Pull-back transformations of special functions

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The topics

- Special functions
  - Hypergeometric functions
  - Heun functions
  - Pull-back transformations

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- Belyi functions
  - Nearly regular Belyi functions
  - Dessins d'enfant

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- Belyi functions
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  - Dessins d'enfant
- Algebraic hypergeometric functions
  - ${}_2\mathrm{F}_1$ dihedral case
  - ${}_2\mathrm{F}_1$ Platonic cases
  - ${}_3\mathrm{F}_2$ Kleinian case

Gauss hypergeometric function

The Gauss hypergeometric function ${}_2\mathrm{F}_1\left(\left. \begin{array}{c} a,\ b \\ c \end{array} \right| z \right) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!}\,z^n$

satisfies the Fuchsian equation

$$\frac{d^2y(z)}{dz^2} + \left(\frac{c}{z} + \frac{a+b-c+1}{z-1} \right) \frac{dy(z)}{dz} + \frac{a\,b}{z\,(z-1)}\,y(z) = 0.$$

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As usual, $(a)_k = a\,(a+1)\cdots(a+k-1)$ is the Pochhammer symbol.

Examples: $\ln(1+z)$, $\arcsin(z)$, complete elliptic integrals,

Chebyshev, Legendre, Jacobi, Krawtchouk polynomials

can be expressed in terms of ${}_2\mathrm{F}_1$ functions.

Local hypergeometric solutions

$$\text{at } z=0: \quad {}_2F_1\left(\left.\begin{array}{c} a,\ b \\ c \end{array}\right|\, z\right), \quad z^{1-c}\,{}_2F_1\left(\left.\begin{array}{c} 1+a-c,\ 1+b-c \\ 2-c \end{array}\right|\, z\right);$$

$$\text{at } z=1: \quad {}_2F_1\left(\left.\begin{array}{c} a,\ b \\ 1+a+b-c \end{array}\right|\, 1-z\right),$$

$$(1-z)^{c-a-b}\,{}_2F_1\left(\left.\begin{array}{c} c-a,\ c-b \\ 1+c-a-b \end{array}\right|\, 1-z\right);$$

$$\text{at } z=\infty: \quad z^{-a}\,{}_2F_1\left(\left.\begin{array}{c} a,\ 1+a-c \\ 1+a-b \end{array}\right|\, \frac{1}{z}\right), \quad z^{-b}\,{}_2F_1\left(\left.\begin{array}{c} b,\ 1+b-c \\ 1-a+b \end{array}\right|\, \frac{1}{z}\right).$$

Local hypergeometric solutions

$$\text{at } z=0: \quad {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, z\right), \quad z^{1-c}\,{}_2F_1\left(\begin{matrix} 1+a-c,\ 1+b-c \\ 2-c \end{matrix}\,\middle|\, z\right);$$

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The Riemann scheme for the hypergeometric equation is

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{matrix}\right\}.$$

Generalized hypergeometric function

The generalized hypergeometric function

$$ {}_p\mathrm{F}_q\left(\left. \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \right| z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k \, k!} \, z^k $$

satisfies the differential equation

$$ \left(z\frac{d}{dz}+a_1\right)\cdots\left(z\frac{d}{dz}+a_p\right)Y(z)=\frac{d}{dz}\left(z\frac{d}{dz}+b_1-1\right)\cdots\left(z\frac{d}{dz}+b_q-1\right)Y(z), $$

keeping in mind $\frac{d}{dz}zY(z) = z\,\frac{d}{dz}Y(z) + \frac{d}{dz}Y(z)$.

Appell functions

Appell's function $F_2\left(\begin{matrix} a;\ b_1, b_2 \\ c_1, c_2 \end{matrix} \,\middle|\, x,\ y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_{n+m}\,(b_1)_n\,(b_2)_m}{(c_1)_n\,(c_2)_m\; n!\; m!}\, x^n\, y^m$

satisfies the holonomic system

$$x(1-x)\frac{\partial^2 F}{\partial x^2} - xy\frac{\partial^2 F}{\partial x \partial y} + (c_1 - (a+b_1+1)x)\,\frac{\partial F}{\partial x} - b_1 y\frac{\partial F}{\partial y} - ab_1 F = 0,$$
$$y(1-y)\frac{\partial^2 F}{\partial y^2} - xy\frac{\partial^2 F}{\partial x \partial y} + (c_2 - (a+b_2+1)y)\,\frac{\partial F}{\partial y} - b_2 x\frac{\partial F}{\partial x} - ab_2 F = 0.$$

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Appell's function $F_3\left(\begin{matrix} a_1, a_2; b_1, b_2 \\ c \end{matrix}\,\middle|\, x, y\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a_1)_n(a_2)_m(b_1)_n(b_2)_m}{(c)_{n+m}\; n!\; m!}\, x^n y^m$

satisfies a similar holonomic system.

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satisfies a similar holonomic system.

$$F_2\left(\begin{matrix} a; -k, -\ell \\ c_1, c_2 \end{matrix} \middle|\, x, y\right) = \frac{(a)_{k+\ell}\,x^k y^\ell}{(c_1)_k (c_2)_\ell}\,F_3\left(\begin{matrix} 1-c_1-k, 1-c_2-\ell; -k, -\ell \\ 1-a-k-\ell-a \end{matrix} \middle|\, \frac{1}{x}, \frac{1}{y}\right).$$

Heun's differential equation

A canonical Fuchsian equation with 4 singularities:

$$\frac{d^2y(x)}{dx^2}+\left(\frac{c}{x}+\frac{d}{x-1}+\frac{a+b-c-d+1}{x-t}\right)\frac{dy(x)}{dx}+\frac{abx-q}{x(x-1)(x-t)}y(x)=0.$$

The 4 regular singular points are $x=0$, $x=1$, $x=t$, $x=\infty$. The parameters a,b,c,d determine the local exponents at them, while q is an *accessory parameter*.

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The Riemann scheme for the Heun equation is

$$P\left\{\begin{array}{ccccc} 0 & 1 & t & \infty & \\ 0 & 0 & 0 & a & x \\ 1-c & 1-d & c+d-a-b & b & \end{array}\right\}.$$

Heun's series

A local solution at $x = 0$ is defined by the series

$$\mathrm{Hn}\left(\begin{array}{c|c|c} t & a,\ b & \\ q & c;\ d & x \end{array} \right) = 1 + \frac{q}{ct}x + \frac{q^2 + (a+b+ct+dt-d+1)q - abct}{2c(c+1)t^2}x^2 + \ldots.$$

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Other local solution at $x = 0$ is

$$x^{1-c}\,\mathrm{Hn}\left(\begin{array}{c|c|c} t & a-c+1,\ b-c+1 & \\ q_1 & 2-c;\ d & \end{array} x \right),$$

with $q_1 = q - (c-1)(a+b-c-d+d\,t+1)$.

Transformations of Heun functions

$$\mathrm{Hn}\left(\begin{matrix} t \\ q \end{matrix} \middle| \begin{matrix} a,\ b \\ c;\ d \end{matrix} \middle| \, x \right) = \mathrm{Hn}\left(\begin{matrix} 1/t \\ q/t \end{matrix} \middle| \begin{matrix} a,\ b \\ c;\ a+b-c-d+1 \end{matrix} \middle| \, \frac{x}{t} \right)$$
$$= \left(1-\frac{x}{t}\right)^{-a} \mathrm{Hn}\left(\begin{matrix} 1-t \\ ac-q \end{matrix} \middle| \begin{matrix} a,\ c+d-b \\ c;\ d \end{matrix} \middle| \, \frac{(1-t)\,x}{x-t} \right)$$
$$= (1-x)^{1-d}\, \mathrm{Hn}\left(\begin{matrix} t \\ q-c\,(d-1)\,t \end{matrix} \middle| \begin{matrix} a-d+1,\ b-d+1 \\ c;\ 2-d \end{matrix} \middle| \, x \right).$$

In total, the set of 24 Kummer's solutions of the hypergeometric equation is replaced by a set of 192 local Heun equations. They related by permutation of the 4 singular points and change the sign of the local exponent difference.

Transformations of special functions

- symmetric squares and other tensor operations; say

  Clausen's identity: ${}_2\mathrm{F}_1\left(\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix} \,\middle|\, z\right)^2 = {}_3\mathrm{F}_2\left(\begin{matrix} 2a,\ 2b,\ a+b \\ 2a+2b,\ a+b+\frac{1}{2} \end{matrix} \,\middle|\, z\right).$

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- Fourier transforms, convolutions and other integral transforms.

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- Fourier transforms, convolutions and other integral transforms.
- Specializations to fewer variables.
- Separation of variables.
- Pull-backs or push-forwards with respect to algebro-geometric morphisms.

An example of separation of variables

The functions $F_2\left(\begin{array}{c} b_1+b_2-\frac{1}{2};\ b_1, b_2 \\ 2b_1,\ 2b_2 \end{array} \middle| -\frac{4t}{(t-1)^2}, \frac{(1-s)(st^2-1)}{s(t-1)^2} \right)$

and $(1-t)^{2b_1+2b_2-1}\,{}_2\mathrm{F}_1\left(\begin{array}{c} b_1+b_2-\frac{1}{2}, b_2 \\ b_1+\frac{1}{2} \end{array} \middle| \, st^2 \right)$ satisfy

the same second order Fuchsian equation with respect to t for any s.

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The functions $F_2\left(\begin{matrix} b_1+b_2-\frac12;\; b_1, b_2 \\ 2b_1,\, 2b_2 \end{matrix} \,\middle|\, -\frac{4t}{(t-1)^2}, \frac{(1-s)(st^2-1)}{s(t-1)^2}\right)$

and $(1-t)^{2b_1+2b_2-1}\,{}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac12, b_2 \\ b_1+\frac12 \end{matrix} \,\middle|\, st^2\right)$ satisfy

the same second order Fuchsian equation with respect to t for any s. Using a symmetry between s, t,

$$F_2\left(\begin{matrix} b_1+b_2-\frac12;\; b_1, b_2 \\ 2b_1, 2b_2 \end{matrix} \,\middle|\, \frac{4(1-s^2)\,t}{(1+s+t-st)^2}, \frac{4(1-t^2)\,s}{(1+s+t-st)^2}\right) =$$

$$\left(\frac{1+s+t-st}{1-s}\right)^{2b_1+2b_2-1} {}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac12, b_2 \\ 2b_2 \end{matrix} \,\middle|\, \frac{4s}{s^2-1}\right) {}_2\mathrm{F}_1\left(\begin{matrix} b_1+b_2-\frac12, b_2 \\ b_1+\frac12 \end{matrix} \,\middle|\, t^2\right).$$

Pull-back transformations of differential equations

A pull-back transformation of a differential equation for $y(z)$ in d/dz has the form

$$z \longmapsto \varphi(x), \qquad y(z) \longmapsto Y(x) = \theta(x)\, y(\varphi(x)),$$

where $\varphi(x)$ is a rational function, and $\theta(x)$ is a product of powers of rational functions. Geometrically, the transformation *pulls-back* a differential equation on the projective line $\mathbb{P}^1_z$ to a differential equation on the projective line $\mathbb{P}^1_x$, with respect to the covering $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ determined by the rational function $\varphi(x)$.

Pull-back transformations induce identities like

$${}_2\mathrm{F}_1\left(\left. \begin{array}{c} a,\ b \\ a-b+1 \end{array} \right| x \right) = (1+x)^{-a}\, {}_2\mathrm{F}_1\left(\left. \begin{array}{c} \frac{a}{2},\ \frac{a+1}{2} \\ a-b+1 \end{array} \right| \frac{4x}{(1+x)^2} \right).$$

More hypergeometric formulas

$$
{}_2F_1\left(\left.\begin{array}{c} a,\ \frac{2a+1}{2} \\ \frac{4a+2}{3} \end{array}\right|\, x\right) = \left(1-\frac{3x}{4}\right)^{-a} {}_2F_1\left(\left.\begin{array}{c} \frac{a}{3},\ \frac{a+1}{3} \\ \frac{4a+5}{6} \end{array}\right|\, \frac{27\,x^2\,(1-x)}{(4-3x)^3}\right),
$$

$$
{}_2F_1\left(\left.\begin{array}{c} \frac{4a}{3},\ \frac{4a+1}{3} \\ \frac{4a+1}{2} \end{array}\right|\, x\right) = \left(1-\frac{8x}{9}\right)^{-a} {}_2F_1\left(\left.\begin{array}{c} \frac{a}{3},\ \frac{a+1}{3} \\ \frac{4a+5}{6} \end{array}\right|\, \frac{64\,x^3\,(1-x)}{(9-8x)^3}\right),
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$$

$$
{}_2F_1\left(\left.\begin{matrix} 5/42,\ 19/42 \\ 5/7 \end{matrix}\,\right|\, x\right) = \left(1-\frac{19}{9}x-\frac{343}{243}x^2+\frac{16807}{6561}x^3\right)^{-1/28} \times
$$

$$
{}_2F_1\left(\left.\begin{matrix} 1/84,\ 29/84 \\ 6/7 \end{matrix}\,\right|\, \frac{x^2\,(1-x)\,(49x-81)^7}{4\,(16807x^3-9261x^2-13851x+6561)^3}\right),
$$

$$
{}_2F_1\left(\left.\begin{matrix} 3/20,\ 7/20 \\ 3/4 \end{matrix}\,\right|\, x\right) = \left(1-\frac{8(4+3i)}{25}x\right)^{-1/8} \times
$$

$$
{}_2F_1\left(\left.\begin{matrix} 1/40,\ 9/40 \\ 3/4 \end{matrix}\,\right|\, \frac{4i\,x\,(x-1)\,(4x-2-11i)^4}{(8x-4+3i)^5}\right).
$$

Heun-to-hypergeometric reductions

$$\mathrm{Hn}\left(\begin{matrix} -1 \\ 0 \end{matrix} \left| \begin{matrix} 2a, 2b \\ 2c-1; a+b-c+1 \end{matrix} \right| x \right) = {}_2\mathrm{F}_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \middle| x^2 \right),$$

$$\mathrm{Hn}\left(\begin{matrix} \frac{1}{2} \\ 2ab \end{matrix} \left| \begin{matrix} 2a,\ 2b \\ c;\ c \end{matrix} \right| x \right) = {}_2\mathrm{F}_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \middle| 4x(1-x) \right),$$

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$$\mathrm{Hn}\left(\begin{matrix}\frac12\\2ab\end{matrix}\,\middle|\,\begin{matrix}2a,\ 2b\\c;\ c\end{matrix}\,\middle|\,x\right)={}_2\mathrm{F}_1\left(\begin{matrix}a,\ b\\c\end{matrix}\,\middle|\,4x(1-x)\right),$$

$$\mathrm{Hn}\left(\begin{matrix}\frac89\\q_1\end{matrix}\,\middle|\,\begin{matrix}3a,2a+b\\2a+2b-\frac13;a+b+\frac13\end{matrix}\,\middle|\,x\right)=$$
$$\left(1-\frac{9x}{8}\right)^{-2a}{}_2\mathrm{F}_1\left(\begin{matrix}a,\ b\\a+b+\frac13\end{matrix}\,\middle|\,\frac{27x^2(x-1)}{(8-9x)^2}\right).$$

$$\mathrm{Hn}\left(\begin{matrix}\frac98\\q_2\end{matrix}\,\middle|\,\begin{matrix}4a,\ 3a+b\\3a+3b-\frac12;a+b+\frac12\end{matrix}\,\middle|\,x\right)=$$
$$\left(1-\frac{8x}{9}\right)^{-3a}{}_2\mathrm{F}_1\left(\begin{matrix}a,\ b\\a+b+\frac12\end{matrix}\,\middle|\,\frac{64x^3(1-x)}{(9-8x)^3}\right).$$

with $q_1=4a^2+4ab-2a/3,\ q_2=9a^2+9ab-3a/2.$

Transformation of singularities

A pull-back of a Fuchsian equation is a Fuchsian equation again. We are interested in pullbacks of the hypergeometric equation to Heun's equation.

Suppose we have a pull-back transformation $E_2 \leftarrow E_1$ of Fuchsian equations with respect to a covering $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ of degree D.

If $Q \in \mathbb{P}^1_x$ lies above a singular point $P \in \mathbb{P}^1_z$ of E_1
then Q is a singular point of E_2 *unless* the local exponent difference at P is equal to $\pm 1/k$, where k is an integer, and the branching order at Q is equal to k.

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If $Q \in \mathbb{P}^1_x$ lies above a non-singular point $P \in \mathbb{P}^1_z$ of E_1, and Q is a branching point, then Q is an apparent singularity at best.

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If $Q \in \mathbb{P}^1_x$ lies above a non-singular point $P \in \mathbb{P}^1_z$ of E_1, and Q is a branching point, then Q is an apparent singularity at best.

If E_1 has 3 singular points, there are at least $D+2$ distinct points on $\mathbb{P}^1_x$ above those 3 points, by Hurwitz formula.

Pull-backs of hypergeometric equation

To keep the number of singularities of E_2 low, we usually can allow the covering φ to branch only above the singular points of E_1.
If φ branches only above 3 points, it is a *Belyi covering*.

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$$\frac{2}{D} + \frac{1}{k} \geq 1.$$

We get $(k, D) \in \{(2,3),\ (2,4),\ (3,3)\}$ as interesting cases.

Keeping one parameter fixed

Restriction of 2 local exponent differences to $1/k$, $1/\ell$ leads to

$$D + 2 - \left\lfloor \frac{D}{k} \right\rfloor - \left\lfloor \frac{D}{\ell} \right\rfloor \leq 4.$$

After dropping the rounding we get the weaker Diophantine inequality

$$\frac{2}{D} + \frac{1}{k} + \frac{1}{\ell} \geq 1.$$

We can have $(k, \ell, D_{\max}) \in \{(2,3,12),\ (2,4,8),\ (2,5,6),\ (2,6,6),$ $(3,3,6),\ (3,4,4),\ (4,4,4)\}$, ignoring $\max(k,\ell) \leq 2$.

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This gives a list of 81 possible branching patterns. There are no coverings for 27 of them. We get 26 composite coverings, and the remaining 28 patterns give indecomposable coverings.

Classification of Heun-to-hypergeometric transformations

The Heun-to-hypergeometric transformations with a parameter are classified by R.V. and Galina Philipuk in [Funkcialaj Ekvacioj, 56 (2013)]. There are 61 transformations with a free parameter, of maximal degree 12.

38 of the parametric transformations apply to hypergeometric equations with two exponent differences restricted to $1/2, 1/3$. They involve different Belyi maps, that also occur in Herfurtner's classification (1991) of rational elliptic surfaces with 4 singular fibers.

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Parametric hypergeometric-to-hypergeometric transformations (of degree at most 6) were found by Goursat (1881). A full classification of ${}_2\mathrm{F}_1$-to-${}_2\mathrm{F}_1$ transformations is given in [Funkcialaj Ekvacioj, 52 (2009)].

Non-parametric Heun-to-hypergeometric pull-backs

There are surely infinitely many transformations between algebraic ${}_2\mathrm{F}_1$ functions. They involve hypergeometric equations $E(1/k, 1/\ell, 1/m)$ with $1/k + 1/\ell + 1/m > 1$.

Solutions of $E(1/2, 1/4, 1/4)$, $E(1/2, 1/3, 1/6)$, $E(1/3, 1/3, 1/3)$ are indefinite elliptic integrals on $j = 1728$ or $j = 0$ elliptic curves. Isogenies on those curves give infinitely many transformations of hypergeometric functions. For example:

$$
\begin{aligned}
{}_2\mathrm{F}_1\left(\left. \begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix} \right| z \right) &= \frac{z^{-1/4}}{2} \int_{1/\sqrt{z}}^{\infty} \frac{dx}{\sqrt{x^3 - x}} \\
&= \frac{1 - z/(1+2i)}{1 - (1+2i)z} \; {}_2\mathrm{F}_1\left(\left. \begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix} \right| \frac{z(z-1-2i)^4}{((1+2i)z-1)^4} \right).
\end{aligned}
$$

Non-parametric Heun-to-hypergeometric pull-backs

There are 9 cases of transformations (of degree up to 24) of $E(1/k, 1/\ell, 1/m)$ with $1/k + 1/\ell + 1/m < 1$.

In particular, let $\Phi_{24} = \dfrac{1728x(x-1)F_1^7}{G_1^3}$ with

$$F_1 = x^3 - 8x^2 + 5x + 1,$$
$$G_1 = (x^2 - x + 1)(x^6 + 229x^5 + 270x^4 - 1695x^3 + 1430x^2 - 235x + 1).$$

Then ${}_2\mathrm{F}_1\left(\left.\begin{array}{c} 2/7,\ 4/7 \\ 6/7 \end{array}\right| x\right) = G_1^{-1/28}\, {}_2\mathrm{F}_1\left(\left.\begin{array}{c} 1/84,\ 29/84 \\ 6/7 \end{array}\right| \Phi_{24}\right).$

Non-parametric Heun-to-hypergeometric pull-backs

Together with Mark van Hoeij (in [Journal of Algebra 441 (2015)]), I classified all transformations from $E(1/k, 1/\ell, 1/m)$ to Heun equations, with k, ℓ, m positive integers and $1/k + 1/\ell + 1/m < 1$. There are 366 different (Galois orbits of) these transformations, of maximal degree 60. For example, $E(\frac{1}{2}, \frac{1}{3}, \frac{1}{7}) \xleftarrow{60} HE(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$, $E(\frac{1}{2}, \frac{1}{3}, \frac{1}{9}) \xleftarrow{24} HE(\frac{1}{9}, \frac{1}{9}, \frac{2}{9}, \frac{2}{9})$, $E(\frac{1}{2}, \frac{1}{3}, \frac{1}{13}) \xleftarrow{18} HE(\frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{2}{13})$, $E(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}) \xleftarrow{10} HE(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$, $E(\frac{1}{3}, \frac{1}{3}, \frac{1}{5}) \xleftarrow{7} HE(\frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5})$, etc.

Non-parametric Heun-to-hypergeometric pull-backs

Together with Mark van Hoeij (in [Journal of Algebra 441 (2015)]), I classified all transformations from $E(1/k, 1/\ell, 1/m)$ to Heun equations, with k, ℓ, m positive integers and $1/k + 1/\ell + 1/m < 1$. There are 366 different (Galois orbits of) these transformations, of maximal degree 60. For example, $E(\frac{1}{2}, \frac{1}{3}, \frac{1}{7}) \stackrel{60}{\longleftarrow} HE(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$, $E(\frac{1}{2}, \frac{1}{3}, \frac{1}{9}) \stackrel{24}{\longleftarrow} HE(\frac{1}{9}, \frac{1}{9}, \frac{2}{9}, \frac{2}{9})$, $E(\frac{1}{2}, \frac{1}{3}, \frac{1}{13}) \stackrel{18}{\longleftarrow} HE(\frac{1}{13}, \frac{1}{13}, \frac{1}{13}, \frac{2}{13})$, $E(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}) \stackrel{10}{\longleftarrow} HE(\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$, $E(\frac{1}{3}, \frac{1}{3}, \frac{1}{5}) \stackrel{7}{\longleftarrow} HE(\frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5})$, etc.
It remains to classify the coverings from $E(1/k, 1/\ell, 1/m)$ to Heun equations, with k, ℓ, m positive integers and $1/k + 1/\ell + 1/m = 1$ and $1/k + 1/\ell + 1/m > 1$.

Rational Belyi functions

Definition

A *rational Belyi function* is a rational function $\varphi(x) \in \mathbb{C}(x)$ such that the set of *critical values* $\{\varphi(x) : \varphi'(x) = 0\}$ is a subset of $\{0, 1, \infty\}$.

Geometrically, a rational Belyi function defines a covering $\varphi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ that branches only above the points $\{0, 1, \infty\}$. Their zeroes are the points above $\varphi = 0$, and the poles are the points above $\varphi = \infty$. The projective line $\mathbb{P}^1(\mathbb{C})$ is the Riemann sphere $\mathbb{C} \cup \infty$.

Belyi functions are explicitly defined over algebraic number fields $\mathbb{Q}(\alpha)$. Arithmetic questions about Belyi functions are very interesting.

Example

The easiest Belyi functions are the power functions $\varphi(x) = x^n$.

A more complicated example is the rational function $\varphi(x) = \dfrac{(2x^2-1)^3}{(3x-2)^4}$.

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We can compute:

$$\varphi(x) - 1 = \frac{(x-1)^3\,(8x^3+24x^2-45x+17)}{(3x-2)^4},$$

$$\varphi'(x) = \frac{12\,(x-1)^2\,(2x^2-1)^2}{(3x-2)^5}.$$

The branching points are $x = \pm\dfrac{1}{\sqrt{2}}, x = \dfrac{2}{3}, x = \infty$ and $x = 1$.

Dessins d'enfant

Generally, a *dessin d'enfant* can be defined as a bi-colored graph (possibly with multiple edges) with a cyclic order of edges around each vertex given.

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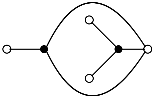
Constructively, given a Belyi map, its dessins d'enfant is the pre-image of the interval segment $[0,1] \subset \mathbb{P}^1_z$ on the Rieman surface, with the vertices above $z=0$ and $z=1$ coloured differently.

For example, the dessin d'enfant for $\varphi(x) = \dfrac{(2x^2-1)^3}{(3x-2)^4}$ is



The *branching pattern* of $\varphi(x)$ is $3+3=4+2=3+1+1+1$.

Regular Belyi functions

Definition

Given positive integers k, ℓ, m, a Belyi function $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is called (k, ℓ, m)-*regular* if all points above $z = 1$ have branching order k, all points above $z = 0$ have branching order ℓ, and all points above $z = \infty$ have branching order m.

Examples of $(2, 3, m)$-regular Belyi functions with $m \in \{3, 4, 5\}$ are the well-known Galois coverings $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 12, 24, 60 with the tetrahedral A_4, octahedral S_4 or icosahedral A_5 monodromy groups, respectively. Platonic solids give their dessins d'enfant [**?**]. For example, the icosahedral covering:

$$\varphi_{60}(x) = \frac{1728\, x^5\, (x^{10} - 11x^5 - 1)^5}{(x^{20} + 228x^{15} + 494x^{10} - 228x^5 + 1)^3}.$$

Nearly regular Belyi functions

Definition

Given yet another positive integer n, a Belyi function $\varphi : \mathbb{P}^1_x \to \mathbb{P}^1_z$ is called (k, ℓ, m)*-minus-n regular* if, with exactly n exceptions, all points above $z = 1$ have branching order k, all points above $z = 0$ have branching order ℓ, and all points above $z = \infty$ have branching order m.

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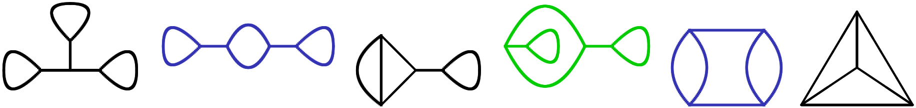
These Belyi functions have rather regular dessins d'enfant.

Definition

A dessin d'enfant is called (k, ℓ, m)*-minus-*n *regular* if, with exactly n exceptions, all white vertices have degree k, all black vertices have degree ℓ, and all cells have degree m.

Beauville coverings of degree 12



$$\frac{64x^3(x^3-1)^3}{8x^3-9}$$

$$\frac{4x^3(x^3-6x+6)^3}{27(x-1)^3(2x-3)^2(x+3)}$$

$$\frac{64x^3(x^3+1)^3}{(8x^3-1)^3}$$

$$\frac{4(x^4-4x^2+1)^3}{27x^2(x^2-4)}$$

$$\frac{4(x^4-x^2+1)^3}{27x^4(x^2-1)^2}$$

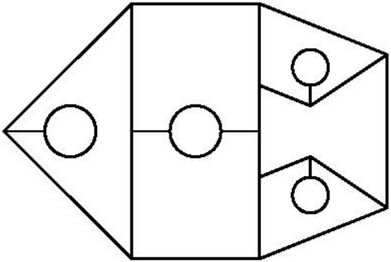
$$\frac{(x^4-12x^3+14x^2+12x+1)^3}{1728x^5(x^2-11x-1)}$$

The degree 54 function

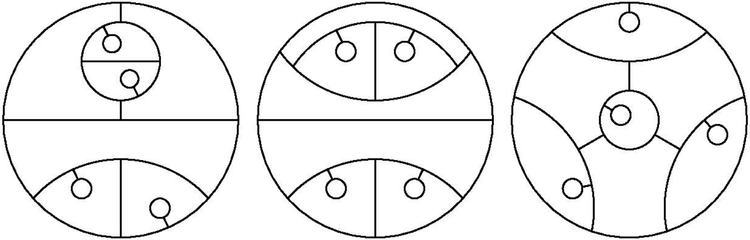
Consider $\varphi(x) = \dfrac{P^3}{864(x-4)(3x^2+1)(x+4)^2 Q^7},$

where $Q = 3x^7 - 7x^6 - 14x^5 - 98x^4 + 147x^3 - 7x^2 + 56x + 16,$

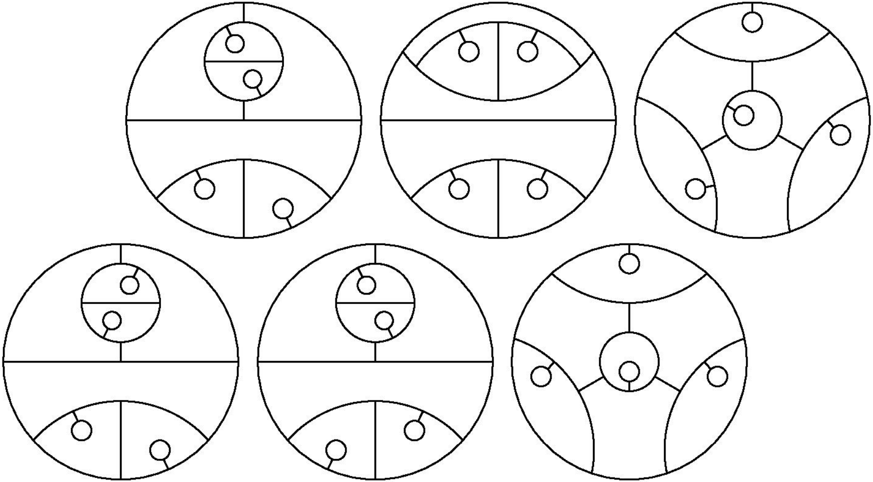
and $P = 47x^{18} - 2028x^{17} + 5502x^{16} + 54540x^{15} - 263535x^{14} - 32592x^{13}$
$+2249268x^{12} - 3436872x^{11} + 14145x^{10} + \ldots + 239616x - 135168.$



The degree 60 coverings



The degree 60 coverings



Algebraic Gauss hypergeometric functions

The Schwarz list of Gauss hypergeometric equations with finite monodromy groups:

- with a finite dihedral monodromy group, contiguous to $E(1/2, 1/2, 1/k)$;
- with the tetrahedral projective monodromy group, contiguous to $E(1/2, 1/3, 1/3)$ or $E(1/3, 1/3, 2/3)$;

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- with the octahedral projective monodromy group, contiguous to $E(1/2, 1/3, 1/4)$ or $E(2/3, 1/4, 1/4)$;
- with the icosahedral projective monodromy group, contiguous to $E(1/2, 1/3, 1/5)$, $E(1/2, 1/3, 2/5)$, $E(1/2, 1/5, 2/5)$, $E(1/3, 1/3, 2/5)$, $E(1/3, 2/3, 1/5)$, $E(2/3, 1/5, 1/5)$, $E(1/3, 2/5, 3/5)$, $E(1/3, 1/5, 3/5)$, $E(1/5, 1/5, 4/5)$ or $E(2/5, 2/5, 2/5)$.

Darboux evaluations

These pull-backs transform hypergeometric functions with a finite monodromy group to power functions a cyclic monodromy group:

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} 1/4,\ 7/12 \\ 4/3 \end{array} \middle|\, \frac{x\,(x+4)^3}{4(2x-1)^3}\right) = \frac{1}{1+\frac{1}{4}x}\,(1-2x)^{3/4},
$$

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} 5/24,\ 13/24 \\ 5/4 \end{array} \middle|\, \frac{108\,x\,(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1}{1-x}\,\left(1+14x+x^2\right)^{5/8},
$$

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} 13/60, -7/60 \\ 3/5 \end{array} \middle|\, \frac{1728\,x\,(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3}\right)
= \frac{1-7x}{\left(1-228x+494x^2+228x^3+x^4\right)^{7/20}}.
$$

Darboux evaluations of genus 1

For some icosahedral cases, Darboux evaluations are on genus 1 curves. For example, consider the curve $y^2 = x(1+x)(1+16x)$ and the function

$$\Phi_{12} = -\frac{54(y+5x)^3\,(1-2y+6x)^5}{(1-16x^2)\,(y-5x)^2\,(1-2y-14x)^5}.$$

Then

$${}_2F_1\left(\left.\begin{array}{c} 8/15, -1/15 \\ 4/5 \end{array}\right| \Phi_{12}\right) = \frac{(1+4x)^{8/15}\,(y+5x)^{1/6}\,x^{1/15}}{(1-2y-14x)^{1/3}\,(y-3x)^{3/10}}$$

around the point $(x, y) = (0, 0)$.

Simplest dihedral ${}_2\mathrm{F}_1$ functions

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} \frac{a}{2},\ \frac{a+1}{2} \\ \frac{1}{2} \end{array} \middle|\, z\right) = \frac{(1-\sqrt{z})^{-a}+(1+\sqrt{z})^{-a}}{2},
$$

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} \frac{a+1}{2},\ \frac{a+2}{2} \\ \frac{3}{2} \end{array} \middle|\, z\right) = \frac{(1-\sqrt{z})^{-a}-(1+\sqrt{z})^{-a}}{2\,a\,\sqrt{z}} \quad (a\neq 0),
$$

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} \frac{1}{2},\ 1 \\ \frac{3}{2} \end{array} \middle|\, z\right) = \frac{\log(1+\sqrt{z})-\log(1-\sqrt{z})}{2\,\sqrt{z}},
$$

$$
{}_2\mathrm{F}_1\left(\begin{array}{c} \frac{a}{2},\ \frac{a+1}{2} \\ a+1 \end{array} \middle|\, z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-a}.
$$

Hypergeometric equations with a dihedral monodromy

Hypergeometric equations with a dihedral monodromy group have the local exponent differences $k+1/2, \ell+1/2, a$.

With a quadratic change of the variable we can pullback them to a Fuchsian equation with the local exponent differences $2k+1, 2\ell+1, a, a$. The mondromy representation is reducible, and the monodromy group is cyclic.

Its solutions can be expressed as $F_2\left(\left. {a;-k,-\ell \atop -2k,-2\ell} \right| x, 2-x \right)$ terminating series.

A general formula for dihedral ${}_2\mathrm{F}_1$ functions

$$\frac{\left(\frac{a+1}{2}\right)_\ell}{\left(\frac{1}{2}\right)_\ell}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}\frac{a}{2},\,\frac{a+1}{2}+\ell\\ \frac{1}{2}-k\end{array}\right|\,z\right)=$$

$$\frac{(1+\sqrt{z})^{-a}}{2}\,F_2\left(\left.\begin{array}{c}a;-k,-\ell\\ -2k,-2\ell\end{array}\right|\,\frac{2\sqrt{z}}{1+\sqrt{z}},\frac{2}{1+\sqrt{z}}\right)$$

$$+\frac{(1-\sqrt{z})^{-a}}{2}\,F_2\left(\left.\begin{array}{c}a;-k,-\ell\\ -2k,-2\ell\end{array}\right|\,\frac{2\sqrt{z}}{\sqrt{z}-1},\frac{2}{1-\sqrt{z}}\right).$$

A general formula for dihedral ${}_2\mathrm{F}_1$ functions

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$$\frac{(1+\sqrt{z})^{-a}}{2}\,F_2\left(\left.\begin{array}{c}a;-k,-\ell\\ -2k,-2\ell\end{array}\right|\,\frac{2\sqrt{z}}{1+\sqrt{z}},\frac{2}{1+\sqrt{z}}\right)$$

$$+\frac{(1-\sqrt{z})^{-a}}{2}\,F_2\left(\left.\begin{array}{c}a;-k,-\ell\\ -2k,-2\ell\end{array}\right|\,\frac{2\sqrt{z}}{\sqrt{z}-1},\frac{2}{1-\sqrt{z}}\right).$$

The basic reason: the functions $F_2\left(\left.\begin{array}{c}a;\;b_1,b_2\\ 2b_1,2b_2\end{array}\right|\,x,2-x\right)$ and $(x-2)^{-a}\,{}_2\mathrm{F}_1\left(\left.\begin{array}{c}\frac{a}{2},\frac{a+1}{2}-b_2\\ b_1+\frac{1}{2}\end{array}\right|\,\frac{x^2}{(2-x)^2}\right)$ satisfy the same ODE.

More general identities for dihedral ${}_2\mathrm{F}_1$ functions

Other identities:

$$\frac{\left(\frac{a+1}{2}\right)_k \left(\frac{a}{2}\right)_{k+\ell+1}}{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_{k+1} \left(\frac{1}{2}\right)_\ell} (-1)^k \, z^{k+\frac{1}{2}} \, {}_2\mathrm{F}_1\left(\left. \begin{matrix} \frac{a+1}{2}+k, \; \frac{a}{2}+k+\ell+1 \\ \frac{3}{2}+k \end{matrix} \right| z \right) =$$

$$\frac{(1+\sqrt{z})^{-a}}{2} F_2\left(\left. \begin{matrix} a; -k, -\ell \\ -2k, -2\ell \end{matrix} \right| \frac{2\sqrt{z}}{1+\sqrt{z}}, \frac{2}{1+\sqrt{z}} \right)$$

$$-\frac{(1-\sqrt{z})^{-a}}{2} F_2\left(\left. \begin{matrix} a; -k, -\ell \\ -2k, -2\ell \end{matrix} \right| \frac{2\sqrt{z}}{\sqrt{z}-1}, \frac{2}{1-\sqrt{z}} \right),$$

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$$\frac{(1+\sqrt{z})^{-a}}{2} F_2\!\left(\left.\begin{matrix} a; -k, -\ell \\ -2k, -2\ell \end{matrix}\right|\, \frac{2\sqrt{z}}{1+\sqrt{z}}, \frac{2}{1+\sqrt{z}}\right)$$

$$-\frac{(1-\sqrt{z})^{-a}}{2} F_2\!\left(\left.\begin{matrix} a; -k, -\ell \\ -2k, -2\ell \end{matrix}\right|\, \frac{2\sqrt{z}}{\sqrt{z}-1}, \frac{2}{1-\sqrt{z}}\right),$$

$${}_2\mathrm{F}_1\!\left(\left.\begin{matrix} \frac{a-\ell}{2},\ \frac{a+\ell+1}{2} \\ a+k+1 \end{matrix}\right|\, 1-z\right) = \left(\frac{1+\sqrt{z}}{2}\right)^{-a-k} z^{k/2} \times$$

$$F_3\!\left(\left.\begin{matrix} k+1, \ell+1; -k, -\ell \\ a+k+1 \end{matrix}\right|\, \frac{\sqrt{z}-1}{2\sqrt{z}}, \frac{1-\sqrt{z}}{2}\right).$$

Generalizing Clausen's identity

$$ {}_2F_1\left(\left. \begin{array}{c} a,\ b \\ a+b+\frac{1}{2} \end{array} \right| z \right)^2 = {}_3F_2\left(\left. \begin{array}{c} 2a,\ 2b,\ a+b \\ 2a+2b,\ a+b+\frac{1}{2} \end{array} \right| z \right). $$

Generalizing Clausen's identity

$$ {}_2F_1\left(\left. \begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix} \right| z \right)^2 = {}_3F_2\left(\left. \begin{matrix} 2a,\ 2b,\ a+b \\ 2a+2b,\ a+b+\frac{1}{2} \end{matrix} \right| z \right). $$

If $a+b=-k$, then dihedral functions:

$$ (1-z)^{-a}\, {}_3F_2\left(\left. \begin{matrix} -k,\ a, -a-2k \\ -2k,\ \frac{1}{2}-k \end{matrix} \right| \frac{z}{z-1} \right) = $$

$$ {}_2F_1\left(\left. \begin{matrix} \frac{a}{2},\ \frac{a+1}{2} \\ \frac{1}{2}-k \end{matrix} \right| z \right)^2 - \frac{2^{4k}\, k!^4\, (a)_{2k+1}^2}{(2k)!^2 (2k+1)!^2}\, z^{2k+1}\, {}_2F_1\left(\left. \begin{matrix} \frac{a+1}{2}+k,\ \frac{a}{2}+k+1 \\ \frac{3}{2}+k \end{matrix} \right| z \right)^2. $$

Generalizing Clausen's identity

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ a+b+\frac{1}{2} \end{matrix} \,\middle|\, z \right)^2 = {}_3F_2\left(\begin{matrix} 2a,\, 2b,\ a+b \\ 2a+2b,\, a+b+\frac{1}{2} \end{matrix} \,\middle|\, z \right). $$

If $a+b=-k$, then dihedral functions:

$$ (1-z)^{-a}\,{}_3F_2\left(\begin{matrix} -k,\ a,\, -a-2k \\ -2k,\ \frac{1}{2}-k \end{matrix} \,\middle|\, \frac{z}{z-1} \right) = $$

$$ {}_2F_1\left(\begin{matrix} \frac{a}{2},\ \frac{a+1}{2} \\ \frac{1}{2}-k \end{matrix} \,\middle|\, z \right)^2 - \frac{2^{4k}\,k!^4\,(a)_{2k+1}^2}{(2k)!^2(2k+1)!^2}\, z^{2k+1}\,{}_2F_1\left(\begin{matrix} \frac{a+1}{2}+k,\ \frac{a}{2}+k+1 \\ \frac{3}{2}+k \end{matrix} \,\middle|\, z \right)^2. $$

A further generalization involving ${}_2F_1\left(\begin{matrix} \frac{a}{2},\ \frac{a+1}{2}+\ell \\ \frac{1}{2}-k \end{matrix} \,\middle|\, z \right)^2$

involves terminating Kampé de Fériet series $F^{2:1;1}_{1:1;1}$.

Tetrahedral solutions

$$
{}_2F_1\left(\left.\begin{array}{c} \frac{n}{2}+\frac{1}{4}, -\frac{n}{2}-\frac{1}{12} \\ 2/3 \end{array}\right| \frac{x(x+4)^3}{4(2x-1)^3}\right) = (1-2x)^{-\frac{3n}{2}-\frac{1}{4}}\,{}_2F_1\left(\left.\begin{array}{c} -n, -2n-\frac{1}{3} \\ 2/3 \end{array}\right| 2x\right),
$$

$$
{}_2F_1\left(\left.\begin{array}{c} -\frac{m}{2}+\frac{1}{4}, -\frac{m}{2}-\frac{1}{12} \\ 2/3-m \end{array}\right| \frac{x(x+4)^3}{4(2x-1)^3}\right) = (1-2x)^{-\frac{3m}{2}-\frac{1}{4}}\,{}_2F_1\left(\left.\begin{array}{c} -2m, -2m-\frac{1}{3} \\ 2/3-m \end{array}\right| -\frac{x}{4}\right).
$$

For general tetrahedral functions ${}_2F_1\left(\left.\begin{array}{c} \frac{n-m}{2}+\frac{1}{4}, \frac{-n-m}{2}-\frac{1}{12} \\ 2/3-m \end{array}\right| \frac{x(x+4)^3}{4(2x-1)^3}\right)$ we need more complicated double terminating sums.

The Kleinian group with 168 elements

Algebraic hypergeometric functions ${}_p\mathrm{F}_q$ are classified by Beukers and Heckman (1989).

There are ${}_3\mathrm{F}_2$ functions with the famous group $\mathrm{PSL}_2(\mathbb{F}_7)$ as the projective monodromy group. That group is isomorphic to $\mathrm{GL}_3(\mathbb{F}_2)$, has 168 elements, and is the symmetry group of Klein's curve $X^3Y + Y^3Z + Z^3X = 0$.

Representative hypergeometric functions are

$$ {}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{9}{14}, \frac{11}{14}, \frac{15}{14} \\ \frac{4}{3}, \frac{5}{3} \end{array}\right| z\right), \; {}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{3}{14}, \frac{5}{14}, \frac{13}{14} \\ \frac{1}{2}, \frac{3}{4} \end{array}\right| z\right), \; {}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{14}, \frac{1}{14}, \frac{5}{14} \\ \frac{1}{7}, \frac{5}{7} \end{array}\right| z\right). $$

Degree 24 transformations

Recaling the degree 24 map $\Phi_{24} = \dfrac{1728x(x-1)F_1^7}{G_1^3}$,

we get the Darboux evaluations:

$$
{}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{42}, \frac{13}{42}, \frac{9}{14} \\ \frac{4}{7}, \frac{6}{7} \end{array}\right| \Phi_{24}\right) = (1-x)^{1/7}\, G_1^{-1/14},
$$

$$
{}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{5}{42}, \frac{19}{42}, \frac{11}{14} \\ \frac{5}{7}, \frac{8}{7} \end{array}\right| \Phi_{24}\right) = (1-x)^{2/7}\, F_1^{-1}\, G_1^{5/14},
$$

$$
{}_3\mathrm{F}_2\left(\left.\begin{array}{c} \frac{17}{42}, \frac{31}{42}, \frac{15}{14} \\ \frac{9}{7}, \frac{10}{7} \end{array}\right| \Phi_{24}\right) = (1-x)^{-3/7}\, F_1^{-3}\, G_1^{17/14}.
$$

Other transformed form

With $\omega = \exp(2\pi/3)$,

$$F_2^{1/6}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c} -\frac{1}{42},\ \frac{5}{42},\ \frac{17}{42} \\ \frac{1}{3},\ \frac{2}{3} \end{array}\right| \Phi_{24}\left(\frac{x+\omega+1}{\omega(1-x)}\right)^{-1}\right)$$
$$= \frac{1}{3}\,(1-x)^{-1/42}\,(1-\omega x)^{5/42}\,(1-\omega^2 x)^{17/42}$$
$$+\frac{1}{3}\,(1-x)^{5/42}(1-\omega x)^{17/42}(1-\omega^2 x)^{-1/42}$$
$$+\frac{1}{3}\,(1-x)^{17/42}(1-\omega x)^{-1/42}(1-\omega^2 x)^{5/42}.$$

Twisting the summands by ω, ω^2 we get other hypergeometric solutions around the point $\Phi = \infty$.

Degree 21 map

We can use the degree 21 map

$$\Psi_{21}=\frac{x^3(3x^2-7)^3(2x^2-7x+7)^3(11x^2-35x+28)^3}{1728(x^3-7x+7)^7}$$

to transform the hypergeometric equation with the Kleinian monodromy to a third order equation with a dihedral projective monodromy (of 8 elements).

The function $\left(x^3-7x+7\right)^{1/6}\,{}_3\mathrm{F}_2\left(\left.\begin{array}{c}-\frac{1}{42},\ \frac{5}{42},\ \frac{17}{42}\\ \frac{1}{3},\ \frac{2}{3}\end{array}\right|\Psi_{21}\right)$

satisfies the same differential equation as

$$\sqrt{2x-3}\quad\text{and}\quad\left(21-8x^2\pm4\sqrt{28-21x^2+4x^4}\right)^{1/4}.$$