = 0$).
- The same rule works for ADs and β-functions.
- If Z is a (G-scheme) renormalization constant then one should not only nullify all even zetas in $Z(\hat{a})$ but also replace every odd zeta term in it with its "hatted" counterpart.

$\hat{G}$-scheme: useful properties and benefits

1. All 2-point (masless, but not necessarily SI) correlators (at least to 5 loops), β-functions and ADs (at least to 6 loops) are π-free in $\hat{G}$-scheme

2. It is more or less obvious that a change of scheme from $\hat{G}$ one to any other π-free(!) scheme will not induce any π-dependence in correlators. Thus, with the help of the $\hat{G}$-scheme the no-π-conjecture is upgraded to a

BIG No-π Theorem

Let F be any L-loop massless correlator and all L-loop p-integrals form a π-safe class. Then F is π-free in any (massless) renormalization scheme for which corresponding β-function and AD γ are both π-free at least at the level of $L+1$ loops.

$\hat{G}$-scheme: constraints on even zetas

Suppose we know a result for an AD $\hat{\gamma} := (\gamma)_{\hat{G}-\text{scheme}}$ as well as the precise way how hatted zetas are related to the normal ones. The infromation should be enough to construct the result in normal, say, $\overline{\text{MS}}$-scheme Thus, all terms proportional to even zetas in γ should be possible to recover. To do this let us consider the relation between $\hat{a}$ and a:

$$\hat{a} = a \left(1 + \sum_{1 \leq i \leq L} c_i \, a^i \right),$$

As the bare charge must not depend on the choice of the renormalization scheme the coefficients c_i are fixed by requiring that

$$Z_a \, a = \hat{Z}_a(\hat{a})\hat{a}$$

For simplicity we start from the case of 4 loops. On general grounds we can write

$$\beta = \beta_1 a + \beta_2 a^2 + (r_3 + \beta_3^{\zeta_3} \, \zeta_3) \, a^3 + (r_4 + \beta_4^{\zeta_3} \, \zeta_3 + \beta_4^{\zeta_4} \, \zeta_4 + \beta_4^{\zeta_5} \, \zeta_5) \, a^4$$

where r_i is β_i with all zetas set to zero

The corresponding RCs Z_a and $\hat{Z}_a$ read:

$$
\begin{aligned}
Z_a &= 1 + \frac{a\beta_1}{\epsilon} + a^2\,(\frac{1}{2\epsilon}\,\beta_2 + \frac{1}{\epsilon^2}\,\beta_1^2) + a^3\,(\frac{1}{3\epsilon}\,(r_3 + \beta_3^{\zeta_3}\,\zeta_3) + \frac{7}{6\epsilon^2}\,\beta_1\,\beta_2 + \frac{1}{\epsilon^3}\,\beta_1^3) \\
&\quad + a^4\,(\frac{1}{4\epsilon}\,(r_4 + \beta_4^{\zeta_3}\,\zeta_3 + \beta_4^{\zeta_4}\,\zeta_4 + \beta_4^{\zeta_5}\,\zeta_5) + \frac{1}{\epsilon^2}\,(\frac{5}{6}\,\beta_1\,r_3 + \frac{5}{6}\,\beta_1\,\beta_3^{\zeta_3}\,\zeta_3 + \frac{3}{8}\,\beta_2^2) \\
&\quad + \frac{23}{12\epsilon^3}\,\beta_1^2\,\beta_2 + \frac{1}{\epsilon^4}\,\beta_1^4) \qquad (1)
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{Z}_a &= 1 + \frac{\hat{a}}{\epsilon}\,\beta_1 + \hat{a}^2\,(\frac{1}{2\epsilon}\,\beta_2 + \frac{1}{\epsilon^2}\,\beta_1^2) + \hat{a}^3\,(\frac{1}{3\epsilon}\,(r_3 + \beta_3^{\zeta_3}\,\hat{\zeta}_3) + \frac{7}{6\epsilon^2}\,\beta_1\,\beta_2 + \frac{1}{\epsilon^3}\,\beta_1^3) \\
&\quad + \hat{a}^4\,(\frac{1}{4\epsilon}\,(r_4 + \beta_4^{\zeta_3}\,\hat{\zeta}_3 + \beta_4^{\zeta_5}\,\hat{\zeta}_5) + \frac{1}{\epsilon^2}\,(\frac{5}{6}\,\beta_1\,r_3 + \frac{5}{6}\,\beta_1\,\beta_3^{\zeta_3}\,\hat{\zeta}_3 + \frac{3}{8}\,\beta_2^2) \\
&\quad + \frac{23}{12\epsilon^3}\,\beta_1^2\,\beta_2 + \frac{1}{\epsilon^4}\,\beta_1^4). \qquad (2)
\end{aligned}
$$

Equation for c_i can be now easily solved with the result

$$
\begin{aligned}
c_1 &= c_2 = 0, \\
c_3 &= -\frac{1}{2}\beta_3^{\zeta_3}\zeta_4 + \frac{5\epsilon^2}{6}\beta_3^{\zeta_3}\zeta_6 - \frac{7\epsilon^4}{2}\beta_3^{\zeta_3}\zeta_8, \\
c_4 &= \frac{1}{4\epsilon}(\beta_4^{\zeta_4} - \beta_1\beta_3^{\zeta_3})\zeta_4 - \frac{3}{8}\beta_4^{\zeta_3}\zeta_4 - \frac{5}{8}\beta_4^{\zeta_5}\zeta_6 \\
&\quad + \frac{5\epsilon}{12}\beta_1\beta_3^{\zeta_3}\zeta_6 + \epsilon^2(\frac{5}{8}\beta_4^{\zeta_3}\zeta_6 + \frac{35}{16}\beta_4^{\zeta_5}\zeta_8) - \frac{7\epsilon^3}{4}\beta_1\beta_3^{\zeta_3}\zeta_8 - \frac{21\epsilon^4}{8}\beta_4^{\zeta_3}\zeta_8
\end{aligned}
$$

As the coefficients c_i have to be finite at $\epsilon \to 0$ we arrive at the exact connection

$$\boldsymbol{\beta_4^{\zeta_4} = \beta_1\beta_3^{\zeta_3}}$$

Repeating the same reasoning for L=5 and 6 (and similar one for the case of an AD) we arrive at a host of new exact identities for even zetas terms

Model independent predictions for β and γ for any 1-charge theory

$$\beta_4^{\zeta_4} = \beta_1 \beta_3^{\zeta_3}$$

$$\beta_5^{\zeta_4} = \frac{1}{2}\beta_3^{\zeta_3}\beta_2 + \frac{9}{8}\beta_1\beta_4^{\zeta_3}$$

$$\beta_5^{\zeta_6} = \frac{15}{8}\beta_1\beta_4^{\zeta_5}$$

$$\beta_5^{\zeta_3\zeta_4} = 0$$

$$\beta_6^{\zeta_4} = \frac{3}{4}\beta_2\beta_4^{\zeta_3} + \frac{6}{5}\beta_1\beta_5^{\zeta_3}$$

$$\beta_6^{\zeta_6} = \frac{5}{4}\beta_2\beta_4^{\zeta_5} + 2\beta_1\beta_5^{\zeta_5} - \beta_1^3\beta_3^{\zeta_3}$$

$$\beta_6^{\zeta_3\zeta_4} = \frac{12}{5}\beta_1\beta_5^{\zeta_3^2}$$

$$\gamma_4^{\zeta_4} = -\frac{1}{2}\beta_3^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_1\gamma_3^{\zeta_3}$$

$$\gamma_5^{\zeta_4} = -\frac{3}{8}\beta_4^{\zeta_3}\gamma_1 + \frac{3}{2}\beta_2\gamma_3^{\zeta_3} - \beta_3^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_1\gamma_4^{\zeta_3}$$

$$\gamma_5^{\zeta_6} = -\frac{5}{8}\beta_4^{\zeta_5}\gamma_1 + \frac{5}{2}\beta_1\gamma_4^{\zeta_5}$$

$$\gamma_5^{\zeta_3\zeta_4} = 0$$

$$\gamma_6^{\zeta_4} = \frac{3}{2}\beta_3^{(1)}\gamma_3^{\zeta_3} - \frac{3}{10}\beta_5^{\zeta_3}\gamma_1 - \frac{3}{4}\beta_4^{\zeta_3}\gamma_2 + \frac{3}{2}\beta_2\gamma_4^{\zeta_3} - \frac{3}{2}\beta_3^{\zeta_3}\gamma_3^{(1)} + \frac{3}{2}\beta_1\gamma_5^{\zeta_3}$$

$$\gamma_6^{\zeta_6} = -\frac{1}{2}\beta_5^{\zeta_5}\gamma_1 - \frac{5}{4}\beta_4^{\zeta_5}\gamma_2 + \frac{5}{2}\beta_2\gamma_4^{\zeta_5} + \frac{5}{2}\beta_1\gamma_5^{\zeta_5} + \frac{3}{2}\beta_1^2\beta_3^{\zeta_3}\gamma_1 - \frac{5}{2}\beta_1^3\gamma_3^{\zeta_3}$$

$$\gamma_6^{\zeta_3\zeta_4} = -\frac{3}{5}\beta_5^{\zeta_3^2}\gamma_1 + 3\beta_1\gamma_5^{\zeta_3^2}$$

$$\beta_6^{\zeta_8} = \frac{14}{5}\beta_1\beta_5^{\zeta_7}$$

$$\beta_6^{\zeta_3\zeta_6} = 0$$

$$\beta_6^{\zeta_4\zeta_5} = 0$$

$$\gamma_6^{\zeta_8} = -\frac{7}{10}\beta_5^{\zeta_7}\gamma_1 + \frac{7}{2}\beta_1\gamma_5^{\zeta_7}$$

$$\gamma_6^{\zeta_3\zeta_6} = 0$$

$$\gamma_6^{\zeta_4\zeta_5} = 0$$

The above constraints have been sucessfully checked on the following examples:

L=4 and 5: numerous QCD RG functions (including gauge-dependent ones taken in the Landau gauge) recently computed in

/K.Ch, Falcioni, Herzog and J Vermaseren [1709.08541] .

L=4,5 and 6: β-function and ADs of $O(n)$ ϕ^4 model recently computed in
Batkovich, K. Ch. and Kompaniets, [1601.01960] (γ_2 only)
Schnetz, [1606.08598] ($\beta, \gamma_2, \gamma_m$)
Kompaniets and Panzer, [1705.06483] ($\beta, \gamma_2, \gamma_m$)

Predictions for 6-loop QCD RG functions:

$$\beta_6 \overset{\pi}{=} \boxed{\frac{608}{405} n_f^5 \zeta_4} + n_f^4 \left(\frac{164792}{1215}\zeta_4 - \frac{1840}{27}\zeta_6\right) + n_f^3 \left(-\frac{4173428}{405}\zeta_4 + \frac{1800280}{243}\zeta_6\right)$$
$$+ n_f^2 \left(\frac{68750632}{405}\zeta_4 - \frac{13834700}{81}\zeta_6\right) + n_f \left(-\frac{146487538}{135}\zeta_4 + \frac{40269130}{27}\zeta_6\right)$$
$$+ 99\,(44213\,\zeta_4 - 64020\,\zeta_6)$$

$$\gamma_6^m \overset{\pi}{=} \boxed{\frac{320}{243} n_f^5 \zeta_4 + n_f^4 \left(-\frac{90368}{405}\zeta_4 + \frac{22400}{81}\zeta_6\right)}$$
$$+ n_f^3 \left(-\frac{92800}{27}\zeta_3\zeta_4 - \frac{2872156}{405}\zeta_4 + \frac{503360}{243}\zeta_6\right)$$
$$+ n_f^2 \left(\frac{661760}{9}\zeta_3\zeta_4 + \frac{155801234}{405}\zeta_4 - \frac{378577520}{729}\zeta_6 + \frac{12740000}{81}\zeta_8\right)$$
$$+ n_f \left(-\frac{1413280}{3}\zeta_3\zeta_4 - \frac{4187656168}{1215}\zeta_4 + \frac{5912758120}{729}\zeta_6 - \frac{96071360}{27}\zeta_8\right)$$
$$+ 3194400\,\zeta_3\zeta_4 + \frac{272688530}{81}\zeta_4 - \frac{6778602160}{243}\zeta_6 + 15889720\,\zeta_8$$

boxed terms are in FULL AGREEMENT with the well-known results by
/Gracey (1996)/ and /Ciuchini, Derkachov, Gracey and Manashov (1999-2000)/
all other terms are new

New developments: 6 and 7 loops

I will discuss below new things, please do not be too critical to my reasonings. No real mathematical rigour!

In particular I will assume that all MZVs as well as other transcendental objects are independent if the opposite is not established in literature,,, Let's come back to the hatted representation of L=5 p-integrals

$$\hat{\zeta}_3 := \zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6 + \frac{21\epsilon^5}{2}\zeta_8\,, \qquad \hat{\zeta}_5 := \zeta_5 + \frac{5\epsilon}{2}\zeta_6 - \frac{35\epsilon^3}{4}\zeta_8\,,$$

$$\hat{\zeta}_7 := \zeta_7 + \frac{7\epsilon}{2}\zeta_8\,, \qquad \hat{\varphi} := \varphi - 3\epsilon\,\zeta_4\,\zeta_5 + \frac{5\epsilon}{2}\zeta_3\,\zeta_6 \qquad \text{and} \qquad \hat{\zeta}_9 := \zeta_9,$$

where

$$\varphi := \frac{3}{5}\,\zeta_{5,3} + \zeta_3\,\zeta_5 - \frac{29}{20}\,\zeta_8 = \zeta_{6,2} - \zeta_{3,5}$$

Now MZVs start to appear and the question what is π-(in)dependence of a given expession stops to be trivial (like it was for the case of SZVs before
/in the case of $L \leq 4$/)

In fact, related questions were investigated and significantly clarified in somewhat different context of (Feynman) periods (by which we understand finite D=4 p-integrals
/or, equivalently, leading $1/\epsilon$ singularities of primitively divergent Feynman integrals/)

The field is (at least) 25 years old, was started from D. Broadhurst, Multiloop calculations without subtractions: 5-loop propagator in ϕ^4- theory, unpublished notes (1993).

and continued by remarkable works

D. Broadhurst, D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops. Phys. Lett. B393, 403, (1997)
D. Broadhurst, D. Kreimer, Knots and numbers in 4 theory to 7 loops and beyond. Int. J. Mod. Phys. C6, 519, (1995)

where the suppession of π in massless periods was discovered for L=5, namely the combination $\zeta_{5,3} - \frac{29}{20}\zeta_8$ was found to kill all powers of π from both subdivergence-free diagrams that contribute to the *six-loop* β-function of the φ^4 model.

David and Dirk also observed the same suppression of π for the case of 3 subdivergence-free 7-loop diagrams of the ϕ^4-theory if one uses the combination

$$\zeta_{5,3,3} + 45\zeta_2\zeta_9 + 3\zeta_4\zeta_7 - \frac{5}{2}\zeta_5\zeta_6$$

instead of just $\zeta_{5,3,3}$ which appears in these counterterms

Later the theory of periods as well as new methods of their calculation and classification were strongly advanced in many remarkable works made by Y. André, C. Bogner, F. Brown, D. Broadhurst, D. Kreimer, E, Panzer, O. Schnetz ..., ★

In particularly, at higher loop numbers, Oliver Schnetz has found that the combination

$$\zeta_{7,3} - \frac{793}{94}\zeta_{10}$$

suppresses the appearance of π^{10} at L=6 (that is L=7 for primitively divergent periods). Even at 7 loops /means L=8 for AD's/ he encountered *only one problem*, namely in hiding explicitly π-dependent structure π^{12}!

All these facts hint that our program of "Hatization" (that is selfconsistent hiding explicitly π-proportional pieces of p-integrals inside of the *not* π-proportional ones has some chances to be successful for $L > 5$

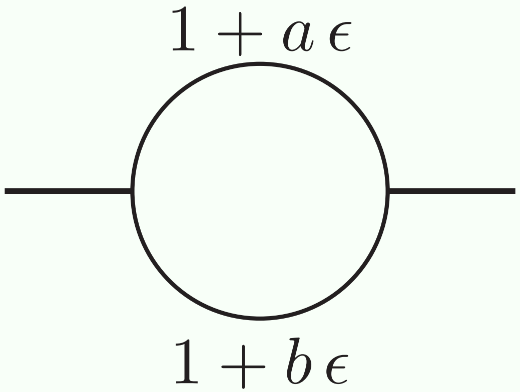
★ **Very important and useful for the classification of MZVs were studies of J. Blümlein, D. Broadhurst, J. Vermaseren (e.g. "The MZV Data Mine"!!!)**

PROBLEM:

direct way: finding and evaluation of master p-integrals was fully implemented only for L=4, and (semi)-fully at L=5. The case with $L > 5$ is excuded for the moment due to their overwhelming complexity!

Hopeless? NO!

A lot of information can be get from 1LR diagrams like



$$\sim G(\alpha,\beta) = \frac{\Gamma(\alpha+\beta-2+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(2-\alpha-\epsilon)\,\Gamma(2-\beta-\epsilon)}{\Gamma(4-\alpha-\beta-2\epsilon)}$$

$$\epsilon\, G(1,1+\epsilon) = \frac{1}{2}+\frac{1}{2}\epsilon+\frac{3}{2}\epsilon^2+\left(\frac{9}{2}-3\zeta_3\right)\epsilon^3+\left(-3\zeta_3+\frac{27}{2}-\frac{\pi^4}{20}\right)\epsilon^4+\left(-9\zeta_3-21\zeta(5)+\frac{81}{2}-\frac{\pi^4}{20}\right.$$

$$+$$

$$\left(-27\zeta_3+9\zeta_3^2-21\zeta(5)+\frac{243}{2}+-\frac{3\pi^4}{20}-\frac{\pi^6}{21}\right)\epsilon^6$$

$$+$$

$$\left(-81\zeta_3+\frac{3\pi^4\zeta_3}{10}+9\zeta_3^2-63\zeta(5)-147\zeta(7)+\frac{729}{2}-\frac{9\pi^4}{20}-\frac{\pi^6}{21}\right)\epsilon^7+\mathcal{O}(\epsilon^8)$$

Summary of 1LR case:

Hatted repesentations of *all* normal (that is SZV) odd zetas can be found from just expanding deeply in ϵ $G(1, 1+\epsilon)$ for arbitrary large number of loops

$\ldots$

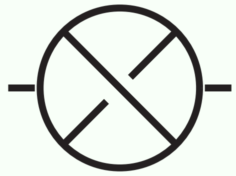
Thus, if we just assume that the $L = 4$ case is described completely by SVZ's (which is true!) then the corresponding hatted representation

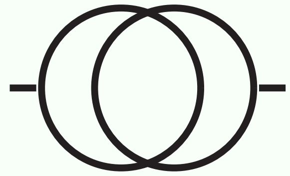
$$\hat{\zeta}_3 := \zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6, \ \hat{\zeta}_5 := \zeta_5 + \frac{5\epsilon}{2}\zeta_6 \quad \text{and} \quad \hat{\zeta}_7 := \zeta_7.$$

can be derived just from the properties of $G(1, 1 + a\,\epsilon)$!

But no MZV's ever appear! (because they aer absent in the normal Γ-function expanded around integer values of its argument)

Next step: consider 3-loop case: then MZV's do show up in corresponding two 1LI masters:





but the resulting 2 eqs. for every π dependent term do fix the hatted form the only MZV, namely $\zeta(5,3)$ appearing at 5-loops (or, equivalently, φ) in full agreement to the result of /Georgoudis, Goncalves, Panzer, Pereira, [1802.00803]/ as it should be (as what we take into account is a small, essentially trivial, subset of around 150 higly nontrivial 5-loop masters considered there).

Lesson: higher orders in ϵ of 3-loop masters do "know" everything about π-structure of 4- and 5-loop masters!

What about 4-loop case? Luckily, many orders in ϵ are known from
R. N. Lee, A. V. Smirnov and V. A. Smirnov, *Master Integrals for Four-Loop Massless Propagators up to Transcendentality Weight Twelve, Nucl. Phys.* B856 (2012) 95–110,

Hatted form for the 6-loop case /transcendental level ≤ 11/

$$\underbrace{\hat{\zeta}_3 := \boxed{\zeta_3} + \frac{3\epsilon}{2}\zeta_4}_{L=3} \quad \underbrace{-\frac{5\epsilon^3}{2}\zeta_6}_{\delta(L=4)} \quad \underbrace{+\frac{21\epsilon^5}{2}\zeta_8}_{\delta(L=5)} \quad \underbrace{-\frac{153\epsilon^7}{2}\zeta_{10}}_{\delta(L=6)}, \tag{3}$$

$$\underbrace{\hat{\zeta}_5 := \boxed{\zeta_5} + \frac{5\epsilon}{2}\zeta_6}_{(L=4)} \quad \underbrace{-\frac{35\epsilon^3}{4}\zeta_8}_{\delta(L=5)} \quad \underbrace{+63\epsilon^5\zeta_{10}}_{\delta(L=6)}, \tag{4}$$

$$\underbrace{\hat{\zeta}_7 := \boxed{\zeta_7}}_{L=4} \quad \underbrace{+\frac{7\epsilon}{2}\zeta_8}_{\delta(L=5)} \quad \underbrace{-21\epsilon^3\zeta_{10}}_{\delta(L=6)}, \tag{5}$$

$$\underbrace{\hat{\varphi} := \boxed{\varphi} - 3\epsilon\,\zeta_4\,\zeta_5 + \frac{5\epsilon}{2}\zeta_3\,\zeta_6}_{L=5} \quad \underbrace{-\frac{24\,\epsilon^2}{47}\zeta_{10} + \epsilon^3\left(-\frac{35}{4}\zeta_3\zeta_8 + 5\zeta_5\zeta_6\right)}_{\delta(L=6)}, \tag{6}$$

$$\underbrace{\hat{\zeta}_9 := \boxed{\zeta_9}}_{L=5} \quad \underbrace{+\frac{9}{2}\epsilon\,\zeta_{10}}_{\delta(L=6)}, \tag{7}$$

$$\underbrace{\hat{\zeta}_{7,3} := \boxed{\zeta_{7,3} - \frac{793}{94}\zeta_{10}} + 3\epsilon(-7\zeta_4\zeta_7 - 5\zeta_5\zeta_6)}_{L=6}, \tag{8}$$

$$\underbrace{\hat{\zeta}_{11} := \boxed{\zeta_{11}}}_{L=6}, \tag{9}$$

$$\underbrace{\hat{\zeta}_{5,3,3} := \boxed{\zeta_{5,3,3} + 45\zeta_2\zeta_9 + 3\zeta_4\zeta_7 - \frac{5}{2}\zeta_5\zeta_6}}_{L=6}. \tag{10}$$

The boxed terms are in agreement with the results of F. Brown, D. Broadhurst, D. Kreimer, E, Panzer, O. Schnetz ...

Now we can upgrade our formulas for π-dependent terms in AD's and β-functions at the next 7-loop level!

$$\beta_7^{\zeta_4} = \frac{3}{8}\,\beta_4^{\zeta_3}\beta_3^{(1)} + \frac{9}{10}\,\beta_2\beta_5^{\zeta_3} - \frac{1}{2}\,\beta_3^{\zeta_3}\beta_4^{(1)} + \frac{5}{4}\,\beta_1\beta_6^{\zeta_3},$$

$$\beta_7^{\zeta_6} = \frac{5}{8}\,\beta_4^{\zeta_5}\beta_3^{(1)} + \frac{3}{2}\,\beta_2\beta_5^{\zeta_5} + \frac{25}{12}\,\beta_1\beta_6^{\zeta_5} - 2\beta_1^2\beta_3^{\zeta_3}\beta_2 - \frac{5}{4}\,\beta_1^3\beta_4^{\zeta_3},$$

$$\beta_7^{\zeta_3\zeta_4} = \frac{9}{5}\,\beta_2\beta_5^{\zeta_3^2} - \frac{1}{8}\,\beta_3^{\zeta_3}\beta_4^{\zeta_3} + \frac{5}{2}\,\beta_1\beta_6^{\zeta_3^2},$$

$$\beta_7^{\zeta_8} = \frac{21}{10}\,\beta_2\beta_5^{\zeta_7} + \frac{35}{12}\,\beta_1\beta_6^{\zeta_7} - \frac{7}{24}\,\beta_1(\beta_3^{\zeta_3})^2 + \frac{7}{4}\,\beta_1^2\beta_5^{\zeta_3^2} - \frac{35}{8}\,\beta_1^3\beta_4^{\zeta_5},$$

$$\beta_7^{\zeta_3\zeta_6} = \frac{5}{8}\beta_3^{\zeta_3}\beta_4^{\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\zeta_3\zeta_5} + \frac{25}{12}\beta_1\beta_6^{\phi},$$

$$\beta_7^{\zeta_4\zeta_5} = -\frac{1}{2}\beta_3^{\zeta_3}\beta_4^{\zeta_5} + \frac{5}{4}\beta_1\beta_6^{\zeta_3\zeta_5} - \frac{5}{2}\beta_1\beta_6^{\phi},$$

$$\beta_7^{\zeta_{10}} = \frac{15}{4}\beta_1\beta_6^{\zeta_9},$$

$$\beta_7^{\zeta_4\zeta_3^2} = \frac{15}{4}\beta_1\beta_6^{\zeta_3^3},$$

$$\beta_7^{\zeta_4\zeta_7} = \beta_7^{\zeta_5\zeta_6} = \beta_7^{\zeta_3\zeta_8} = 0.$$

Tests of pur predictions for AD's at L=7 loop: I

We have checked that the π-dependent contributions to the terms of order $n_f^6 \alpha_s^7$ in the the QCD β-function as well as to the terms of order $n_f^6 \alpha_s^7$ and of order $n_f^5 \alpha_s^7$ contributing to the quark mass AD, all computed in

J. Gracey, *The QCD Beta function at $\mathcal{O}(1/N_f)$*, *Phys.Lett.* **B373** (1996) 178–184, [hep-ph/9602214].

M. Ciuchini, S. E. Derkachov, J. Gracey and A. Manashov, *Quark mass anomalous dimension at O(1/N(f)**2) in QCD*, *Phys.Lett.* **B458** (1999) 117–126, [hep-ph/9903410].

M. Ciuchini, S. E. Derkachov, J. Gracey and A. Manashov, *Computation of quark mass anomalous dimension at O(1 / N**2(f)) in quantum chromodynamics*, *Nucl.Phys.* **B579** (2000) 56–100, [hep-ph/9912221].

are in agreement with our predictions

Tests of pur predictions for AD's at L=7 loop, cont-ed

Significantly more complicated test is provided by the recent calculation of the full 7-loop RG functions in the φ^4-model

O. Schnetz, *Numbers and Functions in Quantum Field Theory*, *Phys. Rev.* **D97 (2018) 085018, [**1606.08598**]**

We have reproduced successfully all π-dependent constants appearing in the β-function and anomalous dimensions γ_m and γ_2 of the $O(n)$ φ^4 at 7 loops

Oliver Schnetz, PRD 97 (2018): 7! loop result for ϕ^4 RG functions:

$$\beta = \Big(\frac{195654269}{23040} + \frac{15676169}{720}\zeta(3) - \frac{316009}{3840}\pi^4 \frac{18326039}{480}\zeta(5) - \frac{129631}{5040}\pi^6$$
$$+ \frac{516957}{20}\zeta(3)^2 - \frac{4453}{60}\pi^4\zeta(3) + \frac{1536173}{20}\zeta(7) - \frac{20425591}{1260000}\pi^8$$
$$+ 116973\zeta(3)\zeta(5) + \frac{947214}{25}\zeta(5,3) - \frac{1010}{63}\pi^6\zeta(3) + \frac{613}{5}\pi^4\zeta(5) + 4176\zeta(3)^3$$
$$+ \frac{547118}{3}\zeta(9) - \frac{45106}{43659}\pi^{10} - 48\pi^4\zeta(3)^2 + \frac{84231}{2}\zeta(3)\zeta(7) - \frac{273030}{7}\zeta(5)^2$$
$$+ \frac{8460}{7}\zeta(7,3) - \frac{174}{25}\pi^8\zeta(3) + \frac{6227}{35}\pi^6\zeta(5) - \frac{56043}{25}\pi^4\zeta(7)$$
$$- 504387\pi^2\zeta(9) + 46845\zeta(3)^2\zeta(5) + 27216\zeta(3)\zeta(5,3) - \frac{336258}{5}\zeta(5,3,3)$$
$$+ \frac{52756839}{10}\zeta(11) + 24P_{7,11}\Big) g^8 + \ldots$$

All π-dependent terms follow from $\beta/.\pi \to 0$: first (partial) check of both the 7-loop β for the ϕ^4-model and on the hatted representation of $\mathcal{P}_7$. The same is true for t γ_m, γ_2 and the 6-loop self-energy

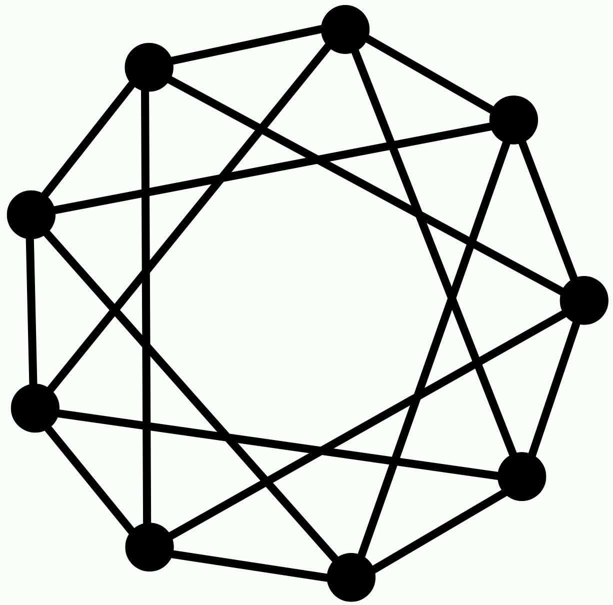


Figure 1: The completed graph $P_{7,11}$

Hatted form for the 7-loop case /transcendental level ≤ 13/

We have succeeded in hiding all π-dependent terms except for the only one, namely, π^{12} (as was expected from our expierence with 8-loop periods).

New hatted representations for the MZV of weight 12 and 12 read:

$$\hat{\zeta}_{9,3} := \boxed{\zeta_{9,3}} + \epsilon\,(-\frac{81}{2}\zeta_4\zeta_9 - 21\zeta_5\zeta_8 - \frac{75}{2}\zeta_6\zeta_7)$$

and similar for

$\hat{z}_{6,4,1,1}$, $\hat{z}_{7,3,3}$ **and** $\hat{z}_{5,5,3}$

We believe that our formulas *may* **lead to predictions for all π-dependent terms in 8-loop AD's and β-functions except for the one proportional to π^{12}.**

Caveats

We need extra info about the hatted form of $P_{7,11}$ and, possibly, other non MZV constants of weight 10 nd 11, which we can not reach in our approach.

However, if we limit ourselves with the π-dependent terms of weight strictly less than 12 then already available results should be enough for predictions of π-dependent contributions to 8-loop AD's and β-functions

Unfortunately, there is not much data about 8-loop RG functions

Conclusions

- all π-dependent terms in a generic (L+1)-loop $\overline{\text{MS}}-$ (or, equivalently, G-) anomalous dimension γ are completely fixed by π-independent contributions to γ (and corresponding β) with loop number L or less *provided* the (all) L-loop p-master integrals are π-safe

- The π-safeness holds for L=4 and L=5 and, probably, for L=6. It is known that for L=7 the property (partially) stops to be valid★ and, thus, our predictions should be modified (at astronomically large for QCD level of L=8 RG functions)

- All available results at 5 (QCD), and 6 and 7 loops (large n_f QCD and the ϕ^4-model) do meet all our constraints

- The no-π conjecture for all one-scale RG-invariant Euclidean correlators first suggested Jamin and Miravitllas less than a year ago has been proved and extended to a case of generic Euclidean correlators