DESY Zeuthen

Regularization-scheme dependence

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what most of us do most of the time:

$$\int \frac{d^4 k_{[4]}}{(2\pi)^4}\ F(A^\mu_{[4]}, \gamma^\mu_{[4]}, k^\mu_{[4]}, g^{\mu\nu}_{[4]} \ldots) \to \int \frac{d^d k_{[d]}}{(2\pi)^d}\ F(A^\mu_{[d]}, \gamma^\mu_{[d]}, k^\mu_{[d]}, g^{\mu\nu}_{[d]} \ldots)$$

with $d = 4 - 2\epsilon$ conventional dimensional regularization CDR

- CDR is consistent regularization scheme (unitarity)
- regularizes IR and UV singularities
- compatible with many (but not all) symmetries
- sometimes incompatible with computational techniques
- can we do better?
- naively: for regularization not all 4 need to be d

warning:

- this is a very old topic
- people use different names for the same thing . . .
- . . . and the same names for different things
- people try things that sometimes work in special cases . . .
- . . . but often is inconsistent in general
- ⇒ conflicting statements in the literature

a desperate attempt to put things on a solid foundation, unify terminology (and even notation):

To d or not to d: recent developments and comparisons of regularization schemes [Gnendiger et al. 1705.01827]

introduction	the obvious
schemes	vector spaces split of vector field definition of common schemes
toy computation @ NLO $(e^+e^- \to)\ \gamma^* \to q\bar{q}$	in CDR in HV in FDH in DRED
beyond NLO	renormalization in FDH IR structure NNLO scheme dependence NNLO anomalous dimension from SCET
what about γ_5	same old or FDF

- perturbative calculations have UV and IR singularities in intermediate results
- requires regularization, many possibilities (in principle)
- intermediate results depend on chosen regularization
- final, physical, finite results must be regularization-scheme independent
- might want to use different schemes for different parts
- this talk is not about renormalization
- and only about dimensional schemes (i.e. not IREG, FDR etc.)

what about "$4 < d < d_s = 4$" ??? [Stöckinger '05]

$S_{[4]}$	$\subset$	$QS_{[d]}$	$\subset$	$QS_{[d_s]}$	$\equiv$	$QS_{[d]} \oplus$	$QS_{[n_\epsilon]}$
strictly 4-dim.		quasi d-dim.		quasi d_s-dim.			'evanescent'
unregularized		actually ∞-dim.		usually $d_s = 4$			space

$$g_{[4]}^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$(g_{[d]})^\mu{}_\mu = d$$

$$(g_{[d]} g_{[4]})^\mu{}_\nu = (g_{[4]})^\mu{}_\nu$$

$$(g_{[d]} g_{[n_\epsilon]})^\mu{}_\nu = 0$$

$QS_{[X]}$ is always an ∞-dimensional space, strictly speaking no explicit representation of spinors, polarization vectors etc.

vector fields in $QS_{[d_s]}$: $A^\mu_{[d_s]} = A^\mu_{[d]} + A^\mu_{[n_\epsilon]}$

$A^\mu_{[d]}$: gauge field

$A^\mu_{[n_\epsilon]}$: "ϵ-scalars", nothing to do with gauge fields

$$\text{QED:} \qquad D^\mu_{[d_s]}\,\psi_i = \partial^\mu_{[d]}\,\psi_i + i\Big(e\,A^\mu_{[d]} + e_e\,A^\mu_{[n_\epsilon]}\Big)\,Q\,\psi_i$$

$$\text{QCD:} \qquad D^\mu_{[d_s]}\,\psi_i = \partial^\mu_{[d]}\,\psi_i + i\Big(g_s\,A^{\mu,a}_{[d]} + g_e\,A^{\mu,a}_{[n_\epsilon]}\Big)\,T^a_{ij}\,\psi_j$$

$e_e,\ g_e$: nothing to do with gauge couplings

$$\beta \;= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\Big(\frac{e}{4\pi}\Big)^2 = -\Big(\frac{e}{4\pi}\Big)^4\Big[-\frac{4}{3}N_F\Big] + \ldots$$

$$\beta_e = \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\Big(\frac{e_e}{4\pi}\Big)^2 = -\Big(\frac{e_e}{4\pi}\Big)^4\big[-4-2\,N_F\big] - \Big(\frac{e}{4\pi}\Big)^2\Big(\frac{e_e}{4\pi}\Big)^2\big[+6\,\big] + \ldots$$

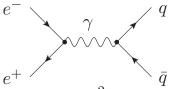
most common dimensional schemes

	CDR	HV	FDH	DRED
singular vector fields (1PI; soft / collinear in initial / final state)	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[d_s]}$	$g^{\mu\nu}_{[d_s]}$
regular vector fields (all other VFs)	$g^{\mu\nu}_{[d]}$	$g^{\mu\nu}_{[4]}$	$g^{\mu\nu}_{[4]}$	$g^{\mu\nu}_{[d_s]}$

- DRED (with arbitrary n_ϵ) covers all elements of the other DS
- nothing in DRED is 4-dimensional !
- what does it mean: "do the algebra in 4 dimensions" ??

$e^+e^- \to \gamma^* \to q\bar{q}$ at NLO in CDR

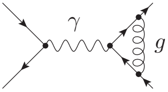
LO



$\sim e^2$

$$M^{(0)}_{\rm CDR} \simeq e^4 (d-2) \quad \Rightarrow \quad \sigma^{(0)} = \frac{\Phi_2}{8\pi} M^{(0)}_{\rm CDR}\Big|_{d\to 4}$$

virtual

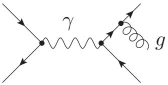


$\sim e^2\, g_s^2$

$$M^{(1)}_{\rm CDR} \simeq M^{(0)}_{\rm CDR} \left(\frac{\alpha_s}{\pi}\right) \left[-\frac{1}{\epsilon^2} - \frac{3}{2\,\epsilon} - 4\right] + \mathcal{O}(\epsilon)$$

$$\sigma^{(v)}_{\rm CDR} \simeq \sigma^{(0)} \frac{\alpha_s}{\pi} C_F \left[-\frac{1}{\epsilon^2} - \frac{1}{2\,\epsilon} - \frac{5-\pi^2}{2}\right]$$

real



$\sim e^2\, g_s$

$$M^{(0)}_{\rm CDR}(q\bar{q}g) \simeq e^4\, g_s^2\, (d-2) \times f(s_{ij}, d)$$

$$\sigma^{(r)}_{\rm CDR} \simeq \sigma^{(0)} \frac{\alpha_s}{\pi} C_F \left[\frac{1}{\epsilon^2} + \frac{1}{2\,\epsilon} + \frac{13-2\,\pi^2}{4}\right]$$

e does not renormalize in QCD (Ward Id)

$e^+e^- \to \gamma^* \to q\bar{q}$ at NLO in HV

- photon γ^* is now 4 dim: $M^{(0)}_{\rm HV} \simeq e^4\, 2 \quad \Rightarrow \quad \sigma^{(0)} = \frac{\Phi_2}{8\pi}\, M^{(0)}_{\rm HV}\Big|_{d\to 4}$
- virtual: trivial changes

$$M^{(1)}_{\rm HV} \simeq M^{(0)}_{\rm HV} \left(\frac{\alpha_s}{\pi}\right) \left[-\frac{1}{\epsilon^2} - \frac{3}{2\,\epsilon} - 4\right] + \mathcal{O}(\epsilon)$$

$$\sigma^{(v)}_{\rm HV} \simeq \sigma^{(0)}\, \frac{\alpha_s}{\pi}\, C_F \left[-\frac{1}{\epsilon^2} - \frac{3}{2\,\epsilon} - \frac{8-\pi^2}{2}\right]$$

- real: reg(ular) gluon $g \in S_{[4]}$, but if collinear $g \in QS_{[d]}$

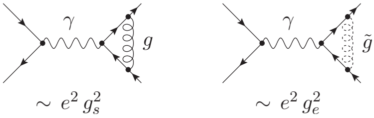
$$M^{(0{\rm reg})}_{\rm HV}(q\bar{q}g) \stackrel{\rm reg}{=} M^{(0)}_{\rm CDR}(q\bar{q}g)\Big|_{d\to 4} \quad \text{but} \quad \sigma^{(r)}_{\rm HV} \neq \int d\Phi_3\, M^{(0{\rm reg})}_{\rm HV}(q\bar{q}g)$$

$$\sigma^{(r)}_{\rm HV} \simeq \sigma^{(0)}\, \frac{\alpha_s}{\pi}\, C_F \left[\frac{1}{\epsilon^2} + \frac{3}{2\,\epsilon} + \frac{19-2\,\pi^2}{4}\right]$$

- “violation” of unitarity [Catani, Seymour, Trocsanyi ’96]

$e^+e^- \to \gamma^* \to q\bar{q}$ at NLO in FDH

- photon γ is now 4 dim: $M^{(0)}_{\rm FDH} = M^{(0)}_{\rm HV}$, does not contain $\alpha_e \sim e_e^2$
- virtual: gluon is quasi d_s dimensional $\Rightarrow$ split into $g \equiv g_{[d]}$ and $\tilde{g} \equiv g_{[n_\epsilon]}$



$$M^{(1)}_{\rm FDH} \simeq M^{(1)}_{\rm HV} + M^{(0)}_{\rm FDH} \frac{\alpha_e}{\pi} \left[\frac{n_\epsilon}{4\,\epsilon}\right]$$

- from practical point of view: set $\alpha_e \to \alpha_s$ and do algebra in 4 dim

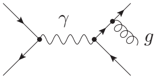
$$\sigma^{(v)}_{\rm FDH} \simeq \sigma^{(0)} \frac{\alpha_s}{\pi} C_F \left[-\frac{1}{\epsilon^2} - \frac{3}{2\,\epsilon} - \frac{8-\pi^2}{2}\right] + \sigma^{(0)} \frac{\alpha_e}{\pi} C_F \left[\frac{n_\epsilon}{4\,\epsilon}\right]$$

$$\Rightarrow \sigma^{(0)} \frac{\alpha_s}{\pi} C_F \left[-\frac{1}{\epsilon^2} - \frac{3}{2\,\epsilon} - \frac{7-\pi^2}{2}\right]$$

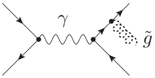
FDH

$e^+e^- \to \gamma^* \to q\bar{q}$ at NLO in FDH

- real: need ϵ-scalar gluon $\tilde{g}$ in collinear (singular) region



$\sim e^2\, g_s$



$\sim e^2\, g_e$

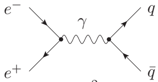
$$M^{(0)}_{\rm FDH}(q\bar{q}g) + M^{(0)}_{\rm FDH}(q\bar{q}\tilde{g}) = M^{(0)}_{\rm HV}(q\bar{q}g) + e^4\, g_e^2\, n_\epsilon\, f(s_{ij})$$

- cannot integrate 4-dim matrix element over phase space
- NLO: $\alpha_e n_\epsilon \times$ finite terms (and scheme-dependence) drop out

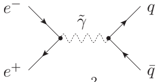
$$\sigma^{(1)} = \sigma^{(0)} + \sigma^{(v)}_{\rm DS} + \sigma^{(r)}_{\rm DS}\Big|_{d\to 4} \simeq \frac{N_c}{3\,s}\left(\frac{e^4}{4\pi}\right)\left[1 + \left(\frac{\alpha_s}{4\pi}\right) 3\, C_F\right]$$

$e^+e^- \to \gamma^* \to q\bar{q}$ at NLO in DRED

- LO: photon is d_s dimensional, $M^{(0)}_{\rm DRED}$ does contain α_e



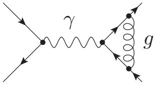
$\sim e^2$



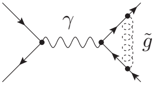
$\sim e_e^2$

$$M^{(0)}_{\rm DRED} = M^{(0,\gamma)}_{\rm DRED} + M^{(0,\tilde{\gamma})}_{\rm DRED}$$
$$\simeq e^4\,(d-2) + e_e^4\, n_\epsilon$$

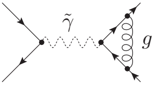
- virtual: photon and gluon are d_s dimensional $\Rightarrow$ split



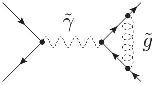
$\sim\ e^2\, g_s^2$



$\sim\ e^2\, g_e^2$

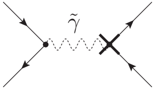


$\sim\ e_e^2\, g_s^2$



$\sim\ e_e^2\, g_e^2$

- (finite) counterterm: α_e renormalizes (no Ward Id !!)

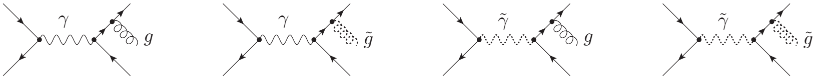


$$\text{CT}_{\rm DRED} \simeq M^{(0,\tilde{\gamma})}_{\rm DRED} \left\{ \frac{\alpha_s}{4\pi}\left[-\frac{6}{\epsilon}\right] + \frac{\alpha_e}{4\pi}\,\frac{4-n_\epsilon}{\epsilon} \right\}$$

DRED

$e^+e^- \to \gamma^* \to q\bar{q}$ at NLO in DRED

- real: photon and gluon are d_s dimensional $\Rightarrow$ split



$\sim e^2 g_s$ $\quad$ $\sim e^2 g_e$ $\quad$ $\sim e_e^2 g_s$ $\quad$ $\sim e_e^2 g_e$

$$M^{(0)}_{\mathrm{DRED}}(q\bar{q}g) = M^{(0,\gamma)}_{\mathrm{DRED}}(q\bar{q}g) + M^{(0,\gamma)}_{\mathrm{DRED}}(q\bar{q}\tilde{g}) + M^{(0,\tilde{\gamma})}_{\mathrm{DRED}}(q\bar{q}g) + M^{(0,\tilde{\gamma})}_{\mathrm{DRED}}(q\bar{q}\tilde{g})$$

- setting $e_e = e$ and $g_e = g_s$:

$$M^{(0)}_{\mathrm{DRED}}(q\bar{q}g)\Big|_{\substack{e_e=e\\ g_e=g_s}} = M^{(0)}_{\mathrm{CDR}}(q\bar{q}g)\Big|_{d=4} \quad \text{and} \quad \sigma^{(r)}_{\mathrm{DRED}} = \int d\Phi_3\, M^{(0)}_{\mathrm{DRED}}(q\bar{q}g)$$

- scheme transitions

DRED $\to$ FDH	drop processes with 'external' (regular) ϵ-scalars
FDH $\to$ HV	drop $n_\epsilon = d_s - d$ terms

conceptually (actually to all orders)

- keep n_ϵ free $\Rightarrow$ physical quantities finite after proper renormalization (no $1/\epsilon$ poles)
- ϵ-scalars contribute $\sim n_\epsilon \times$ finite $= 2\epsilon \times$ finite
- drop out for physical quantities after $\epsilon \to 0$
- $\Rightarrow$ scheme independence of physical quantities

for the FDH practitioner ($=$ DRED if no external VF)

- don't bother with split, do algebra in 4 dimensions $d_s = 4$
- don't bother with regular (tree-level) ϵ-scalars processes
- but for N^{n}LO calculation, need to do split at N^{n-1}LO
- everything that goes "wrong" for DRED, goes "wrong" for FDH one order higher

- general rule for scheme transition CDR → HV → FDH → DRED for virtual amplitude at NLO [Kunszt, AS, Trocsanyi '95]
- massive quarks [Catani, Dittmaier, Trocsanyi '00]
- scheme (in)dependence of real corrections / cross sections at NLO [AS, Stöckinger '08]

  PDF for ϵ-scalars not required

  no issues with factorization in DRED ! [AS, Stöckinger '05]
- CDR and DRED are unitary;

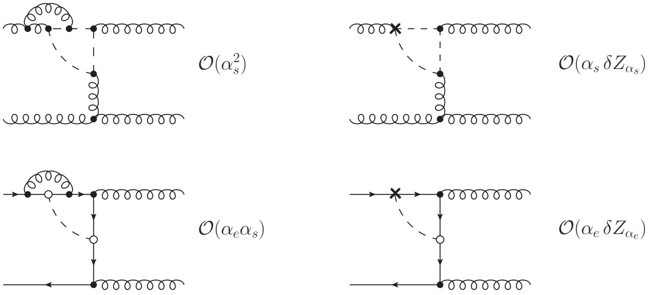
  FDH and HV are not unitary;
  but perfectly consistent if done properly !!
- $e_e,\ g_e$ renormalize differently than $e,\ g_s$! [Jack, Jones, Roberts '94]

  except in SUSY theories (DRED and FDH respect SUSY)

  in FDH @ NLO you get away with incorrect renormalization

beyond NLO

e.g. $gg \to gg$ and $q\bar{q} \to gg$ at NNLO in FDH [Bern, De Freitas, Dixon '03]



- we must distinguish α_s and α_e at NLO, but not at NNLO
- and renormalize properly [Harlander, Kant, Mihaila, Steinhauser '06]
- unless theory is SUSY
- or we are lucky (no α_e dependence at NLO) as in $gg \to gg$

scheme dependence of virtual amplitudes @ NNLO

consider (properly !) UV renormalized amplitudes squared

$$\mathcal{M}^{\rm RS}(\epsilon, n_\epsilon, \{p\}) \equiv 2\,{\rm Re}\,\langle \mathcal{A}_0^{\rm RS}(\epsilon, n_\epsilon, \{p\}) | \mathcal{A}^{\rm RS}(\epsilon, n_\epsilon, \{p\})\rangle$$

structure of IR singularities in CDR: [Gardi, Magnea '09; Becher, Neubert '09]

generalize to other schemes [Broggio, Gnendiger, AS, Stöckinger, Visconti '15]

$$\mathcal{M}^{\rm RS}_{\rm sub}(\epsilon, n_\epsilon, \{p\}, \mu) = 2\,{\rm Re}\langle \mathcal{A}_0^{\rm RS*}(\epsilon, n_\epsilon, \{p\}) | \big(\mathbf{Z}^{\rm RS}\big)^{-1} | \mathcal{A}^{\rm RS}(\epsilon, n_\epsilon, \{p\})\rangle$$

IR subtraction: $$\mathbf{Z}^{\rm RS}(\epsilon, n_\epsilon, \{p\}, \mu) = \mathcal{P} \exp \int_\mu^\infty \frac{d\mu'}{\mu'} \mathbf{\Gamma}^{\rm RS}(n_\epsilon, \{p\}, \mu')$$

anomalous dim: $$\mathbf{\Gamma}^{\rm RS}(\{p\}, \mu) = \sum_{(i,j)} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2}\, \gamma^{\rm RS}_{\rm cusp}\, \ln \frac{\mu^2}{-s_{ij}} + \sum_{i=1}^{n} \gamma_i^{\rm RS}$$

$$\mathcal{M}_{\rm fin}(\{p\},\mu) = \lim_{(N)_\epsilon\to 0} \mathcal{M}^{\rm RS*}_{\rm sub}(\epsilon, n_\epsilon, \{p\},\mu)$$ is scheme independent

all scheme dependence is in: $\gamma^{\rm RS}_{\rm cusp}, \gamma^{\rm RS}_{q}, \gamma^{\rm RS}_{g}$

and $\gamma^{\rm DRED}_{\tilde{g}}$ for DRED

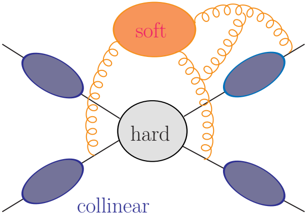
and $\gamma^{\rm RS}_{Q}$ for massive quarks

$x \in \{{\rm cusp}, q, Q, g\}$: $\gamma^{\rm CDR}_x = \gamma^{\rm HV}_x$ and $\gamma^{\rm FDH}_x = \gamma^{\rm DRED}_x$

get anomalous dimensions $\gamma^{\rm RS}_x$ from

- extraction from explicit NNLO calculations (form factors, $2\to 2$ processes) [Kilgore '12; Gnendiger, AS, Stöckinger '14]
- through SCET ($\sim$ process independent) [Broggio, Gnendiger, AS, Stöckinger, Visconti '15]
- agreement is very strong cross check

virtual NNLO

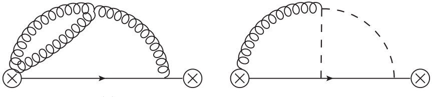


soft function $\Rightarrow$ $\gamma^{\rm RS}_{\sf cusp}, \gamma^{\rm RS}_{W_{\{\rm DY,\ H\}}}$

jet function $\Rightarrow$ $\gamma^{\rm RS}_{\sf cusp}, \gamma^{\rm RS}_{J_{\{q,g\}}}$

$\gamma^{\rm RS}_{\{q,g\}}$ $\Leftarrow$ $\gamma^{\rm RS}_{J_{\{q,g\}}}, \gamma^{\rm RS}_{W_{\{\rm DY,\ H\}}}$

e.g. quark jet function:



(a) (b)

$$\frac{\not{n}}{2}\,\bar{n}\cdot p\,\mathcal{J}_q^{\rm RS}(p^2) = \int d^4x\, e^{ipx}\langle 0|T\{\chi_{hc}(x)\bar{\chi}_{hc}(0)\}|0\rangle$$

$$= \int d^4x\, e^{ipx}\langle 0|T\{\frac{\not{n}\,\not{\bar{n}}}{4}W^\dagger(x)\psi(x)\bar{\psi}(0)W(0)\frac{\not{\bar{n}}\,\not{n}}{4}\}|0\rangle$$

an example: quark anomalous dimension

$$\bar\gamma_q = \left(\frac{\alpha_s}{4\pi}\right)(-3\,C_F) + \left(\frac{\alpha_e}{4\pi}\right) n_\epsilon \frac{C_F}{2}$$
$$+ \left(\frac{\alpha_s}{4\pi}\right)^2 \Big[C_A C_F\Big(-\frac{961}{54} - \frac{11}{6}\pi^2 + 26\zeta_3\Big) + C_F^2\Big(-\frac{3}{2} + 2\pi^2 - 24\zeta_3\Big)$$
$$+ C_F N_F \Big(\frac{65}{27} + \frac{\pi^2}{3}\Big) + n_\epsilon \Big(\frac{167}{108} + \frac{\pi^2}{12}\Big) C_A C_F \Big]$$
$$+ \left(\frac{\alpha_s}{4\pi}\right)\left(\frac{\alpha_e}{4\pi}\right) n_\epsilon \Big[\frac{11}{2} C_A C_F - \Big(2 + \frac{\pi^2}{3}\Big) C_F^2\Big]$$
$$+ \left(\frac{\alpha_e}{4\pi}\right)^2 \Big[- n_\epsilon \frac{3}{4} C_F N_F - n_\epsilon^2 \frac{C_F^2}{8}\Big] + \mathcal{O}(\alpha^3)$$

$$\bar\gamma_q = \gamma_q^{\rm FDH} = \gamma_q^{\rm DRED}; \qquad \bar\gamma_q\big|_{n_\epsilon\to 0} = \gamma_q^{\rm HV} = \gamma_q^{\rm CDR}$$

- FDH: $g^{\mu\nu}_{[d_s]} = g^{\mu\nu}_{[d]} + g^{\mu\nu}_{[n_\epsilon]}, \quad \gamma^{\mu}_{[d_s]} = \gamma^{\mu}_{[d]} + \gamma^{\mu}_{[n_\epsilon]}$ [Gnendiger, AS '18]

- ['t Hooft, Veltman '72] [Breitenlohner, Maison '77]

$$\gamma_5^{\mathrm{BM}} \equiv \frac{i}{4!}\,\varepsilon^{\mu\nu\rho\sigma}_{[4]}\,\big(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\big)_{[d]}$$

- anticommuting γ_5^{AC} with standard cyclic trace:

$$\{\gamma_5^{\mathrm{AC}}, \gamma^{\mu}_{[d]}\} \equiv 0$$

- all (anti)commutators of γ_5^{BM} already fixed by definition:

$$\{\,\gamma_5^{\mathrm{BM}}, \gamma^{\mu}_{[d]}\,\} = 2\,\gamma^{\mu}_{[d-4]}\,\gamma_5^{\mathrm{BM}}$$

$$[\,\gamma_5^{\mathrm{BM}}, \gamma^{\mu}_{[n_\epsilon]}\,] = 0$$

- all (anti)commutators of γ_5^{AC} to be defined:

$$\{\,\gamma_5^{\mathrm{AC}}, \gamma^{\mu}_{[d]}\,\} \equiv 0$$

$$\{\,\gamma_5^{\mathrm{AC}}, \gamma^{\mu}_{[n_\epsilon]}\,\} \equiv 0$$

- same algebraic relations in d and d_s dimensions:

$$\{\,\gamma_5^{\mathrm{BM}}, \gamma^{\mu}_{[d_s]}\,\} = 2\,\gamma^{\mu}_{[d_s-4]}\,\gamma_5^{\mathrm{BM}}$$

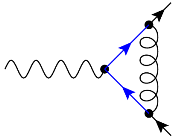
- same algebraic relations in d and d_s dimensions:

$$\{\,\gamma_5^{\mathrm{AC}}, \gamma^{\mu}_{[d_s]}\,\} = 0$$

FDF: strictly four-dim. implementation of FDH algebra at NLO
[Fazio, Mastrolia, Mirabella, Torres Bobadilla '14]

- FDH-regularized $\quad \not{k}_{[d]} = \not{k}_{[4]} + \not{k}_{[d-4]} \equiv \not{k}_{[4]} + i\,\mu\,\gamma_5$

  neglect odd powers of $\mu \quad \not{k}_{[d]}\not{k}_{[d]} = k^2_{[d]} \equiv k^2_{[4]} - \mu^2$



$$\sim \int \frac{d^d k_{[d]}}{(2\pi)^d} \frac{\left[\gamma^\alpha\,(\not{k}_1 + i\,\mu\,\gamma_5)\,\gamma^\mu\,(\not{k}_2 + i\,\mu\,\gamma_5)\,\gamma_\alpha\right]_{[4]}}{\left[\,k_1^2\;k_2^2\;k^2\,\right]_{[d]}}$$

$$= \int \frac{d^d k_{[d]}}{(2\pi)^d} \frac{f_1([4])\;+\;f_2(\mu^2)}{g([d])} \stackrel{f_2=0}{\equiv} \mathcal{M}^{(1)}_{\text{FDH}}$$

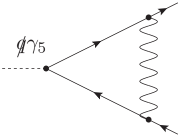
- strictly 4-dim. algebra $\Rightarrow$ generalized-unitarity methods

  (explicit representation of pol. vectors and spinors)

γ_5 in FDF

- FDF algebra realized in strictly four dimensions $\Rightarrow \gamma_5^{\rm BM} = \gamma_5^{\rm AC} \equiv \gamma_5$
- consider pseudoscalar form factor

  one diagram at one-loop:



FDF:

$$\sim \int \dots \left[(\not{k}_1 + i\,\mu\,\gamma_5)\;\not{q}\gamma_5\;(\not{k}_2 + i\,\mu\,\gamma_5)\right]_{[4]} \dots$$

$$\sim \frac{1}{\epsilon} + \frac{7}{2} + \mathcal{O}(\epsilon)$$

FDH + $\gamma_5^{\rm BM}$:

$$\sim \varepsilon^{\mu\nu\rho\sigma}_{[4]} \int \dots \Big(\not{q}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma - \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\not{q}\Big)_{[d]} \dots$$

$$\sim \frac{1 - \frac{n_\epsilon}{2}}{\epsilon} + \frac{9}{2} + \mathcal{O}(\epsilon) \stackrel{n_\epsilon = 2\epsilon}{=} \frac{1}{\epsilon} + \frac{7}{2} + \mathcal{O}(\epsilon)$$

- FDF yields same results as FDH + $\gamma_5^{\rm BM}$
  $\to$ same counterterms needed to restore symmetries
- however: result obtained in a much simpler way
- extension beyond NLO ??

CDR / HV

- most commonly used schemes
- many all-order statements proven rigorously

FDH / DRED: similar status

- equivalence to CDR / $\overline{\text{MS}}$ [Jack, Jones, Roberts '94]
- rigorous definition of vector spaces [Stöckinger '05]
- relation between UV ren. and unitarity [Harlander, Jones, Kant, Mihaila, Steinhauser '09]
- quasi 4-dim. algebra simpler than quasi d-dim. one
- but not much help in connection with γ_5 [Gnendiger, AS '18]

transition rules for UV-ren. amplitudes available up to NNLO

- massless QCD [Broggio, Gnendiger, AS, Stöckinger, Visconti '15]
- massive case [Gnendiger, AS, Visconti '16]