On The Four-Loop Form Factors Of Massless QCD

Robert M. Schabinger

with Andreas von Manteuffel and Erik Panzer, based on Phys. Lett. **B744** (2015) 101; **JHEP** 1502 (2015) 120; Phys. Rev. **D93** (2016) no.12, 125014; and arXiv:1611.00795

Trinity College Dublin

Outline

What We're Doing

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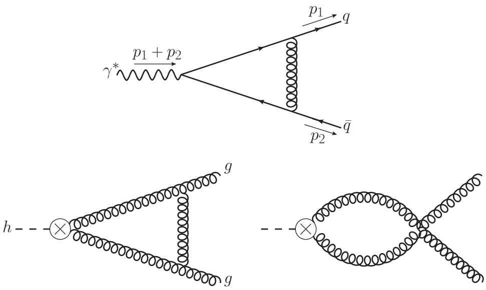

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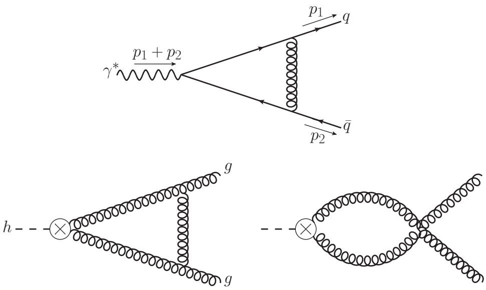

(Yes, we do use Feynman diagrams)

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L. Magnea and G. Sterman, Phys.Rev. D42 (1990) 4222

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$\Longrightarrow$ Γ_4 is the last unknown ingredient needed for N^3LL resummation!

A Dipole Formula For Gauge Theory IR Divergences?

S. Catani, Phys. Lett. **B427** (1998) 161; S. Mert Aybat *et. al.*, Phys. Rev. **D74** (2006) 074004

T. Becher and M. Neubert, **JHEP** 0906 (2009) 081; E. Gardi and L. Magnea, **JHEP** 0903 (2009) 079

The IR divergences of the simplest non-Abelian gauge theory, planar $SU(N_c)$ $\mathcal{N} = 4$ super Yang-Mills, are believed to be of the form:

$$\mathcal{A}_1^{\mathcal{N}=4}(p_1, \ldots, p_n) = \exp\left\{-\frac{1}{2}\sum_{L=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^L \mu_\epsilon^{2L\epsilon}\int_0^{\mu_\epsilon^2} \mathrm{d}\mu^2 \left(\mu^2\right)^{-1-L\epsilon}\right.$$

$$\left.\sum_{\substack{\mathrm{i,j}=1\\ \mathrm{i<j}}}^{n}\left(\Gamma_{1;L}^{\mathcal{N}=4}\ln\left(\frac{\mu^2}{-s_{ij}}\right) + \mathcal{G}_{1;L}^{\mathcal{N}=4}\right)\frac{\mathbf{T}_\mathrm{i}\cdot\mathbf{T}_\mathrm{j}}{N_c}\right\}\sum_{L=0}^{\infty}\mathbf{H}_{1;L}^{\mathcal{N}=4}(\epsilon; p_1, \ldots, p_n)$$

At four points, this structure has been realized explicitly at strong coupling (L. F. Alday and J. Maldacena, **JHEP** 0706 (2007) 064). In principle, the above structure could hold for more general gauge theories like QCD.

When Something Sounds Too Good To Be True...

Although some three-loop evidence was collected by Dixon (Phys. Rev. **D79** (2009) 091501) for the n_f terms, it is now clear that the dipole conjecture fails for QCD due to a Regge limit four-loop calculation and an eikonal three-loop calculation which probe the structure of the soft anomalous dimension matrix.

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In fact, Casimir scaling for the light-like cusp anomalous dimension

$$\Gamma_L^g \stackrel{?}{=} C_A/C_F \Gamma_L^q$$

is still very much an open problem at four loops.

R. Boels *et. al.*, **JHEP** 1302 (2013) 063; Nucl. Phys. **B902** (2016) 387;

A. Grozin *et. al.*, **JHEP** 1601 (2016) 140; J. Henn *et. al.*, JHEP 1605 (2016) 066

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See also: B. Ruijl *et. al.*, **PoS** LL2016 (2016) 071 for a splitting-function approach.

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- Use the reductions to write the raw integrand as a linear combination of scalar master integrals.
- Construct an alternative basis of finite integrals and rewrite everything in terms of it using auxiliary reductions.
- Evaluate all finite master integrals either analytically using `HyperInt` (Erik Panzer's program) or numerically using `FIESTA 4`.
  A. Smirnov, Comput. Phys. Commun. **204** (2016) 189;
  T. Hahn, Comput. Phys. Commun. **168** (2005) 78

Integration By Parts Reduction

F. Tkachov, Phys. Lett. **B100** (1981) 65; K. Chetyrkin and F. Tkachov, Nucl. Phys. **B192** (1981) 159

It is well-known that one can generate recurrence relations by considering families of Feynman integrals and then integrating by parts in d spacetime dimensions, *e.g.*

$$\begin{aligned} 0 &= \int \frac{d^d q}{(2\pi)^d} \frac{\partial}{\partial q_\mu} \left(\frac{q_\mu}{(q^2 - m^2)^a} \right) \\ &= \int \frac{d^d q}{(2\pi)^d} \left(\frac{d}{(q^2 - m^2)^a} - \frac{2aq^2}{(q^2 - m^2)^{a+1}} \right) \\ &= (d - 2a) I(a) - 2am^2 I(a+1) \end{aligned}$$

In this case, the recurrence relation can be directly solved. Usually, one employs Laporta's algorithm (S. Laporta, Int. J. Mod. Phys. **A15** (2000) 5087) to reduce some particular integrals to masters using linear algebra.

What is a finite field?

Just think of the ring of integers restricted modulo some prime number, *e.g.* $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$.

The ring becomes a field because multiplicative inverses now exist for all elements except zero, *e.g.* $2^{-1} = 2$, since $2^2 \equiv 1 \bmod 3$.

To do computations, one can take $\{0, 1, 2\}$ as the elements of the finite field $\mathbb{Z}_3$.

It is worth noting the reason why we restrict modulo a prime number. This is easily seen if one considers $\mathbb{Z}_6$ where $2 \times 3 = 0$.
$3 = 2^{-1} \times 2 \times 3 = 2^{-1} \times 0 = 0 \Longrightarrow 2^{-1}$ does not exist.

The Complexity of Gaussian Elimination

- It is widely believed that the computational complexity of Gaussian elimination is $\mathcal{O}\left(n^3\right)$ for $n \times n$ rational matrices.
- This is far too simplistic and is true only if each arithmetic operation takes essentially the same amount of time.
- In a finite field, the "grade school" algorithm does have $\mathcal{O}\left(n^3\right)$ complexity but, even over the rational numbers, the situation is much worse because the numerators and denominators of the rational numbers typically increase in size after every operation.
- Intermediate expression swell is a severe problem for the grade school Gaussian elimination algorithm and can lead to run-times and run-time storage requirements which are exponential in n!

X. G. Fang and G. Havas, ISSAC '97, 28, (1997)

And It Gets Worse...

- The linear systems that one obtains from Laporta's algorithm will always have *polynomial* entries at the outset and this introduces additional complications.
- Avoiding unrecognized zeros during the course of the elimination procedure requires a very large number of polynomial greatest common divisor (GCD) computations.
- These operations actually account for a substantial fraction of the total run-time of most currently available integration by parts reduction codes.
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Luckily for us, an enormous amount of mathematical research has been devoted to ameliorating these problems!

The General Idea

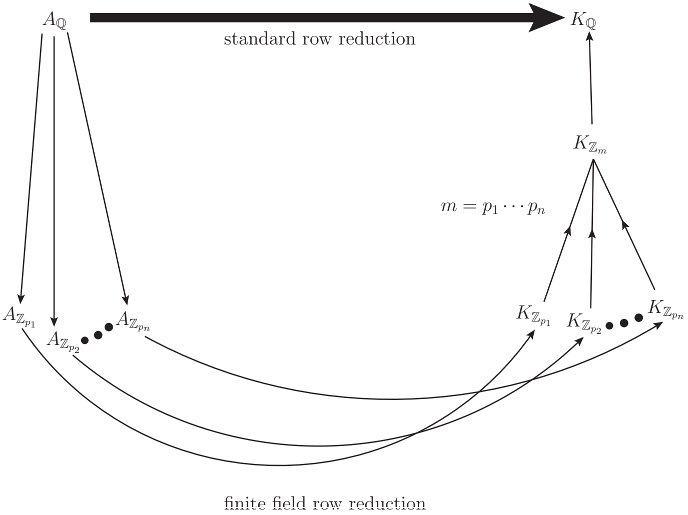
For simplicity, we restrict ourselves in this talk to linear systems with coefficients in $\mathbb{Q}$. Let us stress, however, that a univariate rational function version is implemented and a multivariate version is coming.

- Map system to "machine-sized" finite fields and row reduce over them to avoid intermediate expression swell.
- Sew the solutions together by "Chinese remaindering."
- Reconstruct the rational null space vectors of interest. (amazingly, this is not only possible but very efficient)

P. S. Wang, SYMSAC '81, ACM Press, 212 (1981);

P. S. Wang *et. al.* SIGSAM Bulletin **16**, No. 2, 2 (1982)

A Visualization Of The Algorithm



How Well Does The Algorithm Actually Perform?

- Despite the maturity of the subject, it is surprisingly difficult to find a working public implementation. Manuel Kauers's `Mathematica` package `LinearSystemSolver.m` is the only example known to us. Although not thoroughly documented, we have seen examples where `LinearSystemSolver.m` outperforms the `Reduze 2` linear algebra engine!
- A univariate code, provisionally called `Finred`, was painstakingly developed by my collaborator Andreas von Manteuffel to achieve near-optimal scaling behavior and memory usage.
- Unlike `Reduze 2`, the parallelization is rather trivial since all finite field samples are completely independent of one another.
- On an old desktop, all of the rank five three-loop form factor reductions run through in about 20 minutes.
- The four-loop form factor reductions are under way, first results for the N_f^3 color structure topologies already available.

From Conventional To Finite Integral Bases

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E. Panzer, JHEP **1403** (2014) 071; S. Weinberg, Phys. Rev. **118**, 838, 1960

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- For $x = \Delta d/2$ (the dimension shift divided by two), $y = \nu - N$ (the number of "extra" powers of the propagators or "dots"), and all fixed non-negative integers $n = x + y$, this test is carried out in practice by considering the integrals which correspond to all possible non-negative integer solutions $\{x, y\}$, beginning with the $n = 0$ case corresponding to the basic scalar integral in $d = 4 - 2\epsilon$.

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- Rotate from the old basis to the new basis using auxiliary IBPs.
- The computationally expensive part at this stage is to perform a Tarasov shift (Phys. Rev. **D54** (1996) 6479) on the old basis and then IBP reduce the resulting linear combination of integrals in $d + 2$ with a number of additional dots equal to the loop order. This connects the "conventional" integral bases in d and $d + 2$; it can be used iteratively if multiple dimension shifts are required.

What About The Auxiliary Reductions Needed For The Basis Rotation?

In his classic paper on dimension shifts, Tarasov also points out that one can, for any integral topology, eliminate all irreducible numerators in favor of higher-multiplicity propagators. A single irreducible numerator is eliminated at the cost of adding L additional dots and going from a single integral to a linear combination of integrals.

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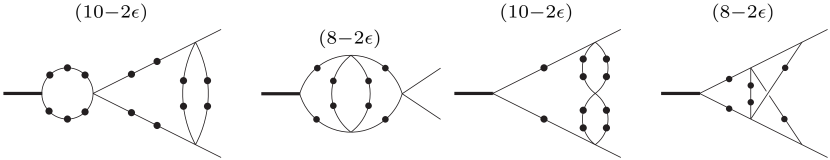

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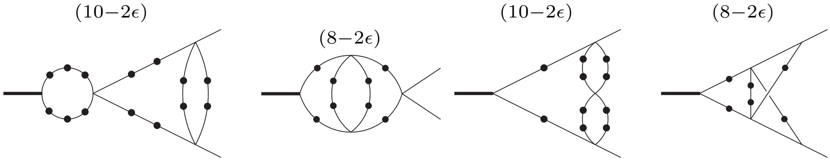
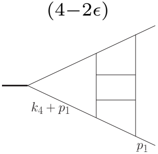


$\Longrightarrow$ Auxiliary reductions not a problem if using Feynman diagrams!

An Illustrative Comparison

J. Henn *et. al.*, JHEP 1605 (2016) 066



$$\epsilon^8 \;[\text{diagram}]^{(4-2\epsilon)}\left[(k_4^2)^2\right] = \frac{1}{576} + \frac{1}{36}\zeta_2\epsilon^2 + \frac{151}{864}\zeta_3\epsilon^3 + \frac{173}{288}\zeta_2^2\epsilon^4$$

$$+ \left(\frac{505}{216}\zeta_2\zeta_3 + \frac{5503}{1440}\zeta_5\right)\epsilon^5 + \left(\frac{6317}{720}\zeta_2^3 + \frac{9895}{2592}\zeta_3^2\right)\epsilon^6 + \mathcal{O}\left(\epsilon^7\right)$$

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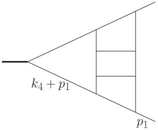
ϵ^8 

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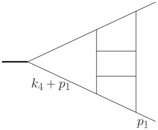
ϵ^8 

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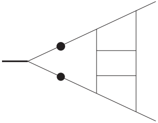
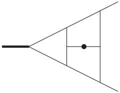
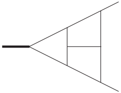
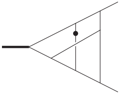
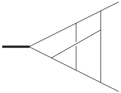
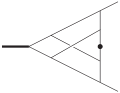
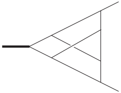
(6−2ϵ)



$$= -\frac{3}{5}\zeta_2^2 + 5\zeta_2\zeta_3 + \frac{25}{2}\zeta_5 - \frac{7}{10}\zeta_2^3 - \frac{3}{10}\zeta_2^2\zeta_3 - \frac{5}{2}\zeta_2\zeta_5 - \frac{147}{16}\zeta_7 + \mathcal{O}\left(\epsilon\right)$$

For Fixed Program Settings, Finite Integral Bases Offer Spectacular Performance Enhancements

Let's continue with the three-loop form factor example from before

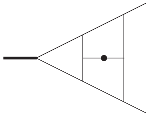
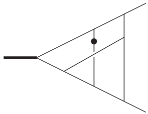
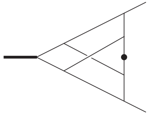
diagram	run time	relative accuracy	diagram	run time	relative accuracy
$(6-2\epsilon)$ 	128 s	5.12×10^{-6}	$(4-2\epsilon)$ 	39094 s	9.91×10^{-4}
$(6-2\epsilon)$ 	192 s	2.68×10^{-6}	$(4-2\epsilon)$ 	19025 s	9.38×10^{-5}
$(6-2\epsilon)$ 	127 s	2.26×10^{-6}	$(4-2\epsilon)$ 	19586 s	1.07×10^{-4}

up to and including contributions of weight six.

How About Three-Loop Form Factors @ Weight 8?

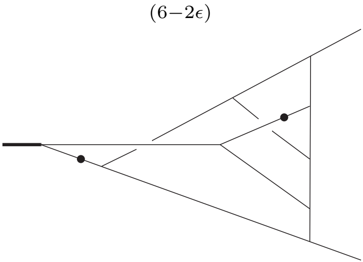
Important for our recent paper and as an answer to

K. G. Chetyrkin *et. al.*, Nucl. Phys. **B742** (2006) 208

diagram	run time	relative accuracy	run time	relative accuracy
$(6-2\epsilon)$ 	128 s	5.12×10^{-6}	491 s	2.22×10^{-5}
$(6-2\epsilon)$ 	192 s	2.68×10^{-6}	761 s	5.84×10^{-6}
$(6-2\epsilon)$ 	127 s	2.26×10^{-6}	485 s	8.45×10^{-6}

Finite Parts Of The Four-Loop Form Factors?

In about a day on an ancient desktop with `FIESTA 4`, we find



$$\approx 3.1808 + 58.829\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

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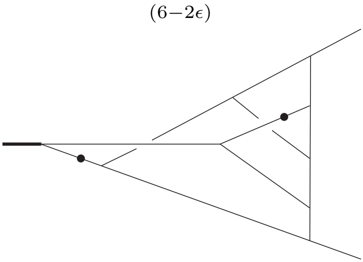
In about a day on an ancient desktop with `FIESTA 4`, we find



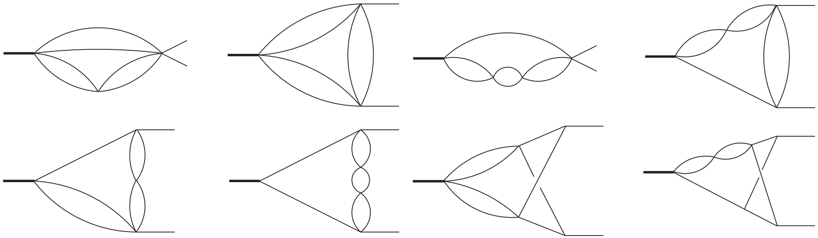
$$\approx 3.1808 + 58.829\epsilon + \mathcal{O}\left(\epsilon^2\right)$$

This is the result through to weight 8 with 4 digit absolute accuracy!

The Master Integrals For The N_f^3 Contributions To The Four-Loop Gluon Form Factor

- From the reductions, it seemed initally that 10 master integrals would appear in the $C_F N_f^3$ and $C_A N_f^3$ color structures.
- Actually, two factorizable topologies drop out of the final results.
- All master integrals can be evaluated to all orders in ϵ.

R. J. Gonsalves, Phys. Rev. **D28** (1983) 1542; Gehrmann *et. al.*, Phys. Lett. **B640** (2006) 252



N_f^3 Part Of The Bare Four-Loop Gluon Form Factor

In the $\overline{\text{MS}}$ scheme, we find

$$\mathcal{F}_4^g(\epsilon)\Big|_{C_F N_f^3} = -\frac{2}{3\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{32\zeta_3}{3} - \frac{145}{9}\right) + \frac{1}{\epsilon}\left(\frac{352\zeta_2^2}{45} + \frac{1040\zeta_3}{9} + \frac{68\zeta_2}{9} - \frac{10003}{54}\right) + \frac{4288\zeta_5}{27} - 64\zeta_3\zeta_2 + \frac{2288\zeta_2^2}{27} + \frac{24812\zeta_3}{27} + \frac{3074\zeta_2}{27} - \frac{508069}{324} + \mathcal{O}(\epsilon)$$

$$\mathcal{F}_4^g(\epsilon)\Big|_{C_A N_f^3} = \frac{1}{27\epsilon^5} + \frac{5}{27\epsilon^4} + \frac{1}{\epsilon^3}\left(-\frac{14\zeta_2}{27} - \frac{55}{81}\right) + \frac{1}{\epsilon^2}\left(-\frac{586\zeta_3}{81} - \frac{70\zeta_2}{27} - \frac{24167}{1458}\right) + \frac{1}{\epsilon}\left(-\frac{802\zeta_2^2}{135} - \frac{5450\zeta_3}{81} - \frac{262\zeta_2}{81} - \frac{465631}{2916}\right) - \frac{14474\zeta_5}{135} + \frac{4556}{81}\zeta_3\zeta_2 - \frac{1418}{27}\zeta_2^2 - \frac{99890\zeta_3}{243} + \frac{38489\zeta_2}{729} - \frac{20832641}{17496} + \mathcal{O}(\epsilon)$$

in both general R_ξ gauge and $\xi = 1$ background field gauge.

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Moch *et. al.*, JHEP **0508** (2005) 049

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- Numerical checks on the expansion coefficients of all masters to part per mille precision using `FIESTA 4`.

Outlook

Overall, our IBP algorithm seems strong enough to significantly ameliorate one of the biggest performance bottlenecks to calculating the four-loop cusp anomalous dimensions in QCD and we have seen that `FIESTA 4` can deliver acceptably precise results for finite non-planar twelve-line four-loop integrals if any relevant masters are inaccessible to us using `HyperInt`-like methods. The methods developed naturally open up several avenues for future research:

- Implement a multivariate version of `Finred`.
- Calculate reduced amplitudes for important $\mathrm{N}^2\mathrm{LO}$ wishlist problems such as two-loop Higgs + jet production with exact top mass dependence. (see T. Peraro, arXiv:1608.01902 for work in this direction on three-jet production)
- Further $\mathrm{N}^2\mathrm{LO}$ calculations using finite integrals numerically, in the spirit of what was done recently for double Higgs production.

S. Borowka *et. al.*, Phys. Rev. Lett. **117** (2016) no. 1, 012001