A Guided Tour through an NNLO Subtraction Scheme

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Introduction

Factorization theorem:

$$\mathrm{d}\sigma = \sum_{ij} \int \mathrm{d}x_1 \mathrm{d}x_2 f_i(x_1, \mu^2) f_j(x_2, \mu^2) \mathrm{d}\hat{\sigma}_{ij}(\{p_i\}, \mu^2, x_1, x_2)$$

$f_{i,j}$: parton distributions

μ : ren./fact. scale

$\hat{\sigma}_{ij}$: partonic cross section

$x_{1,2}$: momentum fractions

$\{p_i\}$: external momenta

Expand partonic cross section in α_s :

$$\mathrm{d}\hat{\sigma} = \mathrm{d}\hat{\sigma}^{\mathrm{LO}} + \mathrm{d}\hat{\sigma}^{\mathrm{NLO}} + \mathrm{d}\hat{\sigma}^{\mathrm{NNLO}} + \ldots$$

Higher order corrections contain:

- **Real corrections → IR singularities after integration.**
- **Virtual** corrections → **explicit** IR singularities.
- **Collinear subtractions** → **explicit** IR singularities.

Goal: Extract and cancel all singularities *prior* to integration.

Overview of subtraction schemes

- Solved at NLO (Catani-Seymour, Frixione-Kunszt-Signer, ...).
- At NNLO:
  - Slicing: $\int |\mathcal{M}|^2 F_J d\phi_d = \int_0^\delta \left[|\mathcal{M}|^2 F_J d\phi_d\right]_{\text{s.c.}} + \int_\delta^1 |\mathcal{M}_J|^2 F_J d\phi_4 + \mathcal{O}(\delta)$

    Born-like NLO+jet

    - qT (Catani, Grazzini '07): VV, VH, HH
    - N-jettiness (Gaunt *et al* '15; Boughezal *et al* '15): digamma, Vj, Hj, VH, single top
  - Subtraction: $\int |\mathcal{M}|^2 F_J d\phi_d = \int \left(|\mathcal{M}_J|^2 F_J - S\right) d\phi_4 + \int S d\phi_d$
    - Antenna (Gehrmann-de Ridder, Gehrmann, Glover '05, …): dijet, Hj, Vj
    - FKS+sector decomposition (Czakon '10, '11; Boughezal, Petriello, Melnikov '12): ttbar, Hj, single top
    - Projection-to-Born (Cacciari *et al* '15): VBF, single top
    - CoLoRFulNNLO (Somogyi, Trocsanyi, Del Duca '05, ...): e+e-

The NNLO Revolution

- Slicing & subtraction tools highly successful: all 2→2 processes known at NNLO.
- BUT: room for improvement. Current subtraction schemes:
  - Are complicated.
  - Obscure the origin of the singularities.
  - Are process-dependent.
- Ideally:
  - Local.
  - Straightforward with clear origin of singularities.
  - Explicit (if possible, analytic) cancellation of poles.
  - Process-independent.
  - Allowing four-dimensional evaluation.

Residue-Improved Sector Decomposition

Residue-improved sector decomposition:

- [Czakon ‘10, ‘11; Boughezal, Melnikov, Petriello ‘12]
- Modification allowing four-dimensional calculation of matrix elements [Czakon, Heymes ‘14].
- Same strategy as FKS, + *sectors* to separate overlapping singularities.
- Expect simplification when recombining (as in FKS) – **not apparent** in above formulations.
- I will present a simplified implementation:
  - Focus on gauge-invariant matrix elements → independent treatment of soft and collinear singularities.
  - Easier recombination of sectors → explicit pole cancellation for different kinematic structures.

Outline

- **NLO:**
  - Notation for limits.
  - Subtraction procedure.
  - Combining real, virtual and collinear subtraction contributions and cancelling poles.
- **NNLO:**
  - Collinear subtraction, virtual-virtual, real-virtual contributions.
  - Real-real contribution: extension of NLO ideas:
    - Partitions and sectors.
    - Soft subtraction terms.
    - Collinear subtraction terms.
  - Combining contributions and pole cancellation.
- Proof-of-principle calculation and comparison.

Outline (II)

Focus on $pp \to V + X$:

- Color singlet final state (Drell-Yann, diboson, *HV*).
- Extension to colored final states straightforward (but complicated).

Consider partonic channel $q\bar{q} \to V + n\ g$:

- Most complicated singular structure.
- Other partonic channels are simplifications of this.

Subtraction at NLO

$$\mathrm{d}\hat{\sigma}^{\mathrm{NLO}} = \mathrm{d}\sigma^{\mathrm{V}} + \mathrm{d}\sigma^{\mathrm{R}}. + \mathrm{d}\sigma^{\mathrm{CV}}$$

Focus on real radiation – process $q\bar{q} \to V + g$:

$$\mathrm{d}\sigma^{\mathrm{R}} = \frac{1}{2s} \int [\mathrm{d}g_4] F_{LM}(1,2,4) \equiv \langle F_{LM}(1,2,4) \rangle.$$

$$F_{LM}(1,2,4) = \mathrm{dLips}_V \; |\mathcal{M}(1,2,4,V)|^2 \; \mathcal{F}_{\mathrm{kin}}(1,2,4,V).$$

$$[\mathrm{d}g_4] = \frac{\mathrm{d}^{d-1} p_4}{(2\pi)^d 2E_4} \theta(E_{\mathrm{max}} - E_4)$$

Lorentz-inv. Phase space for V **(incl. delta-fn)**

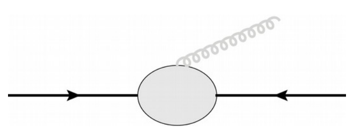
Matrix-element sq.

IR-safe observable

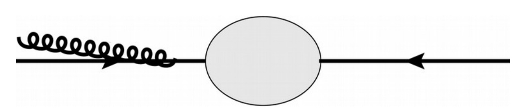
Integration in partonic CoM frame

Arbitrarily large energy parameter

Singular regions:



- g_4 → soft.
- g_4 → collinear to either initial state parton.



Iterative subtraction

Define operators:

$$S_i A = \lim_{E_i \to 0} A \qquad C_{ij} A = \lim_{\rho_{ij} \to 0} A \qquad \rho_{ij} = 1 - \cos\theta_{ij}$$

Rewrite as

$$\begin{aligned}\langle F_{LM}(1,2,4)\rangle = &\langle S_4 F_{LM}(1,2,4)\rangle + \\ &\langle (C_{41}+C_{42})(I-S_4)F_{LM}(1,2,4)\rangle + \\ &\langle (I-C_{41}-C_{42})(I-S_4)F_{LM}(1,2,4)\rangle .\end{aligned}$$

- First term: soft gluon decouples completely → need upper bound: $E_{\rm max}$.
- Second term: collinear and soft+collinear gluon decouples partially or completely.
- Singularities made explicit by **integrating over decoupled gluon.**
- Third term: finite, can be integrated numerically in 4-dimensions.

Soft and soft+collinear limits

Soft limit: eikonal function

$$S_4 F_{\mathrm{LM}}(1,2,4) = g_{s,b}^2 \mathrm{Eik}(1,2,4) F_{LM}(1,2) = g_{s,b}^2 2C_F \frac{p_1 \cdot p_2}{(p_1 \cdot p_4)(p_2 \cdot p_4)} F_{LM}(1,2)$$
$$= g_{s,b}^2 2C_F \frac{1}{E_4^2} \frac{\rho_{12}}{\rho_{14}\rho_{24}} F_{LM}(1,2).$$

Gluon decouples from F_{LM}, so collinear limits act on prefactor only:

$$C_{41} S_4 F_{LM}(1,2,4) = g_{s,b}^2 2C_F \frac{1}{E_4^2} \frac{1}{\rho_{14}} F_{LM}(1,2),$$
$$C_{42} S_4 F_{LM}(1,2,4) = g_{s,b}^2 2C_F \frac{1}{E_4^2} \frac{1}{\rho_{24}} F_{LM}(1,2).$$

$$\Rightarrow S_4(I - C_{41} - C_{42}) F_{LM}(1,2,4) = g_{s,b}^2 2C_F \frac{1}{E_4^2} \left(\frac{\rho_{12}}{\rho_{14}\rho_{24}} - \frac{1}{\rho_{14}} - \frac{1}{\rho_{24}} \right) F_{LM}(1,2) = 0.$$

using $\rho_{12} = 2, \ \rho_{24} = 2 - \rho_{14}$

(for color singlet final state).

We only need to consider **collinear** limits $\langle (C_{41} + C_{42}) F_{LM}(1,2,4) \rangle$.

Collinear limits

The collinear limit is

$$C_{41} F_{LM}(1,2,4) = \frac{g_{s,b}^2}{E_4^2 \rho_{41}} (1-z) P_{qq}(z) \left(\frac{F_{LM}(z \cdot 1, 2)}{z} \right).$$

Flux factor

$$z = 1 - E_4/E_1$$
$$P_{qq}(z) = P_{qq}^{(0)}(z) + \mathcal{O}(\epsilon)$$

Gluonic *angles* decouple:

$$\int_{-1}^{1} \mathrm{d}\cos\theta_{14} \, (\sin\theta_{14})^{d-4} \frac{1}{\rho_{14}} = 2^{-2\epsilon} \int_0^1 \mathrm{d}\eta_{14} (\eta_{14})^{-1-\epsilon} (1-\eta_{41})^{-\epsilon} = -\frac{2^{-2\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}.$$

$$\int \frac{\mathrm{d}\Omega_4^{(d-2)}}{(2\pi)^{d-1}} g_{s,b}^2 = \frac{\alpha_s(\mu)}{2\pi} \frac{\mu^{2\epsilon} e^{\epsilon \gamma_E}}{\Gamma(1-\epsilon)} \equiv [\alpha_s].$$

$$\eta_{ij} = \rho_{ij}/2$$

and changing $\mathrm{d}E_4 \to \mathrm{d}z$:

$$\langle C_{41} F_{LM}(1,2,4) \rangle = -[\alpha_s] \frac{1}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} (2E_1)^{-2\epsilon} \int_{z_{\min}}^{1} \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) \frac{F_{LM}(z \cdot 1, 2)}{z}.$$

$$\mathrm{z}_{\min} = 1 - E_{\max}/E_1$$

But need enough energy to produce *V :* can extend integration region to 0.

Recasting collinear limit

$$\langle C_{41} F_{LM}(1,2,4)\rangle = -[\alpha_s]\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}(2E_1)^{-2\epsilon}\int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z)\frac{F_{LM}(z\cdot 1,2)}{z}.$$

- Collinear pole explicit.
- Soft pole $z\to 1$ regulated.
- Convolution of splitting function with matrix element:
  - Should cancel against collinear subtraction.
  - Rewrite in terms of **plus-prescription:**

$$P_{qq}(z) = \frac{2C_F}{(1-z)} + P_{qq}^{\mathrm{reg}}(z) \quad \Rightarrow \int_0^1 \frac{\mathrm{d}z}{(1-z)^{2\epsilon}} P_{qq}(z) G(z) = \int_0^1 \mathrm{d}z \left[\frac{2C_F}{(1-z)^{1+2\epsilon}} + (1-z)^{-2\epsilon} P_{qq}^{\mathrm{reg}}(z)\right] G(z)$$

$$= -\frac{C_F}{\epsilon} G(1) + \int_0^1 \mathrm{d}z \left[\frac{2C_F}{(1-z)^{1+2\epsilon}}\left(G(z)-G(1)\right) + (1-z)^{-2\epsilon} P_{qq}^{\mathrm{reg}}(z) G(z)\right].$$

Singular as $z\to 1$

Add & integrate

Expand using plus-prescription

Subtract

Finalizing collinear subtractions

Introduce new splitting function $\mathcal{P}_{qq,R}(z) = \hat{P}^{(0)}_{qq}(z) + \epsilon \mathcal{P}^{(\epsilon)}_{qq,R}(z)$,

and write

$$\langle C_{41} F_{LM}(1,2,4) \rangle = -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \times \left[-\left(\frac{C_F}{\epsilon} + \frac{3C_F}{2} \right) \langle F_{LM}(1,2) \rangle + \int\limits_0^1 \mathrm{d}z \mathcal{P}_{qq,R}(z) \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} \right\rangle \right].$$

C_{42} term proceeds analogously, so

$$\hat{O}_{NLO} \equiv (I - S_4)(I - C_{41} - C_{42})$$

$$2s \cdot \mathrm{d}\sigma^{\mathrm{R}} = 2[\alpha_s] s^{-\epsilon} \left(\frac{C_F}{\epsilon^2} + \frac{3C_F}{2\epsilon} \right) \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \langle F_{LM}(1,2) \rangle + \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,4) \rangle$$

$$- \frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \int\limits_0^1 \mathrm{d}z \mathcal{P}_{qq,R}(z) \left\langle \frac{F_{LM}(z \cdot 1, 2)}{z} + \frac{F_{LM}(1, z \cdot 2)}{z} \right\rangle.$$

Combining contributions & cancelling poles

$$2s\cdot \mathrm{d}\sigma^{\mathrm{R}} = 2[\alpha_s]s^{-\epsilon}\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\langle F_{LM}(1,2)\rangle + \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,4)\rangle$$

$$-\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\int_0^1 \mathrm{d}z\mathcal{P}_{qq,R}(z)\left\langle \frac{F_{LM}(z\cdot 1,2)}{z}+\frac{F_{LM}(1,z\cdot 2)}{z}\right\rangle.$$

- Poles in first term cancel with virtual (extracted using [Catani '98]):

$$2s\cdot \mathrm{d}\sigma^{V} = -2[\alpha_s]\cos(\pi\epsilon)\left(\frac{C_F}{\epsilon^2}+\frac{3C_F}{2\epsilon}\right)s^{-\epsilon}\langle F_{LM}(1,2)\rangle + \langle F_{LV}^{\mathrm{fin}}(1,2)\rangle\,.$$

- Poles in third term cancel with collinear subtraction:

$$2s\cdot \mathrm{d}\sigma^{\mathrm{CV}} = \frac{\alpha_s(\mu)}{2\pi\epsilon}\int_0^1 \mathrm{d}z\,\hat{P}_{qq}^{(0)}(z)\left\langle \frac{F_{LM}(z\cdot 1,2)}{z}+\frac{F_{LM}(1,z\cdot 2)}{z}\right\rangle.$$

- Cancellation occurs *within each structure!*

Evaluation in four dimensions

After cancelling poles, we can take the $\epsilon \to 0$ limit and compute everything in four dimensions.

$$2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{NLO}} = \left\langle F_{LV}^{\mathrm{fin}}(1,2) + \frac{\alpha_s(\mu)}{2\pi}\left[\frac{2}{3}\pi^2 C_F F_{LM}(1,2)\right]\right\rangle + \langle \hat{O}_{\mathrm{NLO}} F_{LM}(1,2,4)\rangle +$$
$$+ \frac{\alpha_s(\mu)}{2\pi}\int_0^1 \mathrm{d}z \left[\ln\frac{s}{\mu^2}\hat{P}_{qq}^{(0)}(z) - \mathcal{P}_{qq,R}^{(\epsilon)}(z)\right]\left\langle \frac{F_{LM}(z\cdot 1,2)}{z} + \frac{F_{LM}(1,z\cdot 2)}{z}\right\rangle .$$

Sum of:

- Lower particle-multiplicity terms, with or without convolutions with splitting functions.
- Real emission term, with singular configurations removed by iterated subtraction.
- Finite remainder of virtual corrections.

NNLO: Double-virtual Corrections

$$\mathrm{d}\hat{\sigma}^{\mathrm{NNLO}} = \mathrm{d}\sigma^{\mathrm{VV}} + \mathrm{d}\sigma^{\mathrm{RV}} + \mathrm{d}\sigma^{\mathrm{RR}} + \mathrm{d}\sigma^{\mathrm{CV}} + \mathrm{d}\sigma^{\mathrm{ren.}}.$$

Double-virtual – two-loop $q\bar{q} \to V$.

Pole structure following [Catani '98]:

$$\begin{aligned}\mathrm{d}\sigma^{\mathrm{VV}} =& [\alpha_s]^2 f(s,\epsilon)\langle F_{LM}(1,2)\rangle + [\alpha_s] g(s,\epsilon)\langle F_{LV}^{\mathrm{fin}}(1,2)\rangle \\ & + \langle F_{LV^2}^{\mathrm{fin}}(1,2)\rangle + \langle F_{LVV}^{\mathrm{fin}}(1,2)\rangle.\end{aligned}$$

$f(s,\epsilon),\ g(s,\epsilon)$: Poles $\sim 1/\epsilon^4$ and higher.

$F_{LV^2}^{\mathrm{fin}}(1,2)$: Finite part of 1-loop squared.

$F_{LVV}^{\mathrm{fin}}(1,2)$: Finite part of 2-loop.

NNLO: Real-virtual Corrections

$$\mathrm{d}\hat{\sigma}^{\mathrm{NNLO}} = \mathrm{d}\sigma^{\mathrm{VV}} + \mathrm{d}\sigma^{\mathrm{RV}} + \mathrm{d}\sigma^{\mathrm{RR}} + \mathrm{d}\sigma^{\mathrm{CV}} + \mathrm{d}\sigma^{\mathrm{ren.}}.$$

Real-virtual – one loop $q\bar{q} \to V + g$.

- Same kinematics as NLO → same subtraction strategy.
- Cancellation between soft and soft+collinear no longer manifest.

$$E_4^2 S_4 F_{LRV}(1,2,4) = 2C_F g_{s,b}^2 \left[\frac{\rho_{12}}{\rho_{14}\rho_{24}} F_{LV}(1,2) \right.$$
$$\left. - C_A[\alpha_s] \frac{1}{\epsilon^2} \frac{\Gamma^5(1-\epsilon)\Gamma^3(1+\epsilon)}{\Gamma^2(1-2\epsilon)\Gamma(1+2\epsilon)} \left(\frac{\rho_{12}}{\rho_{14}\rho_{24}} \right)^{1+\epsilon} E_4^{-2\epsilon} 2^{-\epsilon} F_{LM}(1,2) \right].$$

- Soft gluon decouples.
- Integrating over energy requires an **upper bound**: provided by $E_{\max}$.

Real-virtual (II)

Taking collinear and soft+collinear limits and integrating, and making use of 1-loop pole structure:

$$\begin{aligned}\langle F_{LV}(1,2,4)\rangle =&[\alpha_s]^2 s^{-2\epsilon} g_4(s,\epsilon)\,\langle F_{LM}(1,2)\rangle + [\alpha_s] s^{-\epsilon} g_2(s,\epsilon)\,\langle F_{LV}^{\text{fin}}(1,2)\rangle \\ &+ [\alpha_s]^2 s^{-2\epsilon} g_3(s,\epsilon) \int\limits_0^1 \mathrm{d}z \mathcal{P}_{qq,RV_1}(z) \left\langle \frac{F_{LM}(z\cdot 1,2)+F_{LM}(1,z\cdot 2)}{z}\right\rangle \\ &+ [\alpha_s]^2 s^{-2\epsilon} g_1(s,\epsilon) \int\limits_0^1 \mathrm{d}z \mathcal{P}_{qq,RV_2}(z) \left\langle \frac{F_{LV}^{\text{fin}}(z\cdot 1,2)+F_{LV}^{\text{fin}}(1,z\cdot 2)}{z}\right\rangle \\ &+ \left\langle \hat{\mathcal{O}}_{NLO} F_{LV}^{\text{fin}}(1,2,4)\right\rangle + f(s,\epsilon)\left\langle \hat{\mathcal{O}}_{NLO} F_{LM}(1,2,4)\right\rangle.\end{aligned}$$

With functions $g_n(s,\epsilon) \sim 1/\epsilon^n \qquad f(s,\epsilon) \sim 1/\epsilon^2$

- Convolutions of splitting functions with lower multiplicity terms.
- Terms with singular limits removed through iterative subtraction.

NNLO: Collinear subtraction

$$\mathrm{d}\hat{\sigma}^{\mathrm{NNLO}} = \mathrm{d}\sigma^{\mathrm{VV}} + \mathrm{d}\sigma^{\mathrm{RV}} + \mathrm{d}\sigma^{\mathrm{RR}} + \mathrm{d}\sigma^{\mathrm{CV}} + \mathrm{d}\sigma^{\mathrm{ren.}}.$$

Collinear renormalization:

$$\mathrm{d}\sigma^{\mathrm{CV}} = \frac{\alpha_s(\mu)}{2\pi}\left[\Gamma_1 \otimes \mathrm{d}\hat{\sigma}^{\mathrm{NLO}} + \mathrm{d}\hat{\sigma}^{\mathrm{NLO}} \otimes \Gamma_1\right] - \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \left[\Gamma_1 \otimes \mathrm{d}\hat{\sigma}^{\mathrm{LO}} \otimes \Gamma_1 + \Gamma_2 \otimes \mathrm{d}\hat{\sigma}^{\mathrm{LO}} + \mathrm{d}\hat{\sigma}^{\mathrm{LO}} \otimes \Gamma_2\right].$$

$\Gamma_{1,2}$: (convolutions of) Altarelli-Parisi splitting functions.

More complicated, but well-understood:

- Double convolutions $\sim F_{LM}(xy \cdot 1, 2)$.
- Left-right convolutions $\sim F_{LM}(x \cdot 1, y \cdot 2)$.
- Work in CoM frame: particular care in convolutions involving $s^{-\epsilon}$.

NNLO: Real-real Corrections

$$\mathrm{d}\hat{\sigma}^{\mathrm{NNLO}} = \mathrm{d}\sigma^{\mathrm{VV}} + \mathrm{d}\sigma^{\mathrm{RV}} + \mathrm{d}\sigma^{\mathrm{RR}} + \mathrm{d}\sigma^{\mathrm{CV}} + \mathrm{d}\sigma^{\mathrm{ren.}}.$$

Real-real corrections – process $q\bar{q} \to V + gg.$

$$2s \cdot \mathrm{d}\sigma^{\mathrm{RR}} = \frac{1}{2!}\int [\mathrm{d}g_4][\mathrm{d}g_5] F_{LM}(1,2,4,5).$$

Singularity structure much more complicated:

- g_4 or g_5 → soft.
- g_4 or g_5 → collinear to initial state partons.
- g_4 or g_5 → collinear to each other.
- Combination of the above – can approach **each limit in different ways!**

Separating the singularities is the name of the game!

Treatment of real-real singularities

- **Step 0:** New limit operators.

$$\mathbb{S}A = \lim_{E_4,E_5\to 0} A, \text{ at fixed } E_5/E_4,$$

$$\mathbb{C}_i A = \lim_{\rho_{4i},\rho_{5i}\to 0} A, \text{ with non vanishing } \rho_{4i}/\rho_{5i}, \rho_{45}/\rho_{4i}, \rho_{45}/\rho_{5i},$$

and recall $S_i A = \lim_{E_i\to 0} A \qquad C_{ij} A = \lim_{\rho_{ij}\to 0} A.$

- **Step 1:** Order gluon energies $E_4 > E_5$.

$$2s \cdot \mathrm{d}\sigma^{\mathrm{RR}} = \int [\mathrm{d}g_4][\mathrm{d}g_5]\theta(E_4 - E_5) F_{LM}(1,2,4)$$
$$\equiv \big\langle F_{LM}(1,2,4,5) \big\rangle.$$

- Gluon energies bounded by $E_{\max}$.
- Energies defined in CoM frame.

Soft singularities

RECALL**:** we are dealing with gauge-invariant matrix elements (as opposed to individual Feynman diagrams) – can regulate soft and collinear singularities independently.

- **Step 2:** Regulate the soft singularities:

$$\left\langle F_{LM}(1,2,4,5)\right\rangle = \left\langle \mathcal{S}\!\!\!S F_{LM}(1,2,4,5)\right\rangle + \left\langle S_5(I-\mathcal{S}\!\!\!S)F_{LM}(1,2,4,5)\right\rangle + \left\langle (I-S_5)(I-\mathcal{S}\!\!\!S)F_{LM}(1,2,4,5)\right\rangle.$$

- First term: both g_4 and g_5 soft.
- Second term: g_5 soft, soft singularities in g_4 are regulated.
- Third term: regulated against all soft singularities.
- All three terms contain **(overlapping)** collinear singularities.

Phase-space partitioning

- **Step 3:** Introduce **phase-space partitions**

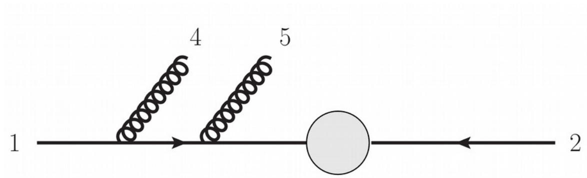
$$1 = w^{14,15} + w^{24,25} + w^{14,25} + w^{15,24}.$$

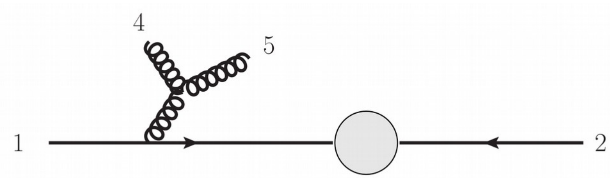
with

$$C_{42} w^{14,15} = C_{52} w^{14,15} = 0 \quad \Rightarrow \quad w^{14,15} \text{ contains } C_{41},\ C_{51},\ C_{45}$$

$$C_{41} w^{24,25} = C_{51} w^{24,25} = 0 \qquad\quad w^{24,25} \text{ contains } C_{42},\ C_{52},\ C_{45}$$

Triple collinear
partition



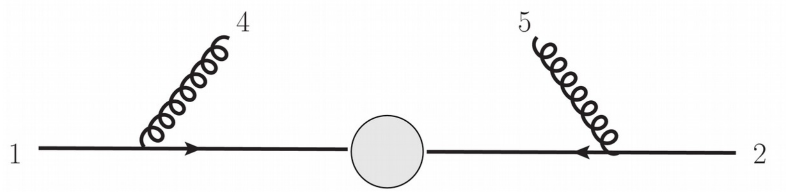


and

$$C_{42} w^{14,25} = C_{51} w^{14,25} = C_{45} w^{14,25} = 0 \quad \Rightarrow \quad w^{14,25} \text{ contains } C_{41},\ C_{52}$$

$$C_{41} w^{15,24} = C_{52} w^{15,24} = C_{45} w^{15,24} = 0 \qquad\quad w^{15,24} \text{ contains } C_{42},\ C_{51}$$

Double collinear
partition



Sector Decomposition

- **Step 4: Sector decomposition:**

- Triple collinear sectors still have **overlapping** singularities.
- Define angular ordering to separate singularities.

$$1 = \theta\left(\eta_{51} < \frac{\eta_{41}}{2}\right) + \theta\left(\frac{\eta_{41}}{2} < \eta_{51} < \eta_{41}\right) \qquad \eta_{ij} = \rho_{ij}/2$$

$$+ \theta\left(\eta_{41} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{41} < \eta_{51}\right) \equiv \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}.$$

- Thus the limits are $\theta^{(a)} : C_{51}$ $\quad \theta^{(b)} : C_{45}$

$$\theta^{(c)} : C_{41} \qquad \theta^{(d)} : C_{45}$$

Sectors *a,c* and *b,d* same to *4* ↔ *5*, but recall <u>energy ordering.</u>

Removing collinear singularities

Then we can write soft regulated term as

$$\left\langle (I-S_5)(I-\mathbb{S})F_{LM}(1,2,4,5)\right\rangle = \left\langle F_{LM}^{s_r c_s}(1,2,4,5)\right\rangle + \left\langle F_{LM}^{s_r c_t}(1,2,4,5)\right\rangle + \left\langle F_{LM}^{s_r c_r}(1,2,4,5)\right\rangle,$$

with

$$\langle F_{LM}^{s_r c_r}(1,2,4,5)\rangle = \sum_{(ij)\in dc} \Big\langle [I-\mathbb{S}][I-S_5]\Big[(I-C_{5j})(I-C_{4i})\Big][\mathrm{d}g_4][\mathrm{d}g_5]w^{i4,j5}F_{LM}(1,2,4,5)\Big\rangle$$
$$+\sum_{i\in tc}\Big\langle [I-\mathbb{S}][I-S_5]\Big[\theta^{(a)}[I-\mathbb{C}_i][I-C_{5i}]+\theta^{(b)}[I-\mathbb{C}_i][I-C_{45}]$$
$$+\theta^{(c)}[I-\mathbb{C}_i][I-C_{4i}]+\theta^{(d)}[I-\mathbb{C}_i][I-C_{45}]\Big][\mathrm{d}g_4][\mathrm{d}g_5]w^{i4,i5}$$
$$\times F_{LM}(1,2,4,5)\Big\rangle.$$

- All singularities removed through iterated subtractions – evaluated in 4-dimensions.
- Only term involving fully-resolved matrix element $F_{LM}(1,2,4,5)$.

Removing collinear singularities

$$\Big\langle (I-S_5)(I-\mathbb{S})F_{LM}(1,2,4,5)\Big\rangle = \Big\langle F_{LM}^{s_r c_s}(1,2,4,5)\Big\rangle + \Big\langle F_{LM}^{s_r c_t}(1,2,4,5)\Big\rangle + \Big\langle F_{LM}^{s_r c_r}(1,2,4,5)\Big\rangle.$$

Remaining two terms contain singularities:

$$\langle F_{LM}^{s_r c_s}\rangle = \sum_{(ij)\in dc}\Bigg\langle [I-\mathbb{S}]\,[I-S_5]\Big[C_{4i}[\mathrm{d}g_4]+C_{5j}[\mathrm{d}g_5]\Big]w^{i4,j5}F_{LM}(1,2,4,5)\Bigg\rangle$$
$$+\sum_{i\in tc}\Bigg\langle [I-\mathbb{S}]\,[I-S_5]\Big[\theta^{(a)}C_{5i}+\theta^{(b)}C_{45}+\theta^{(c)}C_{4i}+\theta^{(d)}C_{45}\Big]$$
$$\times [\mathrm{d}g_4][\mathrm{d}g_5]w^{i4,i5}F_{LM}(1,2,4,5)\Bigg\rangle.$$

- Soft-regulated single-collinear subtraction.
- Partitioning factors and sectors: **one collinear singularity** in each term.

Removing collinear singularites

$$\left\langle (I-S_5)(I-\mathbb{S})F_{LM}(1,2,4,5)\right\rangle = \left\langle F_{LM}^{s_r c_s}(1,2,4,5)\right\rangle + \left\langle F_{LM}^{s_r c_t}(1,2,4,5)\right\rangle + \left\langle F_{LM}^{s_r c_r}(1,2,4,5)\right\rangle$$

and

$$\langle F_{LM}^{s_r c_t}(1,2,4,5)\rangle = -\sum_{(ij)\in dc}\left\langle [I-\mathbb{S}]\,[I-S_5]C_{4i}C_{5j}[\mathrm{d}g_4][\mathrm{d}g_5]w^{i4,j5}F_{LM}(1,2,4,5)\right\rangle + \sum_{i\in tc}\left\langle [I-\mathbb{S}]\,[I-S_5]\left[\theta^{(a)}\mathbb{C}_i\,[I-C_{5i}] + \theta^{(b)}\mathbb{C}_i\,[I-C_{45}] + \theta^{(c)}\mathbb{C}_i\,[I-C_{4i}] + \theta^{(d)}\mathbb{C}_i\,[I-C_{45}]\right][\mathrm{d}g_4][\mathrm{d}g_5]w^{i4,i5}F_{LM}(1,2,4,5)\right\rangle.$$

- Triple-collinear subtraction – all other singularities regulated.

Treating singular limits

We have four singular subtraction terms:

$$\left\langle \mathbb{S} F_{LM}(1,2,4,5) \right\rangle \quad \left\langle S_5 (I - \mathbb{S}) F_{LM}(1,2,4,5) \right\rangle \quad \left\langle F_{LM}^{s_r c_s}(1,2,4,5) \right\rangle \quad \left\langle F_{LM}^{s_r c_t}(1,2,4,5) \right\rangle$$

We know how to treat them:

- Gluon(s) decouple partially or completely.
- Decouple completely:
  - Integrate over gluonic angles and energy.
- Decouple partially:
  - Integrate over gluonic angles.
  - Integral(s) over energy → integrals over splitting function in *z.*
- Results in **lower particle multiplicity terms** convoluted with (new) splitting functions.

Soft subtraction terms

<u>Double soft subtraction:</u> $\left\langle S\!\!\!S F_{LM}(1,2,4,5)\right\rangle$

- Both gluons decouple: $S\!\!\!S F_{LM}(1,2,4,5) = g_{s,b}^4 \mathrm{Eik}(1,2,4,5) F_{LM}(1,2).$

Double eikonal function

- Overall energy factorizes → integrand *independent* of partonic energy.
- Integral is **constant** for color-singlet production.
- Abelian contribution: product of NLO structures.
- Non-abelian: more complicated.
- Integrate over relative energies and over gluonic angles *numerically.*

$$\left\langle S\!\!\!S F_{LM}(1,2,4,5)\right\rangle = [\alpha_s]^2 \left\langle E_{\max}^{-4\epsilon} F_{LM}(1,2)\right\rangle \left(\frac{c_4}{\epsilon^4} + \frac{c_3}{\epsilon^3} + \frac{c_2}{\epsilon^2} + \frac{c_1}{\epsilon} + c_0\right).$$

c_i : Constants for color-singlet production

Single-soft subtraction term

Single-soft subtraction: $\langle (I - \mathbb{S}) S_5 F_{LM}(1,2,4,5) \rangle$

- **Gluon 5** decouples – integrate over it:

$$\langle (I - \mathbb{S}) S_5 F_{LM}(1,2,4,5) \rangle = \frac{[\alpha_s]}{\epsilon^2} \langle E_4^{-2\epsilon} f(\rho_{12}, \rho_{14}, \rho_{24}, \epsilon) \; (I - S_4) F_{LM}(1,2,4) \rangle.$$

- **This term is not integrable:** contains NLO-like collinear singularities.
- Regulate the singularities as done at NLO:

$$\Big\langle \big[I - \mathbb{S} \big] S_5 F_{LM}(1,2,4,5) \Big\rangle = \frac{[\alpha_s]}{\epsilon^2} \Big\langle E_4^{-2\epsilon} f(\rho_{12}, \rho_{14}, \rho_{24}, \epsilon)$$

$$\times \big[I - C_{41} - C_{42} \big] \big[I - S_4 \big] F_{LM}(1,2,4) \Big\rangle$$

$$- \frac{[\alpha_s]^2 s^{-2\epsilon}}{\epsilon^3} f(\epsilon) \int_0^1 \mathrm{d}z \, \mathcal{P}_{qq,RR_1}(z) \left\langle \frac{F_{LM}(z \cdot 1, 2) + F_{LM}(1, z \cdot 2)}{z} \right\rangle.$$

- First term: Regulated through iterative subtractions.
- Second term: LO matrix element convoluted with splitting function.

Collinear subtractions

General structure: splitting functions with **explicit poles** convoluted with lower multiplicity terms:

$$F_{LM}(z\cdot 1,2,4) \qquad F_{LM}(1,z\cdot 2,4) \qquad F_{LM}(1,2,4)$$

$$F_{LM}(z\cdot 1,2) \qquad F_{LM}(1,z\cdot 2) \qquad F_{LM}(1,2)$$

Further singularities regulated →

$$\left\langle \hat{\mathcal{O}}_{NLO} F_{LM}(z\cdot 1,2,4)\right\rangle \qquad \left\langle \hat{\mathcal{O}}_{NLO} F_{LM}(1,z\cdot 2,4)\right\rangle \qquad \left\langle \hat{\mathcal{O}}_{NLO} F_{LM}(1,2,4)\right\rangle$$

$$\langle F_{LM}(z\cdot 1,\bar{z}\cdot 2)\rangle \qquad \langle F_{LM}(z\cdot 1,2)\rangle \qquad \langle F_{LM}(1,z\cdot 2)\rangle \qquad \langle F_{LM}(1,2)\rangle$$

- At NLO, pole cancellation achieved in *each structure.*
- Recombine structures from different sectors/partitions.

Double-collinear partition

In single-collinear subtraction:

$$DC = \Big\langle [I - \mathbb{S}]\,[I - S_5]\Big[(C_{41}[\mathrm{d}g_4] + C_{52}[\mathrm{d}g_5])\, w^{14,25} + (C_{42}[\mathrm{d}g_4] + C_{51}[\mathrm{d}g_5])\, w^{24,15}\Big] \times F_{LM}(1,2,4,5)\Big\rangle.$$

Limit acts on phase space!

Consider one term: $\langle [I - \mathbb{S}]\,[I - S_5]\,C_{41}[\mathrm{d}g_4] w^{14,25} F_{LM}(1,2,4,5)\rangle$:

- Write $C_{41} w^{14,25} = \tilde{w}^{14,25}_{4||1}$ – independent of gluon 4.
- Soft & collinear limits commute:

$$\mathbb{S}(I - S_5) C_{41} F_{LM}(1,2,4,5) \sim \mathbb{S} F_{LM}(1-4,2,5) - \mathbb{S} S_5 F_{LM}(1-4,2,5) = 0.$$

- Limit reduces to $C_{41}(I - S_5) F_{LM}(1,2,4,5) = \dfrac{g^2_{s,b}}{E_4^2 \rho_{41}} \mathcal{P}_{qq}(z)\,[I - S_5]\,F_{LM}(z\cdot 1, 2, 5).$

Double-collinear partition

Limit acting on phase space: $C_{41} \int_0^1 \mathrm{d}\eta_{14} \left(\eta_{14}(1-\eta_{41})\right)^{-\epsilon} \to \int_0^1 \mathrm{d}\eta_{14} (\eta_{14})^{-\epsilon}$

Angular integration is then: $2^{(1-2\epsilon)} \int_0^1 \mathrm{d}\eta_{14} \eta_{14}^{-\epsilon} \frac{1}{2\eta_{41}} = -\frac{2^{-2\epsilon}}{\epsilon}.$

Limits of integration: $z = 1 - E_4/E_1$ and $E_4 > E_5 \Rightarrow z < 1 - E_5/E_1 \equiv z_{\max}(E_5).$

Find:

$$\left\langle [I - \mathbb{S}]\,[I - S_5] C_{41} [\mathrm{d}g_4] w^{14,25} F_{LM}(1,2,4,5) \right\rangle$$

$$= -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \int_{z_{\min}}^{z_{\max}(E_5)} \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}} \mathcal{P}_{qq}(z) \left\langle \tilde{w}_{4||1}^{14,25} [I - S_5] F_{LM}(z \cdot 1, 2, 5) \right\rangle.$$

- Lower limit can be extended to 0, but upper limit cannot be extended to 1.

Double-collinear partition

$$DC = \Big\langle [I - \mathbb{S}]\,[I - S_5]\Big[\left(C_{41}[\mathrm{d}g_4] + C_{52}[\mathrm{d}g_5]\right) w^{14,25} + \left(C_{42}[\mathrm{d}g_4] + C_{51}[\mathrm{d}g_5]\right) w^{24,15}\Big] \times F_{LM}(1,2,4,5)\Big\rangle.$$

Now consider the term $\langle [I - \mathbb{S}]\,[I - S_5]\, C_{51}[\mathrm{d}g_5] w^{24,15} F_{LM}(1,2,4,5)\rangle$:

- Proceeds analogously, but with:
  - $z = 1 - E_5/E_1$ and $E_4 > E_5 \Rightarrow z > 1 - E_4/E_1 \equiv z_{\min}(E_4)$.
  - Collinear and soft limit *both* act on gluon 5

    $\rightarrow$ different splitting function to include soft subtraction.

Get

$$\langle [I - \mathbb{S}]\,[I - S_5]\, C_{51}[\mathrm{d}g_5] w^{24,15} F_{LM}(1,2,4,5)\rangle$$

$$= -\frac{[\alpha_s] s^{-\epsilon}}{\epsilon} \int\limits_{z_{\min}(E_4)}^{1} \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}} \hat{\mathcal{P}}^{(-)}_{qq}(z) \langle \tilde{w}^{24,15}_{5||1} F_{LM}(z \cdot 1, 2, 4)\rangle.$$

$$\hat{\mathcal{P}}^{(-)}_{qq} f(z) \equiv \mathcal{P}_{qq}(z) f(z) - 2C_F f(1)$$

Combining partitions

Take first double collinear term and **rename** the resolved gluon 4:

$$\left\langle \left[I-\mathbb{S}\right]\left[I-S_5\right] C_{41}[\mathrm{d}g_4] w^{14,25} F_{LM}(1,2,4,5)\right\rangle$$

$$= -\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\int\limits_0^{z_{\max}(E_4)} \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}}\mathcal{P}_{qq}(z)\left\langle \tilde{w}^{15,24}_{5||1}\left[I-S_4\right]F_{LM}(z\cdot 1,2,4)\right\rangle.$$

Using $z_{\max}(E_4) = 1 - E_4/E_1 = z_{\min}(E_4)$ combine with second term:

$$\left\langle \left[I-\mathbb{S}\right]\left[I-S_5\right]\left[C_{41}[\mathrm{d}g_4] w^{14,25} + C_{51}[\mathrm{d}g_4] w^{15,24} F_{LM}(1,2,4,5)\right\rangle$$

$$= -\frac{[\alpha_s]s^{-\epsilon}}{\epsilon}\int\limits_0^{1} \frac{\mathrm{d}z}{(1-z)^{1+2\epsilon}}\left\langle \tilde{w}^{15,24}_{5||1}\left(\hat{\mathcal{P}}^{(-)}_{qq}(z)\left[I-S_4\right]F_{LM}(z\cdot 1,2,4)+\right.\right.$$

$$\left.\left.\theta(z_4-z)2C_F\left[I-S_4\right]F_{LM}(1,2,4)+\theta(z_4-z)\hat{\mathcal{P}}^{(-)}_{qq}(z)S_4F_{LM}(z\cdot 1,2,4)\right)\right\rangle.$$

Similar simplifications on combining terms from **double** & **triple** collinear partitions.

Triple-collinear subtraction term

Terms with double-unresolved collinear limits:

$$\langle F_{LM}^{s_r c_t} \rangle = - \sum_{(ij)\in dc} \Big\langle [I - \mathbb{S}]\,[I - S_5]\, C_{4i} C_{5j} [dg_4][dg_5] w^{i4,j5} F_{LM}(1,2,4,5) \Big\rangle$$

$$+ \sum_{i \in tc} \Big\langle [I - \mathbb{S}]\,[I - S_5] \Big[\theta^{(a)} \mathbb{C}_i [I - C_{5i}] + \theta^{(b)} \mathbb{C}_i [I - C_{45}]$$

$$+ \theta^{(c)} \mathbb{C}_i [I - C_{4i}] + \theta^{(d)} \mathbb{C}_i [I - C_{45}] \Big] [dg_4][dg_5] w^{i4,i5} F_{LM}(1,2,4,5) \Big\rangle$$

- For triple collinear sectors, limits have complicated triple collinear splitting function:
  - Integration is non-trivial.
  - Expand the *integrand* in ϵ.
  - Evaluate numerically.
  - Produces $1/\epsilon$ pole & finite term.

Double-real cross section: recap

We have now written complete double-real cross section as:

- Splitting functions convoluted with LO matrix elements – including explicit $1/\epsilon^4$ (and higher) poles.

$$\langle F_{LM}(z\cdot 1, \bar{z}\cdot 2)\rangle\,,\ \langle F_{LM}(z\cdot 1, 2)\rangle\,,\ \langle F_{LM}(1, z\cdot 2)\rangle\,,\ \langle F_{LM}(1,2)\rangle$$

- Splitting functions convoluted with NLO matrix elements, regulated by iterative subtraction – including explicit $1/\epsilon^2$ (and higher) poles.

$$\langle \mathcal{O}_{NLO} F_{LM}(z\cdot 1, 2, 4)\rangle\,,\ \langle \mathcal{O}_{NLO} F_{LM}(1, z\cdot 2, 4)\rangle\,,\ \langle \mathcal{O}_{NLO} F_{LM}(1, 2, 4)\rangle$$

- NNLO matrix elements, regulated by iterative subtraction – **finite.**
- All singularities made explicit.
- Evaluate in **four dimensions.**

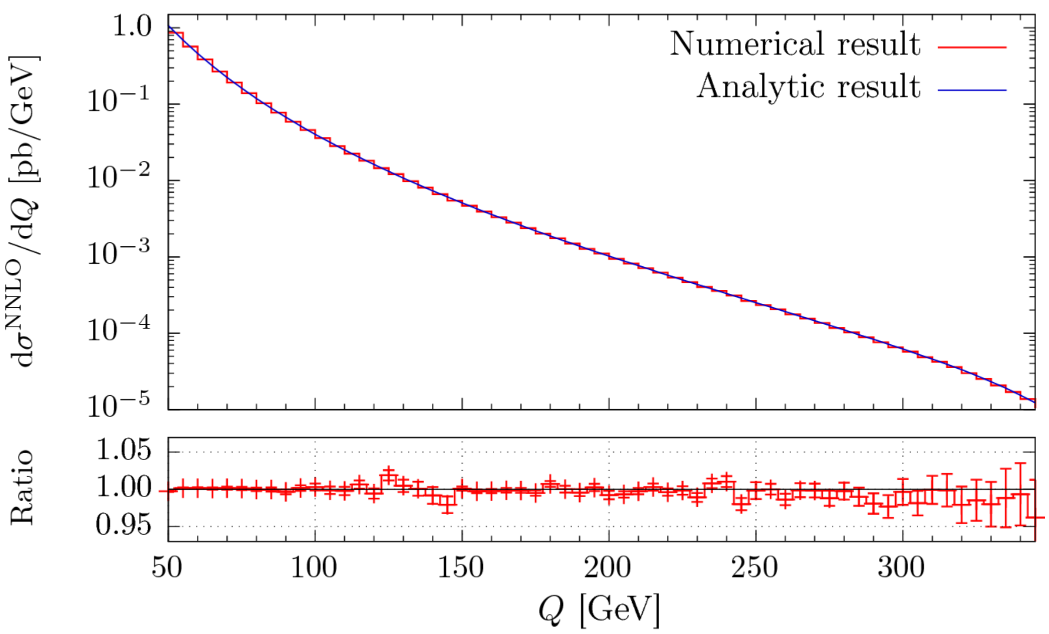
Pole cancellation

- Combine poles from real-real, real-virtual, virtual-virtual, collinear subtraction.
- Poles *must* cancel for each structure F_{LM} :
  - ✔ $\langle \mathcal{O}_{NLO} F_{LM}(z \cdot 1, 2, 4) \rangle$, $\langle \mathcal{O}_{NLO} F_{LM}(1, z \cdot 2, 4) \rangle$, $\langle \mathcal{O}_{NLO} F_{LM}(1, 2, 4) \rangle$ cancel *analytically*.
  - ✔ $\left\langle F_{LV}^{\text{fin}}(1,2) \right\rangle$ cancels *analytically*.
  - ✔ $\langle F_{LM}(z \cdot 1, \bar{z} \cdot 2) \rangle$, $\langle F_{LM}(z \cdot 1, 2) \rangle$, $\langle F_{LM}(1, z \cdot 2) \rangle$, $\langle F_{LM}(1, 2) \rangle$ cancels *numerically* (double-soft and triple-collinear computed numerically).

Proof-of-principle

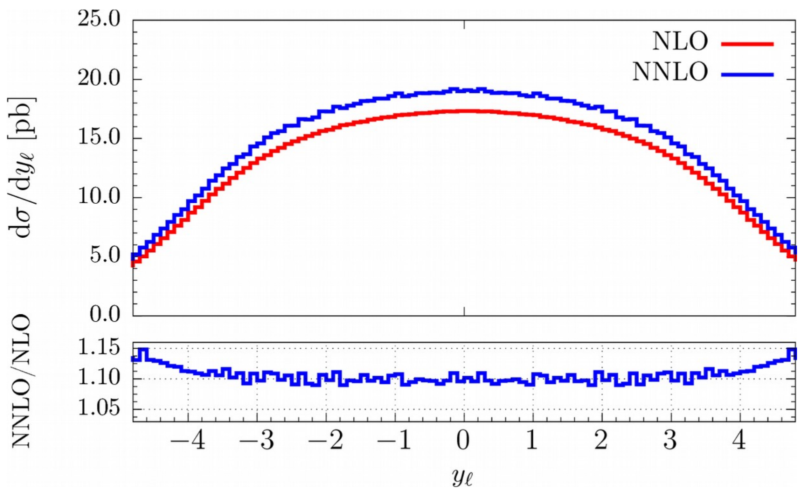
- Calculate $pp \to \gamma^* + X \to e^+e^- + X$ to NNLO (using $q\bar{q} \to \gamma^* + ng$ channel only).
- Lepton pairs with invariant mass $50 \text{ GeV} \leq Q \leq 350 \text{ GeV}$.
- Extract results from [Hamberg, Matsuura, van Neerven '91] to compare (analytic in *Q).*
- NNLO *contributions.*

$$\mathrm{d}\sigma^{\mathrm{NNLO}} = 14.471(4) \text{ pb}$$
$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{\mathrm{analytic}} = 14.470 \text{ pb}$$

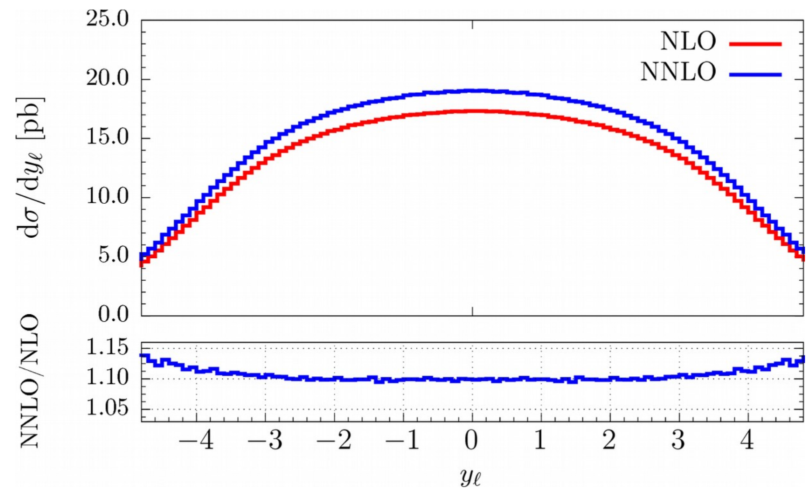


- Sub per-mille agreement in cross sections.
- Per-mille to percent agreement across **5 orders of magnitude** in *Q.*

Differential distributions (I)



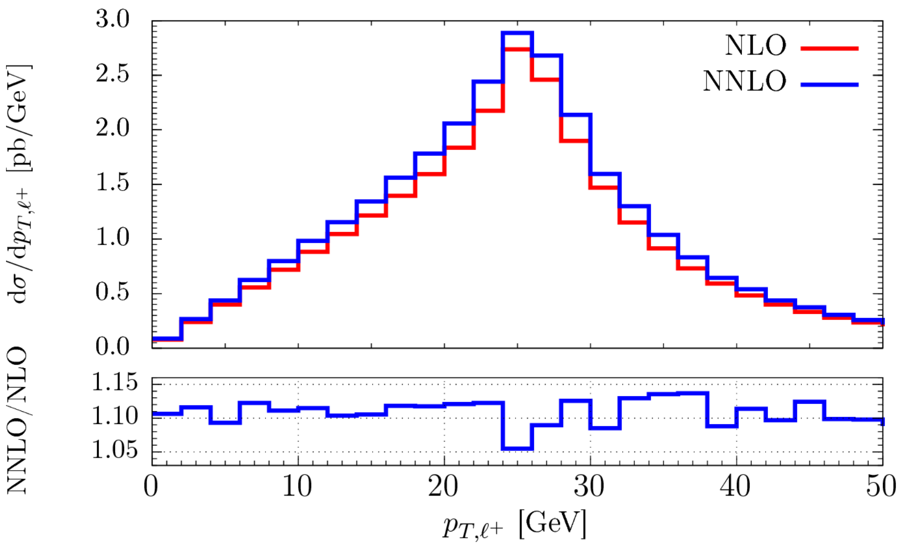
O(10 CPU hours) runtime



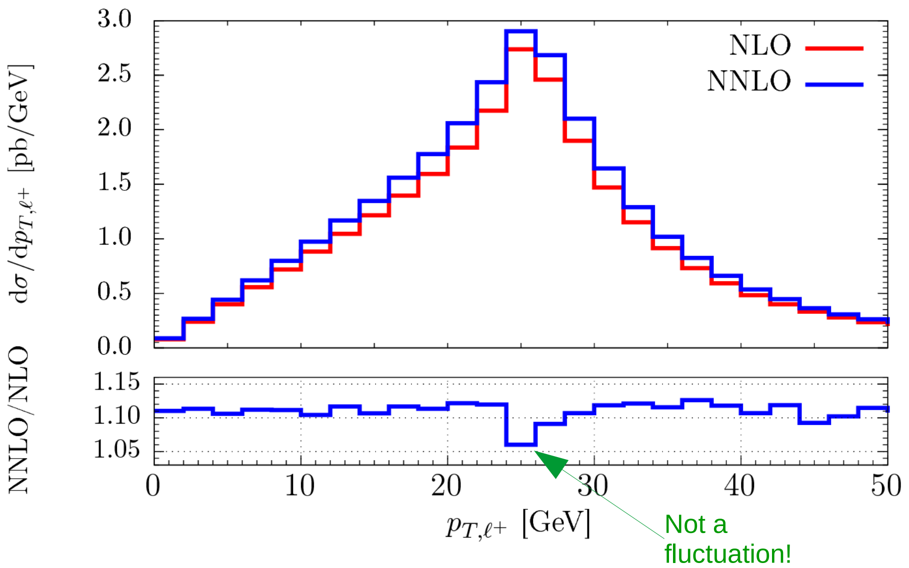
O(100 CPU hours) runtime

- Lepton rapidity.
- O(10 CPU hours): percent-level bin-to-bin fluctuations.
- O(100 CPU hours): per-mille bin-to-bin fluctuations.

Differential distributions (II)



O(10 CPU hours) runtime



O(100 CPU hours) runtime

- Lepton transverse momentum.
- O(100 CPU hours): percent-level bin-to-bin fluctuations.
- Delicate observable: receives contributions from large range of invariant masses.
  - Improves once introduce Z boson propagator.
  - Competitive with state-of-the-art NNLO codes.

Other partonic channel

- Other partonic channels (*qg, gq, gg, qq→qq*) follow same strategy, but fewer limits.
- All calculated within this approach for DY.
  - Similar agreement with analytic results, including for numerically tiny channels.
- Extension to different color singlets a work in progress, requires:
  - Amplitudes.
  - Phase space for color singlet.

Summary

- Modification of residue-improved sector decomposition method of handling NNLO subtraction.
- Characterized by decoupling of soft and collinear limits.
- Develop iterative subtraction procedure:
  - Manifestly regulated finite term.
  - Integrated subtraction terms: convolutions of splitting function with explicit poles with lower multiplicity processes.
  - Pole cancellation <u>independent</u> of matrix elements.
- **Process independent** (colorless final state).
- Tested in dilepton production through virtual photon *for all partonic channels*:
  - Excellent agreement with analytic results.
- Future work:
  - Efficient implementation in numerical integration.
  - Extension to colored final states.
  - Include alpha-parameters.
  - …

End of tour

Thank you and questions?