Elliptic Integrals and the Two-Loop QCD Matrix Elements for t-tbar Production

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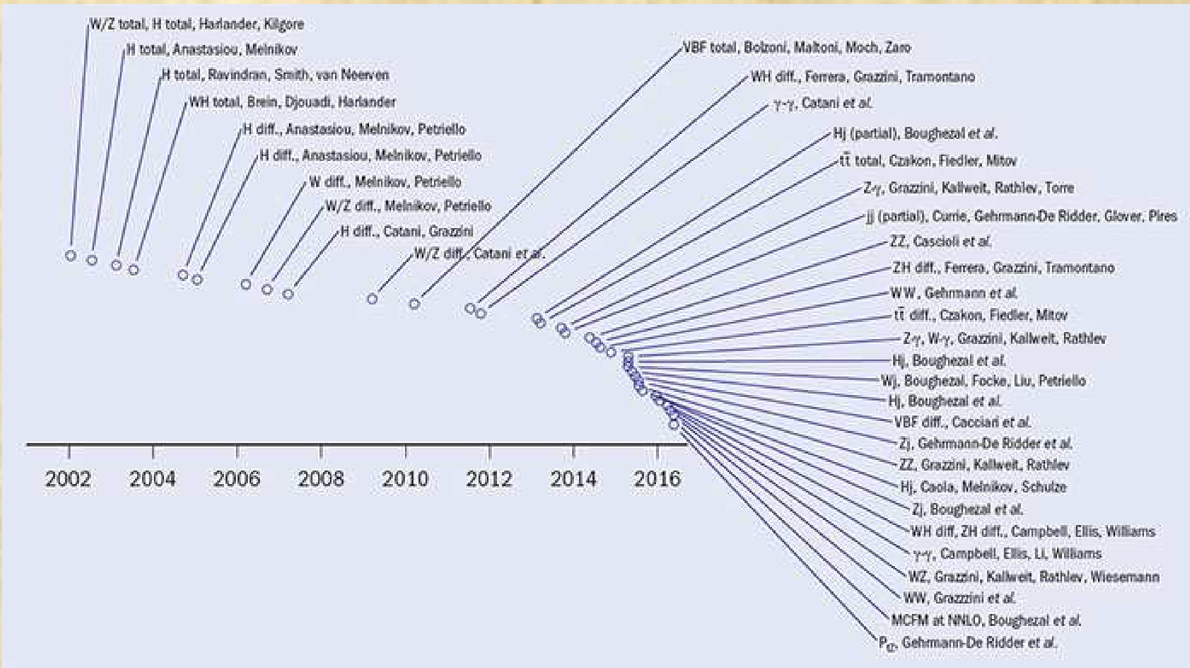
Plan of the Talk

- General Introduction
  - Feynman Integrals and Differential Equations
  - Disentangled systems and generalized polylogarithms
  - Non-Disentangled systems: Some Examples
- Top-Antitop Production at Hadron Colliders
  - Two-Loop QCD Corrections: Structure of the amplitude
  - Analytic calculation of matrix elements
  - The “Elliptic” Color Coefficients
- Conclusions and Outlooks

Collider Physics at NNLO



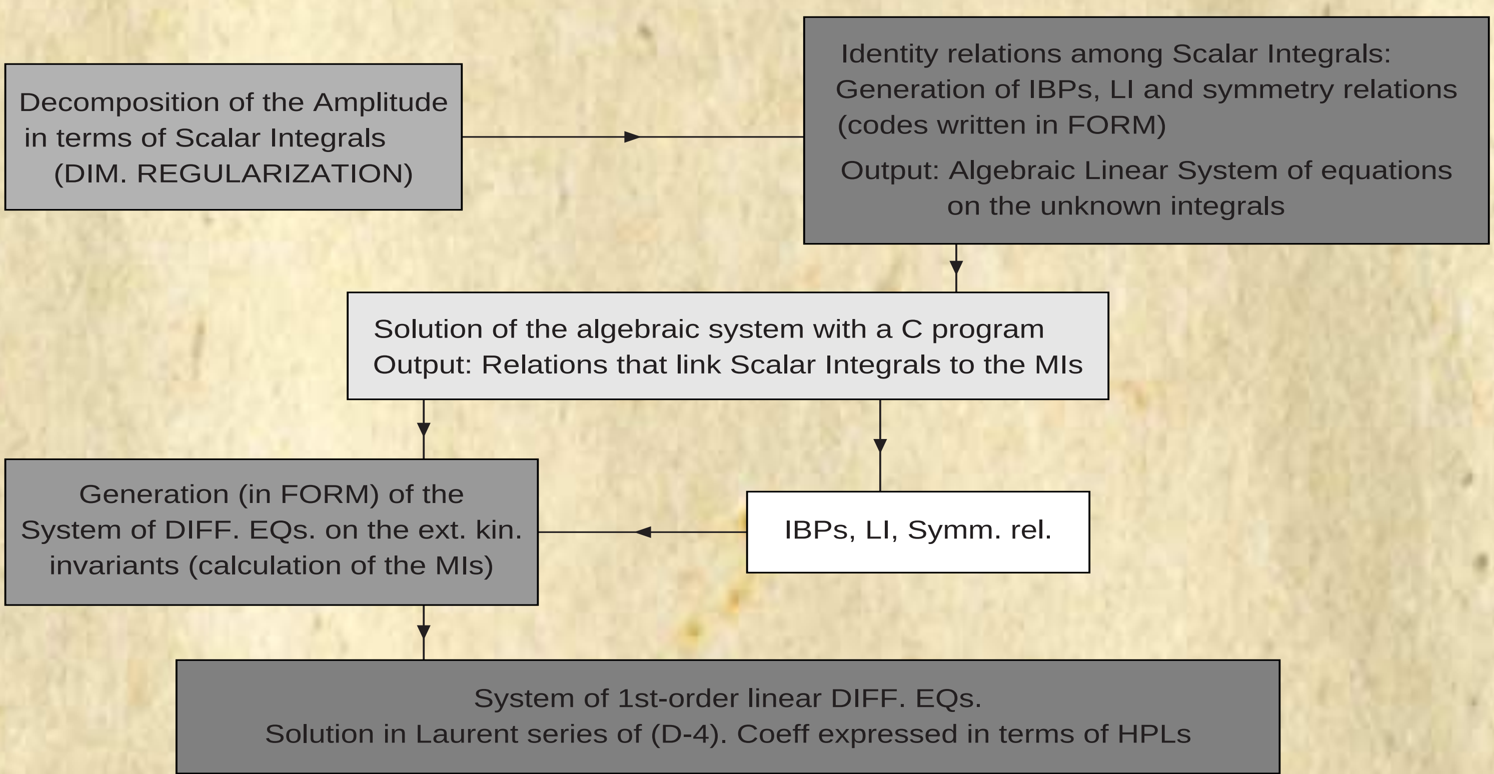
This “Explosion” was possible thanks to the progress in different areas:

- Evaluation of the matrix elements, at One and Two Loops (and more ...)
- IR structure and subtraction terms
- Pdf’s

“The Two-Loop Explosion”, G. Zanderighi, CERN Courier April 2017

Differential Equations Method

- One of the more successful techniques for the computation of multi-loop Feynman diagrams in the last years is the Differential Equations Method



V. Kotikov, *Phys. Lett.* **B254** (1991) 158; **B259** (1991) 314; **B267** (1991) 123.
E. Remiddi, *Nuovo Cim.* **110A** (1997) 1435.

Integration-by-Parts and Laporta Algorithm

One of the building blocks of the method is constituted by the REDUCTION PROCEDURE

- Using IBP's identities and Lorentz Identites the scalar integrals in terms of which our observable is expressed are "REDUCED" to a set of linearly independent ones: the MASTER INTEGRALS
- Different algorithms used for this goal

PUBLIC PROGRAMS

- AIR – Maple package (C. Anastasiou, A. Lazopoulos, JHEP **0407** (2004) 046)
- FIRE – Mathematica package (A. V. Smirnov, JHEP **0810** (2008) 107)
- REDUZE – REDUZE2 C++/GiNaC packages (C. Studerus, Comput. Phys. Commun. **181** (2010) 1293; A. von Manteuffel and C. Studerus, arXiv:1201.4330 [hep-ph].)
- LiteRed – Mathematica package (R. N. Lee arXiv:1212.2685 [hep-ph])
- Kira – C++/GiNaC (P. Maierhöfer, J. Usivitsch, P. Uwer, arXiv:1705.05610)

Differential Equations for the MIs

The Master Integrals are function of the Mandelstam invariants ($x = s/m^2, t/m^2, ...$)

$$F_i = \int d^D k_1 d^D k_2 \, \frac{S_1^{n_1} \cdots S_q^{n_q}}{D_1^{m_1} \cdots D_t^{m_t}} = F_i(x)$$

They obey systems of first-order linear differential equations in the invariants

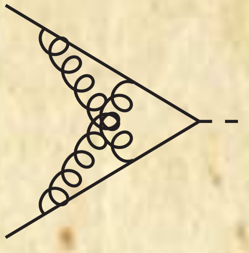
$$\frac{dF_i}{dx} = \sum_j h_j(x, D) \; F_j + \Omega_i(x, D)$$

where $i, j = 1, ..., N_{MIs}$ and $\Omega_i(x, D)$ involves subtopologies.

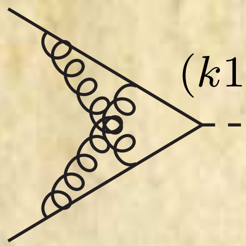
- The choice of the masters is arbitrary, but important for the solution of the system!
- We look for solutions in $(D - 4) \sim 0$ (Laurent expansion)
- The system can be solved analytically (but also numerically ...)
- Analytical solutions need a suitable functional basis, that depends on the problem

V. Kotikov, *Phys. Lett.* **B254** (1991) 158; **B259** (1991) 314; **B267** (1991) 123.
E. Remiddi, *Nuovo Cim.* **110A** (1997) 1435.

Example



$$= \left(\frac{\mu^2}{a}\right)^{2\epsilon} \sum_{i=-1}^{0} \epsilon^i R_i + \mathcal{O}(\epsilon)$$



$(k1 \cdot k2)$

$$= \left(\frac{\mu^2}{a}\right)^{2\epsilon} S_0 + \mathcal{O}(\epsilon)$$

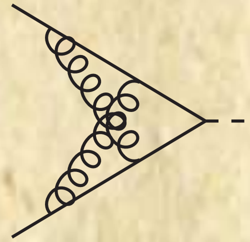
$$a^2 R_{-1} = -\frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1-x)^2} + \frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right][\zeta(3) + \zeta(2)H(0,x) + 2H(0,0,0,x)$$
$$+2H(0,1,0,x) - 2H(0,-1,0,x)]$$

$$a^2 R_0 = -\frac{1}{4}\left[\frac{1}{(1-x)} - \frac{1}{(1-x)^2} + \frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right]\left[\frac{37\zeta^2(2)}{10} + \zeta(3)(H(0,x) - 4H(-1,x) + H(1,x))\right.$$
$$-2\zeta(2)H(0,0,x) - 4\zeta(2)H(-1,0,x) - 2\zeta(2)H(0,-1,x) - 2\zeta(2)H(0,1,x) + 4\zeta(2)H(1,0,x)$$
$$+12H(0,0,0,0,x) + 8H(-1,0,-1,0,x) - 8H(-1,0,0,0,x) - 8H(-1,0,1,0,x)$$
$$+20H(0,-1,-1,0,x) - 16H(0,-1,0,0,x) - 12H(0,-1,1,0,x) - 24H(0,0,-1,0,x)$$
$$-16H(0,0,1,0,x) - 12H(0,1,-1,0,x) + 8H(0,1,0,0,x) + 4H(0,1,1,0,x)$$
$$\left.-8H(1,0,-1,0,x) + 8H(1,0,0,0,x) + 8H(1,0,1,0,x)\right]$$

$$aS_0 = \left[\frac{1}{(1+x)} - \frac{1}{(1-x)}\right]\left\{\frac{\zeta^2(2)}{10} - \zeta(3)H(0,x) + \zeta(2)(2H(1,0,x) + 3H(0,-1,x))\right.$$
$$\left.+\frac{1}{2}H(0,0,0,0,x) + H(0,-1,0,0,x) + H(0,0,-1,0,x) + H(0,1,0,0,x) + 2H(1,0,0,0,x)\right\}$$

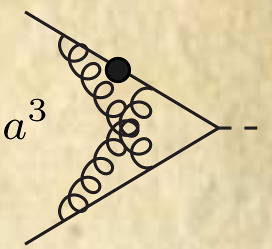
UT or non-UT

Since the choice of the Masters is arbitrary, let us analyze the following two expressions:



$=$

$$= -\frac{1}{\epsilon}\Big\{\frac{1}{4p^2(p^2+4a)}[\zeta(3)+\zeta(2)H(0,x)$$
$$+2H(0,0,0,x)+2H(0,1,0,x)-2H(0,-1,0,x)]\Big\}$$
$$-\frac{1}{4p^2(p^2+4a)}\Big[\frac{37\zeta^2(2)}{10}+\zeta(3)(H(0,x)-4H(-1,x)$$
$$+H(1,x))-2\zeta(2)H(0,0,x)-4\zeta(2)H(-1,0,x)$$
$$-2\zeta(2)H(0,-1,x)-2\zeta(2)H(0,1,x)+4\zeta(2)H(1,0,x)$$
$$+12H(0,0,0,0,x)+8H(-1,0,-1,0,x)$$
$$-8H(-1,0,0,0,x)-8H(-1,0,1,0,x)$$
$$+20H(0,-1,-1,0,x)-16H(0,-1,0,0,x)$$
$$-12H(0,-1,1,0,x)-24H(0,0,-1,0,x)$$
$$-16H(0,0,1,0,x)-12H(0,1,-1,0,x)$$
$$+8H(0,1,0,0,x)+4H(0,1,1,0,x)$$
$$-8H(1,0,-1,0,x)+8H(1,0,0,0,x)+8H(1,0,1,0,x)\Big]$$

a^3  $=$

$$= \frac{1}{\epsilon}\Big\{\frac{1}{32(1-x)^2}-\frac{1}{32(1-x)}+\frac{1}{16(x+1)^2}-\frac{1}{16(x+1)}$$
$$+\Big[\frac{1}{32(1-x)^3}-\frac{3}{64(1-x)^2}+\frac{1}{128(1-x)}+\frac{1}{128(x+1)}\Big]\zeta(2)$$
$$+\Big[\frac{1}{32(1-x)^4}-\frac{1}{16(1-x)^3}+\frac{3}{128(1-x)^2}+\frac{1}{128(1-x)}$$
$$-\frac{1}{128(x+1)^2}+\frac{1}{128(x+1)}\Big]\zeta(3)+\Big[\frac{1}{16(1-x)^3}-\frac{3}{32(1-x)^2}$$
$$+\frac{1}{64(1-x)}+\frac{1}{64(x+1)}\Big](H(0,0,x)+H(1,0,x)-H(-1,0,$$
$$+\Big[\frac{1}{16(1-x)^3}-\frac{3}{32(1-x)^2}+\frac{1}{32(1-x)}+\frac{1}{16(x+1)^3}$$
$$-\frac{3}{32(x+1)^2}+\frac{1}{32(x+1)}+\Big(\frac{1}{32(1-x)^4}-\frac{1}{16(1-x)^3}$$
$$+\frac{3}{128(1-x)^2}+\frac{1}{128(1-x)}-\frac{1}{128(x+1)^2}\ldots\Big\}+finite$$

The Canonical Form

In 2013, J. M. Henn proposed to base on this property (UT) the search for the "good" basis of Master Integrals and the solution of the system of differential equations.

The idea is based on

- Analytic solution of the full system of diff eqs, not topology-by-topology. This is made possible because of the extreme simplification of the system in terms of UT integrals.
  - A UT integral is an integral that, order-by-order in ϵ, is expressed ONLY in terms of functions of the same weight.
  - The UT integrals obey a very special system of first order linear diff eqs. If f is a vector of UT MIs, depending on the variables x_i, in $D = 4 - 2\epsilon$ dimensions, we have

$$df(\epsilon, x_i) = \epsilon \, dA(x_i) \, f(\epsilon, x_i)$$

    $\Longrightarrow$ the dimensional parameter is totally factorized and the matrix of the system depends only on the kinematics! This makes possible a strightforward solution of the system, order-by-order in ϵ
  - This strategy concerns (for the moment) GHPL-like Master Integrals, for which we can define the "weight" of the repeated integrations

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601

How to Choose The Canonical Basis?

There is at the moment no GENERAL algorithm to choose the canonical basis. However, there are some interesting partial results

- Integrals with constant leading singularities (maximal cuts) are observed to satisfy canonical differential equations

  Cachazo '08, Arkani-Hamed-Bourjaily-Cachazo-Trnka '12

- If the system has rational alphabet there is an algorithm to choose the canonical basis

  Lee '14, Meyer '15,'17

- If the system can be brought to a form in which the matrix $A(x)$ is linear in ϵ

$$\frac{\partial}{\partial x_m} f(x,\epsilon) = \left(A(x) + \epsilon B(x)\right) f(x,\epsilon)$$

  therefore the term in ϵ^0, $A(x)$, can be removed arriving at a canonical form

  Argeri et al. '14

- In some particular case the non canonical parts of the system can be removed sistematically re-defining the masters and solving block-diagonal linear differential equations

  Gehrmann-von Manteuffel-Tancredi-Weihs '14

Functional Basis for the Solutions

If the system of differential equations can be cast in canonical form (triangularized in ep), then

- when all possible square roots are removed (with changes of variables), the appropriate functional basis for the analytic solutions is the one of Multiple Polylogarithms (MPLs)

$$G(a_1, a_2, ..., a_n, x) = \int_0^x \frac{1}{t - a_1} G(a_2, ..., a_n, t) dt$$

Goncharov '98, Remiddi-Vermaseren '99,
Ablinger-Bluemlein-Schneider '13, Duhr-Gangl-Rhodes '12

  - MPLs (or GPLs) can be evaluated numerically with dedicated C++ fast and precise numerical routines

    Vollinga-Weinzierl '05

- In the case the alphabet cannot be fully linearized, we can find a solution in terms of repeated integrals that involve square roots. In particular, we can find a solution at weight 2 in terms of logarithms and Li_2 functions. The weight 3 will be an integration over known functions, while the weight 4 would involve a two-fold integration. However, integrating by parts we can make in such a way that we are left with a single one-fold integration to be done numerically.

  Henn-Caron Huot '14

Decoupling and Non-Decoupling Systems

Although the number of problems that can be solved using the idea of systems that order-by-order in ϵ exhibit a triangular matrix for the homogeneous system

$$\partial_x h(x) = \begin{pmatrix} a_{1,1} & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} h(x) + \text{non homogeneous terms}$$

is big, not all the systems follow this behaviour.

In some cases we are in the situation in which the simplification of the system cannot be better than this

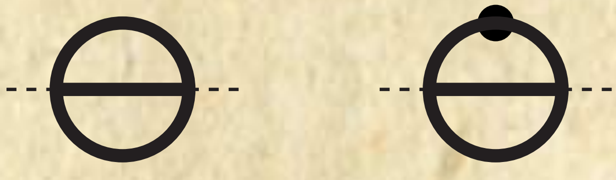
$$\partial_x h(x) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & 0 & a_{4,4} \end{pmatrix} h(x) + \text{non homogeneous terms}$$

- In this case, although two of the masters can be solved using only first order differential equations, the other two are coupled and their sub-system is equivalent to a second order differential equation

Two-Point Functions

The first case of Master Integrals that cannot be expressed in terms of generalized polylogarithms is the two-loop equal masses Sunrise

- Reducing the corresponding topology we find two MIs that obey a coupled system of first order linear differential equations in the dimensionless variable $z = p^2/m^2$



- The second-order linear diff eq for the scalar diagram in d dimensions is:

$$\frac{d^2}{dz^2}F + \frac{(3(4-d)z^2 + 10(6-d)z + 9d}{2z(z+1)(z+9)}\frac{d}{dz}F + \frac{(d-3)[(d-4)z - d - 4]}{2z(z+1)(z+9)}F = \Omega(z,d)$$

- Expanding in $(d-4)$ we find

$$F = -\frac{3}{8(d-4)^2} + \frac{(z+18)}{32(d-4)} + F_0 + ...$$

- The solution of F_0, F_1, etc ... is more easily found from the 2-dimensional solution using Tarasov's dimensional relations

Two-Point Functions

- The solutions of the homogeneous equation in $d = 2$ are given in terms of complete elliptic integral of the first kind

$$\psi_1(z) = \frac{K(m(z))}{[(z+1)^3(z+9)]^{\frac{1}{4}}} \qquad \psi_2(z) = \frac{K(1-m(z))}{[(z+1)^3(z+9)]^{\frac{1}{4}}}$$

where

$$K(m) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-mx^2)}} \qquad m = \frac{z^2+6z-3+\sqrt{(z+1)^3(z+9)}}{2\sqrt{(z+1)^3(z+9)}}$$

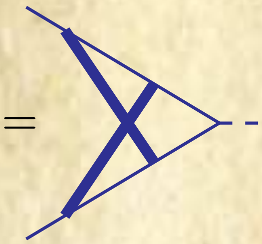
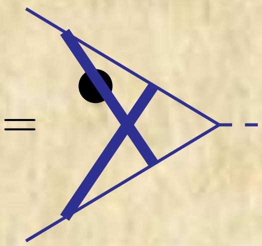
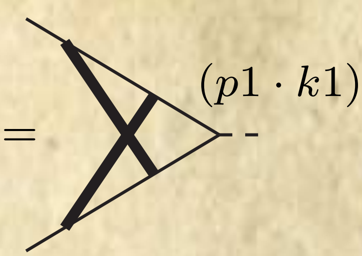
- Therefore, the particular solution is expressed via Euler's variation of constants in terms of integrals over the elliptic kernel represented by the homogeneous solutions

- Very recently proposal of expressing the solution in terms of Elliptic Polylogarithms, repeated integrations in the nome of the elliptic curve (not in the Mandelstam variable)

S. Laporta and E. Remiddi, '05
L. Adams, C. Bogner, S. Weinzierl, '12
E. Remiddi and L. Tancredi, '17

Three-Point Functions

Also three-point functions can exhibit an "elliptic behaviour". The two-massive exchange has three MIs

$F_1 =$  $F_2 =$  $F_3 =$ 

With this choice, the third one decouples from the other two. Therefore, we can write a second order differential equation for F_1 (for instance)

$$\frac{d^2 F_1}{dx^2} + \left[\frac{3}{x} + \frac{1}{x+1} + \frac{1}{x-8}\right]\frac{dF_1}{dx} + \left[\frac{1}{x^2} + \frac{9}{8x} - \frac{4}{3(x+1)} + \frac{5}{24(x-8)}\right] F_1 = \Omega(x)$$

Since the $d = 2$ homogeneous equations for the Sunrise $S(z)$ was

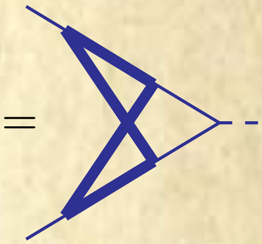
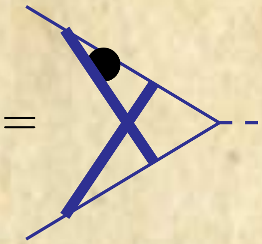
$$\frac{\partial^2}{\partial z^2} S(z) + \left[\frac{1}{z} + \frac{1}{z+1} + \frac{1}{z+9}\right]\frac{\partial}{\partial z} S(z) + \left[\frac{1}{3z} - \frac{1}{4(z+1)} - \frac{1}{12(z+9)}\right] S(z) = 0$$

it means that there is a simple relation between $S(z)$ and $F_1(z)$: $S(z) = -(z+1)\, F_1(-z-1)$

Aglietti-B.-Grassi-Remiddi, '08

Three-Point Functions

Very recently another elliptic three-point function was studied in detail

$F_1 =$  $F_2 =$ 

- The homogeneous solutions for the two masters are expressed in terms of the complete elliptic integrals of the first and second kind

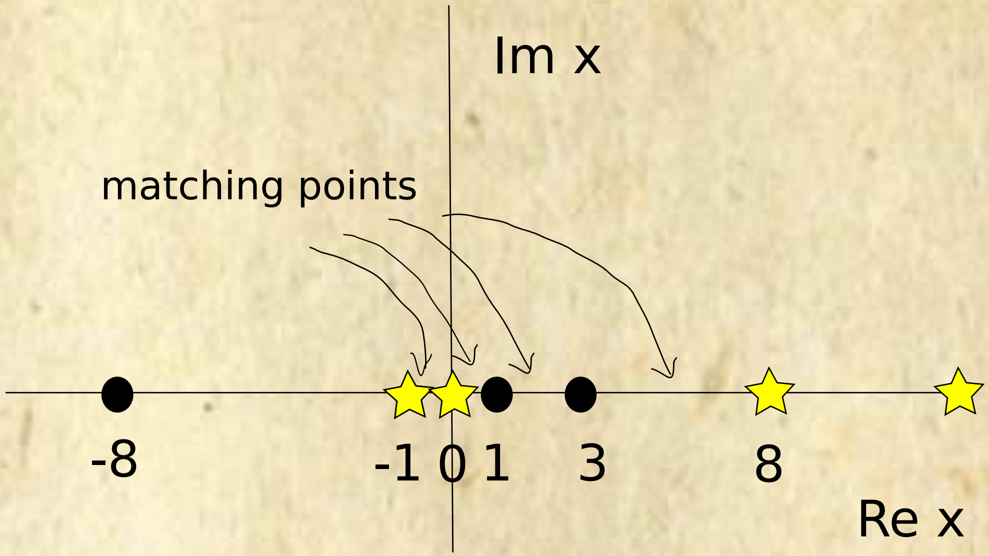
$$K(f(x)) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-fx^2)}} \qquad E(f(x)) = \int_0^1 \frac{\sqrt{1-fx^2}}{\sqrt{1-x^2}}dx$$

- The complete solution is found integrating in the different kinematic regions the non homogeneous part of the differential equation (previously expressed in terms of GPLs) over the elliptic homogeneous solutions
- Excellent numerical performance

von Manteuffel-Tancredi, '17

Semi-Numerical Evaluation

In the case of One dimensionless variable, one can adopt a Semi-Numerical evaluation of the masters, based on the differential equation

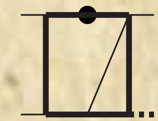
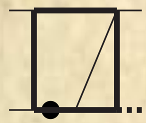
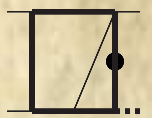


- We expand the diff eq and the solution in series of x around the singular points: $x = 0, 8, \infty, -1$. Every series depends on 2 arbitrary constants $\Rightarrow$ we impose the matching conditions expressing all of them in terms of 2 of them.
- Imposing the initial conditions we find the solution.
- In so doing, we construct a Fortran routine that gives $F_1(x)$ for every value of x with the desired precision.

S. Pozzorini and E. Remiddi, '06
U. Aglietti, R.B., L. Grassi, E. Remiddi, '08

Elliptic Box Diagram

- For instance, we have 4 coupled 6-denominator master integrals

$h = \Big\{ s^{\frac{3}{2}}\epsilon^4$  $, \epsilon^4$  $, \epsilon^3$  $, \epsilon^4$ $(k_2+p_1)^2$  $\Big\}$

for which the system becomes

$$\partial_x h(x) = C(x) h(x) + \text{non homogeneous terms}$$

where now the matrix C is not decoupled

$$C(x) = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 \\ a_{4,1} & a_{4,2} & 0 & a_{4,4} \end{pmatrix}.$$

- We have to solve a Second Order linear Differential Equations (for instance for $h_1(x)$!!!!

The Differential equations

- The second order linear differential equation is

$$\partial_x^2 h_1(x) + P(x)\partial_x h_1(x) + Q(x)h_1(x) = r(x)$$

- We rescale the three dimensionless variables with a non physical parameter (we parametrize the path of integration)

$$x = \{x_1, x_2, x_3\} \quad \rightarrow \quad x(\alpha) = \{x_1\alpha, x_2\alpha, x_3\alpha\}$$

and we find the differential equation w.r.t. α:

$$\partial_\alpha^2 h_1(\alpha) + P(\alpha, x_i)\partial_\alpha h_1(\alpha) + Q(\alpha, x_i)h_1(\alpha) = r(\alpha, x_i)$$

where the functions $P(\alpha, x_i)$ and $Q(\alpha, x_i)$ are

$$\begin{aligned} P(\alpha) &= \frac{2x_1\left(\alpha\, x_1\,(x_2 - x_3)^2 - 4\,(x_2\,(x_1 - x_3) + x_3\,(x_1 + x_3))\right)}{d_1(\alpha)} \\ Q(\alpha) &= \frac{x_1^2\,(x_2 - x_3)^2}{4d_1(\alpha)} \\ d_1(\alpha) &= x_1^2\,\alpha^2\,(x_2 - x_3)^2 - 8x_1\,\alpha\,(x_2(x_1 - x_3) + x_3(x_1 + x_3)) + 16(x_1 + x_3)^2\,. \end{aligned} \tag{1}$$

Solution of the Second Order Diff Eq

- Three singular points: the two roots of $d_1(\alpha) = 0$ and the point at infinity.

$$h_1(\alpha) = c_1\, y_1(\alpha) + c_2\, y_2(\alpha) - y_1(\alpha) \int_0^\alpha dz \frac{r(z)}{w(z)} y_2(z) + y_2(\alpha) \int_0^\alpha dz \frac{r(z)}{w(z)} y_1(z)$$

- The homogeneous solutions are

$$y_1(\alpha) = K\left(\frac{1}{2} - \frac{k(\alpha)}{2}\right) \qquad y_2(\alpha) = K\left(\frac{1}{2} + \frac{k(\alpha)}{2}\right)$$

where $K(z)$ is the complete elliptic integral of the first kind and the function $k(z)$ is

$$k(z) = \frac{(x_2 - x_3)^2\, x_1\, z - 4\,(x_2(x_1 - x_3) + x_3(x_1 + x_3))}{8\sqrt{x_1\, x_3\, x_2\,(x_1 + x_3 - x_2)}}$$

- Using the other first order differential equations we can solve the remaining MIs of the topology (deriving h_1 ...)

Structure of the Functional Basis

- The solution of the differential equation is found as a combination of repeated integrations over the elliptic kernels

$$K^{(1)}(\alpha) = K\left(\frac{1}{2} + \frac{k(\alpha)}{2}\right), \qquad K^{(-1)}(\alpha) = K\left(\frac{1}{2} - \frac{k(\alpha)}{2}\right),$$
$$E^{(1)}(\alpha) = E\left(\frac{1}{2} + \frac{k(\alpha)}{2}\right), \qquad E^{(-1)}(\alpha) = E\left(\frac{1}{2} - \frac{k(\alpha)}{2}\right).$$

as

$$h \sim \int_0^1 \mathcal{F}(\alpha)\left\{K(i)(\alpha), E^i(\alpha)\right\} d\alpha$$

where $\mathcal{F}(t)$ denotes a linear combination of pure weight-two and weight-three functions, belonging to the subtopologies

- The goal would be to find a structure behind these repeated integrations, like the algebra of the generalized polylogarithms. However for the moment many points remain unsolved
  - The notion of canonical basis in the elliptic sector is not yet defined. One can hope to find a sort of minimal complication

Cuts and Solutions of the Homogeneous Eq

- Another possible approach to the solution of the Homogeneous Diff Eq is the direct calculation of the maximal cut:
  - Simultaneously replace propagators with their δ-functions

$$\frac{1}{(p^2+m^2)} \to \delta(p^2+m^2)$$

  - If the propagator is squared, we cut it in the IBP sense (reduction to integrals with single prop and scalar prods)
  - The observation is based on the fact that if the masters under consideration obey a system

$$\partial_x M_i(\epsilon, x) = A_{ij}(\epsilon, x) M_j(\epsilon, x) + \Omega_i(\epsilon, x)$$

    then

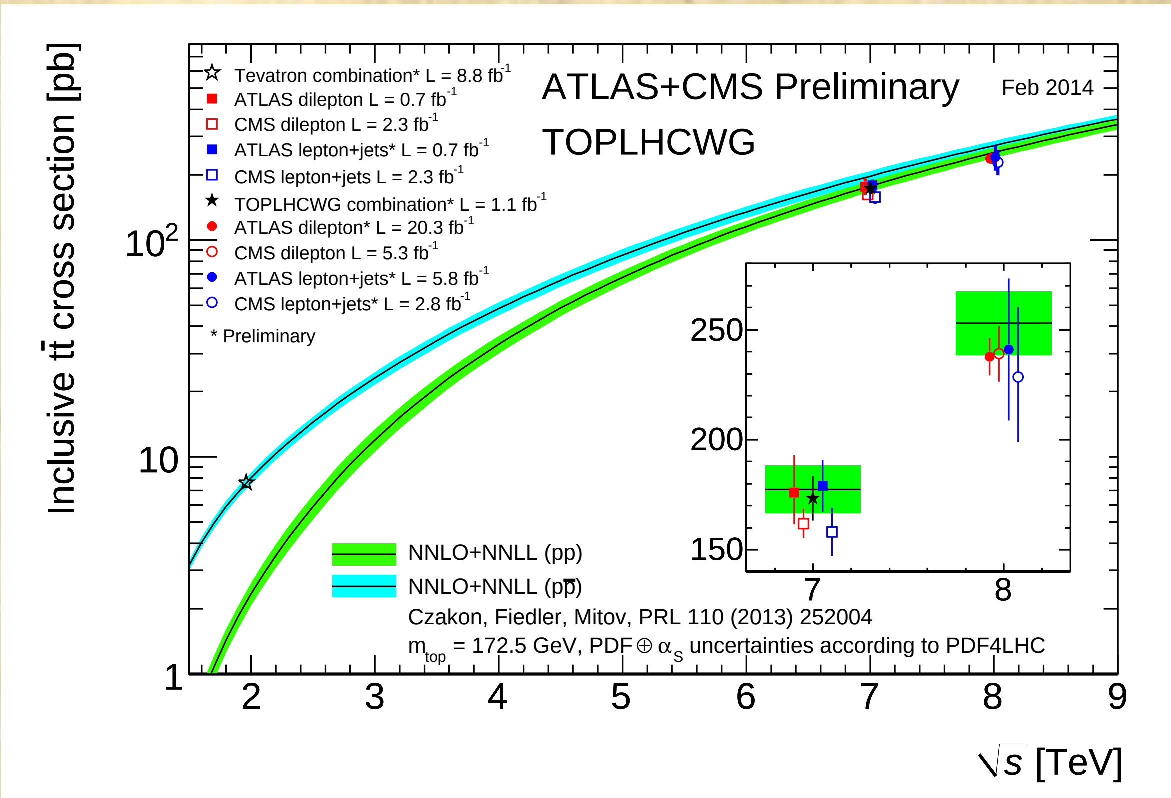
$$\partial_x Cut(M_i(\epsilon, x)) = A_{ij}(\epsilon, x) Cut(M_j(\epsilon, x))$$

    because $Cut(\Omega_i(\epsilon, x) = 0 \quad \Longrightarrow$ the MaxCut is solution of the Hom Eq
  - Itegrate directly finite MaxCut can help to solve the system of Diff Eqs

R. N. Lee and V. A. Smirnov, JHEP 12 (2012) 104.
A. Primo and L. Tancredi, NPB 916 (2017) 94.

Top Production at Hadron Colliders

$t\bar{t}$ cross section measurements in perfect agreement with theoretical predictions at NNLO+NNLL



Recently also distributions and NLO EW corrections included in the analysis (Czakon et al. '17)

P. Bärnreuther, M. Czakon and A. Mitov, Phys. Rev. Lett. **109** (2012) 132001

M. Czakon and A. Mitov, JHEP **1212** (2012) 054, JHEP **1301** (2013) 080

M. Czakon, P. Fiedler and A. Mitov, Phys. Rev. Lett. **110** (2013) 252004

Theoretical Framework: pQCD

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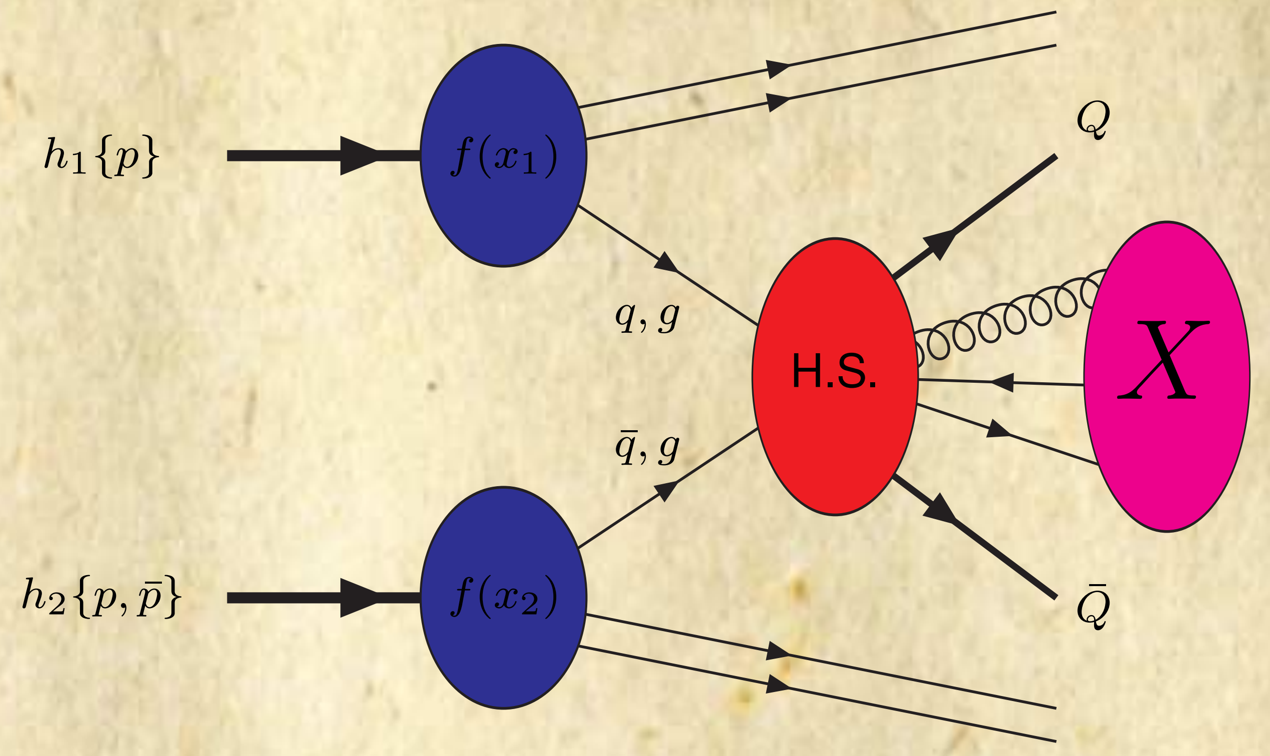
Let us consider the heavy-quark production in hadron collisions $h_1 + h_2 \to Q\bar{Q} + X$

According to the FACTORIZATION THEOREM the process can be sketched as follows:



$$\sigma_{h_1,h_2} = \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2 f_{h_1,i}(x_1, \mu_F) f_{h_2,j}(x_2, \mu_F)\ \hat{\sigma}_{ij}\left(\hat{s}, m_t, \alpha_s(\mu_R), \mu_F, \mu_R\right)$$

$$s = \left(p_{h_1} + p_{h_2}\right)^2 \ , \ \hat{s} = x_1 x_2 s$$

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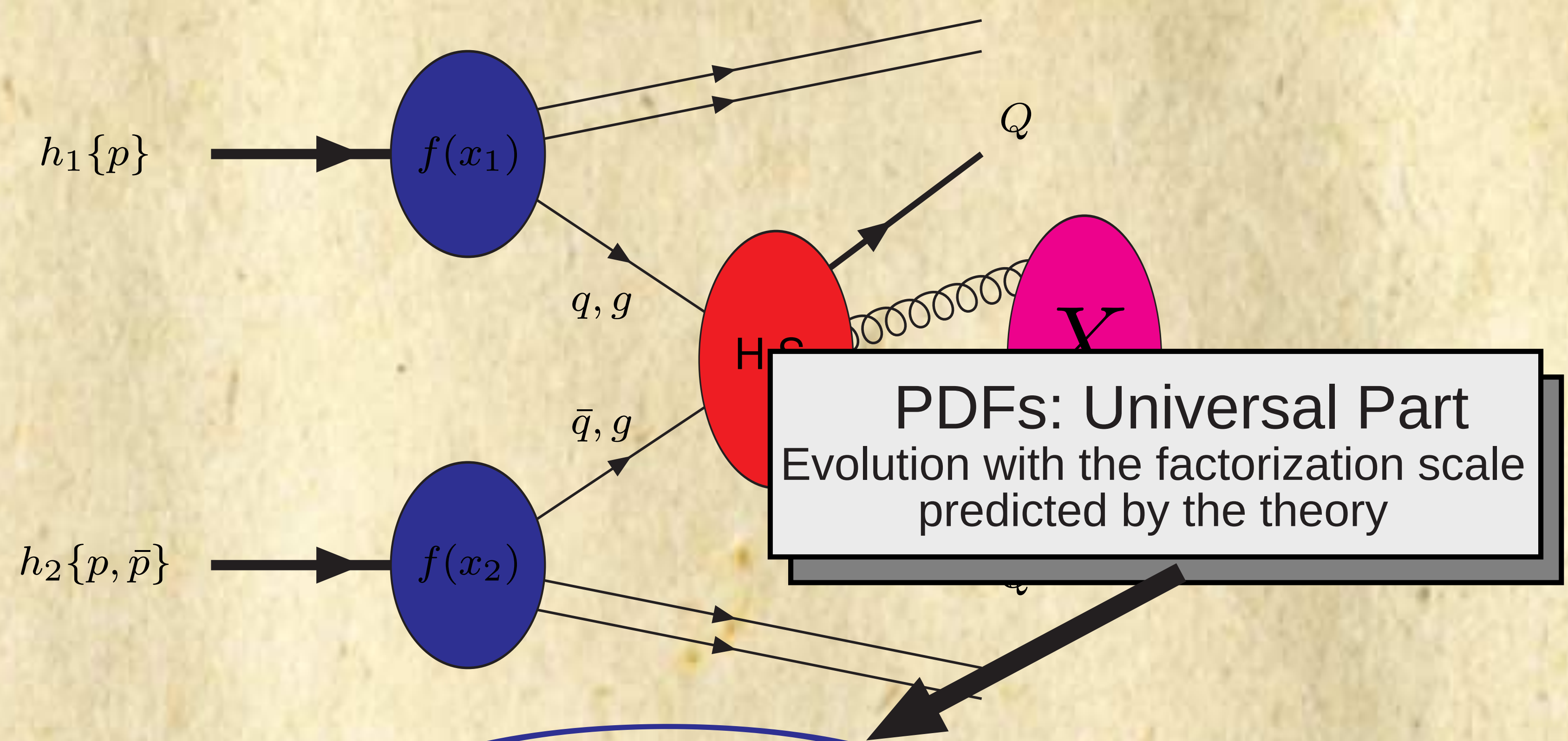

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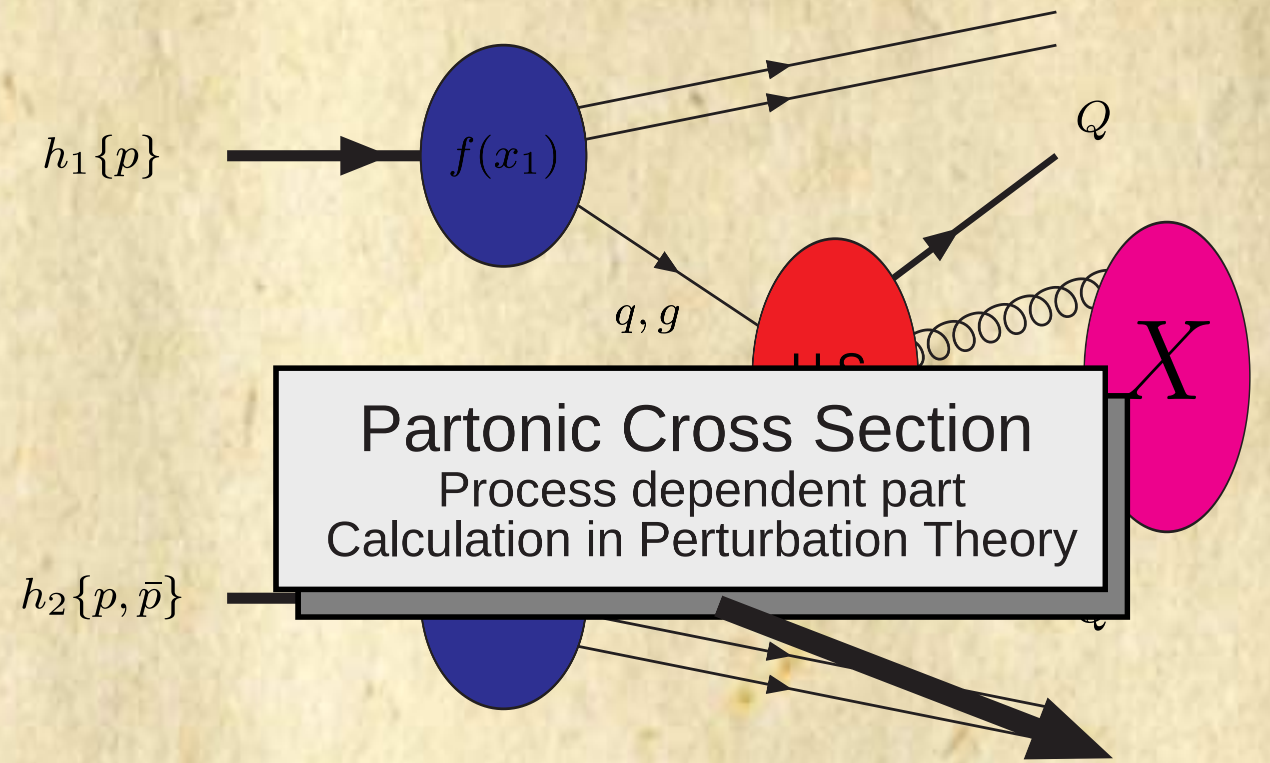

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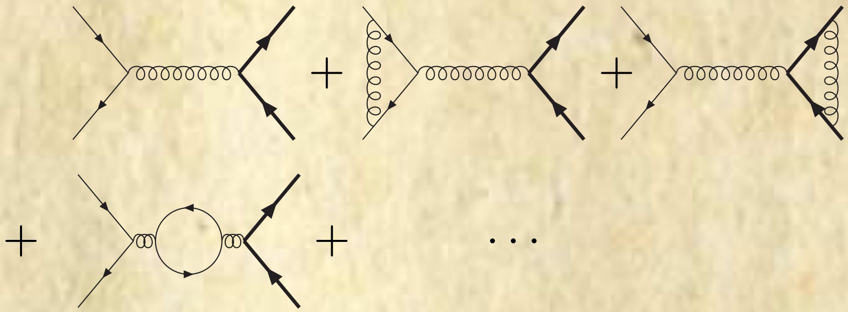
$$s = \left(p_{h_1} + p_{h_2}\right)^2 \ , \ \hat{s} = x_1 x_2 s$$

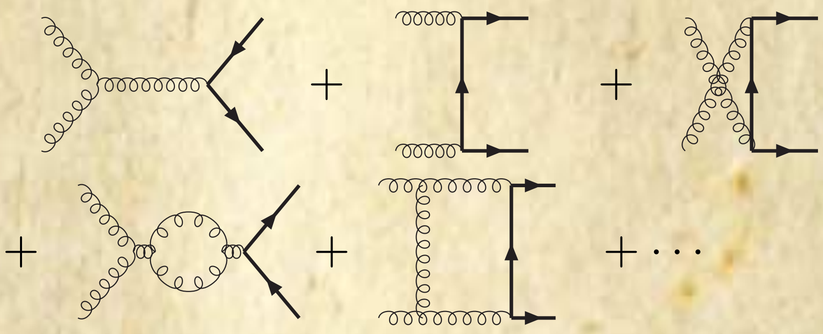
Partonic Cross Section: PT Expansion

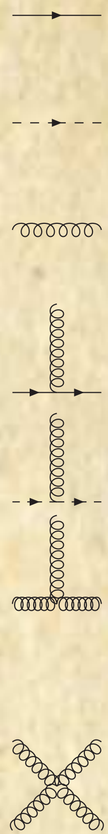
$$\hat{\sigma}_{ij}^{Q\bar{Q}} \propto \left|\mathcal{M}_{ij}^{Q\bar{Q}}\right|^2 = \left|\mathcal{M}_{ij,0}^{Q\bar{Q}} + \alpha_S\,\mathcal{M}_{ij,1}^{Q\bar{Q}} + \alpha_S^2\,\mathcal{M}_{ij,2}^{Q\bar{Q}} + \cdots\right|^2$$

$$\mathcal{M}_{q\bar{q}}^{Q\bar{Q}} \quad =$$



$$\mathcal{M}_{g\bar{g}}^{Q\bar{Q}} \quad =$$





$$\rightarrow \quad \frac{\delta_{ij}(-i\,\not{k}+m)}{k^2+m^2-i\epsilon}$$

$$\rightarrow \quad \frac{\delta_{ab}}{k^2 - i\epsilon}$$

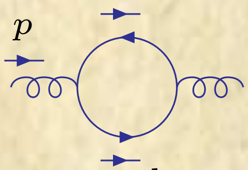
$$\rightarrow \quad \frac{\delta_{\mu\nu}\,\delta_{ab}}{k^2 - i\epsilon}$$

$$\rightarrow \quad i g_S t^a_{ij}\,\gamma^\mu$$

$$\rightarrow \quad -i g_S f^{cab} p^\mu$$

$$\rightarrow \quad i g_S f^{abc}[\delta_{\mu\nu}(p_\sigma - q_\sigma) + \delta_{\nu\sigma}(q_\mu - k_\mu) + \delta_{\mu\sigma}(k_\nu - p_\nu)]$$

$$\rightarrow \quad -g_S^2[f^{gac} f^{gbd}(2\delta_{\mu\nu}\delta_{\sigma\tau} - \delta_{\mu\sigma}\delta_{\nu\tau} - \delta_{\mu\tau}\delta_{\nu\sigma}) + \cdots$$



$$\propto \frac{\alpha_S}{\pi}\int d^4k\,\frac{tr\{t^a t^b\}\,tr\{\gamma^\mu(-i\,\not{k}+m)\gamma^\nu[i(\not{p}-\not{k})+m]\}}{(k^2+m^2)[(p-k)^2+m^2]}$$

Analytic Calculation: $t\bar{t}$ @ NNLO

The NNLO calculation of the top-quark pair hadro-production requires several ingredients:

- Virtual Corrections
  - two-loop matrix elements for $q\bar{q} \to t\bar{t}$ and $gg \to t\bar{t}$

    Czakon '08, R. B., Ferroglia, Gehrmann, Maitre, von Manteuffel, Studerus '08-'13, Ferroglia, Neubert, Pecjak, Yang '09
  - interference of one-loop diagrams

    Körner et al. '05-'08; Anastasiou and Aybat '08
- Real Corrections
  - one-loop matrix elements for the hadronic production of $t\bar{t}$ + 1 parton
  - tree-level matrix elements for the hadronic production of $t\bar{t}$ + 2 partons

    Dittmaier, Uwer and Weinzierl '07-'08, Bevilacqua, Czakon, Papadopoulos, Worek '10, Melnikov, Schulze '10
- Subtraction Terms
  - Both matrix elements known for $t\bar{t} + j$ calculation, BUT subtraction up to 1 unresolved parton, while in a complete NNLO computation of $\sigma_{t\bar{t}}$ we need subtraction terms with up to 2 unresolved partons.

    Need of an extension of the subtraction methods at the NNLO

    Gehrmann-De Ridder, Ritzmann '09, Daleo et al. '09, Boughezal et al. '10, Glover, Pires '10, Del Duca, Somogyi, Trocsanyi '13, Catani et al. '07, '15

    Double real in $\sigma_{t\bar{t}}$

    Czakon '10, Anastasiou, Herzog, Lazopoulos '10

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

$$|\mathcal{M}|^2 (s,t,m,\varepsilon) = \frac{4\pi^2\alpha_s^2}{N_c}\left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right)\mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right)\right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times 0)} \quad = \quad & N_c C_F \Big[N_c^2 A + B + \frac{C}{N_c^2} + N_l \left(N_c D_l + \frac{E_l}{N_c} \right) \\ & + N_h \left(N_c D_h + \frac{E_h}{N_c} \right) + N_l^2 F_l + N_l N_h F_{lh} + N_h^2 F_h \Big]\end{aligned}$$

218 two-loop diagrams contribute to the 10 different color coefficients

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Czakon '08.

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  Czakon '08.

- The coefficients D_i, E_i, F_i, and A are known analytically (agreement with num res)

  R. B., Ferroglia, Gehrmann, Maitre, and Studerus '08-'09

- The coefficients B and C can be calculated analytically (with the same techniques)

  R. B., Ferroglia, Gehrmann, von Manteuffel, in progress

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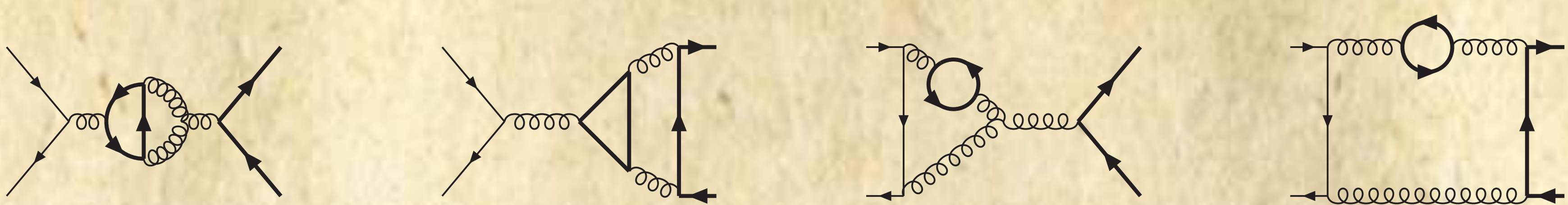
- The poles of $\mathcal{A}_2^{(2\times 0)}$ (and therefore of B and C) are known analytically

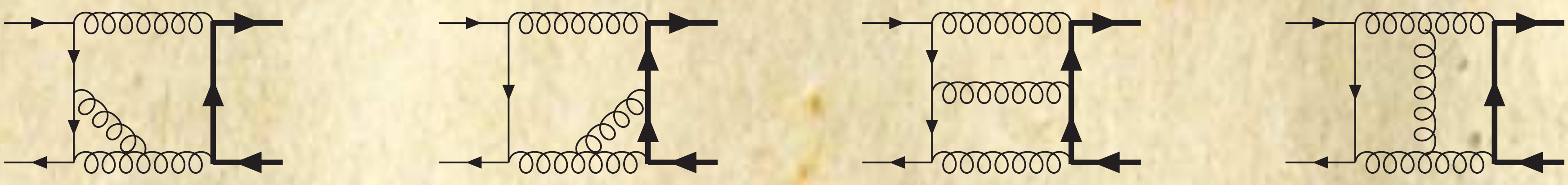
  Ferroglia, Neubert, Pecjak, and Li Yang '09

Two-Loop Corrections to $q\bar{q} \to t\bar{t}$

- D_i, E_i, F_i come from the corrections involving a closed (light or heavy) fermionic loop:



- A the leading-color coefficient, comes from the planar diagrams:



- The calculation is carried out analytically using:
  - Laporta Algorithm for the reduction of the dimensionally-regularized scalar integrals (in terms of which we express the $|\mathcal{M}|^2$) to the Master Integrals (MIs)
  - Differential Equations Method for the analytic solution of the MIs

Master Integrals for N_l and N_h

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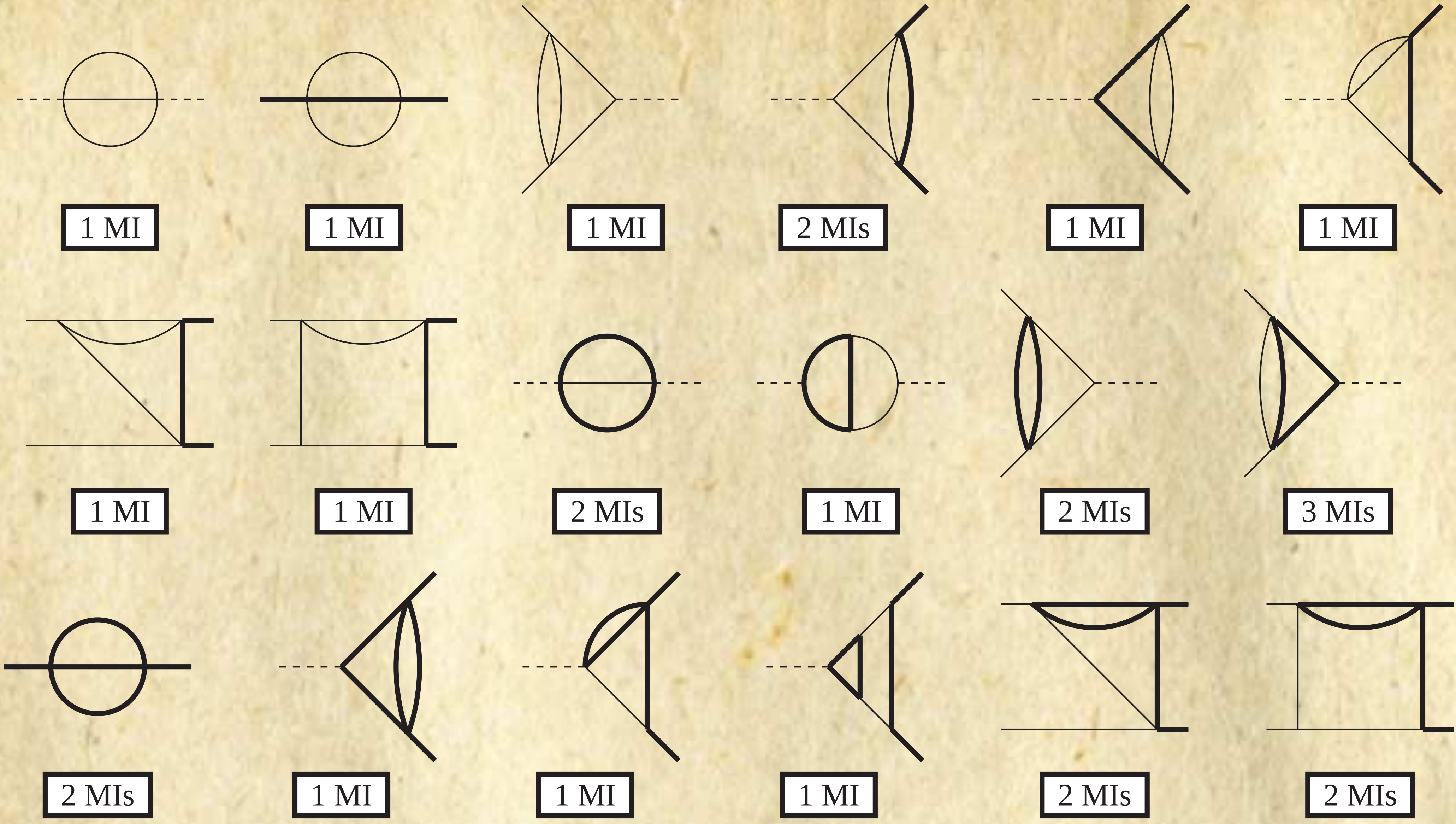

18 irreducible two-loop topologies (26 MIs)

R. B., A. Ferroglia, T. Gehrmann, D. Maitre, and C. Studerus, JHEP **0807** (2008) 129.

Master Integrals for the Leading Color Coeff

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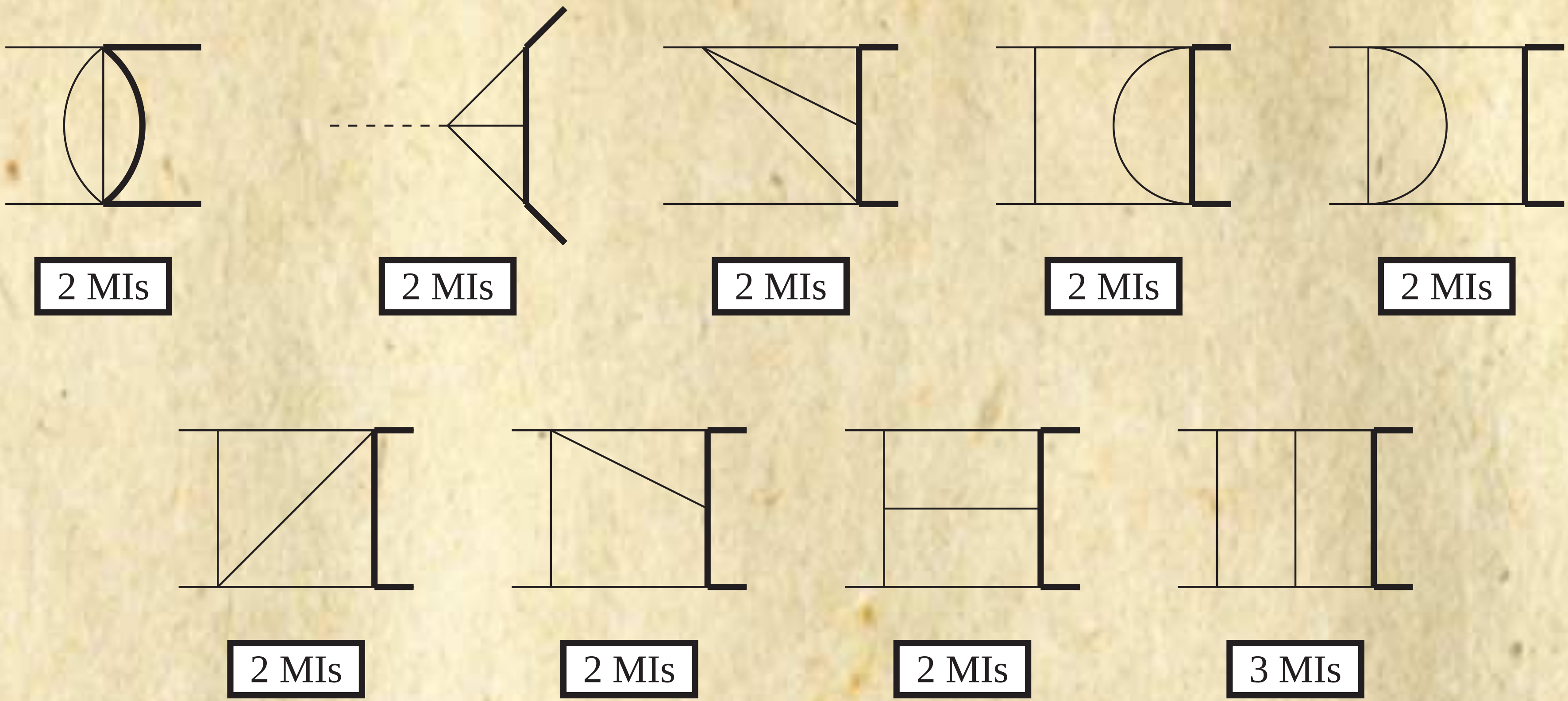

For the leading color coefficient there are 9 additional irreducible topologies (19 MIs)

R. B., A. Ferroglia, T. Gehrmann, and C. Studerus, JHEP **0908** (2009) 067.

Example: Box for the Leading Color Coeff

$$= \frac{1}{m^6} \sum_{i=-4}^{-1} A_i \epsilon^i + \mathcal{O}(\epsilon^0)$$

$$A_{-4} = \frac{x^2}{24(1-x)^4(1+y)},$$

$$A_{-3} = \frac{x^2}{96(1-x)^4(1+y)} \Big[-10G(-1;y) + 3G(0;x) - 6G(1;x) \Big],$$

$$A_{-2} = \frac{x^2}{48(1-x)^4(1+y)} \Big[-5\zeta(2) - 6G(-1;y)G(0;x) + 12G(-1;y)G(1;x) + 8G(-1,-1;y) \Big],$$

$$\begin{aligned}
A_{-1} = {} & \frac{x^2}{48(1-x)^4(1+y)} \Big[-13\zeta(3) + 38\zeta(2)G(-1;y) + 9\zeta(2)G(0;x) + 6\zeta(2)G(1;x) - 24\zeta(2)G(-1/y;x) \\
& +24G(0;x)G(-1,-1;y) - 24G(1;x)G(-1,-1;y) - 12G(-1/y;x)G(-1,-1;y) \\
& -12G(-y;x)G(-1,-1;y) - 6G(0;x)G(0,-1;y) + 6G(-1/y;x)G(0,-1;y) + 6G(-y;x)G(0,-1;y) \\
& +12G(-1;y)G(1,0;x) - 24G(-1;y)G(1,1;x) - 6G(-1;y)G(-1/y,0;x) + 12G(-1;y)G(-1/y,1;x) \\
& -6G(-1;y)G(-y,0;x) + 12G(-1;y)G(-y,1;x) + 16G(-1,-1,-1;y) - 12G(-1,0,-1;y) \\
& -12G(0,-1,-1;y) + 6G(0,0,-1;y) + 6G(1,0,0;x) - 12G(1,0,1;x) - 12G(1,1,0;x) + 24G(1,1,1;x \\
& -6G(-1/y,0,0;x) + 12G(-1/y,0,1;x) + 6G(-1/y,1,0;x) - 12G(-1/y,1,1;x) + 6G(-y,1,0;x) \\
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\end{aligned}$$

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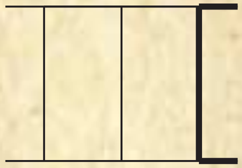

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$$+12G(-1;y)G(1,0;x) - 24G(-1;y)G(1,1;x) - 6G(-1;y)G(-1/y,0;x) + 12G(-1;y)G(-1/y,1;x)$$

$$-6G(-1;y)G(-y,0;x) \ldots 12G(-1;y)G(-y,1;x) + 16G(-1,-1,-1;y) - \ldots(-1,0,-1;y)$$

$$-12G(0,-1,-1;y) + 6G(0,0,-1;y) + 6G(1,0,0;x) - 12G(1,0,1;x) - 12G(1,1,0;x) + 24G(1,1,1;x$$

$$-6G(-1/y,0,0;x) + 12G(-1/y,0,1;x) + 6G(-1/y,1,0;x) - 12G(-1/y,1,1;x) + 6G(-y,1,0;x)$$

$$-12G(-y,1,1;x)\Big]$$

$$\rho = \frac{-4m^2 t}{\hat{s}} \to 1$$

1- and 2-dim GHPLs

GHPLs

- One- and two-dimensional Generalized Harmonic Polylogarithms (GHPLs) are defined as repeated integrations over set of basic functions. In the case at hand

$$f_w(x) = \frac{1}{x-w}\,, \quad \text{with} \quad w \in \left\{0, 1, -1, -y, -\frac{1}{y}, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}\right\}$$

$$f_w(y) = \frac{1}{y-w}\,, \quad \text{with} \quad w \in \left\{0, 1, -1, -x, -\frac{1}{x}, 1 - \frac{1}{x} - x\right\}$$

- The weight-one GHPLs are defined as

$$G(0;x) = \ln x\,, \qquad G(w;x) = \int_0^x dt f_w(t)$$

- Higher weight GHPLs are defined by iterated integrations

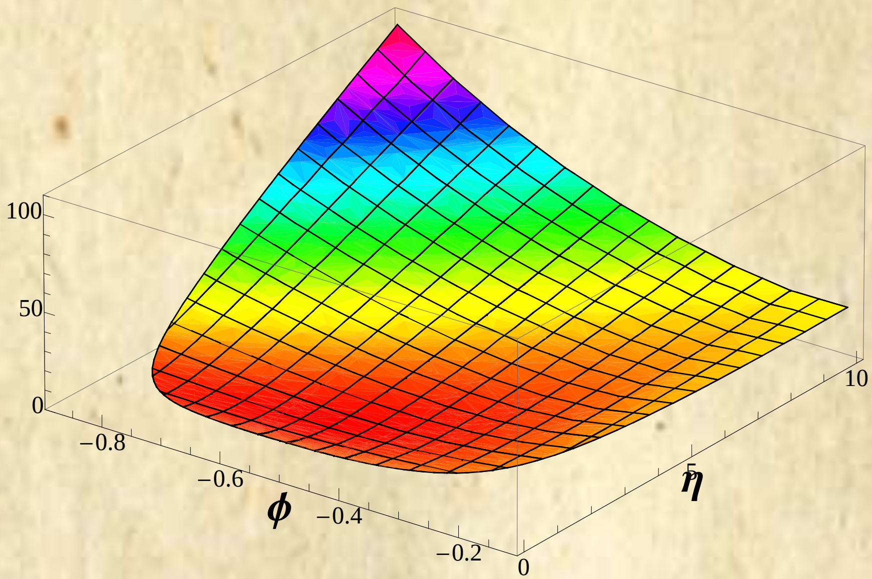
$$G(\underbrace{0,0,\cdots,0}_{n};x) = \frac{1}{n!}\ln^n x\,, \qquad G(w,\cdots;x) = \int_0^x dt f_w(t) G(\cdots;t)$$

- Shuffle algebra. Integration by parts identities

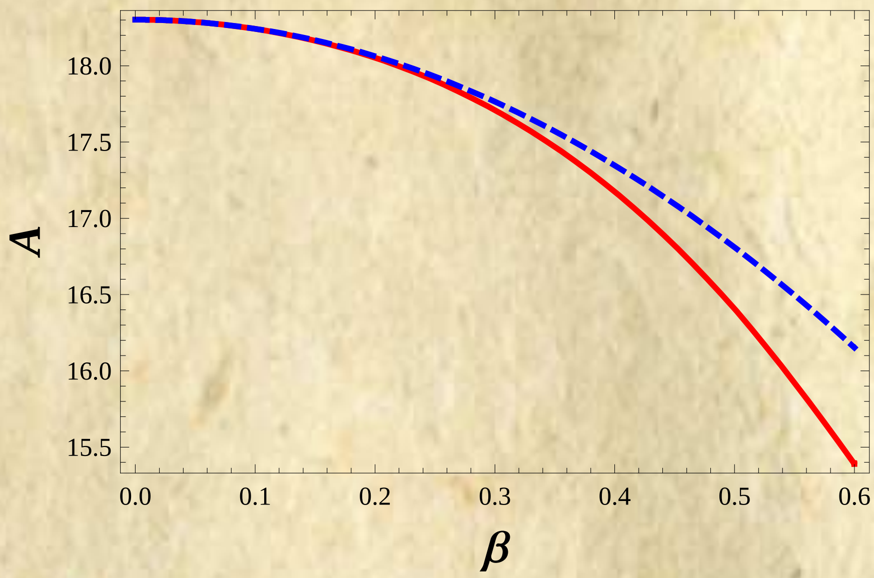
Goncharov ’98, Remiddi and Vermaseren ’99, Gehrmann and Remiddi ’01-’02, Vollinga and Weinzierl ’04

Coefficient A

Finite part of A



Threshold expansion versus exact result



$$\eta = \frac{s}{4m^2} - 1\,, \quad \phi = -\frac{t-m^2}{s}$$

$$\beta = \sqrt{1 - \frac{4m^2}{s}}$$

partonic c.m. scattering angle $= \frac{\pi}{2}$

Numerical evaluation of the GHPLs with GiNaC C++ routines (Vollinga and Weinzierl '04).

R. B., A. Ferroglia, T. Gehrmann, and C. Studerus, JHEP **0908** (2009) 067.

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$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\begin{aligned}\mathcal{A}_2^{(2\times0)} \quad = \quad & (N_c^2-1)\Big(N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h \\ & + N_l F_l + N_h F_h + \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h \\ & + N_c N_l N_h H_{lh} + \frac{N_l^2}{N_c} I_l + \frac{N_h^2}{N_c} I_h + \frac{N_l N_h}{N_c} I_{lh}\Big)\end{aligned}$$

789 two-loop diagrams contribute to 16 different color coefficients

- Numeric result for $\mathcal{A}_2^{(2\times0)}$ recently published

  P. Bärnreuther, M. Czakon and P. Fiedler, '14

- The poles of $\mathcal{A}_2^{(2\times0)}$ are known analytically

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- The leading color A, and light-quark E_l–I_l coefficients are known analytically

  R. B., Ferroglia, Gehrmann, von Manteuffel and Studerus '11, '13

Two-Loop Corrections to $gg \to t\bar{t}$

$$|\mathcal{M}|^2 (s,t,m,\varepsilon) = \frac{4\pi^2 \alpha_s^2}{N_c} \left[\mathcal{A}_0 + \left(\frac{\alpha_s}{\pi}\right) \mathcal{A}_1 + \left(\frac{\alpha_s}{\pi}\right)^2 \mathcal{A}_2 + \mathcal{O}\left(\alpha_s^3\right) \right]$$

$$\mathcal{A}_2 = \mathcal{A}_2^{(2\times 0)} + \mathcal{A}_2^{(1\times 1)}$$

$$\mathcal{A}_2^{(2\times 0)} = (N_c^2 - 1)\Big(N_c^3 A + N_c B + \frac{1}{N_c} C + \frac{1}{N_c^3} D + N_c^2 N_l E_l + N_c^2 N_h E_h$$

$$+ N_l F_l + N_h F_h + \frac{N_l}{N_c^2} G_l + \frac{N_h}{N_c^2} G_h + N_c N_l^2 H_l + N_c N_h^2 H_h$$

$$+ N_c N_l N_h H_{lh} + \frac{N_l^3}{N_c} I \dots$$

For the leading-color coefficient

NO additional MI

789 two-loop diagrams contribute to 16 different c…

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- For the light-fermion contrib

11 additional MIs

different color coefficients

ublished

P. BÃďrnreuther, M. Czakon and P. Fiedler, '14

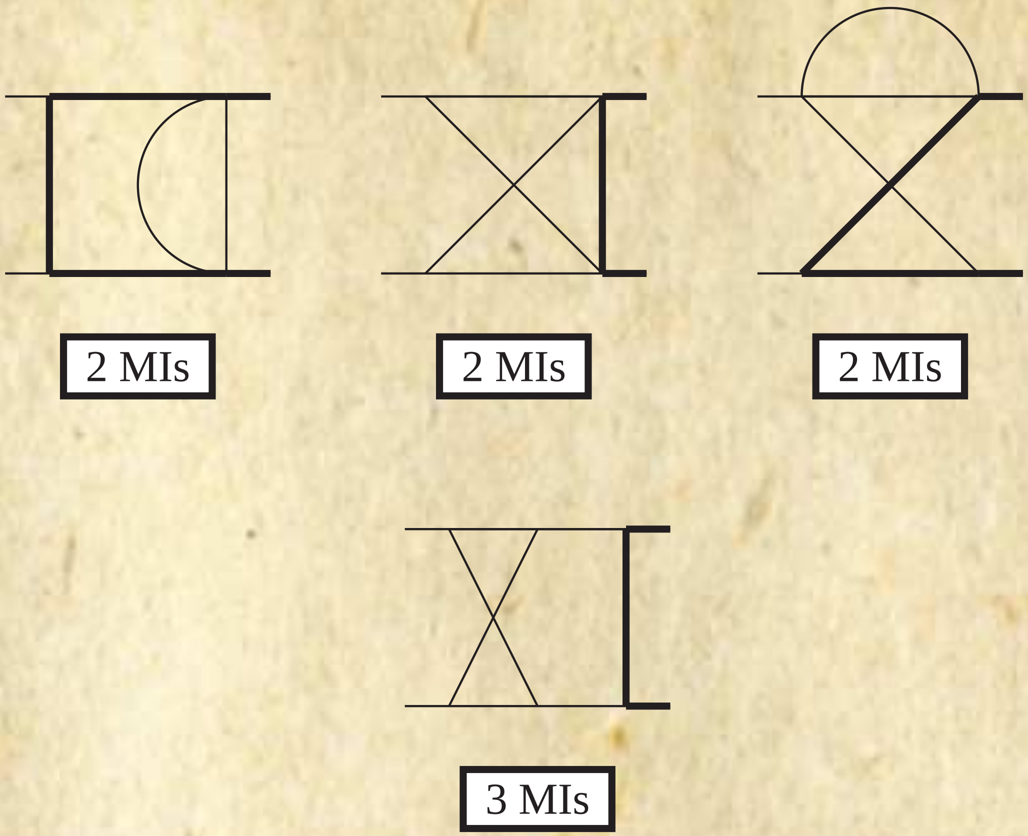
$\mathcal{A}_2$ lytically

Ferroglia, Neubert, Pecjak, and Li Yang '09

- The leading color A, and light-quark E_l–I_l coefficients are known analytically

R. B., Ferroglia, Gehrmann, von Manteuffel and Studerus '11, '13

Additional Master Integrals for the N_l Coeff

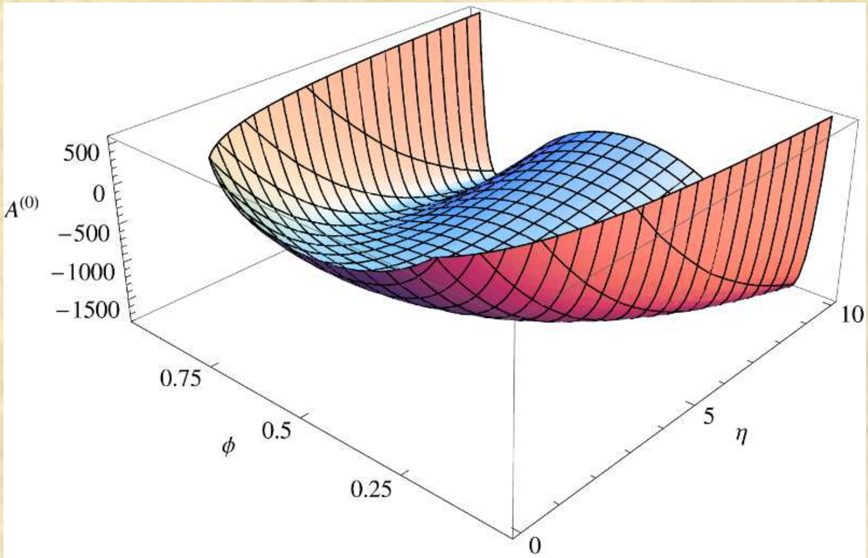


For the N_l coefficients in the gg channel there are 5 additional irreducible topologies (11 MIs)

A. von Manteuffel and C. Studerus, JHEP **1310** (2013) 037

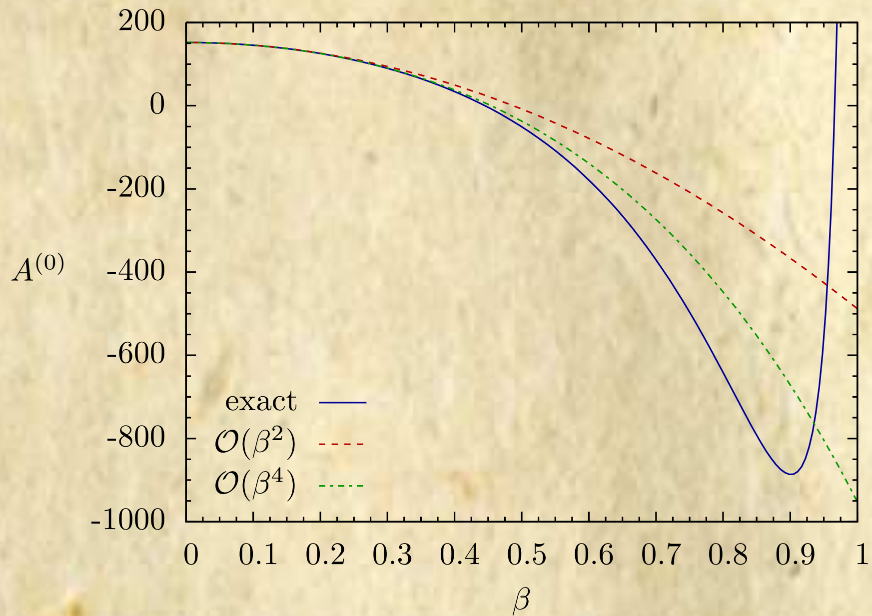
Coefficient A in gg

Finite part of A



$$\eta = \frac{s}{4m^2} - 1\,, \quad \phi = -\frac{t-m^2}{s}$$

Threshold expansion versus exact result



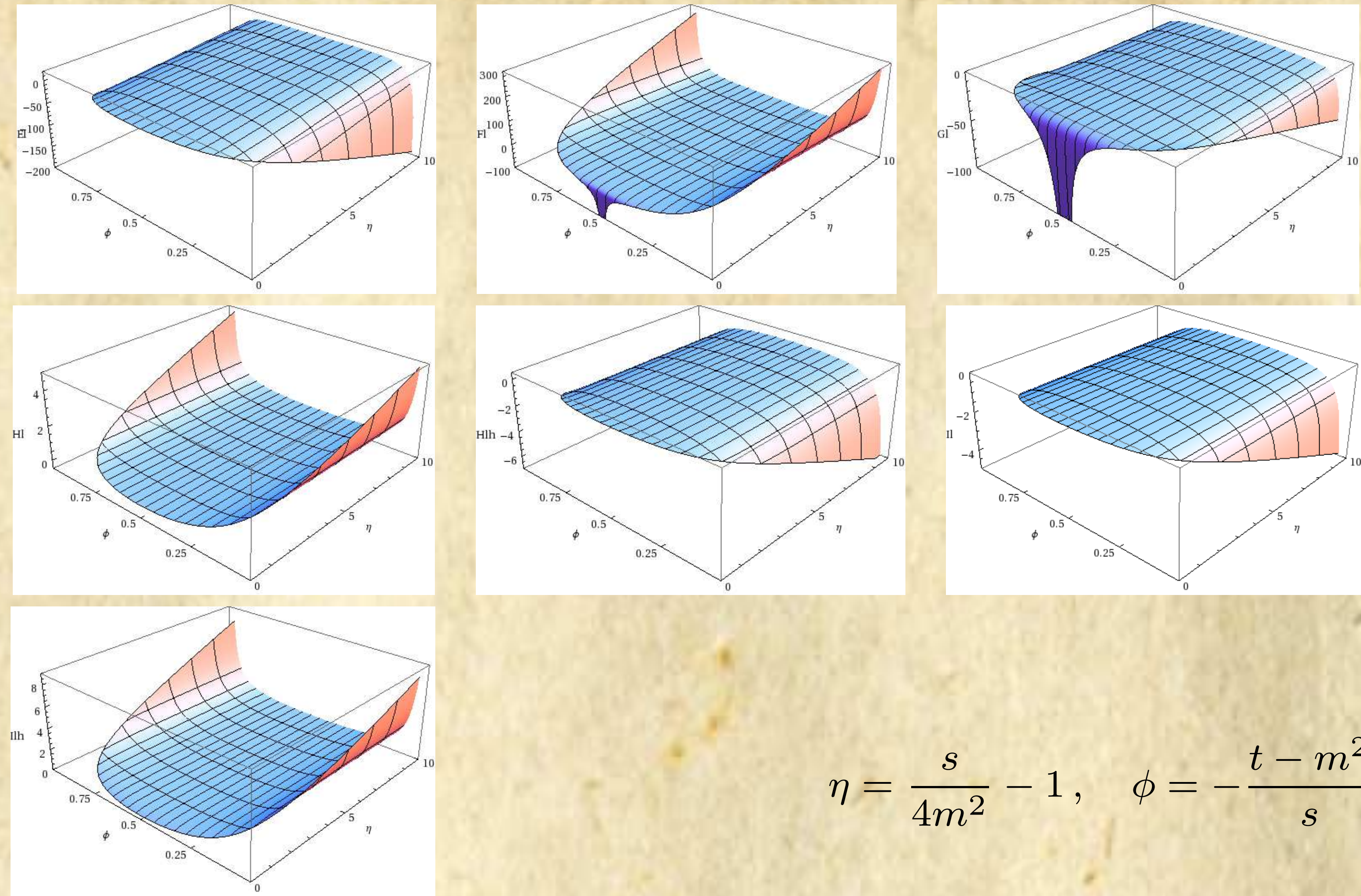
$$\beta = \sqrt{1 - \frac{4m^2}{s}}$$

partonic c.m. scattering angle $= \frac{\pi}{2}$

Numerical evaluation of the GHPLs with GiNaC C++ routines (Vollinga and Weinzierl '04).

R. B., A. Ferroglia, T. Gehrmann, A. von Manteuffel, and C. Studerus, JHEP **1101** (2011) 102

Coefficients E_l in gg



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R. B., A. Ferroglia, T. Gehrmann, A. von Manteuffel and C. Studerus, JHEP **1312** (2013) 038

Light Quark Coefficients in gg

Some considerations concerning the functional basis in which to express our analytic results are in order:

- The result can be written in terms of 289 GHPLs up to weight 4. They can be reduced to 221 using the algebra (3 MB of analytic formula)
- Alphabet in the naive case:

$$G(...;y) \quad \in \quad \left\{ -1, 0, -\frac{1}{x}, -x, -\frac{(1+x^2)}{x}, -\frac{(1-x+x^2)}{x} \right\}$$

$$G(...;x) \quad \in \quad \left\{ -1, 0, 1, [1+o^2], [1-o+o^2] \right\}$$

- NOTE: in this basis, 200 s for the numerical evaluation of a single phase space point! Hopeless! No way to use it in a Monte Carlo. What to do?

From complicated functions of simple arguments x, y → To simpler functions of complicated arguments

R. B., A. Ferroglia, T. Gehrmann, A. von Manteuffel, and C. Studerus, JHEP **1312** (2013) 038

Optimized Functional Basis

- It turns actually out that a good choice is to express the result in terms ONLY of logarithms, polylogarithms Li_n with $n = 2, 3, 4$, and a single type of multiple polylogarithms, the $\mathrm{Li}_{2,2}$:

$$\mathrm{Li}_n(x) = -G(\underbrace{0, \cdots, 0, 1}_{n}; x)\,, \qquad \mathrm{Li}_{2,2}(x_1, x_2) = G\left(0, \frac{1}{x_1}, 0, \frac{1}{x_1 x_2}; 1\right)$$

of arguments

$$\pm x,\ \pm x^2,\ -\frac{1}{y},\ -y,\ -\frac{y}{x},\ -x(x+y),\ \frac{x+y}{y},\ -\frac{x+z(x,y)}{x+y}, \cdots$$

these arguments are such that the multiple polylogarithms are real valued in the Minkowski region

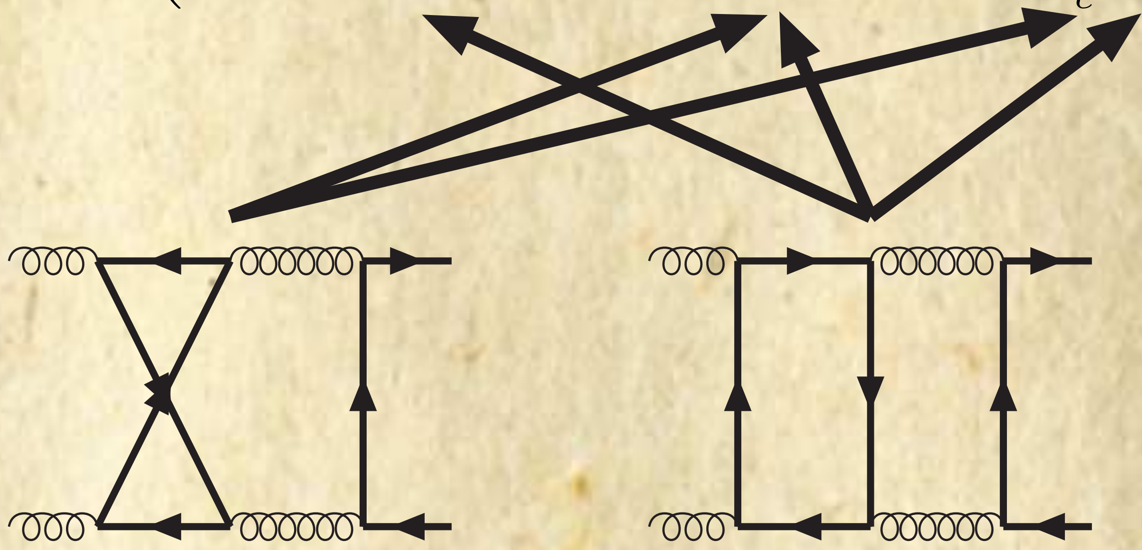
- We find again 225 multipole polylogarithms, out of which 57 $\mathrm{Li}_{2,2}$. Moreover the size of the analytic expression is always about 3 MB. However, the numerical evaluation now takes a fraction of a second!!
- Part of this transformation was done using symbols and co-products (Duhr, Gangl, Rhodes '12)

R. B., A. Ferroglia, T. Gehrmann, A. von Manteuffel, and C. Studerus, JHEP **1312** (2013) 038

Heavy-Quark Loop Coefficients

The color structure of the heavy-quark loop coefficients is the following

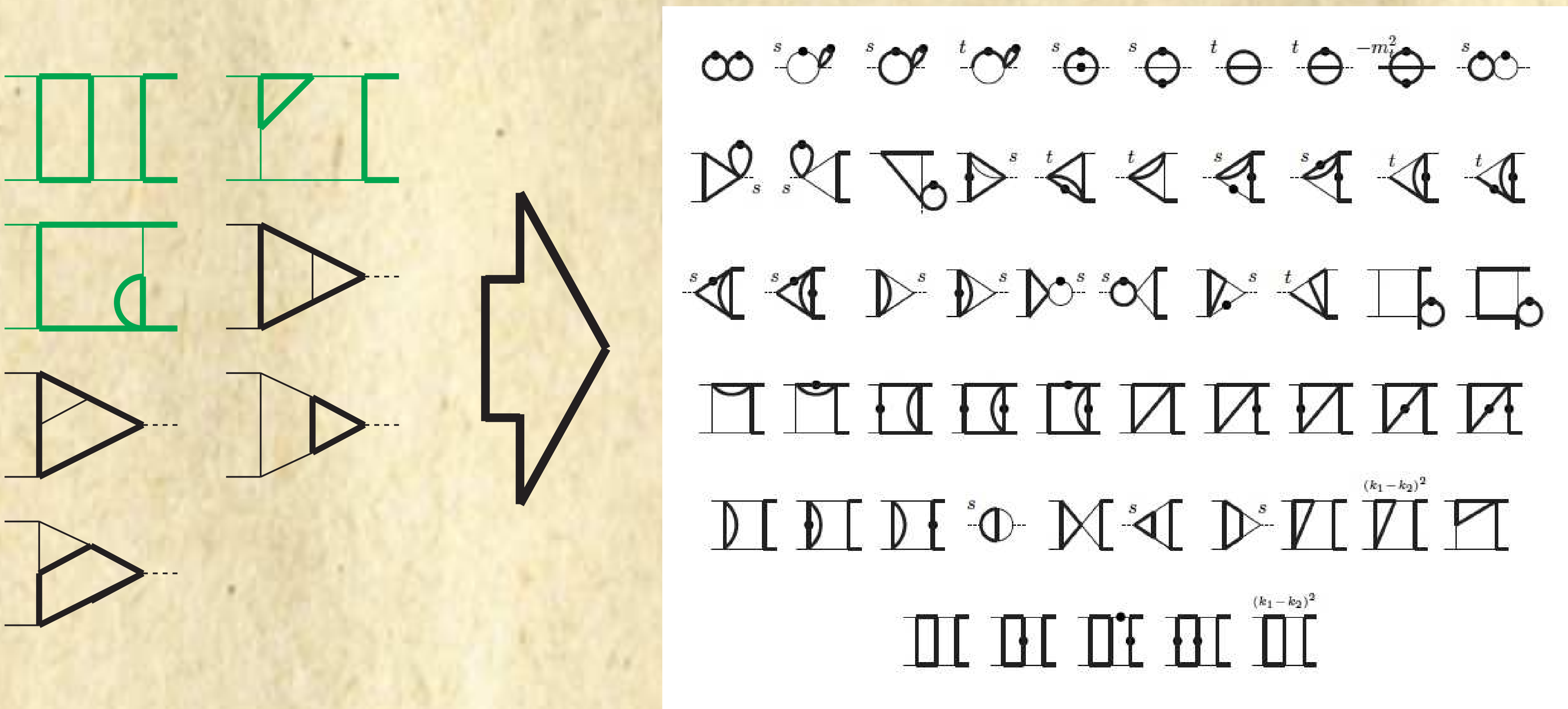
$$A_2^{2\times 0} = (N_c^2 - 1)\left(N_c^2 N_h \, E_h + N_h \, F_h + \frac{N_h}{N_c^2} \, G_h \right)$$



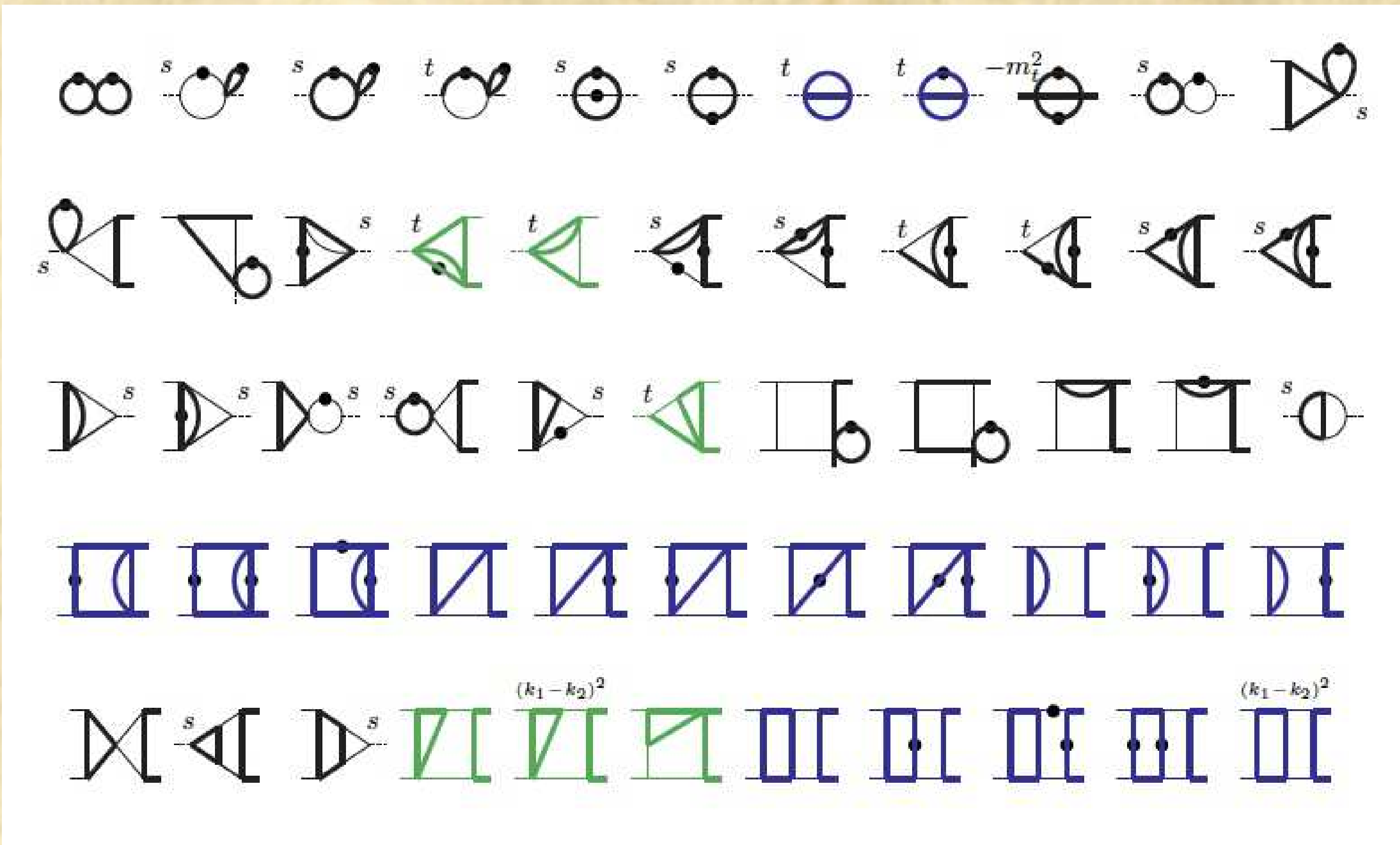
- The planar diagrams contribute to all the three color factors, while the crossed diagrams only to two of them
- $\Longrightarrow$ calculation of planar diagrams gives one gauge independent color factors out of three

Planar Corrections

- The planar Feynman diagrams can be described in terms of dim-reg scalar integrals belonging to 7 topologies: 2 at 7 denominators and 5 at 6 denominators
- The 7-denom topologies are reduced to a set of 55 Master Integrals using IBP’s
- The MIs are calculated with the Diff Eqs Method



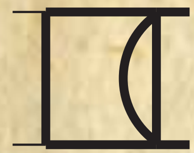
Planar Master Integrals



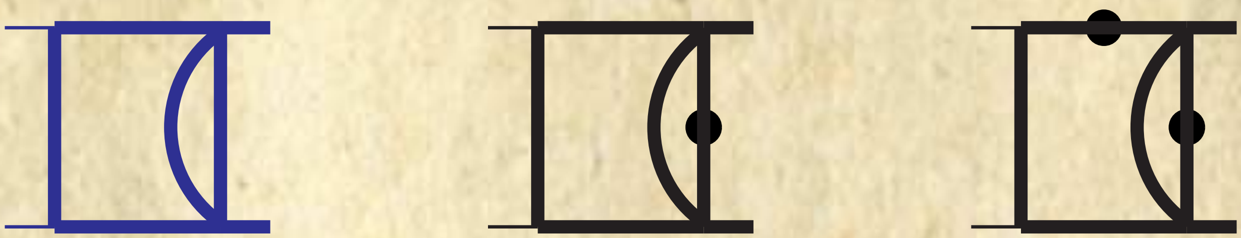
- Blue diagrams have homogeneous solutions expressed in terms of Elliptic Integrals
- Green diagrams contain non-homogeneous elliptic terms

5-Den Elliptic Box

The first unknown four-point function is the 5-denominator Elliptic Box



- The reduction procedure gives three MIs
  With the following choice we succeed to disentangle one of them:



- The system of first order differential equations becomes, at each order in epsilon, constituted by a single first order equation and two coupled equations (equivalent to a second order diff eq)
- We contruct the second order differential equation for one of the two masters (we choose the second) in s and t. We find the two independent solutions of the homogeneous equation
- We compute the Wronskian and we determinate the particular solution via Euler's variation of constants

Second order homogeneous differential equation

The equations in s and t of the master integral are ($m_t = 1$):

$$\frac{d^2}{ds^2}F + p(s,t)\frac{d}{ds}F + q(s,t)F = 0$$
$$\frac{d^2}{dt^2}F + r(s,t)\frac{d}{dt}F + u(s,t)F = 0$$

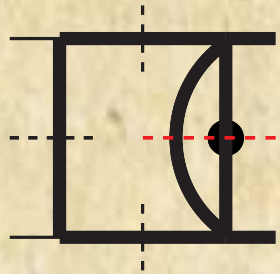
$$p(s,t) = -\frac{1}{(s-4)} - \frac{2}{s} - \frac{1}{(s - 4\frac{t-1}{t-9})} - \frac{1}{(s + \frac{(t-1)^2}{t})} + \frac{1}{(s + 4\frac{t+1}{t+3})}$$

$$q(s,t) = -\frac{1}{4s^2} - \frac{(t-9)^5}{(256(t-3)^3(4-9s-4t+st))} - \frac{(3+t)^5}{(64(-4+3s+4t+st)(-3-2t+t^2)^2)}$$
$$+\frac{(5-10t+2t^2)}{(4s(t-1)^2)} + \frac{(-25-77t-27t^2+t^3)}{(128(-4+s)(1+t)^2)}$$
$$-\frac{((t-9)^2(-1971+1944t-534t^2+48t^3+t^4))}{(256(4+s(t-9)-4t)(t-3)^3(t-1))} + \frac{(9t^2+6t^3+2t^4-6t^5+t^6)}{((t-3)^2(t-1)^2(1+t)^2(1-2t+st+t^2))}$$
$$-\frac{((3+t)^2(135+192t-10t^2-72t^3+11t^4))}{(64(t-3)^2(t-1)(1+t)^2(-4+4t+s(3+t)))}$$

and similar coefficients for the equation in t ...

Maximal Cut

We move to "PLAN B" which consists on the calculation of the $d = 4$ maximal cut (Primo and Tancredi), which is solution of the differential equation.



$$Cut(s,t) = \frac{K\left(\frac{16(t-1)(s+t-1)\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}}{4(t-1)^2\left(2\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-1\right)+s\left(t^2+8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}t-6t-8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-3\right)}\right)}{2s\sqrt{\frac{4(t-1)^2\left(2\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-1\right)+s\left(t^2+8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}t-6t-8\sqrt{\frac{s(t^2+(s-2)t+1}{(t-1)^2(s+t-1)^2}}-3\right)}{s}}}$$

The two solutions are then

$$F_{1,0} = \frac{1}{R(s,t)}K(\omega) \qquad F_{2,0} = \frac{1}{R(s,t)}K(1-\omega)$$

The decoupled Masters

In principle, once the solution of the coupled masters is found, the problem is completely solved

- We solve the second order linear diff eq for one of the coupled MIs (homogeneous solutions and particular solution as repeated integrations over the elliptic kernel)
- The solution of the other coupled MI comes just performing derivatives
- The ϵ-decoupled MIs of the same set can be calculated solving a first order linear diff eq

However, this implies an additional integration over the solution of the coupled MIs

$\Longrightarrow$ even more complicated functional structure!

- Since the set of Masters can be chosen freely, we can find different basis in which we decouple one master and solve a second order diff eq for one of the coupled.
- We found two basis constituted by (F_1, F_2, F_3) and (F_1, F_2, F_4), with F_2, F_3 and F_4 constituting a basis finite in 4 dimensions. Having solved F_2, we can get the solutions of F_3 and F_4 just by derivatives

We calculated numerically the finite part of F_1 in the Euclidean region and found agreement with FIESTA4 (5 digits)

Conclusions

- Analytic computations received a big boost in the last years. In particular the reduction to the MIs and the method of differential equations for their calculation seams to be very powerful (many calculations more and more complicated)
- The paradigm at the moment seams to be the following
  - The masters that can be expressed in terms of multiple polylogarithms satisfy a system of diff eqs in canonical form
  - Increasing the complexity of the calculations, we start to find cases in which the system does not decouple in ϵ. In these cases, higher-order differential equations (for the moment second-order) have to be solved. The basis of functions involved points in the direction of generalized hypergeometric functions (and particular subcases)
- We discussed the calculation of the planar corrections to $gg \to t\bar{t}$ that involve a closed heavy-quark loop, in perturbative QCD. We afforded the calculation of 55 MIs: 31 are expressed in term of multiple polylogarithms (or more in general repeated integrations over a limited alphabet); 24 of them involves elliptic integrals.
- For the masters involving elliptic integrals, we calculated the homogeneous solutions for the corresponding second order differential equations using the maximal cut in $d = 4$ dimensions.
- The study of the structure of the new functions just started ....