Universität Zürich[UZH]

Four-loop beta functions in the Standard Model: Leading contributions and their impact on vacuum stability

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Outline

I. Introduction: Vacuum stability and the evolution of couplings

II. SM β-functions at 4 loops

1. Details of the calculation: Renormalization, IR regularization

2. The problem of γ_5-treatment

3. Leading 4 loop contributions to β_{g_s} [JHEP 1602 (2016) 095] and leading 4 loop contributions to β_{y_t} and β_λ [arXiv:1604.00853]

III. Connection to pure QCD results

IV. Current status of the the vacuum stability question in the SM

I. Vacuum stability and the evolution of couplings

The Standard Model: $\mathcal{L} = \mathcal{L}_{QCD} + \mathcal{L}_{EW} + \mathcal{L}_{Yukawa} + \mathcal{L}_{\Phi}$

gauge group: $\mathbf{SU}_C(3) \times \mathbf{SU}(2) \times \mathbf{U}_Y(1) \quad \xrightarrow{\text{SSB}} \quad \mathbf{SU}_C(3) \times \mathbf{U}_Q(1)$

- fermions: quarks and leptons:

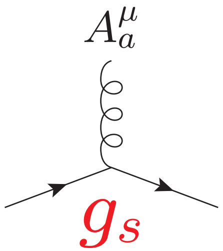
  $Q_L = \begin{pmatrix} t \\ b \end{pmatrix}_L, \; t_R, \; b_R; \quad L_L = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \; \tau_R; \quad$ (+ light generations)

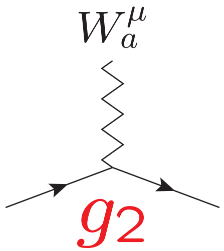
- gauge bosons:
  A^a_μ (SU$_C$(3), $a = 1, \ldots, 8$)
  W^a_μ (SU(2), $a = 1, 2, 3$)
  B_μ (U$_Y$(1))

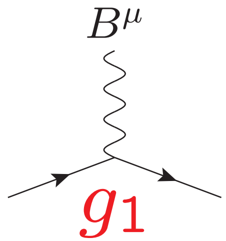
  $$D^\mu = \partial^\mu - i g_1 Y_f B^\mu - i \frac{g_2}{2} \sigma^a W^{a\,\mu} - i g_s T^a A^{a\,\mu}$$

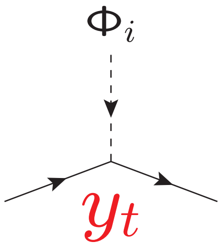
- scalar SU(2)-doublet: $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \xrightarrow{\text{SSB}} \begin{pmatrix} \Phi^+ \\ \frac{1}{\sqrt{2}}(v + H + i\chi) \end{pmatrix}$

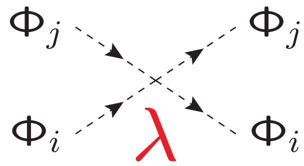
Standard Model interactions:

- strong: 

electroweak: 



- Yukawa: 

Higgs self-interaction: 

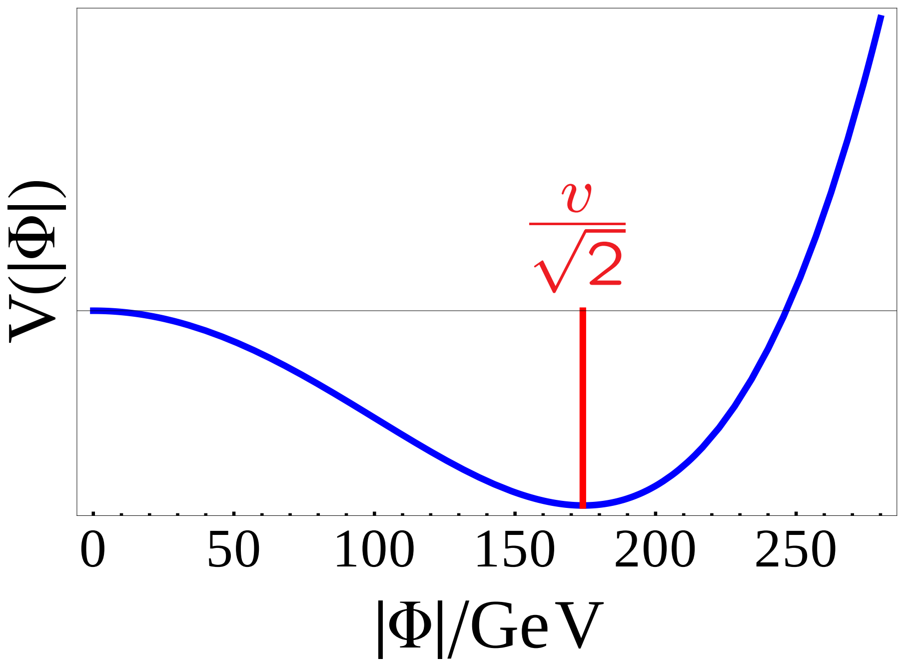
Classical Higgs potential:

$$V(\Phi) = m^2\, \Phi^\dagger \Phi + \lambda\, \left(\Phi^\dagger \Phi\right)^2$$

$$|\Phi(x)| = \frac{1}{\sqrt{2}}(v + H(x))$$

$$\Phi_{\text{cl}}(x) \equiv \langle 0|\, \Phi(x)\, |0\rangle = \frac{v}{\sqrt{2}} \neq 0$$

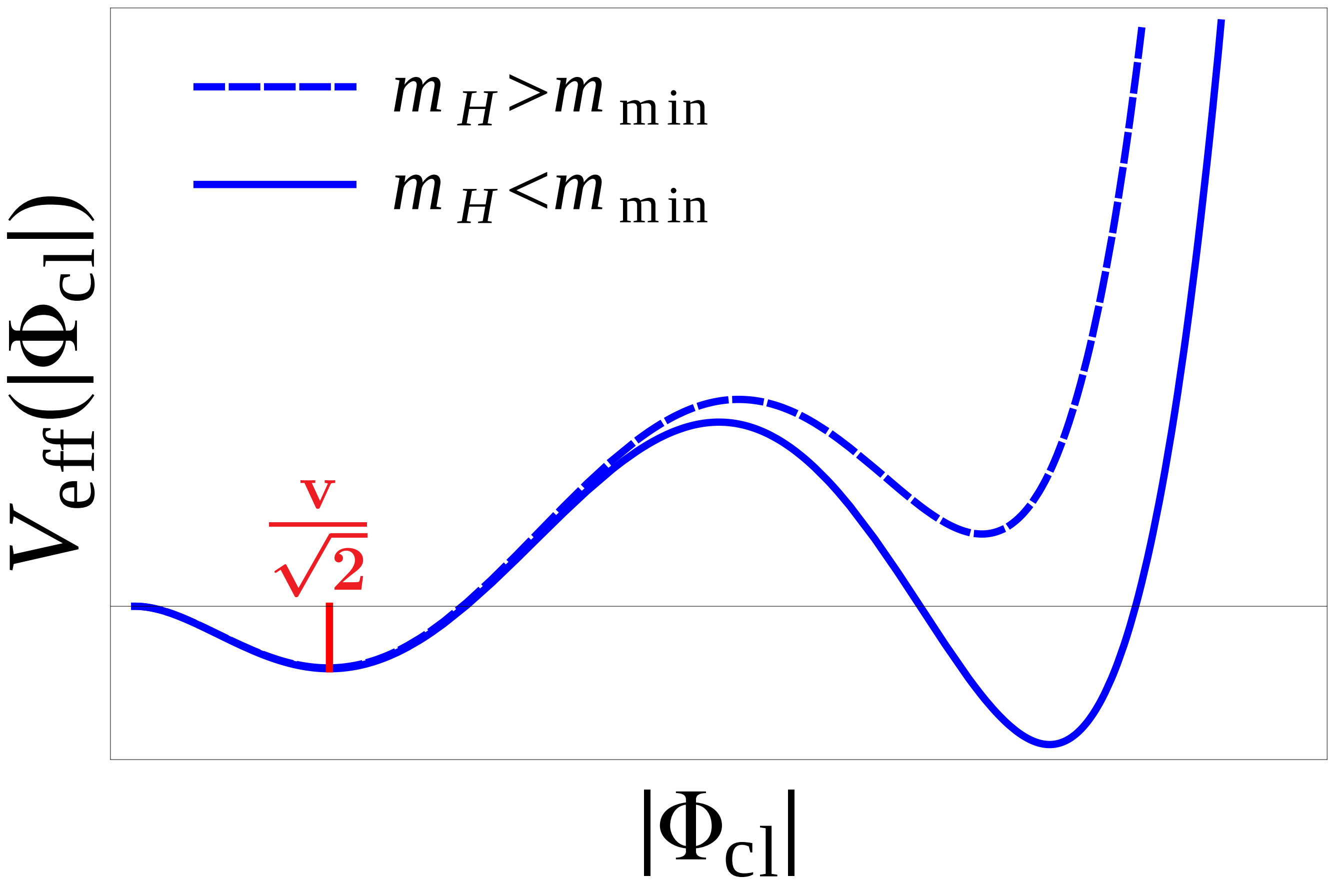
$v \approx 246$ GeV, $M_H^2 = -2m^2 = 2\lambda v^2$



The effective Potential

- Radiative corrections $\Rightarrow$ Evolution of couplings λ, g_1, g_2, g_s, y_t, ... and fields Φ, ...
- Higgs potential $\rightarrow$ $\boxed{V_{\text{eff}}\,(\lambda(\Lambda), g_i(\Lambda), y_t(\Lambda), \ldots)\,[\Phi(\Lambda)]}$ [Coleman, Weinberg]

  (Λ: scale up to which the SM is valid)



For $\Phi_{\rm cl} \sim \Lambda \gg v$:

$$\boxed{V_{\rm eff}^{\rm RG}[\Phi] = \lambda(\Lambda)\Phi^4(\Lambda) + \mathcal{O}(\lambda^2(\Lambda), g_i^2(\Lambda))}$$

$$\Phi(\Lambda) = \Phi_{\rm cl} \cdot \exp\left(- \int_0^{\log\left(\frac{\Lambda^2}{\mu_0^2}\right)} dt' \gamma_\Phi(\lambda(t'), g_i(t')) \right)$$

(Λ: scale up to which the SM is valid, μ_0: starting point for running, e.g. $\mu_0 = M_t$)

[Altarelli, Isidori; Ford, Jack, Jones]

$$\text{Stability of SM vacuum} \Leftrightarrow \lambda(\Lambda) > 0$$

[Cabibbo; Sher; Lindner; Ford]

Evolution of couplings $X \in \{\lambda, g_1, g_2, g_s, y_t, \ldots\}$

β-functions: $$\mu^2 \frac{d}{d\mu^2} X(\mu^2) = \beta_X[\lambda(\mu^2), y_t(\mu^2), g_i(\mu^2), \ldots]$$

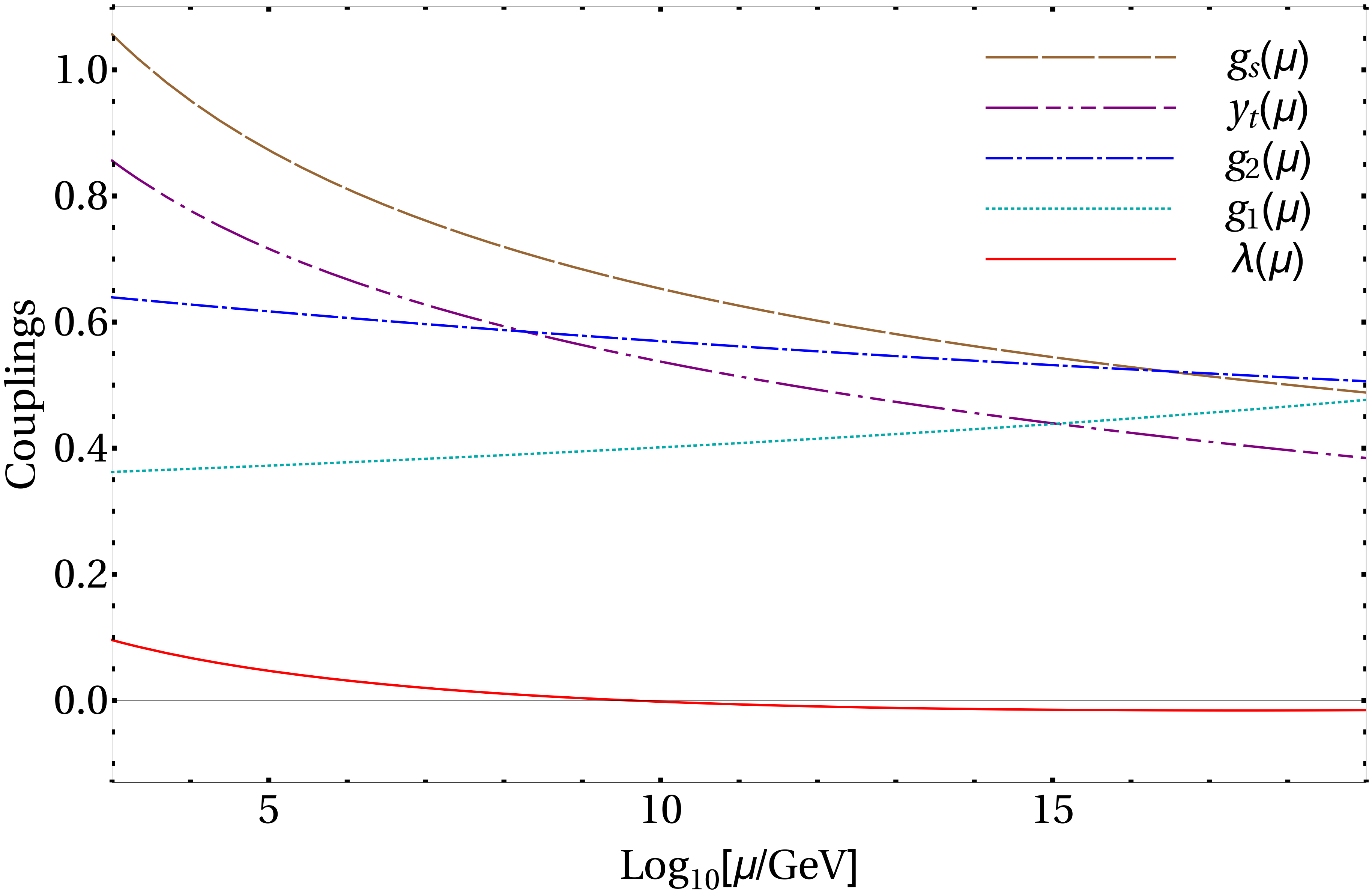
$\Rightarrow$ **Coupled system of differential equations with initial conditions:**

$$\underbrace{\begin{aligned}
\mu^2 \frac{d}{d\mu^2}\lambda(\mu^2) &= \beta_\lambda[\lambda(\mu^2), y_t(\mu^2), g_i(\mu^2)], \\
\mu^2 \frac{d}{d\mu^2}y_t(\mu^2) &= \beta_{y_t}[\lambda(\mu^2), y_t(\mu^2), g_i(\mu^2)], \\
\mu^2 \frac{d}{d\mu^2}g_s(\mu^2) &= \beta_{g_s}[\lambda(\mu^2), y_t(\mu^2), g_i(\mu^2)], \\
\mu^2 \frac{d}{d\mu^2}g_2(\mu^2) &= \beta_{g_2}[\lambda(\mu^2), y_t(\mu^2), g_i(\mu^2)], \\
\mu^2 \frac{d}{d\mu^2}g_1(\mu^2) &= \beta_{g_1}[\lambda(\mu^2), y_t(\mu^2), g_i(\mu^2)],
\end{aligned}} \qquad \underbrace{\begin{aligned}
\lambda(\mu_0^2) &= \lambda_0, \\
y_t(\mu_0^2) &= y_{t\,0}, \\
g_s(\mu_0^2) &= g_{s\,0}, \\
g_2(\mu_0^2) &= g_{2\,0}, \\
g_1(\mu_0^2) &= g_{1\,0}
\end{aligned}}$$

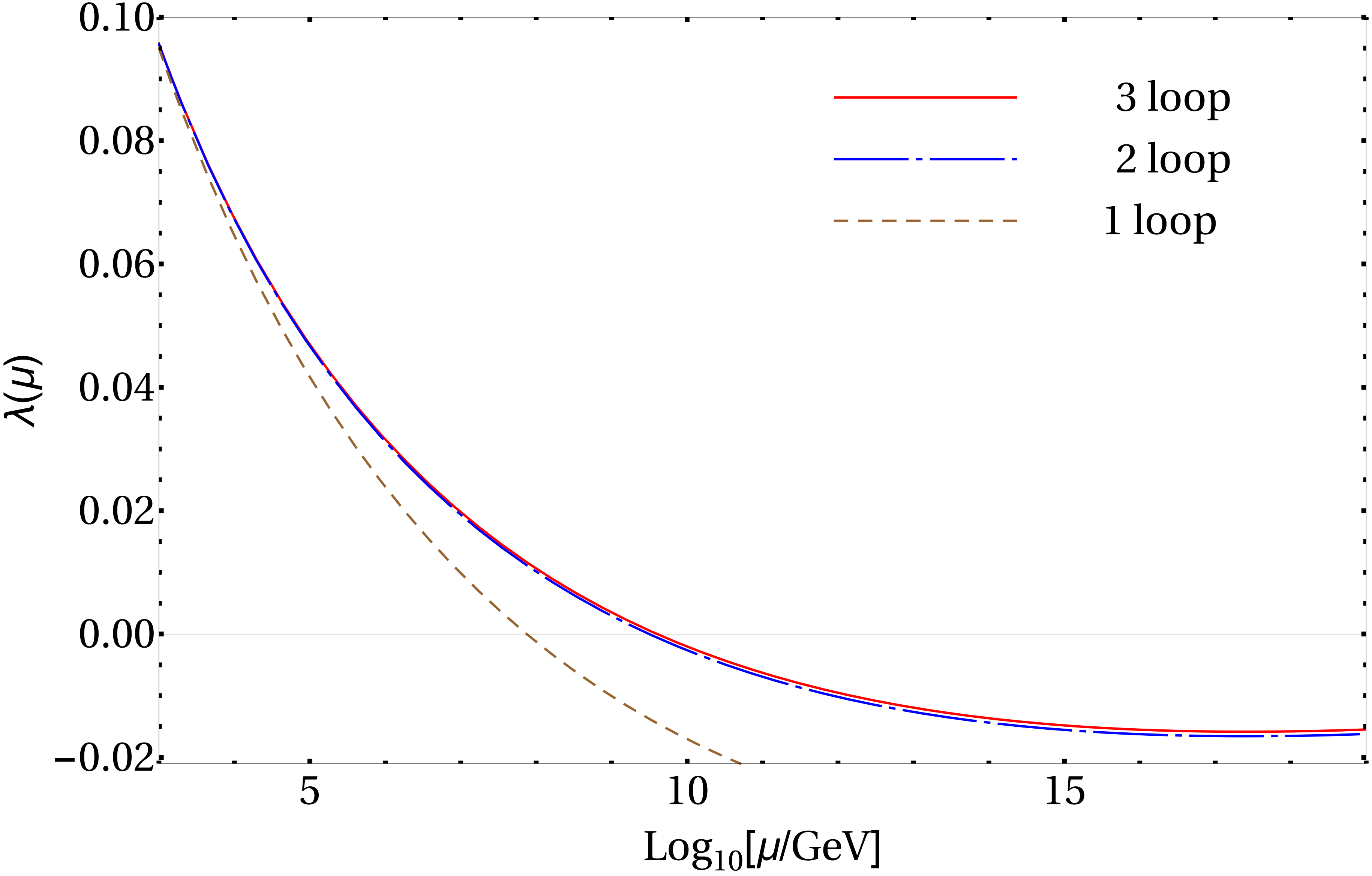
Calculated in $\overline{\text{MS}}$-scheme, power series in couplings

Experimental data matched to $\overline{\text{MS}}$-scheme

Evolution of SM couplings



Evolution of $\lambda(\mu)$



II. Standard Model β-functions up to 4 loops

2 loop

[M. Fischler, C. Hill (1981); D. Jones (1982); M. Machacek, M. Vaughn (1983,1984,1985); I. Jack, H. Osborn (1984,1985)] [M. Fischler, J. Oliensis (1982); M. Machacek, M. Vaughn (1984); C. Ford, I. Jack, D. Jones (1992); M. Luo, Y. Xiao (2003)]

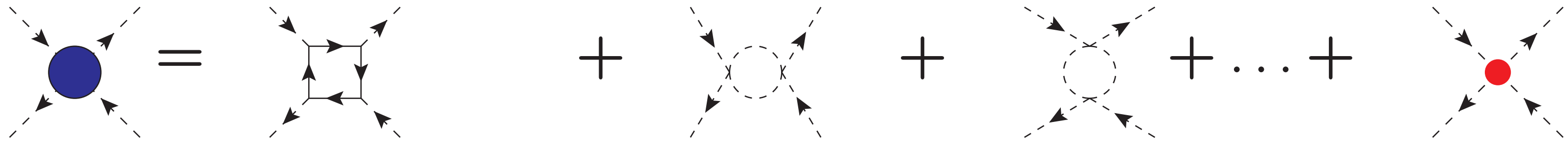
3 loop

- for gauge couplings g_1, g_2, g_s [L. Mihaila, J. Salomon, M. Steinhauser (2012); A. Bednyakov, A. Pikelner, Velizhanin (2012)]
- for Yukawa couplings y_t, y_b, y_τ, etc. [K. Chetyrkin, M.Z. (2012); A. Bednyakov, A. Pikelner, Velizhanin (2013)]
- for the Higgs self-coupling λ (and the mass parameter m^2) [K. Chetyrkin, M.Z. (2012 and 2013); A. Bednyakov, A. Pikelner, Velizhanin (2013)]

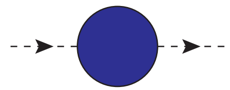
4 loop

- $\beta_{g_s}(g_s)$ [T. van Ritbergen, J. Vermaseren, S. Larin (1997); M. Czakon (2005)]
- $\beta_{g_s}(g_s, y_t, \lambda)$ [A. Bednyakov, A. Pikelner (2015); M.Z. (2015)]
- $\beta_\lambda \propto y_t^4 g_s^6$ [S. Martin (2016); K. Chetyrkin, M.Z. (2016)]
- $\beta_{y_t} \propto y_t g_s^8$ and $\beta_{m^2} \propto y_t^2 g_s^6$ [K. Chetyrkin, M.Z. (2016)]

Calculation of $\beta_\lambda(\lambda, y_t, g_s, g_2, g_1)$



$= \text{finite}$, with the one-loop contributions $\propto \frac{y_t^4}{(16\pi^2)}$ and $\propto \frac{\lambda^2}{(16\pi^2)}$, and counterterm $\delta Z_1^{(4\Phi)}$



$= \text{finite}$ $\Rightarrow$ Field strength renormalization constant $Z_2^{(2\Phi)}$

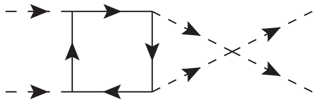
$$\mathcal{L}_{4\Phi} = (-\lambda + \delta Z_1^{(4\Phi)}) \left(\Phi^\dagger \Phi\right)^2$$

$$\stackrel{!}{=} -\lambda_B\,[\Phi_B^\dagger \Phi_B]^2 = -(\lambda + \delta Z_\lambda)\left[Z_2^{(2\Phi)} \Phi^\dagger \Phi\right]^2$$

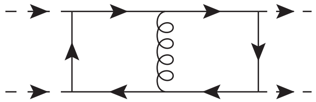
$$\Rightarrow \quad \boxed{\delta Z_\lambda = \frac{\lambda - \delta Z_1^{(4\Phi)}}{\left(Z_2^{(2\Phi)}\right)^2} - \lambda} \quad = \sum_{n=1}^{\infty} \frac{a_n(\lambda, y_t, g_s, g_2, g_1)}{\varepsilon^n}$$

use $\mu^2 \frac{d}{d\mu^2} \lambda_B \equiv 0 \;\Rightarrow\; \boxed{\beta_\lambda = \left[\lambda \frac{\partial}{\partial \lambda} + \frac{1}{2} \sum_i g_i \frac{\partial}{\partial g_i} - 1\right] a_1}$

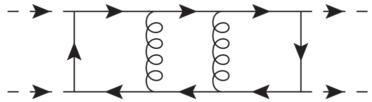
2 and 3 loop orders:



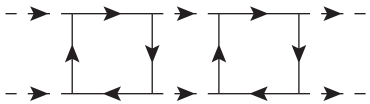
$\propto \frac{y_t^4 \lambda}{(16\pi^2)^2}$



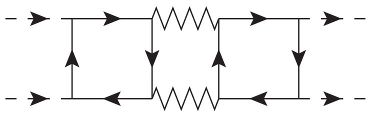
$\propto \frac{y_t^4 g_s^2}{(16\pi^2)^2}$



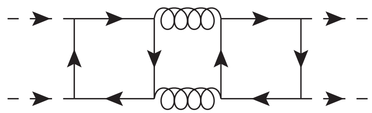
$\propto \frac{y_t^4 g_s^4}{(16\pi^2)^3}$



$\propto \frac{y_t^8}{(16\pi^2)^3}$

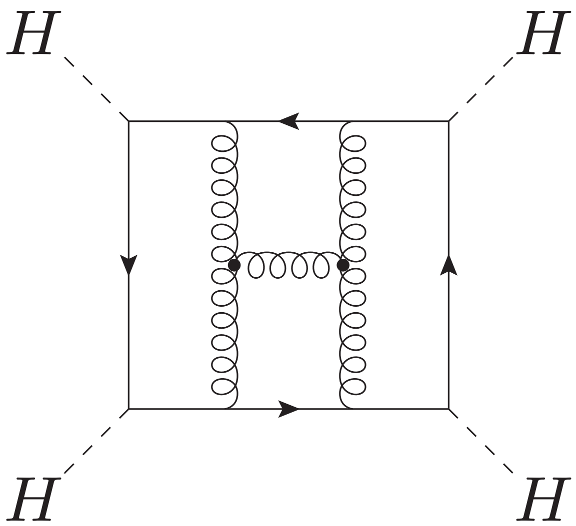


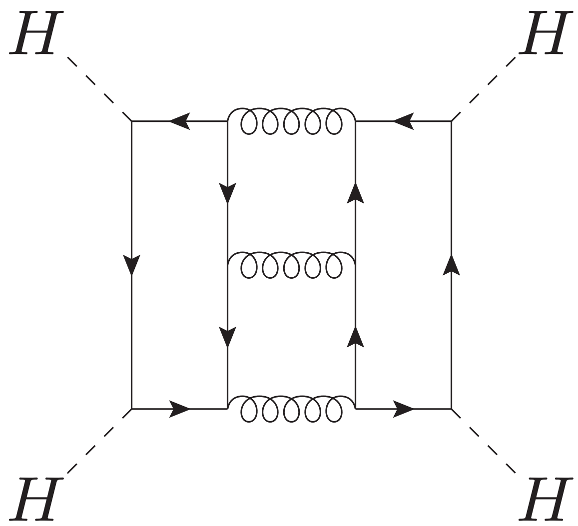
$\propto \frac{y_t^4 g_2^4}{(16\pi^2)^3}$



$\propto \frac{y_t^4 g_s^4}{(16\pi^2)^3}$

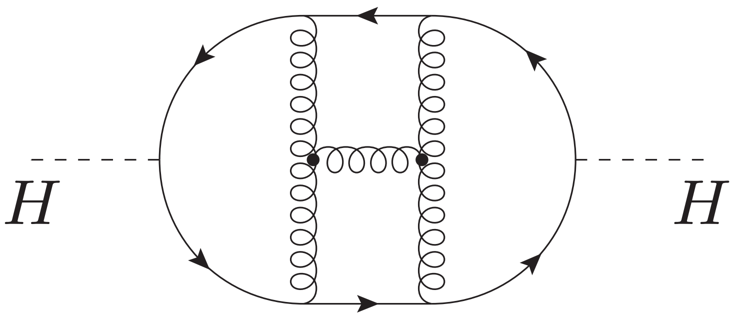
4 loop order:

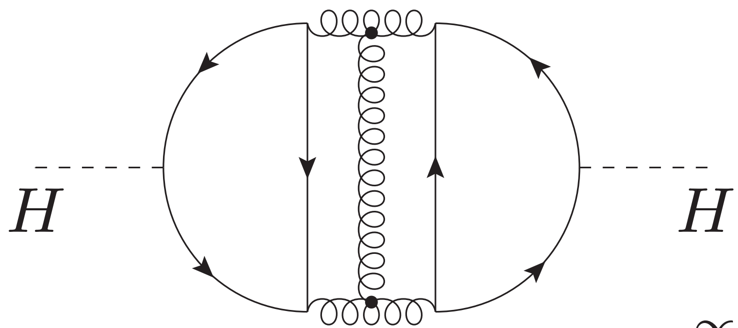




$\propto \frac{y_t^4 g_s^6}{(16\pi^2)^4}$

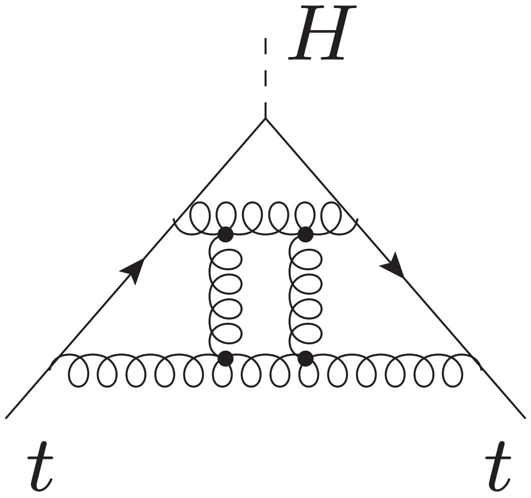
Analogously the Higgs propagator:



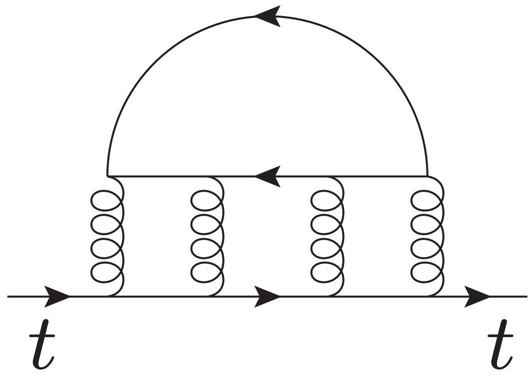


$$\propto \frac{y_t^2 g_s^6}{(16\pi^2)^4}$$

And for the top-Yukawa vertex and top propagator:



$\propto \frac{y_t g_s^8}{(16\pi^2)^4}$

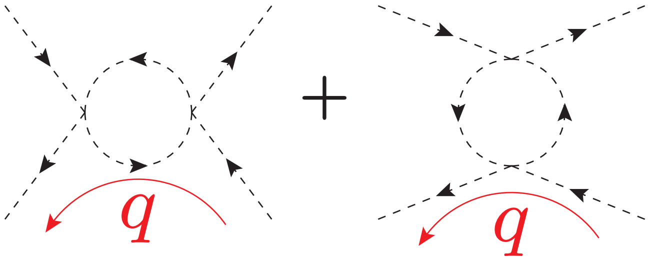


$\propto \frac{g_s^8}{(16\pi^2)^4}$

Regularization of IR singularities

[Misiak and Münz (1995); Larin, van Ritbergen, Vermaseren (1996); Chetyrkin, Misiak and Münz (1998)]

Problem: IR divergencies in diagrams with 4 external Φ:



- Introduce auxiliary mass M^2 in every propagator denominator:

  $\frac{1}{(q+p)^2} \rightarrow \frac{1}{(q+p)^2 - M^2}$.

  (p external, q internal momenta)

- Expand in external momenta p
  $\Rightarrow$ massive tadpole integrals

Exact decomposition of propagators: (p external, q internal momenta)

$$\frac{1}{(q+p)^2} = \frac{1}{q^2 - M^2} + \frac{-p^2 - 2q{\cdot}p - M^2}{q^2 - M^2}\frac{1}{(q+p)^2}$$

Recursion:

$$\frac{1}{(q+p)^2} = \underbrace{\frac{1}{q^2 - M^2} + \frac{-p^2 - 2q{\cdot}p}{(q^2 - M^2)^2} + \frac{(-p^2 - 2q{\cdot}p)^2}{(q^2 - M^2)^3}}_{\text{used in this method}}$$

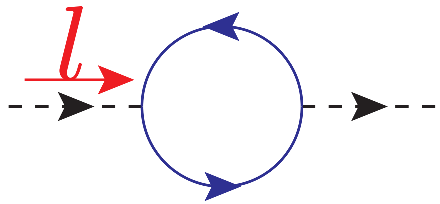
$$\underbrace{- \frac{M^2}{(q^2 - M^2)^2} + \frac{M^2(M^2 + 2p^2 + 4q{\cdot}p)}{(q^2 - M^2)^3}}_{\propto M^2} + \underbrace{\frac{(-p^2 - 2q{\cdot}p - M^2)^3}{(q^2 - M^2)^3}\frac{1}{(q+p)^2}}_{\text{contributes only to finite part}}$$

Exact result independent of M^2 $\Rightarrow$ restore terms $\propto M^2$ in exact decomposition by introducing all possible M^2-counterterms in our expansion

$$\boxed{\frac{M^2}{2}\delta Z^{(2g)}_{M^2}\, A^a_\mu A^{a\,\mu}, \quad \frac{M^2}{2}\delta Z^{(2W)}_{M^2}\, W^a_\mu W^{a\,\mu}, \quad \frac{M^2}{2}\delta Z^{(2B)}_{M^2}\, B_\mu B^\mu, \quad \frac{M^2}{2}\delta Z^{(2\Phi)}_{M^2}\, \Phi^\dagger\Phi}$$

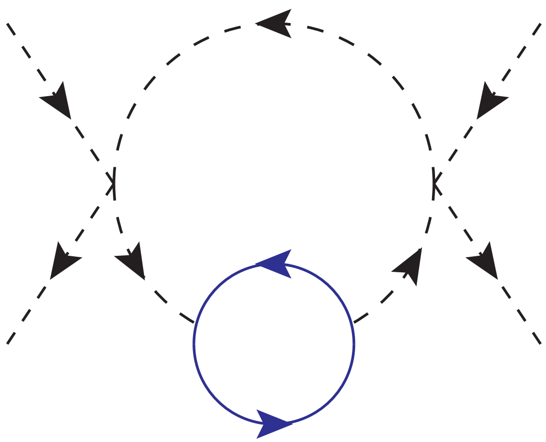
Example:

At 1 loop:

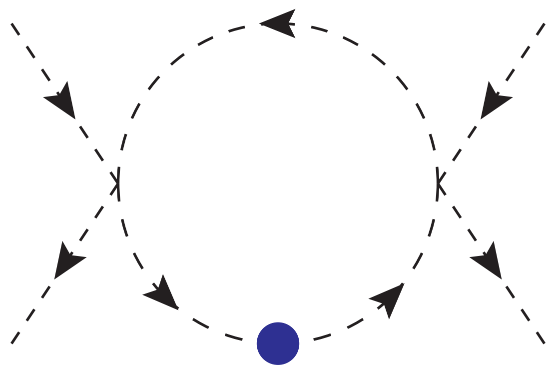


$$= \frac{l^2}{\varepsilon} C_{l^2} + \frac{M^2}{\varepsilon} C_{M^2} + \text{ finite}$$

At 2 loop:



$+$

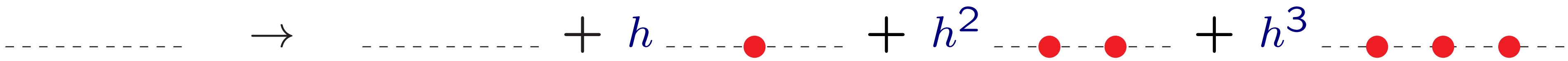


$$M^2 \delta Z^{(2\Phi)}_{M^2} = -\frac{M^2}{\varepsilon} C_{M^2}$$

Renormalization: all counterterms needed at lower loop orders

In lower loop diagrams:

- Vertex V multiplied by $(1 + h\,\delta Z_V)$
- Wavefunction and auxiliary mass counterterms in propagators:



- for fermions in the full SM: $(P_{\mathsf{L}}\,\delta Z_{\mathsf{L}} + P_{\mathsf{R}}\,\delta Z_{\mathsf{R}})$

  with $P_{\mathsf{L}} = \frac{1}{2}(1 - \gamma_5)$, $\quad P_{\mathsf{R}} = \frac{1}{2}(1 + \gamma_5)$

Automation

- Generation of diagrams $\to$ QGRAF [Noguira]

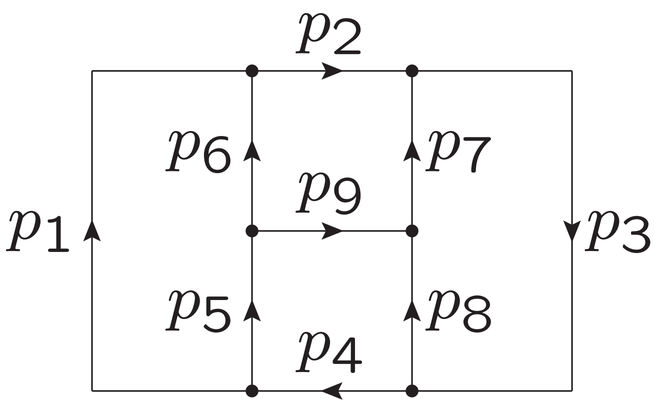
- SU(2) $\times$ U$_Y$(1) group factors $\to$ Form code [M.Z.]

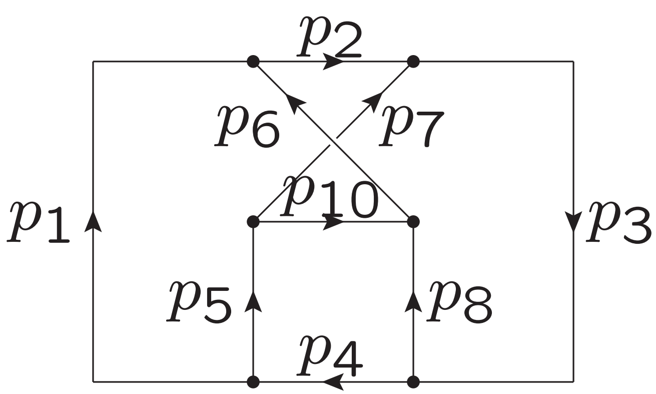
- SU$_C$(3) group factors $\to$ COLOR [Van Ritbergen, Schellekens, Vermaseren]

- find topologies $\to$ GEFICOM [Chetyrkin, M.Z.] and
  Q2E, EXP [Seidesticker, Harlander, Steinhauser]

- Feynman rules, projectors, counterterms, fermion traces, expansion in external momenta $\to$ FORM [Vermaseren] code [Chetyrkin, M.Z.]

- massive tadpole integrals $\to$ up to 3 loop: MATAD [Steinhauser];
  Reduction at 4 loop: FIRE5 (C++ version) [Smirnov] (based on IBP)
  $\Rightarrow$ use 19 Master integrals [Czakon et al]





planar topology: tad4lp   non-planar topology: tad4lnp

independent loop momenta: p_1, p_2, p_3, p_4

$$\Rightarrow p_5 = p_4 - p_1, \quad p_6 = p_2 - p_1, \quad p_7 = p_3 - p_2,$$
$$p_8 = p_3 - p_4, \quad p_9 = p_4 - p_2, \quad p_{10} = p_4 + p_2 - p_1 - p_3$$

Scalar products: $\boxed{p_1 \cdot p_2 = -\frac{1}{2}\left(p_6^2 - p_1^2 - p_2^2\right)}$ etc.

$\boxed{p_j^2 \to M^2 - \frac{i}{D_j}}$ with scalar propagators $D_i = \frac{1}{i}\frac{1}{M^2 - p_i^2}$

All scalar four-loop diagrams are linear combinations of

$$\boxed{\text{TAD4l}(n_1, \ldots, n_{10}) := \int \mathrm{d}^D p_1 \int \mathrm{d}^D p_2 \int \mathrm{d}^D p_3 \int \mathrm{d}^D p_4 \prod_{i=1}^{10} D_i^{n_i}}$$

Results

$$\mu^2 \frac{d}{d\mu^2}\lambda(\mu) = \beta_\lambda = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n}\beta_\lambda^{(n)}$$

(in the $\overline{\text{MS}}$-scheme)

$$\begin{aligned}
\beta_\lambda^{(4)} =& y_t^4 g_s^6 d_R \Bigg\{ C_F^3 \left(-\frac{2942}{3} + 160\zeta_5 + 288\zeta_4 + 48\zeta_3 \right) \\
&+ T_F C_F^2 \left(-64 + n_f \left(+\frac{562}{3} - 160\zeta_4 + \frac{32}{3}\zeta_3 \right) \right) \\
&+ C_A C_F^2 \left(\frac{3584}{3} + 720\zeta_5 + 32\zeta_4 - \frac{3304}{3}\zeta_3 \right) \\
&+ C_A T_F C_F \left(\frac{5888}{9} - 160\zeta_5 + 352\zeta_3 + n_f \left(-\frac{2644}{243} + 128\zeta_4 + 16\zeta_3 \right) \right) \\
&+ C_A^2 C_F \left(-\frac{121547}{243} - 520\zeta_5 - 88\zeta_4 + \frac{1880}{3}\zeta_3 \right) \\
&+ T_F^2 C_F \left(-\frac{256}{9} n_f + n_f^2 \left(-\frac{128}{3}\zeta_3 + \frac{10912}{243} \right) \right) \Bigg\} \\
&+ \mathcal{O}(y_t^6) + \mathcal{O}(\lambda) + \mathcal{O}(g_2) + \mathcal{O}(g_1)
\end{aligned}$$

Numerics

for $M_{\rm t} = 173.39$ GeV, $\alpha_s(M_{\rm Z}) = 0.1181$, $M_{\rm H} = 125.09$ GeV

At $\mu = M_t$:

$$
\begin{aligned}
\frac{\beta_\lambda^{(1)}(\mu = M_{\rm t})}{(16\pi^2)} &= -1.0 \times 10^{-2}, \\
\frac{\beta_\lambda^{(2)}(\mu = M_{\rm t})}{(16\pi^2)^2} &= -2.3 \times 10^{-5}, \\
\frac{\beta_\lambda^{(3)}(\mu = M_{\rm t})}{(16\pi^2)^3} &= +1.1 \times 10^{-5}, \\
\frac{\beta_\lambda^{(4)}(\mu = M_{\rm t})}{(16\pi^2)^4} &= +1.3 \times 10^{-5}
\end{aligned}
$$

Numerics

for $M_\mathrm{t} = 173.39$ GeV, $\alpha_s(M_\mathrm{Z}) = 0.1181$, $M_\mathrm{H} = 125.09$ GeV At $\mu = 10^9$GeV:

$$\begin{aligned}
\frac{\beta_\lambda^{(1)}(\mu = 10^9\mathrm{GeV})}{(16\pi^2)} &= -1.4 \times 10^{-3}, \\
\frac{\beta_\lambda^{(2)}(\mu = 10^9\mathrm{GeV})}{(16\pi^2)^2} &= +8.0 \times 10^{-8}, \\
\frac{\beta_\lambda^{(3)}(\mu = 10^9\mathrm{GeV})}{(16\pi^2)^3} &= +3.1 \times 10^{-7}, \\
\frac{\beta_\lambda^{(4)}(\mu = 10^9\mathrm{GeV})}{(16\pi^2)^4} &= +6.9 \times 10^{-8}
\end{aligned}$$

Results

$$\mu^2 \frac{d}{d\mu^2} y_t(\mu) = \beta_{y_t} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \beta_{y_t}^{(n)}$$ (in the $\overline{\text{MS}}$-scheme)

$$\begin{aligned}
\beta_{y_t}^{(4)} = & y_t g_s^8 \Bigg\{ \frac{d_F^{abcd} d_A^{abcd}}{d_R} (32 - 240\zeta_3) + n_f \frac{d_F^{abcd} d_F^{abcd}}{d_R} (-64 + 480\zeta_3) \\
& + C_F^4 \left(\frac{1261}{8} + 336\zeta_3 \right) - C_A C_F^3 \left(\frac{15349}{12} + 316\zeta_3 \right) \\
& + C_A^2 C_F^2 \left(\frac{34045}{36} - 440\zeta_5 + 152\zeta_3 \right) + C_A^3 C_F \left(-\frac{70055}{72} + 440\zeta_5 - \frac{1418}{9}\zeta_3 \right) \\
& + n_f T_F C_F^3 \left(\frac{280}{3} + 480\zeta_5 - 552\zeta_3 \right) + n_f C_A T_F C_F^2 \left(\frac{8819}{27} - 80\zeta_5 + 264\zeta_4 - 368\zeta_3 \right) \\
& + n_f C_A^2 T_F C_F \left(\frac{65459}{162} - 400\zeta_5 - 264\zeta_4 + \frac{2684}{3}\zeta_3 \right) \\
& + n_f^2 T_F^2 C_F^2 \left(-\frac{304}{27} - 96\zeta_4 + 160\zeta_3 \right) + n_f^2 C_A T_F^2 C_F \left(-\frac{1342}{81} + 96\zeta_4 - 160\zeta_3 \right) \\
& + n_f^3 T_F^3 C_F \left(\frac{664}{81} - \frac{128}{9}\zeta_3 \right) \Bigg\} + \mathcal{O}(y_t^3) + \mathcal{O}(\lambda) + \mathcal{O}(g_2) + \mathcal{O}(g_1).
\end{aligned}$$

---

with $d_F^{abcd} = \frac{1}{6} \text{Tr} \left(T^a T^b T^c T^d + T^a T^b T^d T^c + T^a T^c T^b T^d + T^a T^c T^d T^b + T^a T^d T^b T^c + T^a T^d T^c T^b \right)$

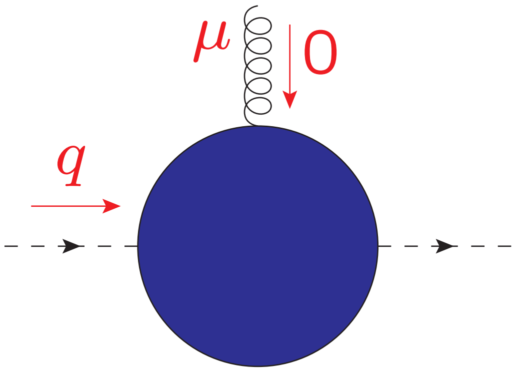
Numerics

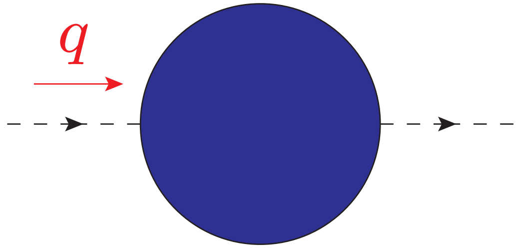
for $M_{\rm t} = 173.39$ GeV, $\alpha_s(M_{\rm Z}) = 0.1181$, $M_{\rm H} = 125.09$ GeV

At $\mu = M_t$:

$$\begin{aligned}
\frac{\beta_{y_t}^{(1)}(\mu = M_{\rm t})}{(16\pi^2)} &= -2.4 \times 10^{-2}, \\
\frac{\beta_{y_t}^{(2)}(\mu = M_{\rm t})}{(16\pi^2)^2} &= -2.9 \times 10^{-3}, \\
\frac{\beta_{y_t}^{(3)}(\mu = M_{\rm t})}{(16\pi^2)^3} &= -1.2 \times 10^{-4}, \\
\frac{\beta_{y_t}^{(4)}(\mu = M_{\rm t})}{(16\pi^2)^4} &= +5.9 \times 10^{-6}
\end{aligned}$$

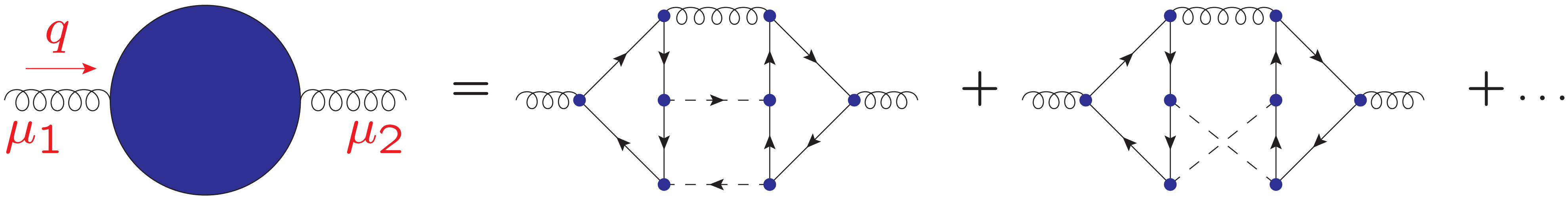
β_{g_s} at 4 loop from the ghost-gluon-vertex





projector: $\frac{q^\mu}{q^2}$

already scalar $\propto q^2$



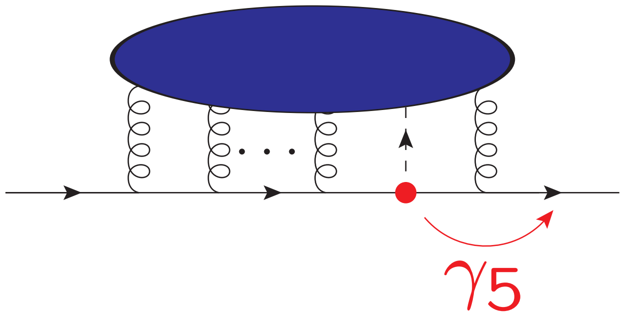
projector: $\frac{1}{D-1}\, g^{\mu_1\mu_2} \cdot \text{Ptr} + q^{\mu_1} q^{\mu_2} \cdot \text{Plong}$

Treatment of γ_5

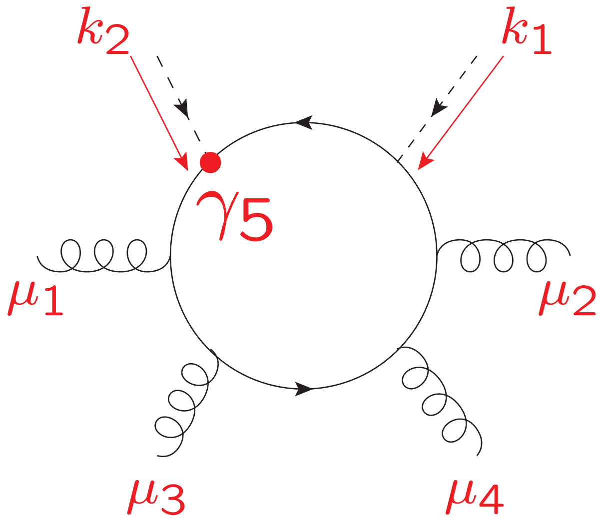
$$\boxed{\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma}$$

in $D = 4$: $\{\gamma_5, \gamma^\mu\} = 0$ and $\gamma_5^2 = 1$

- external fermion line:



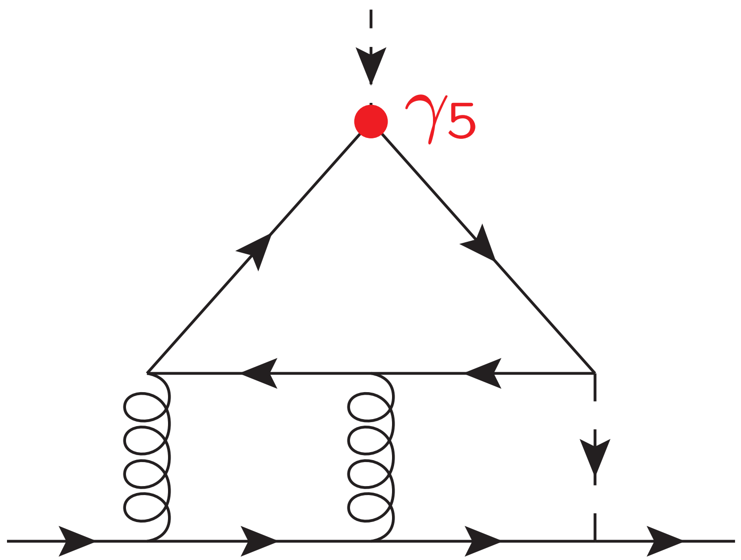
- fermion loop (internal line):



$$\mathrm{Tr}(\ldots) \propto \#\,\varepsilon_{\mu_1\mu_2\mu_3\mu_4} + \#\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,k_1^\alpha\,k_2^\beta + \ldots$$

$\Rightarrow$ At least 4 free Lorentz structures needed, else diagram=0

Treatment of γ_5 in the Z_{y_t} calculation



$\propto P_L = \frac{1}{2}(1 - \gamma_5)$

use projector $\propto \gamma_5$ on external fermion line, apply

$$\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \text{ with } \varepsilon_{0123} = 1$$

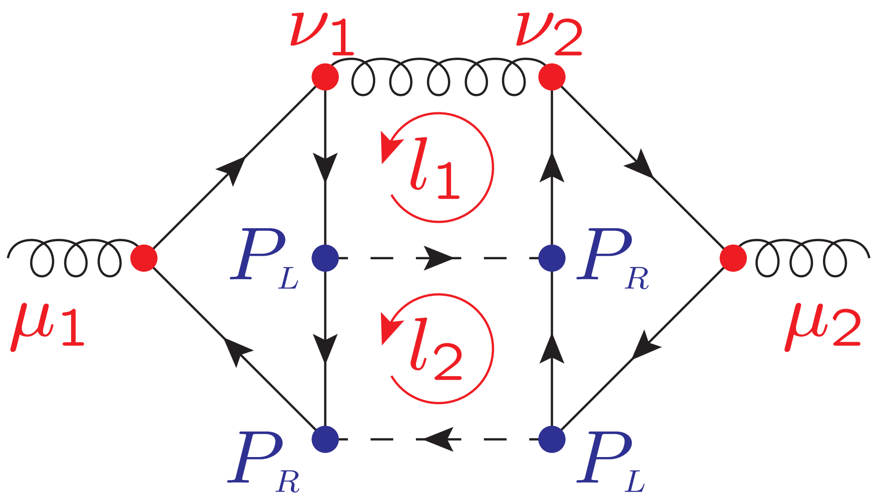
and use

$$\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon_{\nu_1\nu_2\nu_3\nu_4} = -g^{[\mu_1}_{\ \nu_1}g^{\mu_2}_{\ \nu_2}g^{\mu_3}_{\ \nu_3}g^{\mu_4]}_{\ \nu_4}$$

(error of $O(\varepsilon)$)

$\Rightarrow$ Pole part OK if Feynman integrals have only $\frac{1}{\varepsilon}$ poles. ($\surd$)

Treatment of γ_5 in the gluon propagator

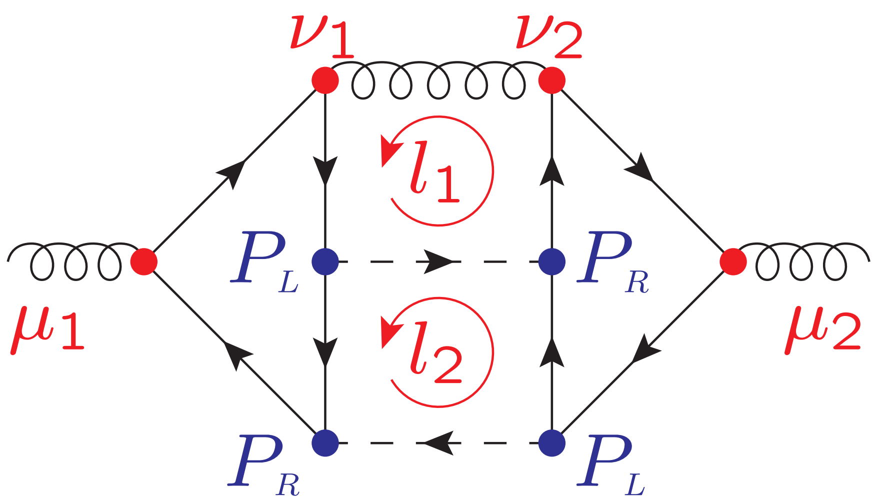


- Use $\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\varepsilon_{\nu_1\nu_2\nu_3\nu_4} = -g^{[\mu_1}_{\nu_1}g^{\mu_2}_{\nu_2}g^{\mu_3}_{\nu_3}g^{\mu_4]}_{\nu_4}\,(1 + \varepsilon \cdot \text{LABEL})$
  $\Rightarrow$ LABEL drops out in final result, only $\frac{1}{\varepsilon}$ poles remain.

- Anticommuting γ_5 to different points in different diagrams changes result in this case!

- Move γ_5 to reading point, e. g. external vertex [Körner, Kreimer, Schilcher]

Treatment of γ_5 in the gluon propagator



Non-naive contribution to gluon self-energy from terms with one γ_5 on each fermion line:

- Move γ_5 to external vertices $\rightarrow$ $\frac{1}{\varepsilon} g_s^4 y_t^4 T_F^2 \left(\frac{4}{3} + 8\zeta_3 \right)$
- Same result for $\gamma_5\,\gamma^{\mu_{\text{ext}}}$ or $\gamma^{\mu_{\text{ext}}}\,\gamma_5$ or $\frac{1}{2}(\gamma_5\,\gamma^{\mu_{\text{ext}}} - \gamma^{\mu_{\text{ext}}}\,\gamma_5)$ [Larin]
  (but same prescription in all 72 affected diagrams)
- Ward identity manifest in the transversal structure of the gluon self-energy respected!

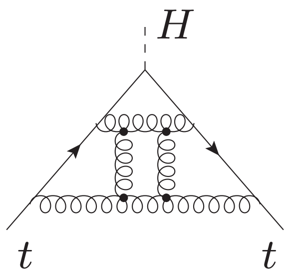
Results

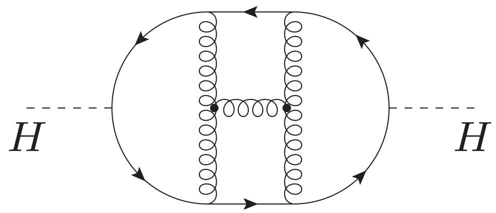
$$\mu^2 \frac{d}{d\mu^2} g_s(\mu) = \beta_{g_s} = \sum_{n=1}^{\infty} \frac{1}{(16\pi^2)^n} \beta_{g_s}^{(n)}$$ (in the $\overline{\text{MS}}$-scheme)

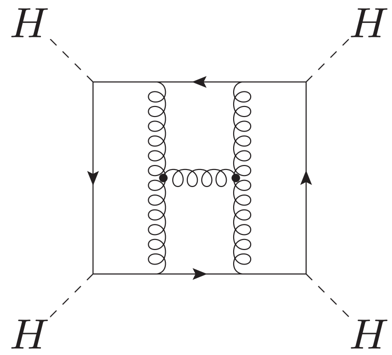
$$\begin{aligned}\frac{\beta_{g_s}^{(4)}}{g_s} = &+ g_s^8 \Big(-\frac{149753}{12} + \frac{1078361}{324} n_f - \frac{50065}{324} n_f^2 - \frac{1093}{1458} n_f^3 \\ &-1782\zeta_3 + \frac{3254}{27}\zeta_3 n_f - \frac{3236}{81}\zeta_3 n_f^2 \Big) \\ &+ g_s^6 y_t^2 \Big(-\frac{9959}{18} + \frac{1625}{54} n_f + 136\zeta_3 \Big) \\ &+ g_s^4 y_t^4 \Big(\frac{423}{2} + \frac{2}{3} - 60\zeta_3 + 4\zeta_3 \Big) \qquad \leftarrow \text{non-naive part} \\ &+ g_s^2 y_t^6 \Big(-\frac{423}{8} - 3\zeta_3 \Big) - 15\, g_s^2 y_t^4 \lambda + 18\, g_s^2 y_t^2 \lambda^2\end{aligned}$$

$$\begin{aligned}\frac{\beta_{g_s}^{(4)}}{\beta_{g_s}^{(1)} (16\pi^2)^3} = &\underbrace{2.26\times 10^{-4}}_{g_s^8} \underbrace{+2.47\times 10^{-5}}_{g_s^6 y_t^2} \underbrace{-1.06\times 10^{-5}}_{g_s^4 y_t^4 (\text{naive})} \underbrace{+4.17\times 10^{-7}}_{g_s^4 y_t^4 (\text{non-naive})} \\ &\underbrace{+2.77\times 10^{-6}}_{g_s^2 y_t^6} \underbrace{+1.06\times 10^{-7}}_{g_s^2 y_t^4 \lambda} \underbrace{-1.82\times 10^{-8}}_{g_s^2 y_t^2 \lambda^2}\end{aligned}$$

III. Connection to pure QCD results







$$\mathcal{L}_{eff} = \mathcal{L}_{QCD} - \frac{y_t}{\sqrt{2}} H\bar{t}t - \frac{\lambda}{4} H^4$$

only external H $\Rightarrow$ insertion of scalar current $j_s = \bar{t}t$ in QCD diagrams.

$$\beta_{y_t} = y_t \gamma_m(\alpha_s) + \ldots,$$

$$\gamma_2^{\Phi} = \frac{y_t^2}{2} \gamma_q^{SS}(\alpha_s) + \ldots$$

scalar correlator $\Pi^S(q^2, m_t) = i \int \mathrm{d}^D x \, e^{iqx} \langle 0| T\,[j_s(x) j_s(0)]\,|0\rangle$

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \Pi^S = -2\gamma_m \Pi^S + \gamma_q^{SS} Q^2 + \gamma_m^{SS} m_t^2, \quad Q^2 = -q^2$$

The scalar correlator

$$\Pi_B^S(q^2, m_t) = i \int \mathrm{d}^D x\, e^{iqx} \langle 0 | T \left[j_s^B(x) j_s^B(0) \right] | 0 \rangle, \quad j_s^B = \bar{t}_B t_B$$

is renormalized as

$$\Pi^S(q^2, m_t) = Z_m^2 \Pi_B^S(q^2, m_t^B) + \left(Z_2^{SS} Q^2 + Z_m^{SS} m_t^2 \right) \mu^{-2\varepsilon}$$

$$\Rightarrow \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \Pi^S = -2\gamma_m \Pi^S + \gamma_q^{SS} Q^2 + \gamma_m^{SS} m_t^2$$

Anomalous dimensions:

$$\gamma_m = -\frac{d \ln Z_m}{d \ln \mu^2},$$

$$\gamma_q^{SS} = \frac{dZ_2^{SS}}{d \ln \mu^2} + (2\gamma_m - \varepsilon) Z_2^{SS},$$

$$\gamma_m^{SS} = \frac{dZ_m^{SS}}{d \ln \mu^2} + (4\gamma_m - \varepsilon) Z_m^{SS}.$$

And for β_λ?

Should be connected to renormalization of

$$
\begin{aligned}
& (2\pi)^D \delta\,(p_1 + p_2 + p_3 + p_4)\;\Gamma(\{p_i\}, \alpha_s, m_t, \mu) \\
= \quad & Z_m^4\, Z_2^2 \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_4 \left(\prod_i e^{ip_i \cdot x_i}\right) \langle 0 | T\,[j_s(x_1)\, j_s(x_2)\, j_s(x_3)\, j_s(x_4)]\,|0\rangle .
\end{aligned}
$$

Start with generating functional of (connected) Green's functions

$$
W(J) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n\, G^{(n)}(x_1, \ldots, x_n)\, J(x_1) \ldots J(x_n)
$$

defined in

$$
Z(\mathcal{L}, J) = e^{iW(J)} = \int \mathcal{D}\,\Phi e^{iS(\Phi) + \int \Phi \cdot J \mathrm{d}^4 x}, \quad S(\Phi) = \int \mathcal{L}(\Phi) \mathrm{d}^4 x.
$$

Quantum Action Principle [Lowenstein; Lam; Breitenlohner] relates properties of (regularized) Lagrangian and the full Green's functions:

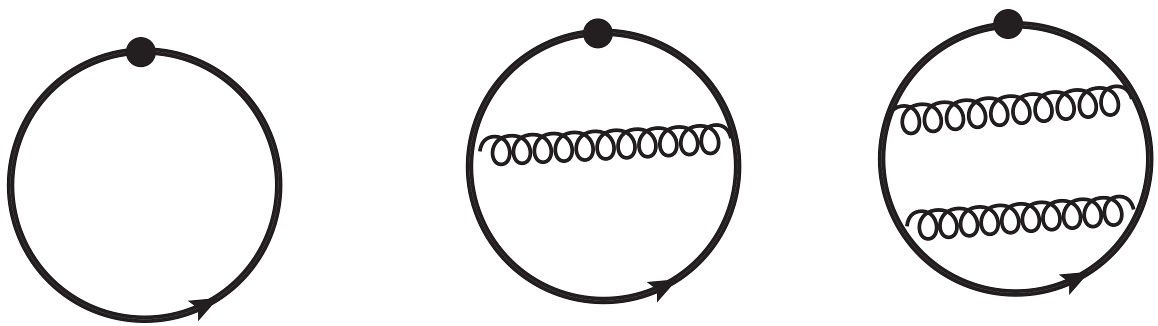
$$
\frac{\partial}{\partial m_t} W(J) \equiv \left(\int \mathcal{D}\,\Phi e^{iS(\Phi) + \int \Phi \cdot J \mathrm{d}^4 x} \frac{\partial}{\partial m_t} S(\Phi) \right) / Z(\mathcal{L}, J),
$$

Renormalized QCD Lagrangian with n_l massless quarks and a massive top:

$$\begin{aligned}\mathcal{L}_{QCD} = & -\frac{1}{4}Z_3\,(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}g\,Z_1^{3g}\,(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)\,(A_\mu \times A_\nu)^a \\ & - \frac{1}{4}g^2\,Z_1^{4g}\,(A_\mu \times A_\nu)^2 + Z_2 \sum_{i=1}^{n_l} \bar{\psi}_i\,(\mathrm{i}\partial\!\!\!/ + gZ_1^{\psi\psi g}Z_2^{-1}A\!\!\!/)\,\psi_i \\ & + Z_2\,\bar{t}\,(\mathrm{i}\partial\!\!\!/ + gZ_1^{\psi\psi g}Z_2^{-1}A\!\!\!/ - Z_m m_t)\,t.\end{aligned}$$

But $\frac{\partial}{\partial m_t}W(J)$ already not finite at $J = 0$!

$$\left(\int \mathcal{D}\,\Phi e^{iS(\Phi)+\Phi\cdot J} \left(\int Z_m Z_2\,\bar{t}t(x)dx \right) \right) / Z(\mathcal{L}, J)$$



Full QCD Lagrangian [Spiridonov]:

$$\mathcal{L}_{QCD}^{full} = \mathcal{L}_{QCD} - E_0^B$$

with vacuum energy

$$E_0^B = \mu^{-2\epsilon}(E_0(\mu) - Z_0(\alpha_s) m_t^4(\mu)),$$

$$\frac{\partial^4}{\partial m_t^4} W(J) \text{ with } \mathcal{L} = \mathcal{L}_{QCD}^{full}$$

$$\Rightarrow \Gamma(\{p_i\}, \alpha_s, m_t, \mu) - i\, 4! Z_0 = \text{finite}$$

In all orders in α_s:

$$\beta_\lambda = y_t^4\,\gamma_0(\alpha_s) \;+\text{higher orders in } y_t,\ g_2,\ g_1,\ \ldots$$

with the anomalous dimension of the vacuum energy

$$\gamma_0 = (4\gamma_m - \varepsilon)\,Z_0 + (\beta(\alpha_s) - \varepsilon)\,\alpha_s \frac{\partial Z_0}{\partial \alpha_s}$$

defined as

$$\frac{dE_0}{d\ln\mu^2} = \gamma_0(\alpha_s)\,m_t^4.$$

Relation between 2-point and 4-point scalar correlator at $q = 0$ via the action principle:

$$\Rightarrow \qquad \gamma_0 = \frac{1}{12}\gamma_m^{SS}.$$

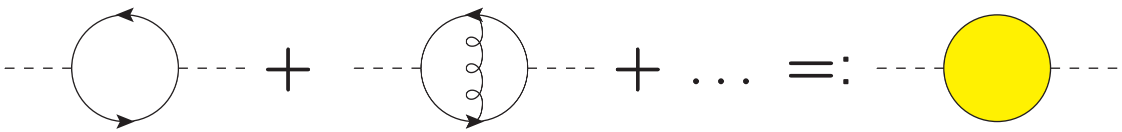
IV. Current status of vacuum stability

Starting values for SM couplings

[Sirlin, Zucchini; Hempfling, Kniehl; Jegerlehner et al; Bezrukov et al; Buttazzo et al; Kniehl, Veretin]

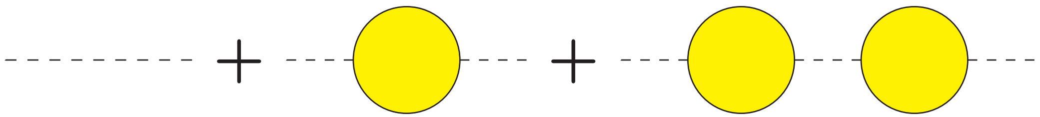
Example: The Higgs propagator

1 PI self energy:



$=: \Sigma(p^2, m_H, g_i)$

full propagator:



$= \frac{i}{p^2 - m_H^2 - \Sigma(p^2, m_H, g_i)}$

Physical mass $M_H \Leftrightarrow$ pole of full propagator

For $\boxed{p^2 = M_H^2} \Rightarrow \boxed{m_H^2 - \Sigma(p^2, m_H, g_i) = M_H^2}$ with $\boxed{m_H = \sqrt{2\,\lambda}\,v}$

Same for M_t, M_W, M_Z, $G_F \Rightarrow$ Solve for λ, v, y_t, etc.

Starting values for SM couplings

Experimental input

$M_{\rm t} = M_t = 174.6^{\rm pole} \pm 1.9$ GeV.

$M_{\rm t}^{\rm MC} \approx 173.34 \pm 0.76$ GeV $\Rightarrow$ $M_{\rm t} = 173.39^{+1.12}_{-0.98}$ GeV

[Moch et al] via MSR mass [Hoang et al]

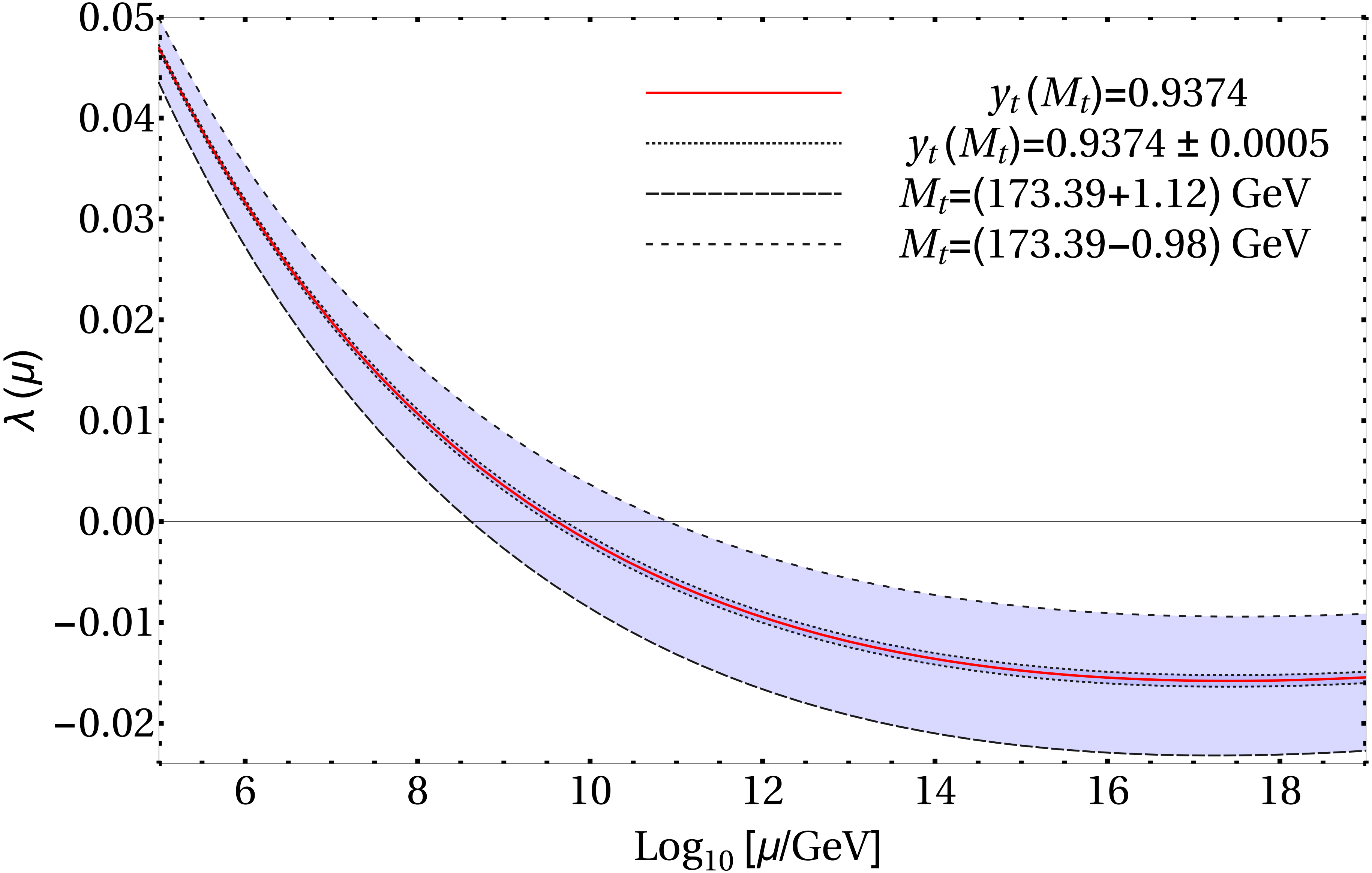
$M_H \approx 125.09 \pm 0.24$ GeV

$\alpha_s \approx 0.1181 \pm 0.0013$

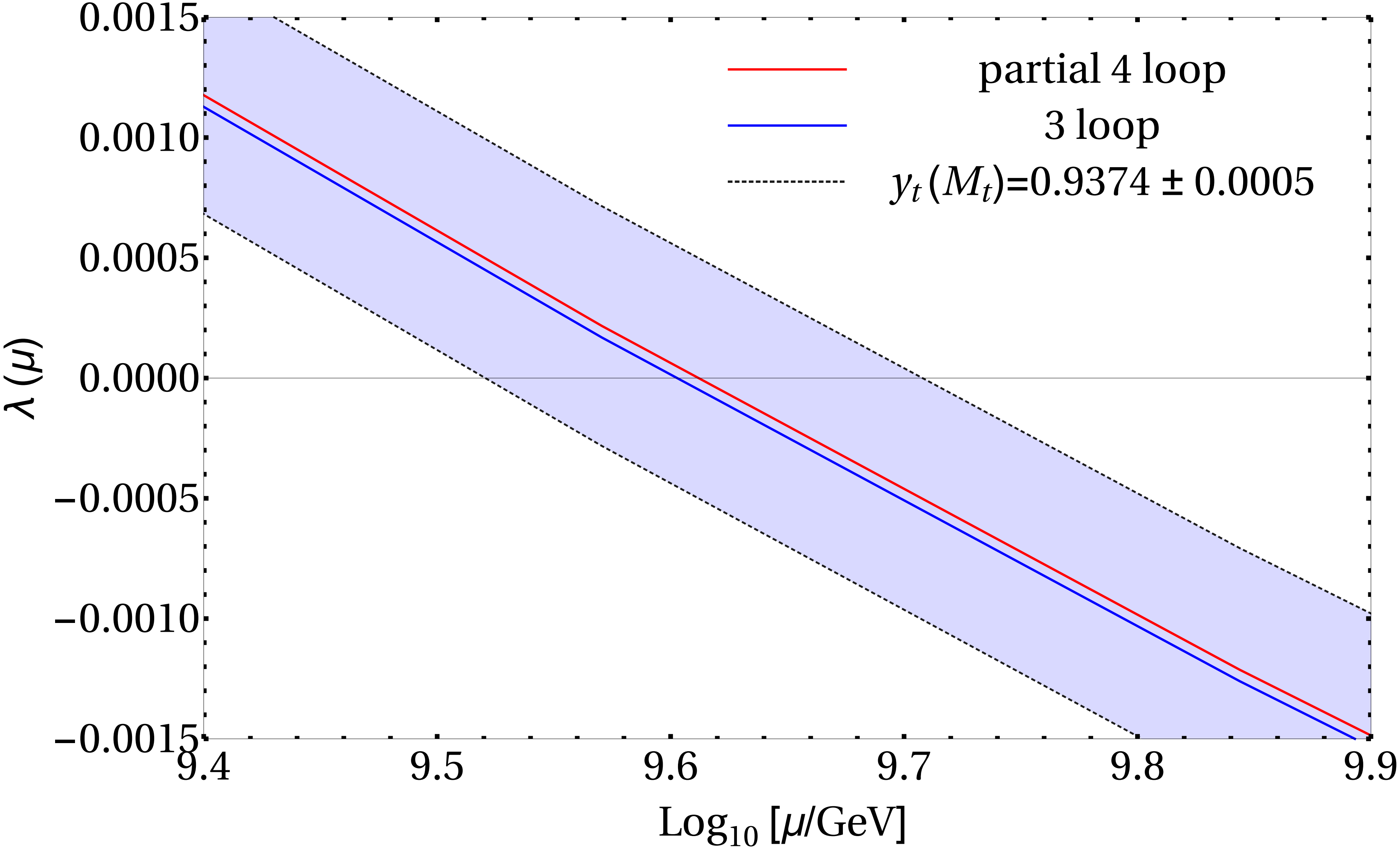
$\overline{\rm MS}$-couplings:

$$
\begin{aligned}
g_s(M_{\rm t}) &= 1.1652 \pm 0.0035(\text{exp}), \\
y_t(M_{\rm t}) &= 0.9374^{+0.0063}_{-0.0062}(\text{exp}) \pm 0.0005 \text{ (2 loop matching)}, \\
\lambda(M_{\rm t}) &= 0.1259 \pm 0.0005(\text{exp}) \pm 0.0003 \text{ (2 loop matching)}, \\
g_2(M_{\rm t}) &= 0.6483, \\
g_1(M_{\rm t}) &= 0.3587
\end{aligned}
$$

Evolution of $\lambda(\mu)$



Evolution of $\lambda(\mu)$



Summary

- Stability of SM vacuum $\leftrightarrow$ $\boxed{\lambda > 0}$

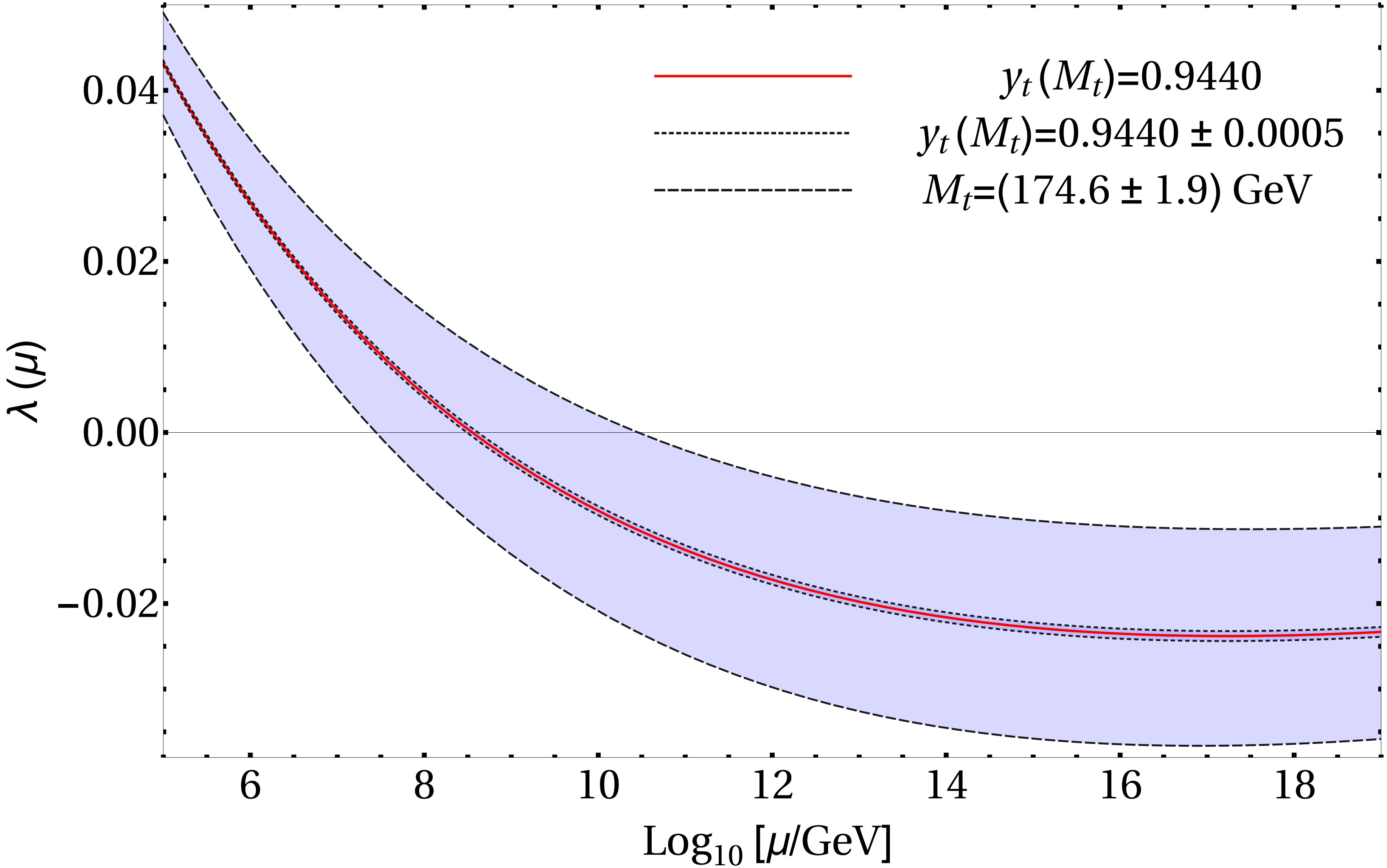
- β-functions in the SM at 3 loops for full SM and now partial 4 loop

- Leading contribution to β_λ and β_{y_t} connected to pure QCD current correlators to all orders.

- Limiting problem for full 4 loop β-functions and 3 loop matching in the SM: γ_5

- SM vacuum looks not stable (metastable), M_t largest source of uncertainty

Evolution of $\lambda(\mu)$



Evolution of $\lambda(\mu)$