The Sudakov form factor to four loops in $\mathcal{N} = 4$ super Yang-Mills theory

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Based on

R. Boels, G. Yang, TH	1705.03444
R. Boels, B. Kniehl, G. Yang	1508.03717, NPB
R. Boels, B. Kniehl, O. Tarasov, G. Yang	1211.7028, JHEP
Th. Gehrmann, J.M. Henn, TH	1112.4524, JHEP

Theory Seminar, HU Berlin, July 6th, 2017

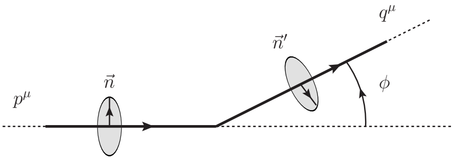
Outline

- The cusp anomalous dimension
- $\mathcal{N} = 4$ super Yang-Mills
- Sudakov from factor in $\mathcal{N} = 4$ SYM
- Integration at four loops
  - Finding a homogeneous integral basis
  - Numerical integration
- Result
- Conclusion and outlook

- Locally supersymmetric Wilson loop operator [Maldacena'98; Rey,Yee'98]

$$W \sim \mathrm{Tr}\left[P \exp\left(i \oint A^\mu \dot{x}_\mu + \oint |dx| \vec{n} \cdot \vec{\Phi}\right)\right]$$

- $\vec{n}$: Vector on S^5, parametrizes coupling of Wilson loop to 6 scalars Φ
- Path: Two segments that form cusp of angle ϕ



- Connection between vev $\langle W \rangle$ of Wilson loop and $\Gamma_{\rm cusp}$ [Polyakov'80; Brandt,Neri,Sato'81] [Korchemsky,Radyushkin'86]

$$\langle W \rangle \sim \exp\left[-\log \frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}} \Gamma_{\rm cusp} + \dots\right]$$

- Light-like limit $\gamma_{\rm cusp}$ of $\Gamma_{\rm cusp}$ via $\phi \to +i\infty$

Light-like cusp anomalous dimension

- $\gamma_{\rm cusp}$ is a universal quantity
- Governs structure of IR divergences in many physical quantities

  $\longrightarrow$ many applications

  - Collider physics: Deep inelastic scattering, ...
  - Governs running of matching coefficients from QCD onto SCET [Bell,Beneke,Li,TH'10]

$$\frac{d}{d\ln\mu} C_i(\mu) = \left[\gamma_{\rm cusp}\ \ln\frac{q^2}{\mu^2} + \gamma' + \gamma_i\right] C_i(\mu)$$

- $\gamma_{\rm cusp}$ has a loop expansion, e.g. in QCD

$$\gamma_{\rm cusp} = \alpha_s\,\gamma_{\rm cusp}^{(1)} + \alpha_s^2\,\gamma_{\rm cusp}^{(2)} + \ldots$$

- In QCD, $\gamma_{\rm cusp}$ is known to three loops [Moch,Vermaseren,Vogt'05]
  - Also appears in massless form factor ($i = q, g$) [Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09] [Gehrmann,Glover,Ikizlerli,Studerus,TH'10] [von Manteuffel,Schabinger,Panzer'15]

$$Poles(F_3^i) = -\frac{11\beta_0^2 C_i \gamma_0^{\rm cusp}}{36\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{5\beta_0 C_i \gamma_1^{\rm cusp}}{36} + \frac{\beta_0^2\gamma_0^i}{3} + \frac{2C_i\gamma_0^{\rm cusp}\beta_1}{9}\right)$$
$$+\frac{1}{\epsilon^2}\left(-\frac{\beta_0\gamma_1^i}{3} - \frac{C_i\gamma_2^{\rm cusp}}{18} - \frac{\beta_1\gamma_0^i}{3}\right) + \frac{\gamma_2^i}{3\epsilon} - \frac{\left(F_1^i\right)^3}{3} + F_2^i F_1^i$$

  - Observe quadratic Casimir scaling [Korchemsky'89]

$$\gamma_{\rm cusp}^{\rm q,QCD} = C_F\ \gamma_{\rm cusp}$$

$$\gamma_{\rm cusp}^{\rm g,QCD} = C_A\ \gamma_{\rm cusp}$$

$$\gamma_{\rm cusp} = 4\left(\frac{\alpha_s}{4\pi}\right) + \left[C_A\left(-8\zeta_2 - \frac{268}{9}\right) - \frac{40}{9}\, n_f\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots$$

    - Verified to three loops

Light-like cusp anomalous dimension

- In general, $\gamma_{\text{cusp}}^{\text{R,QCD}} = C_{2,R}\,\gamma_{\text{cusp}} + C_{4,R}\,\gamma_{\text{cusp}}^{\text{NP}}$
- $C_{4,R} \sim d_R^{abcd}\, d_R^{abcd}$ is the quartic Casimir invariant

$$d_R^{abcd} = \frac{1}{6}\mathrm{Tr}[T_R^a T_R^b T_R^c T_R^d + T_R^a T_R^b T_R^d T_R^c + T_R^a T_R^c T_R^b T_R^d + T_R^a T_R^c T_R^d T_R^b + T_R^a T_R^d T_R^b T_R^c + T_R^a T_R^d T_R^c T_R^b]$$

- $\gamma_{\text{cusp}}^{\text{NP}} \neq 0$ would indicate breakdown of quadratic Casimir scaling
- $\gamma_{\text{cusp}}^{\text{NP}}$ first appears at four loops

Light-like cusp anomalous dimension

- In general, $\gamma_{\rm cusp}^{\rm R,QCD} = C_{2,R}\,\gamma_{\rm cusp} + C_{4,R}\,\gamma_{\rm cusp}^{\rm NP}$
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$$d_R^{abcd} = \frac{1}{6}{\rm Tr}[T_R^a T_R^b T_R^c T_R^d + T_R^a T_R^b T_R^d T_R^c + T_R^a T_R^c T_R^b T_R^d + T_R^a T_R^c T_R^d T_R^b + T_R^a T_R^d T_R^b T_R^c + T_R^a T_R^d T_R^c T_R^b]$$

- $\gamma_{\rm cusp}^{\rm NP} \neq 0$ would indicate breakdown of quadratic Casimir scaling
- $\gamma_{\rm cusp}^{\rm NP}$ first appears at four loops
- Conjectures about quadratic Casimir scaling
  - Holds to all orders in perturbation theory [Becher,Neubert'09] [see also Gardi,Magnea'09; Dixon'09; Dixon,Gardi,Magnea'09; Ahrens,Neubert,Vernazza'12]
  - Breaks down at some finite loop order [Frenkel,Taylor'84]
  - Breaks down at four-loop order [Alday,Maldacena'07]
  - Breaks down at strong coupling [Armoni'06]
  - Breaks down through instanton effects [Korchemsky'17]

- Particle content of $\mathcal{N} = 4$ SYM
  - spin-1 gauge field A_μ
  - left Weyl fermions $\lambda^a_\alpha, \quad a = 1, \ldots, 4$
  - Real scalars $X^i, \quad i = 1, \ldots, 6$
  - They form the $\mathcal{N} = 4$ *Gauge Multiplet* $(A_\mu \; \lambda^a_\alpha \; X^i)$
- Under rotations $SU(4)_R$ in superspace
  - A_μ is a singlet
  - λ^a_α is a **4**, X^i are a rank 2 anti-symmetric **6**
  - Denote them by $\phi_{AB} = \phi^a_{AB} T_a \, , \qquad A, B = 1, \ldots, 4$
- Lagrangian density

$$\mathcal{L} = \mathrm{tr}\Big\{ -\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - \sum_a i \bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a - \sum_i D_\mu X^i D^\mu X^i$$
$$+ \sum_{a,b,i} g C_i^{ab} \lambda_a [X^i, \lambda_b] + \sum_{a,b,i} g \bar{C}_{iab} \bar{\lambda}^a [X^i, \bar{\lambda}^b] + \frac{g^2}{2} \sum_{i,j} [X^i, X^j]^2 \Big\}$$

$\mathcal{N} = 4$ SYM

- Unique maximal supersymmetric gauge theory in $D = 4$
- $\mathcal{N} = 4$ SYM has vanishing β-function
- Related to type IIB superstring on AdS_5 x S^5 background via AdS/CFT
- Related to QCD via maximal transcendentality principle
  - $\mathcal{N} = 4$ SYM is 'hardest part' of QCD

$$\gamma_{\rm cusp}^{\rm QCD} = 4 \left(\frac{\alpha_s}{4\pi}\right) + \left[C_A\left(-8\zeta_2 - \frac{268}{9}\right) - \frac{40}{9}\,n_f\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots$$

$$\gamma_{\rm cusp}^{\mathcal{N}=4} = 4\;2\,g^2 - 8\zeta_2\;2\,g^4 + \ldots$$

  - BES equation predicts planar (leading-in-colour) $\gamma_{\rm cusp}^{\mathcal{N}=4}$ to all orders [Beisert,Eden,Staudacher'04]
    - Verified to four loops [Bern,Dixon,Smirnov'05; Henn,TH'13]

The Sudakov form factor in $\mathcal{N} = 4$ SYM

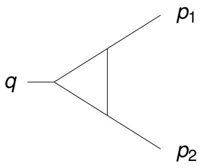
- Introduce bilinear Operator: $\mathcal{O} = \mathrm{Tr}(\phi_{12}\phi_{12})$
- Properties of $\mathcal{O}$
  - $\mathcal{O}$ is a component of the stress-energy supermultiplet of $\mathcal{N} = 4$ SYM
  - $\mathcal{O}$ is a colour singlet and has zero anomalous dimension
- Definition of the form factor

$$\mathcal{F}_{\mathcal{S}} = \langle \phi^a_{34}(p_1)\phi^b_{34}(p_2)\,\mathcal{O}\rangle = \mathrm{Tr}(T^a T^b) F_{\mathcal{S}}$$

  - $\phi^a_{34}(p_i)$: on-shell states in the adjoint representation
  - Kinematics:

$$p_1^2 = p_2^2 = 0$$

$$q^2 = (p_1 + p_2)^2$$



  - Use dimensional regularisation

$$D = 4 - 2\epsilon$$

The Sudakov form factor in $\mathcal{N} = 4$ SYM

- Couplings: $g^2 = \frac{g_{YM}^2 \, N_c}{(4\pi)^2} \, (4\pi)^\epsilon \, e^{-\epsilon \gamma_E}$ 't Hooft coupling: $a = 2g^2$
- Perturbative expansion of the FF

$$F_S = 1 + g^2 \, (-q^2)^{-\epsilon} \, F_S^{(1)} + g^4 \, (-q^2)^{-2\epsilon} \, F_S^{(2)} + g^6 \, (-q^2)^{-3\epsilon} \, F_S^{(3)}$$

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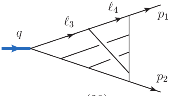
- First non-planar corrections to colour-structure appears at four loops
  - Appearance of quartic Casimir invariant through



(28)

$$\mathrm{Tr}[T_A^a T_A^b T_A^c T_A^d T_A^a T_A^b T_A^c T_A^d]/N_A = \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{1}{24} C_A^4 = \frac{3}{2} N_c^2$$

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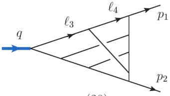

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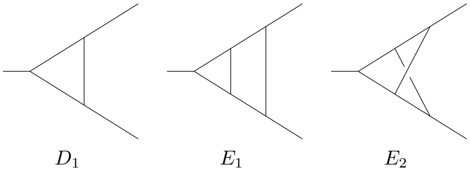
- Relation to light-like cusp AD

$$\log F_S = \sum_{L=1}^{\infty} g^{2L}(-q^2)^{-L\epsilon}\left[-\frac{\gamma_{\rm cusp}^{(L)}}{(2L\epsilon)^2} - \frac{\mathcal{G}_{\rm coll}^{(L)}}{2L\epsilon}\right] + \mathcal{O}\left(\epsilon^0\right)$$



- Form factor to two loops [van Neerven'86]

$$F_S = 1 + a\, x^{\epsilon}\, R_{\epsilon} \cdot 2\, D_1 + a^2\, x^{2\epsilon}\, R_{\epsilon}^2 \cdot [4\, E_1 + E_2]$$

- Have $$x = \frac{\mu^2}{-q^2 - i\eta} \qquad \text{and} \qquad R_{\epsilon} = \frac{e^{\epsilon \gamma_E}}{2\,\Gamma(1-\epsilon)}$$

- Result is remarkably simple
- Crossed diagram already at two loops

$$F_S^{(1)} = R_\epsilon \cdot 2\,D_1$$

$$= -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7\zeta_3}{3}\,\epsilon + \frac{47\pi^4}{1440}\,\epsilon^2 + \epsilon^3\left(\frac{31\zeta_5}{5} - \frac{7\pi^2\zeta_3}{36}\right) + \epsilon^4\left(\frac{949\pi^6}{120960} - \frac{49\zeta_3^2}{18}\right)$$

$$+\,\epsilon^5\left(-\frac{329\pi^4\zeta_3}{4320} - \frac{31\pi^2\zeta_5}{60} + \frac{127\zeta_7}{7}\right) + \epsilon^6\left(\frac{49\pi^2\zeta_3^2}{216} - \frac{217\zeta_3\zeta_5}{15} + \frac{18593\pi^8}{9676800}\right)$$

$$+\,\mathcal{O}(\epsilon^7)\,,$$

$$F_S^{(2)} = R_\epsilon^2 \cdot [4\,E_1 + E_2]$$

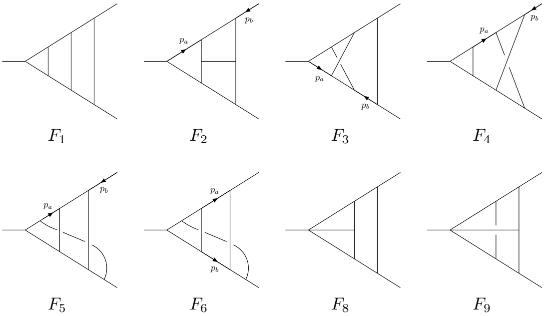
$$= +\frac{1}{2\epsilon^4} - \frac{\pi^2}{24\epsilon^2} - \frac{25\zeta_3}{12\epsilon} - \frac{7\pi^4}{240} + \epsilon\left(\frac{23\pi^2\zeta_3}{72} + \frac{71\zeta_5}{20}\right) + \epsilon^2\left(\frac{901\zeta_3^2}{36} + \frac{257\pi^6}{6720}\right)$$

$$+\,\epsilon^3\left(\frac{1291\pi^4\zeta_3}{1440} - \frac{313\pi^2\zeta_5}{120} + \frac{3169\zeta_7}{14}\right)$$

$$+\,\epsilon^4\left(-66\zeta_{5,3} + \frac{845\zeta_3\zeta_5}{6} - \frac{1547\pi^2\zeta_3^2}{216} + \frac{50419\pi^8}{518400}\right) + \mathcal{O}(\epsilon^5)\,.$$

- No UV divergences. Leading coefficient at L loops is $1/\epsilon^{2L}$
- Each diagram has uniform transcendentality (UT) in its ϵ-expansion

Form factor at three loops



$$F_S = 1 + a\, x^{\epsilon}\, R_{\epsilon} \cdot 2\, D_1 + a^2\, x^{2\epsilon}\, R_{\epsilon}^2 \cdot [4\, E_1 + E_2]$$

$$+\, a^3\, x^{3\epsilon}\, R_{\epsilon}^3 \cdot [8\, F_1 - 2\, F_2 + 4\, F_3 + 4\, F_4 - 4\, F_5 - 4\, F_6 - 4\, F_8 + 2\, F_9]$$

[Gehrmann,Henn,TH'11]

- Also at three loops the result is remarkably simple.
- Single irreducible numerators $(p_a + p_b)^2$

Form factor at three loops

- Again, each diagram has UT in its ϵ-expansion

$$
\begin{aligned}
F_S^{(3)} &= R_\epsilon^3 \cdot [8\,F_1 - 2\,F_2 + 4\,F_3 + 4\,F_4 - 4\,F_5 - 4\,F_6 - 4\,F_8 + 2\,F_9] \\
&= -\frac{1}{6\epsilon^6} + \frac{11\zeta_3}{12\epsilon^3} + \frac{247\pi^4}{25920\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{85\pi^2\zeta_3}{432} - \frac{439\zeta_5}{60}\right) \\
&\quad - \frac{883\zeta_3^2}{36} - \frac{22523\pi^6}{466560} + \epsilon\left(-\frac{47803\pi^4\zeta_3}{51840} + \frac{2449\pi^2\zeta_5}{432} - \frac{385579\zeta_7}{1008}\right) \\
&\quad + \epsilon^2\left(\frac{1549}{45}\zeta_{5,3} - \frac{22499\zeta_3\zeta_5}{30} + \frac{496\pi^2\zeta_3^2}{27} - \frac{1183759981\pi^8}{7838208000}\right) + \mathcal{O}(\epsilon^3)
\end{aligned}
$$

- Leading transcendentality (LT) principle [Kotikov,Lipatov,Onishchenko,Velizhanin'04]
  - Specify QCD quark, gluon FF to a SUSY theory containing a bosonic and fermionic degree of freedom in the same colour representation $(C_A = C_F = 2T_F,\, n_f = 1)$
  - LT pieces of quark, gluon, and $\mathcal{N} = 4$ SYM form factor coincide!
  - Holds at $L = 1, 2, 3$ and in all coefficients to weight 8
  - Holds at $L = 4$ between quark and $\mathcal{N} = 4$ FF to weight 6

- Logarithm of the form factor to three loops

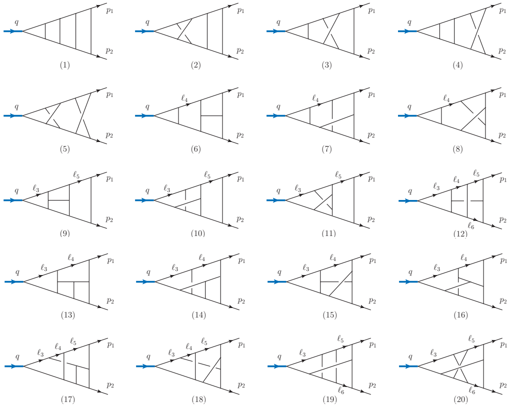
$$\ln(F_S) = \ln\left(1 + a\,x^\epsilon\,F_S^{(1)} + a^2\,x^{2\epsilon}\,F_S^{(2)} + a^3\,x^{3\epsilon}\,F_S^{(3)}\right)$$
$$= a\,x^\epsilon\,F_S^{(1)} + a^2\,x^{2\epsilon}\left[F_S^{(2)} - \frac{1}{2}\left(F_S^{(1)}\right)^2\right] + a^3\,x^{3\epsilon}\left[F_S^{(3)} - F_S^{(1)}F_S^{(2)} + \frac{1}{3}\left(F_S^{(1)}\right)^3\right]$$

- Need lower-loop form factors to higher orders in ϵ-expansion
- At most double poles, expected from exponentiation

$$F_S^{(1)} = -\frac{1}{\epsilon^2} + \frac{\pi^2}{12} + \frac{7\zeta_3}{3}\epsilon + \frac{47\pi^4}{1440}\epsilon^2 + \dots$$
$$F_S^{(2)} - \frac{1}{2}\left(F_S^{(1)}\right)^2 = \frac{\pi^2}{24\epsilon^2} + \frac{\zeta_3}{4\epsilon} + 0 + \epsilon\left(\frac{39\zeta_5}{4} - \frac{5\pi^2\zeta_3}{72}\right) + \dots$$
$$F_S^{(3)} - F_S^{(1)}F_S^{(2)} + \frac{1}{3}\left(F_S^{(1)}\right)^3 = -\frac{11\pi^4}{1620\epsilon^2} + \frac{1}{\epsilon}\left(-\frac{5\pi^2\zeta_3}{54} - \frac{2\zeta_5}{3}\right) - \frac{13\zeta_3^2}{9} - \frac{193\pi^6}{25515} + \dots$$

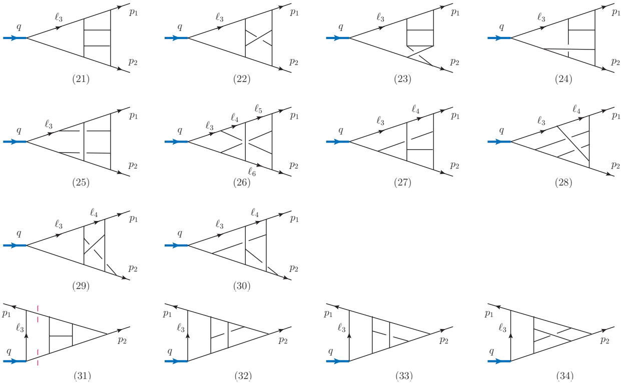
- Poles give correct values of cusp and collinear anomalous dimension
- Finite piece does not exponentiate

- Four-loop integrand was derived from colour-kinematic duality [Boels,Kniehl,Tarasov,Yang'12]
  - Checks via unitarity cuts
- Purely planar diagrams, six (8, 11, 15, 16, 18, 20) have vanishing colour-factor



- 12 propagators and 18 independent scalar products / topology

- Diagrams with non-planar contributions [Boels,Kniehl,Tarasov,Yang'12]
  - Four diagrams (21, 25, 30, 31) have also planar contributions



- All but one topology (26) have internal box(es) or triangle(s)
- Many have one or more graph symmetries

- Complicated integrals with up to double irreducible numerators

Graph	Numerator factor	Color factor	Symmetry factor
(27)	$-(\ell_3\cdot p_1)^2-(\ell_3\cdot p_2)^2-6(\ell_3\cdot p_1)(\ell_3\cdot p_2)$ $+(\ell_3\cdot p_1)(\ell_4\cdot(p_1+5p_2))$ $-(\ell_3\cdot p_2)(\ell_4\cdot(3p_1-p_2))$ $+(p_1\cdot p_2)\big[2\ell_3\cdot(p_{12}+\ell_3-\ell_4)$ $+\ell_4\cdot(p_1-p_2)-p_1\cdot p_2\big]$ $+(\alpha_1+1)\big[(\ell_3\cdot p_{12}-p_1\cdot p_2)^2$ $-\frac{1}{7}(\ell_3\cdot p_{12}-p_1\cdot p_2)(\ell_4\cdot(7p_1-p_2))$ $-\frac{2}{7}(\ell_3\cdot(\ell_3-p_{12})+p_1\cdot p_2)(p_1\cdot p_2)\big]$	$24\,N_c^2\,\delta_{a_1a_2}$	1

- In total $\sim$ 125 numerator terms in non-planar sector
- IBP & Laporta reduction to $\sim$ 280 master integrals achieved with Reduze [Boels,Kniehl,Yang'15]
  - Free parameter α_1 drops out of result
  - Subset of 'rational IBP relations' will prove useful

- Concentrate on non-planar part of form factor. Several possible integration strategies

Integration at four-loop

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- UT basis, purely analytic approach
  - Take p_1^2 off-shell
    apply differential equations in p_1^2/q^2
    transform to UT basis [Meyer'16,'17; Prausa'17; Gituliar,Magerya'17]
    take limit $p_1^2 \to 0$.
    Applied to four-loop QCD case [Henn,Smirnov,Smirnov,Steinhauser'16] [Henn,Lee,Smirnov,Smirnov,Steinhauser'16] [Lee,Smirnov,Smirnov,Steinhauser'17]

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- Finite basis, analytic/numerical approach
  - Take basis of masters with no poles in ϵ
    Requires many dots on propagators and dimensional recurrencies
    Compute finite masters numerically or analytically
    Applied to three-loop QCD case [von Manteuffel,Panzer,Schabinger'15]

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- UT basis, numerical approach: This work
  - Transform single-scale form factor to UT basis
    Integrate UT integrals numerically (FIESTA, SecDec, MB) [Smirnov; Heinrich et al.; Czakon]

UT basis and integration at four loop

- How to find a UT basis in single-scale problems?
- How to know an integral is UT without explicit calculation?

UT basis and integration at four loop

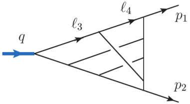
- How to find a UT basis in single-scale problems?
- How to know an integral is UT without explicit calculation?
- dLog form [Arkani-Hamed,Bourjaily,Cachazo,Trnka'14; Bern,Herrmann,Litsey,Stankowicz,Trnka'14]
  - Change variables to scalar parameters a_i, b_i, c_i, d_i in four dimensions

$$l_6 = a_1 p_1 + a_2 p_2 + a_3 q_1 + a_4 q_2 \,, \qquad l_3 = d_1 p_1 + d_2 p_2 + d_3 q_1 + d_4 q_2 \,,$$
$$l_5 = b_1 p_1 + b_2 p_2 + b_3 q_1 + b_4 q_2 \,, \qquad q_i^2 = q_i \cdot p_j = 0 \quad \forall i,j$$
$$l_4 = c_1 p_1 + c_2 p_2 + c_3 q_1 + c_4 q_2 \,, \qquad q_1 \cdot q_2 = -p_1 \cdot p_2$$

$$\int d^4 l_3 \, d^4 l_4 \, d^4 l_5 \, d^4 l_6 \, (\ldots) = (p_1 \cdot p_2)^8 \int \prod_{i=1}^{4} da_i \, db_i \, dc_i \, dd_i \, (\ldots) \,.$$

Example with two boxes:
Topology 28 with numerator

$$(l_3 - l_4 - p_2)^2 \times (l_3 - p_1)^2$$



(28)

UT basis: dLog form

$$\frac{da_1 \ldots dd_4\ (d_1 d_2 - d_3 d_4)(-c_1 + c_1 c_2 - c_3 c_4 + d_1 - c_2 d_1 - c_1 d_2 + d_1 d_2 + c_4 d_3 + c_3 d_4 - d_3 d_4)}{(a_1 a_2 - a_3 a_4)(a_1 a_2 - a_3 a_4 - a_2 b_1 - a_1 b_2 + b_1 b_2 + a_4 b_3 + a_3 b_4 - b_3 b_4)(c_1 c_2 - c_3 c_4)(-c_2 + c_1 c_2 - c_3 c_4)(d_1 + d_1 d_2 - d_3 d_4)(-d_2 + d_1 d_2 - d_3 d_4)}$$

$$\times \frac{1}{(a_1 a_2 - a_3 a_4 - a_2 c_1 - a_1 c_2 + c_1 c_2 + a_4 c_3 + a_3 c_4 - c_3 c_4)(b_1 b_2 - b_3 b_4 - b_2 c_1 - b_1 c_2 + c_1 c_2 + b_4 c_3 + b_3 c_4 - c_3 c_4)}$$

$$\times \frac{1}{(-a_1 + a_1 a_2 - a_3 a_4 + d_1 - a_2 d_1 - a_1 d_2 + d_1 d_2 + a_4 d_3 + a_3 d_4 - d_3 d_4)(b_1 b_2 - b_3 b_4 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)}$$

$$\times \frac{1}{(-b_1 + b_1 b_2 - b_3 b_4 + d_1 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)(c_1 c_2 - c_3 c_4 - c_2 d_1 - c_1 d_2 + d_1 d_2 + c_4 d_3 + c_3 d_4 - d_3 d_4)}$$

$$= -\, dLog[\frac{(c_1 c_2 - c_3 c_4)}{(c_1 c_2 - c_3 c_4 - c_2 d_1 + c_4 d_3)}] dLog[\frac{(-c_2 + c_1 c_2 - c_3 c_4)}{(c_1 c_2 - c_3 c_4 - c_2 d_1 + c_4 d_3)}] dLog[\frac{d_3}{(-1 + d_1 - d_2 + d_1 d_2 - d_3 d_4)}] dLog[\frac{d_4}{(-1 + d_1 - d_2 + d_1 d_2 - d_3 d_4)}]$$

$$\times\, dLog[\frac{((a_1 a_2 - a_3 a_4)(c_3 d_1 - c_1 d_3))}{(-(a_3 c_1 c_2 d_1) + a_1 a_2 c_3 d_1 - a_3 a_4 c_3 d_1 + a_3 c_3 c_4 d_1 + a_3 c_2 d_1^2 - a_2 c_3 d_1^2 - a_1 a_2 c_1 d_3 + a_3 a_4 c_1 d_3 + a_1 c_1 c_2 d_3 - a_1 c_3 c_4 d_3 + a_2 c_1 d_1 d_3 - a_1 c_2 d_1 d_3 + a_4 c_3 d_1 d_3 - a_3 c_4 d_1 d_3 - a_4 c_1 d_3^2 + a_1 c_4 d_3^2)}]$$

$$\times\, dLog[\frac{((a_1 a_2 - a_3 a_4 - a_2 c_1 - a_1 c_2 + c_1 c_2 + a_4 c_3 + a_3 c_4 - c_3 c_4)(c_3 d_1 - c_1 d_3))}{(-(a_3 c_1 c_2 d_1) + a_1 a_2 c_3 d_1 - a_3 a_4 c_3 d_1 + a_3 c_3 c_4 d_1 + a_3 c_2 d_1^2 - a_2 c_3 d_1^2 - a_1 a_2 c_1 d_3 + a_3 a_4 c_1 d_3 + a_1 c_1 c_2 d_3 - a_1 c_3 c_4 d_3 + a_2 c_1 d_1 d_3 - a_1 c_2 d_1 d_3 + a_4 c_3 d_1 d_3 - a_3 c_4 d_1 d_3 - a_4 c_1 d_3^2 + a_1 c_4 d_3^2)}]$$

$$\times\, dLog[\frac{((c_3 d_1 - c_1 d_3)(-a_1 + a_1 a_2 - a_3 a_4 + d_1 - a_2 d_1 - a_1 d_2 + d_1 d_2 + a_4 d_3 + a_3 d_4 - d_3 d_4))}{(-(a_3 c_1 c_2 d_1) + a_1 a_2 c_3 d_1 - a_3 a_4 c_3 d_1 + a_3 c_3 c_4 d_1 + a_3 c_2 d_1^2 - a_2 c_3 d_1^2 - a_1 a_2 c_1 d_3 + a_3 a_4 c_1 d_3 + a_1 c_1 c_2 d_3 - a_1 c_3 c_4 d_3 + a_2 c_1 d_1 d_3 - a_1 c_2 d_1 d_3 + a_4 c_3 d_1 d_3 - a_3 c_4 d_1 d_3 - a_4 c_1 d_3^2 + a_1 c_4 d_3^2)}]$$

$$\times\, dLog[\frac{((c_3 d_1 - c_1 d_3)(a_1 a_2 c_1 - a_3 a_4 c_1 - a_1 c_1 c_2 + a_1 c_3 c_4 - a_1 a_2 d_1 + a_3 a_4 d_1 - a_2 c_1 d_1 + a_1 c_2 d_1 + c_1 c_2 d_1 - c_3 c_4 d_1 + a_2 d_1^2 - c_2 d_1^2 + a_4 c_1 d_3 - a_1 c_4 d_3 - a_4 d_1 d_3 + c_4 d_1 d_3 + a_3 c_1 d_4 - a_1 c_3 d_4 - a_3 d_1 d_4 + c_3 d_1 d_4 + a_1 d_3 d_4 - c_1 d_3 d_4))}{(-(a_3 c_1 c_2 d_1) + a_1 a_2 c_3 d_1 - a_3 a_4 c_3 d_1 + a_3 c_3 c_4 d_1 + a_3 c_2 d_1^2 - a_2 c_3 d_1^2 - a_1 a_2 c_1 d_3 + a_3 a_4 c_1 d_3 + a_1 c_1 c_2 d_3 - a_1 c_3 c_4 d_3 + a_2 c_1 d_1 d_3 - a_1 c_2 d_1 d_3 + a_4 c_3 d_1 d_3 - a_3 c_4 d_1 d_3 - a_4 c_1 d_3^2 + a_1 c_4 d_3^2)}]$$

$$\times\, dLog[\frac{(c_1 c_2 - c_3 c_4 - c_2 d_1 - c_1 d_2 + d_1 d_2 + c_4 d_3 + c_3 d_4 - d_3 d_4)}{(c_1 c_2 - c_3 c_4 - c_2 d_1 + c_4 d_3)}] dLog[\frac{(c_1 c_2 d_1 - c_3 c_4 d_1 - c_2 d_1^2 + c_4 d_1 d_3 + c_3 d_1 d_4 - c_1 d_3 d_4)}{(c_1 c_2 - c_3 c_4 - c_2 d_1 + c_4 d_3)}] dLog[\frac{(d_1 + d_1 d_2 - d_3 d_4)}{(-1 + d_1 - d_2 + d_1 d_2 - d_3 d_4)}] dLog[\frac{(-d_2 + d_1 d_2 - d_3 d_4)}{(-1 + d_1 - d_2 + d_1 d_2 - d_3 d_4)}]$$

$$\times\, dLog[\frac{(a_1 a_2 - a_3 a_4 - a_2 b_1 - a_1 b_2 + b_1 b_2 + a_4 b_3 + a_3 b_4 - b_3 b_4)}{f[b_1, b_2, b_3, b_4]}] dLog[\frac{(b_1 b_2 - b_3 b_4 - b_2 c_1 - b_1 c_2 + c_1 c_2 + b_4 c_3 + b_3 c_4 - c_3 c_4)}{f[b_1, b_2, b_3, b_4]}]$$

$$\times\, dLog[\frac{(b_1 b_2 - b_3 b_4 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)}{f[b_1, b_2, b_3, b_4]}] dLog[\frac{(-b_1 + b_1 b_2 - b_3 b_4 + d_1 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)}{f[b_1, b_2, b_3, b_4]}]$$

✓

Finding UT candidates

$$\frac{(d_1d_2-d_3d_4)(-c_1+c_1c_2-c_3c_4+d_1-c_2d_1-c_1d_2+d_1d_2+c_4d_3+c_3d_4-d_3d_4)}{(a_1a_2-a_3a_4)(a_1a_2-a_3a_4-a_2b_1-a_1b_2+b_1b_2+a_4b_3+a_3b_4-b_3b_4)(c_1c_2-c_3c_4)(-c_2+c_1c_2-c_3c_4)(d_1+d_1d_2-d_3d_4)(-d_2+d_1d_2-d_3d_4)}$$

$$\times\frac{1}{(a_1a_2-a_3a_4-a_2c_1-a_1c_2+c_1c_2+a_4c_3+a_3c_4-c_3c_4)(b_1b_2-b_3b_4-b_2c_1-b_1c_2+c_1c_2+b_4c_3+b_3c_4-c_3c_4)}$$

$$\times\frac{1}{(-a_1+a_1a_2-a_3a_4+d_1-a_2d_1-a_1d_2+d_1d_2+a_4d_3+a_3d_4-d_3d_4)(b_1b_2-b_3b_4-b_2d_1-b_1d_2+d_1d_2+b_4d_3+b_3d_4-d_3d_4)}$$

$$\times\frac{1}{(-b_1+b_1b_2-b_3b_4+d_1-b_2d_1-b_1d_2+d_1d_2+b_4d_3+b_3d_4-d_3d_4)(c_1c_2-c_3c_4-c_2d_1-c_1d_2+d_1d_2+c_4d_3+c_3d_4-d_3d_4)}$$

- Another possibility: Take subsequent residues in all 16 parameters $a_1, \ldots, d_4$
- If other than simple pole appears for any of the remaining parameters: not UT
- Problem: Huge number ($16! = 2 \times 10^{13}$) of possible orders of taking residues

Finding UT candidates

$$\frac{(d_1 d_2 - d_3 d_4)(-c_1 + c_1 c_2 - c_3 c_4 + d_1 - c_2 d_1 - c_1 d_2 + d_1 d_2 + c_4 d_3 + c_3 d_4 - d_3 d_4)}{(a_1 a_2 - a_3 a_4)(a_1 a_2 - a_3 a_4 - a_2 b_1 - a_1 b_2 + b_1 b_2 + a_4 b_3 + a_3 b_4 - b_3 b_4)(c_1 c_2 - c_3 c_4)(-c_2 + c_1 c_2 - c_3 c_4)(d_1 + d_1 d_2 - d_3 d_4)(-d_2 + d_1 d_2 - d_3 d_4)}$$

$$\times \frac{1}{(a_1 a_2 - a_3 a_4 - a_2 c_1 - a_1 c_2 + c_1 c_2 + a_4 c_3 + a_3 c_4 - c_3 c_4)(b_1 b_2 - b_3 b_4 - b_2 c_1 - b_1 c_2 + c_1 c_2 + b_4 c_3 + b_3 c_4 - c_3 c_4)}$$

$$\times \frac{1}{(-a_1 + a_1 a_2 - a_3 a_4 + d_1 - a_2 d_1 - a_1 d_2 + d_1 d_2 + a_4 d_3 + a_3 d_4 - d_3 d_4)(b_1 b_2 - b_3 b_4 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)}$$

$$\times \frac{1}{(-b_1 + b_1 b_2 - b_3 b_4 + d_1 - b_2 d_1 - b_1 d_2 + d_1 d_2 + b_4 d_3 + b_3 d_4 - d_3 d_4)(c_1 c_2 - c_3 c_4 - c_2 d_1 - c_1 d_2 + d_1 d_2 + c_4 d_3 + c_3 d_4 - d_3 d_4)}$$

- Another possibility: Take subsequent residues in all 16 parameters $a_1, \ldots, d_4$
- If other than simple pole appears for any of the remaining parameters: not UT
- Problem: Huge number ($16! = 2 \times 10^{13}$) of possible orders of taking residues
- BUT: Can use this procedure to find candidate UT integrals !!!
- Make product-ansatz for numerator

$$\left[A_1\, l_6^2 + A_2\, l_5^2 + \ldots + A_{19} q^2\right] \times \left[B_1\, l_6^2 + B_2\, l_5^2 + \ldots + B_{19} q^2\right]$$

  - Appearance of other than simple pole in scalar parameter gives constraint on A_i, B_j
  - Use this to find a set of UT candidates

Four-loop non-planar form factor in UT basis

- Make further checks on UT candidates
  - dLog form
  - Write one or two boxes as dLog,
    take all residues of remaining loops ($8! = 40320$)
  - Compute first few orders in ϵ-expansion explicitly
- Use rational IBP relations to write four-loop non-planar form factor in UT basis
  - Linear combination of only 23 integrals with rational pre-factors
  - $\{11, 7, 5\}$ integrals with $\{12, 11, 10\}$ lines
  - Topologies 31 – 34 drop out completely!
  - Most UT integrals come with full $1/\epsilon^8$ pole

Numerical integration of UT integrals

- Use two strategies: Sector decomposition and Mellin Barnes
- Sector decomposition: Mostly FIESTA, also SecDec
  - UT integrals generate considerably fewer integration terms than non-UT siblings of comparable complexity
  - Pole resolution and preparation of integrand still takes $\sim$ two weeks for each integral
  - Subsequent numerical integration takes $\mathcal{O}$(days)

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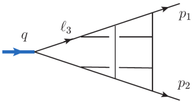
12-line integral in topology 25 with numerator

$$\left[(p_1-\ell_5)^2+2(\ell_4-\ell_5)^2+(\ell_3-\ell_4)^2-(\ell_3-\ell_5)^2-(p_1-\ell_4)^2\right]^2$$
$$-\,4\,(\ell_4-\ell_5)^2\,(p_1-\ell_3+\ell_4-\ell_5)^2$$
$$=\frac{1}{288\epsilon^8}+\frac{\zeta_2}{144\epsilon^6}+\frac{209\zeta_3}{216\epsilon^5}+\frac{623\zeta_4}{48\epsilon^4}$$
$$+\frac{109.8742\pm 0.0034}{\epsilon^3}+\frac{647.6694\pm 0.04375}{\epsilon^2}$$



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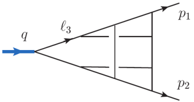
Numerical integration of UT integrals

- Mellin Barnes technique
- Challenge: Find valid MB representations for crossed four-loop topologies
- Using $\mathcal{F}$ and $\mathcal{U}$ graph polynomials of full four-loop integral always works, but generates very high-dim. MB integrands
- Loop-by-loop approach not generally applicable in crossed four-loop topologies

- Mellin Barnes technique
- Challenge: Find valid MB representations for crossed four-loop topologies
- Using $\mathcal{F}$ and $\mathcal{U}$ graph polynomials of full four-loop integral always works, but generates very high-dim. MB integrands
- Loop-by-loop approach not generally applicable in crossed four-loop topologies
- Idea: Use hybrid approach
    - Apply loop-by-loop approach to one or two loops, remaining ones with $\mathcal{F}$ and $\mathcal{U}$ polynomials
    - Examples: $\{2, 2, 1, 0\}$ loops possible before $\mathcal{F}$ and $\mathcal{U}$ method



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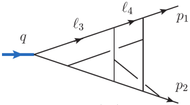

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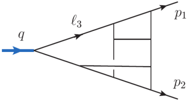

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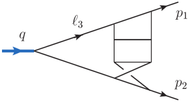

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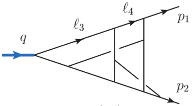
- 12-line integral in topology 30 with numerator

$$(\ell_3 - \ell_4 - p_2)^2 \left[(p_1 - \ell_4)^2 + (\ell_3 - \ell_4)^2 - (\ell_3 - p_1)^2\right]$$



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- Perform two boxes loop-by-loop, remaining two loops with $\mathcal{F}$ and $\mathcal{U}$ polynomials
- Exhibits 13-fold MB representation
- After analytic continuation to $\epsilon = 0$ and application of Barnes lemmas, up to six-fold MB integrals left in numerical evaluation with MB.m [Czakon'05]
- Result

$$\frac{1}{288\epsilon^8} - \frac{\zeta_2}{32\epsilon^6} - \frac{187\zeta_3}{864\epsilon^5} - \frac{403\zeta_4}{288\epsilon^4} - \frac{17.34721164(4)}{\epsilon^3} - \frac{133.31287(3)}{\epsilon^2}$$

- Precision in $1/\epsilon^2$ term typically 3 – 4 digits better than FIESTA

Results

- Non-planar part of four-loop Sudakov form factor in $\mathcal{N} = 4$ SYM

$$F_{S,NP}^{(4)} = 2 \times 24 \times \left[\frac{c_{-8}}{\epsilon^8} + \frac{c_{-7}}{\epsilon^7} + \ldots + \frac{c_{-2}}{\epsilon^2} + \mathcal{O}\left(\epsilon^{-1}\right) \right]$$

ϵ order i	-8	-7	-6	-5	-4	-3	-2
c_i	0	0	0	0	0		
uncertainty	0	0	0	0	0		

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ϵ order i	-8	-7	-6	-5	-4	-3	-2
c_i	0	0	0	0	0	-0.001	
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uncertainty	0	0	0	0	0	$\pm$ 0.021	$\pm$0.24

  - The naive significance of c_{-2} to deviate from zero is 6.4σ
  - If individual errors of are added linearly: $c_{-2} = 1.53 \pm 0.72$

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  - The naive significance of c_{-2} to deviate from zero is 6.4σ
  - If individual errors of are added linearly: $c_{-2} = 1.53 \pm 0.72$

- Relation to non-planar light-like cusp AD

$$\frac{1}{N_c^2} F^{(4)}_{S,NP} = -\frac{\gamma^{(4)}_{\text{cusp, NP}}}{(8\epsilon)^2} + \mathcal{O}\left(\epsilon^{-1}\right)$$

$$\gamma^{(4)}_{\text{cusp, NP}} = -\frac{2 \times 24 \times 64}{N_c^2} \times (1.53 \pm 0.24) \;\sim\; -522 \pm 82$$

- Planar contribution $\quad \gamma^{(4)}_{\text{cusp,P}} = -1752\zeta_6 - 64\zeta_3^2 \sim -1875 \quad$ is 3 – 4 times larger

Conclusion and Outlook

- Conclusion

  - We computed the non-planar Sudakov form factor to four loops in $\mathcal{N} = 4$ SYM
  - Development of an algorithm to find UT integrals in single-scale problems
  - Only 23 UT integrals contribute to non-planar part of form factor at four loops
  - First calculation of non-planar part of light-like cusp AD
  - Result shows strong (numerical) evidence for non-vanishing $\gamma^{(4)}_{\text{cusp, NP}}$
    - Violation of quadratic Casimir scaling

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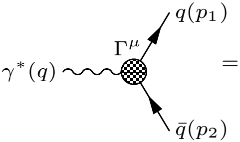
- Outlook
  - Planar form factor, expansion to $\mathcal{O}(\epsilon^{-1})$ and $\mathcal{O}(\epsilon^{0})$
  - Four-loop QCD result in progress

    [von Manteuffel,Schabinger'16]
    [Henn,Smirnov,Smirnov,Steinhauser'16]
    [Henn,Lee,Smirnov,Smirnov,Steinhauser'16]
    [Lee,Smirnov,Smirnov,Steinhauser'17]
  - Implications for factorisation theorems in QCD?
  - Five-loop calculation? $\mathcal{N} = 4$ SYM integrand known [Yang'16]

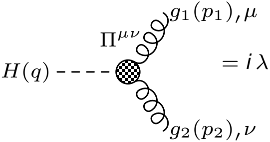
Backup slides

- Quark form factor $\mathcal{F}^q$: $\gamma^* \to q\bar{q}$, massless, on-shell quarks

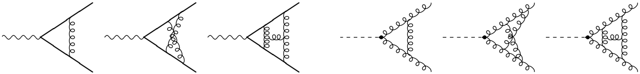


$$= -i\, e\, \bar{u}(p_1)\, \Gamma^\mu_{q\bar{q}}\, u(p_2)\,, \qquad \Gamma^\mu_{q\bar{q}} = \gamma^\mu\, \mathcal{F}^q$$

- Gluon form factor $\mathcal{F}^g$: $H \to gg$, from effective vertex $\mathcal{L}_{eff} = -\frac{\lambda}{4}\, H\, F_a^{\mu\nu}\, F^a_{\mu\nu}$



$$= i\,\lambda\, \Pi^{\mu\nu}_{gg} = i\,\lambda\, \mathcal{F}^g\, \left(g^{\mu\nu}\, p_1 \cdot p_2 - p_1^\nu\, p_2^\mu\right)$$

- Sample diagrams 

History and status of the form factors

- Two-loop form factors through $\mathcal{O}(\epsilon^0)$ known since long
  - $\mathcal{F}_2^q$ [(Gonsalves'83); Kramer,Lampe'87; Matsuura,van Neerven'88] [Matsuura,van der Maarck,van Neerven'89]
  - $\mathcal{F}_2^g$ [Harlander'00; Ravindran,Smith,van Neerven'04]
- Also extension of $\mathcal{F}_2^q$ and $\mathcal{F}_2^g$ to all orders in ϵ [Gehrmann,Maitre,TH'05]
- Three-loop form factors $\mathcal{F}_3^q$ and $\mathcal{F}_3^g$: Pole terms known through $\mathcal{O}(\epsilon^{-1})$, also the finite pieces of the fermionic corrections to $\mathcal{F}_3^q$ [Moch,Vermaseren,Vogt'05]
- $\mathcal{F}_3^q$ and $\mathcal{F}_3^g$ become available through $\mathcal{O}(\epsilon^0)$ [Baikov,Chetyrkin,Smirnov,Smirnov,Steinhauser'09] [Gehrmann,Glover,Ikizlerli,Studerus,TH'10] [von Manteuffel,Schabinger,Panzer'15]
- $\mathcal{F}_3^q$ and $\mathcal{F}_3^g$ through $\mathcal{O}(\epsilon^2)$ shortly after [Lee,Smirnov,Smirnov'10; Lee,Smirnov'10] [Gehrmann,Glover,Ikizlerli,Studerus,TH'10]
- Recently, also progress at four loops
  - n_f^3 terms of quark and gluon from factor [von Manteuffel,Schabinger'16]
  - Planar part and n_f^2-nonplanar part of quark form factor [Henn,Smirnov,Smirnov,Steinhauser'16] [Henn,Lee,Smirnov,Smirnov,Steinhauser'16] [Lee,Smirnov,Smirnov,Steinhauser'17]

- The quark and gluon form factor are the simplest quantities with IR divergences at higher orders in massless QFT
  - Prediction of the IR pole structure of QCD amplitudes
    [Magnea,Sterman'90; Catani'98; Sterman,Tejeda-Yeomans'02; Gehrmann,Gehrmann-de Ridder,Glover'04-'05]
    [Becher,Neubert'09; Gardi,Magnea'09; Dixon'09; Dixon,Gardi,Magnea'09]
  - Relation between form factors, cusp (soft) AD and quark / gluon collinear AD ($i = q, g$ and $C_q = C_F$, $C_g = C_A$)

$$
\begin{aligned}
Poles(F_1^i) &= -\frac{C_i \gamma_0^{\rm cusp}}{2\epsilon^2} + \frac{\gamma_0^i}{\epsilon} \\
Poles(F_2^i) &= \frac{3 C_i \gamma_0^{\rm cusp} \beta_0}{8\epsilon^3} + \frac{1}{\epsilon^2}\left(-\frac{\beta_0 \gamma_0^i}{2} - \frac{C_i \gamma_1^{\rm cusp}}{8}\right) + \frac{\gamma_1^i}{2\epsilon} + \frac{\left(F_1^i\right)^2}{2} \\
Poles(F_3^i) &= -\frac{11 \beta_0^2 C_i \gamma_0^{\rm cusp}}{36\epsilon^4} + \frac{1}{\epsilon^3}\left(\frac{5\beta_0 C_i \gamma_1^{\rm cusp}}{36} + \frac{\beta_0^2 \gamma_0^i}{3} + \frac{2 C_i \gamma_0^{\rm cusp} \beta_1}{9}\right) + \frac{1}{\epsilon^2}\left(-\frac{\beta_0 \gamma_1^i}{3} - \frac{C_i \gamma_2^{\rm cusp}}{18} - \frac{\beta_1 \gamma_0^i}{3}\right) + \frac{\gamma_2^i}{3\epsilon} - \frac{\left(F_1^i\right)^3}{3} + F_2^i F_1^i
\end{aligned}
$$