Feynman integrals and differential equations

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I.: **Precision calculations**

II.: **Differential equations**

III: **Simplifying differential equations**

Precision calculations

Due to the smallness of all coupling constants g, we may compute an observable at high energies reliable in perturbation theory,

$$\sigma = \left(\frac{g}{4\pi}\right)^4 \sigma_{LO} + \left(\frac{g}{4\pi}\right)^6 \sigma_{NLO} + \left(\frac{g}{4\pi}\right)^8 \sigma_{NNLO} + \dots$$

Cross section related to the square of the scattering amplitude: $\sigma \sim |\mathcal{A}|^2$.

Perturbative expansion of the amplitude:

$$\mathcal{A} = g^2 \mathcal{A}^{(0)} + g^4 \mathcal{A}^{(1)} + g^6 \mathcal{A}^{(2)} + \dots,$$

where $\mathcal{A}^{(l)}$ contains l loops.

Loop amplitudes

The computation of the tree amplitude $\mathcal{A}^{(0)}$ poses no conceptional problem.

For loop amplitudes we have to calculate Feynman integrals.

Let us write

$$\mathcal{A}^{(l)} = \sum_j c_j I_j,$$

c_j : coefficients, computation tree-like,

I_j : Feynman integrals

We may take the set of Feynman integrals $\{I_1, I_2, ...\}$ to consist of scalar integrals.

(Tarasov, '96, '97)

Feynman integrals

A Feynman graph with n external lines, r internal lines and l loops corresponds (up to prefactors) in D space-time dimensions to the family of Feynman integrals, indexed by the powers of the propagators ν_j

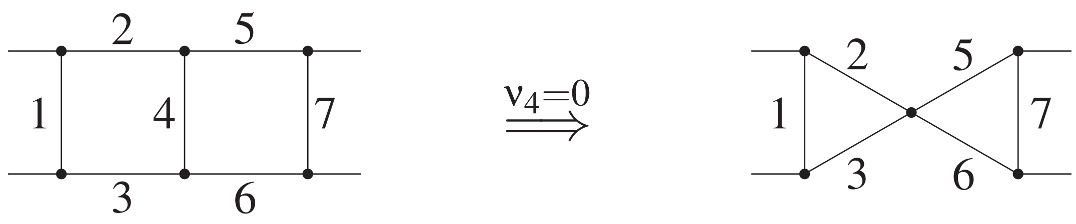
$$I_{\nu_1 \nu_2 \dots \nu_r} = \left(\mu^2\right)^{r-lD/2} \int \prod_{s=1}^{l} \frac{d^D k_s}{i\pi^{\frac{D}{2}}} \prod_{j=1}^{r} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}}$$

The momenta flowing through the internal lines can be expressed through the independent loop momenta k_1, ..., k_l and the external momenta p_1, ..., p_n as

$$q_i = \sum_{j=1}^{l} \lambda_{ij} k_j + \sum_{j=1}^{n} \sigma_{ij} p_j, \qquad \lambda_{ij}, \sigma_{ij} \in \{-1, 0, 1\}.$$

Pinching of propagators

If for some exponent we have $\nu_j = 0$, the corresponding propagator is absent and the topology simplifies:



Integration by parts

Within dimensional regularisation we have for any loop momentum k_i and $v \in \{p_1, ..., p_n, k_1, ..., k_{i-1}, k_{i+1}, ..., k_l\}$

$$\int \prod_{s=1}^{l} \frac{d^D k_s}{i\pi^{\frac{D}{2}}} \; v^\mu \frac{\partial}{\partial k_i^\mu} \; \prod_{j=1}^{r} \frac{1}{(-q_j^2 + m_j^2)^{\nu_j}} \quad = \quad 0.$$

Working out the derivatives leads to relations among integrals with different sets of indices $(\nu_1, ..., \nu_r)$.

This allows us to express most of the integrals in terms of a few master integrals.

Tkachov '81, Chetyrkin '81

Laporta's algorithm

Expressing all integrals in terms of the master integrals requires to solve a rather large linear system of equations.

This system has a block-triangular structure.

Order the integrals by complexity (more propagators $\Rightarrow$ more difficult)

Solve the system bottom-up, re-using the results for the already solved sectors.

Laporta '01

Differential equations

Let t be an external invariant (e.g. $t = (p_i + p_j)^2$) or an internal mass. Let $I_i \in \{I_1, ..., I_N\}$ be a master integral. Carrying out the derivative

$$\frac{\partial}{\partial t} I_i$$

under the integral sign and using integration-by-parts identities allows us to express the derivative as a linear combination of the master integrals.

$$\frac{\partial}{\partial t} I_i = \sum_{j=1}^{N} a_{ij} I_j$$

(Kotikov ’90, Remiddi ’97, Gehrmann and Remiddi ’99)

Differential equations

Let us formalise this:

$$\vec{I} = (I_1, ..., I_N)\,,$$ set of master integrals,

$$\vec{x} = (x_1, ..., x_n)\,,$$ set of kinematic variables the master integrals depend on.

We obtain a system of differential equations of Fuchsian type

$$d\vec{I} \quad = \quad A\vec{I},$$

where A is a matrix-valued one-form

$$A \quad = \quad \sum_{i=1}^{n} A_i dx_i.$$

The matrix-valued one-form A satisfies the integrability condition

$$dA - A \wedge A \quad = \quad 0.$$

Computation of Feynman integrals reduced to solving differential equations!

Classification

Let us distinguish four cases by two criteria:

Case A.1: Single-scale problems evaluating to (multiple) polylogarithms.
Example: Massless $2 \to 2$ scattering.

Case A.2: Multi-scale problems evaluating to multiple polylogarithms
Example: Massless $2 \to 3$ scattering.

Case B.1: Single-scale problems beyond multiple polylogarithms
Example: The massive two-loop sunrise integral.

Case B.2: Multi-scale problems beyond multiple polylogarithms
Example: $2 \to 2$ scattering with internal masses.

Multiple polylogarithms

Definition based on nested sums:

$$\text{Li}_{m_1,m_2,\dots,m_k}(x_1,x_2,\dots,x_k) = \sum_{n_1>n_2>\dots>n_k>0}^{\infty} \frac{x_1^{n_1}}{n_1^{m_1}} \cdot \frac{x_2^{n_2}}{n_2^{m_2}} \cdot \dots \cdot \frac{x_k^{n_k}}{n_k^{m_k}}$$

Definition based on iterated integrals:

$$G(z_1,\dots,z_k;y) = \int_0^y \frac{dt_1}{t_1-z_1} \int_0^{t_1} \frac{dt_2}{t_2-z_2} \dots \int_0^{t_{k-1}} \frac{dt_k}{t_k-z_k}$$

Conversion:

$$\text{Li}_{m_1,\dots,m_k}(x_1,\dots,x_k) = (-1)^k G_{m_1,\dots,m_k}\left(\frac{1}{x_1},\frac{1}{x_1x_2},\dots,\frac{1}{x_1\dots x_k};1\right)$$

Short hand notation:

$$G_{m_1,\dots,m_k}(z_1,\dots,z_k;y) = G(\underbrace{0,\dots,0}_{m_1-1},z_1,\dots,z_{k-1},\underbrace{0\dots,0}_{m_k-1},z_k;y)$$

The ε-form of the differential equation

If we change the basis of the master integrals $\vec{J} = U\vec{I}$, the differential equation becomes

$$d\vec{J} = A'\vec{J}, \qquad A' = UAU^{-1} - U\,dU^{-1}$$

Suppose one finds a transformation matrix U, such that

$$A' \quad = \quad \varepsilon \sum_j C_j \, d \ln p_j(\vec{x}),$$

where

- ε appears only as prefactor,
- C_j are matrices with constant entries,
- $p_j(\vec{x})$ are polynomials in the external variables,

then the system of differential equations is easily solved in terms of multiple polylogarithms.

Henn ’13

Example

Let us consider a simple example: One integral I in one variable x with boundary condition $I(0) = 1$. Consider the differential equation

$$dI = AI, \qquad A = \varepsilon \, d \ln(x-1).$$

Note that

$$d \ln(x-1) = \frac{dx}{x-1}$$

and

$$I(x) = 1 + \varepsilon G(1;x) + \varepsilon^2 G(1,1;x) + \varepsilon^3 G(1,1,1;x) + \ldots$$

Methods to obtain the ε-form

- Basic techniques

  Gehrmann, von Manteuffel, Tancredi, Weihs '14

- Approach based on leading singularities

  Henn '13

- Magnus expansion

  Argeri et al. '14

- Balance transformation for single-scale problems

  Lee '14

- Leinartas decomposition for multi-scale problems

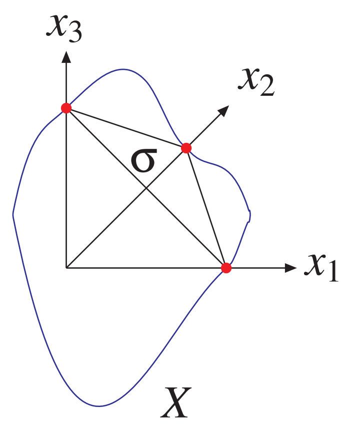
  Meyer '16

Elliptic generalisations of multiple polylogarithms

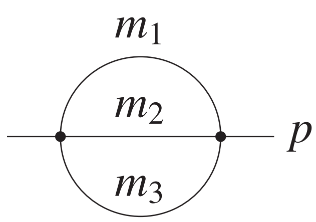
The two-loop sunrise integral with non-zero masses is the first integral, which cannot be expressed in terms of multiple polylogarithms.

In two dimensions: Sunset integral is finite.
Integrand depends only on one graph polynomial.

Graph polynomial corresponds to an elliptic curve.



$$S(t) \quad = $$



$$= \int\limits_{x_j \geq 0} d^3x \, \delta\left(1 - \sum x_j\right) \frac{1}{\mathcal{F}},$$

$$\mathcal{F} = -x_1 x_2 x_3 t + \left(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2\right)\left(x_1 x_2 + x_2 x_3 + x_3 x_1\right), \qquad t = p^2$$

The elliptic dilogarithm

Recall the definition of the classical polylogarithms:

$$\mathrm{Li}_n(x) \quad = \quad \sum_{j=1}^{\infty} \frac{x^j}{j^n}.$$

Generalisation, the two sums are coupled through the variable q:

$$\mathrm{ELi}_{n;m}(x;y;q) \quad = \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{x^j}{j^n} \frac{y^k}{k^m} q^{jk}.$$

Elliptic dilogarithm:

$$\mathrm{E}_{2;0}(x;y;q) \quad = \quad \frac{1}{i}\left[\frac{1}{2}\mathrm{Li}_2(x) - \frac{1}{2}\mathrm{Li}_2\left(x^{-1}\right) + \mathrm{ELi}_{2;0}(x;y;q) - \mathrm{ELi}_{2;0}\left(x^{-1};y^{-1};q\right)\right].$$

Various definitions of elliptic polylogarithms can be found in the literature

Beilinson ’94, Levin ’97, Wildeshaus ’97, Brown, Levin ’11, Bloch, Vanhove ’13, Adams, Bogner, S.W. ’14.

The result for $D = 2$ in terms of elliptic dilogarithms

The result for the two-loop sunset integral in two space-time dimensions with arbitrary masses:

$$S = \underbrace{\frac{4}{\left[\left(t-\mu_1^2\right)\left(t-\mu_2^2\right)\left(t-\mu_3^2\right)\left(t-\mu_4^2\right)\right]^{\frac{1}{4}}}}_{\text{algebraic prefactor}} \quad \underbrace{\frac{K(k)}{\pi}}_{\text{elliptic integral}} \quad \underbrace{\sum_{j=1}^{3} \mathrm{E}_{2;0}\left(w_j; -1; -q\right)}_{\text{elliptic dilogarithms}}$$

t	momentum squared
μ_1, μ_2, μ_3	pseudo-thresholds
μ_4	threshold
$K(k)$	complete elliptic integrals of the first kind
k, q	modulus and nome
w_1, w_2, w_3	points in the Jacobi uniformization

Adams, Bogner, S.W. ’14

The differential equation for the sunrise integral

Can find a transformation such that

$$d\vec{I} \;=\; \left(A^{(0)} + \varepsilon A^{(1)}\right)\vec{I},$$

but cannot find a transformation such that

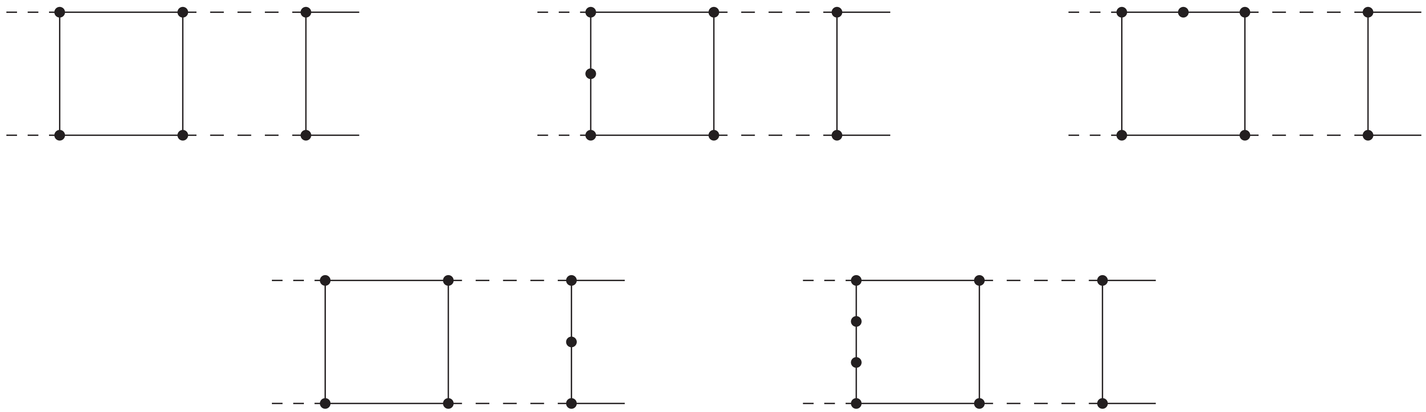
$$d\vec{I} \;=\; \varepsilon A^{(1)}\vec{I}.$$

In the equal mass case we have two master integrals in the sunrise topology.

The system remains coupled at order ε^0 $\Leftrightarrow$ Elliptic integrals have irreducible second-order differential equations.

A more complicated example

Let's look at a two-loop example from $t\bar{t}$-production. In the top topology we have 5 master integrals:



Multi-scale problem ($x_1 = s/m^2$, $x_2 = t/m^2$), contains sunrise as sub-topology.

Do we have to solve at order ε^0 a coupled system of 5 differential equations?

Reduction to a single-scale problem

Let $\alpha = [\alpha_1 : ... : \alpha_n] \in \mathbb{CP}^{n-1}$, without loss of generality take $\alpha_n = 1$.
Consider a path

$$x_i(\lambda) = \alpha_i \lambda, \quad 1 \leq i \leq n.$$

View the master integrals as functions of λ. For the derivative with respect to λ we have

$$\frac{d}{d\lambda}\vec{I} = B\vec{I}, \quad B = \sum_{i=1}^{n} \alpha_i A_i.$$

Let us write

$$B = B^{(0)} + \sum_{j>0} \varepsilon^j B^{(j)}.$$

The Picard-Fuchs operator

Consider the top sector and let us work modulo sub-topologies and ε-corrections.

Let I be one of the master integrals $\{I_1, ..., I_N\}$.

Determine the largest number r, such that the matrix which expresses I, $(d/d\lambda)I$, ..., $(d/d\lambda)^{r-1}I$ in terms of the original set $\{I_1, ..., I_N\}$ has full rank.

It follows that $(d/d\lambda)^r I$ can be written as a linear combination of $I, ..., (d/d\lambda)^{r-1}I$. This defines the Picard-Fuchs operator L_r for the master integral I with respect to λ:

$$L_r I = 0, \quad L_r = \sum_{k=1}^{r} R_k \frac{d^k}{d\lambda^k}.$$

L_r is easily found by transforming to a basis which contains $I, ..., (d/d\lambda)^{r-1}I$.

Factorisation

Suppose L_r factorises as a differential operator

$$L_r \quad = \quad L_{1,r_1} L_{2,r_2} \dots L_{s,r_s},$$

where L_{i,r_i} denotes a differential operator of order r_i.

Then we may convert the system of differential equations at order ε^0 into block triangular form with blocks of size r_1, r_2, ..., r_s. A basis for block i is given by

$$J_{i,j} \quad = \quad \frac{d^{j-1}}{d\lambda^{j-1}} L_{i+1,r_{i+1}} \dots L_{s,r_s} I, \quad 1 \leq j \leq r_i.$$

This decouples the original system into sub-systems of size r_1, r_2, ..., r_s.

Adams, Chaubey, S.W. '17

Lifting

Let us write the transformation to the new basis as

$$\vec{J} = V(\alpha_1, ..., \alpha_{n-1}, \lambda)\vec{I}.$$

Setting

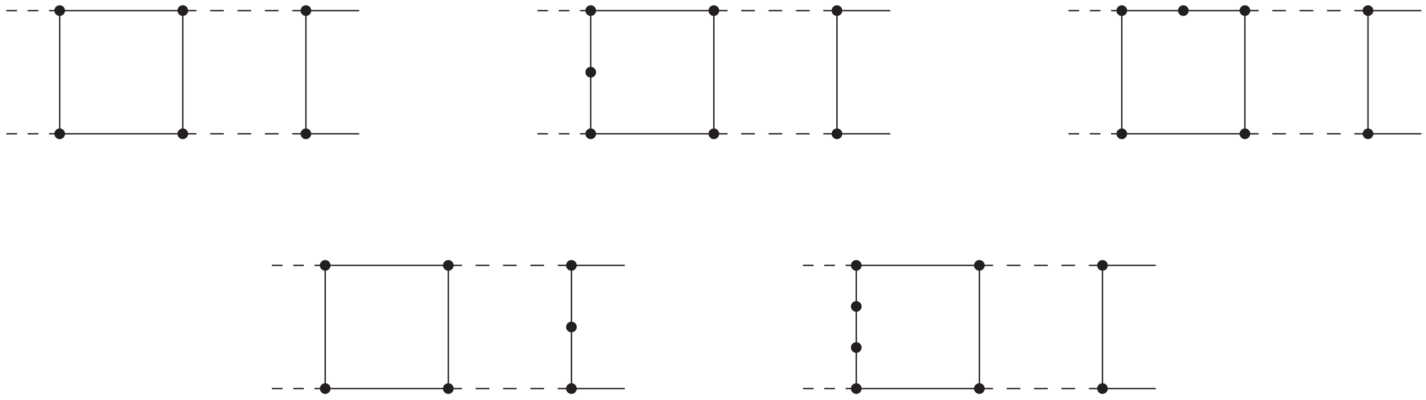
$$U = V\left(\frac{x_1}{x_n}, ..., \frac{x_{n-1}}{x_n}, x_n\right)$$

gives the transformation in terms of the original variables x_1, ..., x_n.

Remark: Terms in the original A of the form $d \ln Z(x_1, ..., x_n)$, where $Z(x_1, ..., x_n)$ is a rational function in $(x_1, ..., x_n)$ and homogeneous of degree zero in $(x_1, ..., x_n)$, map to zero in B. These terms are in many cases easily removed by a subsequent transformation.

Example

Let's return to the example of the double box integral for $t\bar{t}$-production:



Decoupling at ε^0 from the factorisation of the Picard-Fuchs operator:

$$5 \quad = \quad 1+2+1+1.$$

Need to solve only two coupled equations, not five!

Spin-off: The case of linear factors

Suppose all Picard-Fuchs operators factorise into linear factors:

$$L_r = \left(\frac{d}{d\lambda}+R_1\right)\left(\frac{d}{d\lambda}+R_2\right)\dots\left(\frac{d}{d\lambda}+R_r\right)$$

At order ε^0 we obtain first a triangular system.

Can transform to $A^{(0)} = 0$ with transformations obtained from simple integrations.

This gives an alternative algorithm to obtain the ε-form in the multi-scale case.

Example

Let us look at the following sector with two master integrals:

The Picard-Fuchs operator factorises into two linear factors and we may transform to an ε-form:

$$A = \varepsilon \left[\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} d\ln(x_1+1) - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} d\ln(x_1-1) - \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} d\ln(x_2+1) + \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} d\ln(x_1+x_2) + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} d\ln(x_1 x_2+1) \right],$$

with $s = -\frac{m^2(1-x_1)^2}{x_1}$, $t = -m^2 x_2$.

Conclusions

- Differential equations are a powerful tool to compute Feynman integrals.
- A change in the basis of master integrals is analogue to a gauge transformation.
- If a system can be transformed to an ε-form, a solution in terms of multiple polylogarithm is easily obtained.
- There are system, where at order ε^0 two coupled equations remain.
- Factorisation of the Picard-Fuchs operator allows us to find the irreducible blocks.