# Renormalisation circumvents Haag's theorem

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Theory seminar DESY Zeuthen, 31.03.2016

# Outline

# The quest for understanding canonical quantum fields

1. Constructive approaches by Glimm & Jaffe[1]: $d \leq 3$
2. Axiomatic quantum field theory by Wightman & Gårding[2] :
   - axioms to construe quantum fields in terms of operator theory
   - hard work, some results (eg PCT, spin-statistics theorem)
   - interesting: triviality results (no-go theorems)
3. other axiomatic approaches, eg algebraic quantum field theory

**General form of no-go theorems**

$\phi$ quantum field with properties so-and-so $\Rightarrow$ $\phi$ is trivial.

**3 forms of triviality**

(1) $\phi$ free, (2) $\phi = c\mathbb{I}$ or (3) $\phi = 0$

[1] J.Glimm, A.Jaffe, Springer (1981)

[2] A. S. Wightman, L.Gårding, Arkiv för Fysik, 28, 129 - 184 (1964)

# Triviality of sharp-spacetime fields

**Theorem (Wightman, 1964)**

*Let $\varphi(x)$ be a Poincaré-covariant Hermitian scalar field, that is,*

$$U(a,\Lambda)\varphi(x)U(a,\Lambda)^{\dagger} = \varphi(\Lambda x + a)$$

*and suppose it is a well-defined operator with vacuum $\Omega_0$ in its domain. Then there is a constant $c \in \mathbb{C}$ such that*

$$\varphi(x)\Omega_0 = \sqrt{c}\,\Omega_0 \quad \textit{thus} \quad \langle \Omega_0 | \varphi(x_1) ... \varphi(x_n)\Omega_0 \rangle = c^{n/2}.$$

# Canonical computation

## Free canonical field

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \, [e^{-ip\cdot x} a(\mathbf{p}) + e^{ip\cdot x} a^\dagger(\mathbf{p})].$$

QUESTION: $||\varphi(x)\Omega_0|| = ?$

ANSWER: unphysical question, result:

$$||\varphi(x)\Omega_0||^2 = \langle\Omega_0|\varphi(x)\varphi(x)\Omega_0\rangle = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\mathbf{p}^2 + m^2}} = \infty$$

# What to make of it?

1. Implement Heisenberg's uncertainty:

$$\varphi(t,f) = \int d^3x \; f(\mathbf{x})\varphi(t,\mathbf{x}),$$

free fields smoothed out in space, then

$$||\varphi(t,f)\Omega_0||^2 = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{|f(\mathbf{p})|^2}{\sqrt{\mathbf{p}^2+m^2}} < \infty.$$

2. Too strong an assumption: $x \mapsto \langle\Psi|U(x,\mathbb{I})\Phi\rangle$ continuous for all state vectors $\Psi,\Phi \in \mathfrak{H}$, must be weakened.

# Expedient: operator-valued distributions

1. Smearing in both space and time such that

$$f \mapsto \langle \Psi | \varphi(f) \Phi \rangle \qquad \Psi, \Phi \in \mathfrak{H},$$

is distribution on Schwartz space $\mathcal{S}(M)$.

2. Poincaré covariance: for all $f$ Schwartz

$$U(a,\Lambda)\varphi(f)U(a,\Lambda)^\dagger = \varphi(\{a,\Lambda\}f),$$

where $(\{a,\Lambda\}f)(x) = f(\Lambda^{-1}(x-a))$.

3. Sharp-time field

$$\varphi(t,f) := \lim_{\epsilon \to 0} \varphi(\delta_t^\epsilon \otimes f)$$

if existent, $\{\delta_t^\epsilon\}_{\epsilon>0}$ Dirac family centred at $t \in \mathbb{R}$.

# Interaction picture Hamiltonian

**Wightman, 1967**

Hamiltonian $H = H_0 + H_{\text{int}}$, where $H_{\text{int}} = \int d^3x\, \mathcal{H}_{\text{int}}(\mathbf{x})$. Then

$$H_I(t) = e^{iH_0t} H_{\text{int}} e^{-iH_0t} = \int d^3x\, e^{iH_0t} \mathcal{H}_{\text{int}}(\mathbf{x}) e^{-iH_0t}$$

entails $||H_I(t)\Omega_0|| = 0$ for all $t$, thus $H_I(t)\Omega_0 = 0$.

*Proof.* By $e^{-iH_0t}\Omega_0 = \Omega_0$ and translational invariance one has

$$||H_I(t)\Omega_0||^2 = \int d^3x \int d^3y\, \langle\Omega_0|\mathcal{H}_{\text{int}}(\mathbf{x})\mathcal{H}_{\text{int}}(\mathbf{y})\Omega_0\rangle$$
$$= \int d^3x \int d^3y\, \langle\Omega_0|\mathcal{H}_{\text{int}}(0)\mathcal{H}_{\text{int}}(\mathbf{y})\Omega_0\rangle$$

# Response: simply vacuum diagrams

$||H_I(t)\Omega_0||$ not of interest, yet in

**Vacuum expectation of Dyson's series**

$$\langle\Omega_0|S\Omega_0\rangle = \langle\Omega_0|T\{e^{-i\int dt\, H_I(t)}\}\Omega_0\rangle$$
$$= \sum_{n\geq 0}\frac{(-1)^n}{n!}\int dt_1\ldots\int dt_n\langle\Omega_0|\mathrm{T}\{H_I(t_1)\ldots H_I(t_n)\}\Omega_0\rangle$$

there one finds

$$\int dt\int ds\,\langle\Omega_0|\mathrm{T}\{H_I(t)H_I(s)\}\Omega_0\rangle = \text{vacuum diagrams}$$

but cancel out in Gell-Mann-Low formula.

# Canonical (anti)commutation relations (CCR/CAR)

Triviality of fields obeying CCR/CAR

**Theorem (Baumann): scalar field $\varphi(t,f)$ in $d \geq 5$**

Nontrivial commutator given by

$$[\varphi(t,f), \partial_t \varphi(t,g)] = i(f,g) \qquad \Rightarrow \qquad \varphi(t,f) \text{ free}$$

Weyl form of CCR: $d \geq 4$ (Sinha).

**Theorem (Powers): Dirac field $\psi(t,f)$ in $d \geq 3$**

Nontrivial anticommutator given by

$$\{\psi(t,f), \psi^{\dagger}(t,g)\} = i(f,g) \qquad \Rightarrow \qquad \psi(t,f) \text{ free.}$$

Response: so be it.

# Euclidean version of Haag's theorem

## Theorem (Schrader, 1974)

$d\mu_0(\varphi)$ Gaussian measure on distributions $\mathcal{D}'(\mathbb{R}^2)$, interaction term

$$V_\ell(\lambda, \varphi) = \lambda \int_{B_\ell} d^2x : P(\varphi(x)) :$$

$P$ bounded normalised polynomial, $B_\ell = [-\ell/2, \ell/2]^2$ and

$$d\mu_{\ell,\lambda}(\varphi) = e^{-[V_\ell(\lambda,\varphi) - E_\ell(\lambda)]} d\mu_0(\varphi)$$

with $e^{-E_\ell(\lambda)} = \int e^{-V_\ell(\lambda,\varphi)} d\mu_0(\varphi)$. Then $d\mu_{\infty,\lambda}$ exists but

$$d\mu_0 \ \text{ and } \ d\mu_{\infty,\lambda} \ \text{ have mutually disjoint support.}$$

# Wightman framework: axioms (scalar field)

**Axiom 0** (Relativistic Hilbert space) Hilbert space $\mathfrak{H}$, strongly continuous unitary representation $(a, \Lambda) \mapsto U(a, \Lambda)$ of the connected Poincaré group, unique vacuum $\Omega_0 \in \mathfrak{H}$

$$U(a, \Lambda)\Omega_0 = \Omega_0 \qquad \forall (a, \Lambda).$$

**Axiom I** (Spectral condition) generator of translations $P_\mu$ has spectrum in forward lightcone $\sigma(P) \subset \overline{V}_+$.

**Axiom II** (Quantum field)
Poincaré-covariant operator family $\{\varphi(f)\}_{f \in \mathcal{S}(M)}$, common dense and Poincaré-stable domain $\mathfrak{D} \subset \mathfrak{H}$, generate dense subspace $\mathfrak{D}_0 = \mathbb{C}[\varphi(f) : f \in \mathcal{S}(M)]\Omega_0 \subset \mathfrak{H}$.

**Axiom III** (Locality) $[\varphi(f), \varphi(g)] = 0$
if $f, g$ have spacelike-separated support.

# Haag's theorem

**Haag's theorem (Wightman, Jost, Schroer, Reeh, Schlieder)**

1. $\varphi(t,f)$ and $\varphi_0(t,f)$ sharp-time Wightman quantum fields of mass $m > 0$ with Hilbert spaces $\mathfrak{H}$ and $\mathfrak{H}_0$, respectively. In any spacetime dimension $d \geq 2$.
2. Suppose at time $t$ there is a unitary intertwiner $V\colon \mathfrak{H}_0 \to \mathfrak{H}$ such that

$$\varphi(t,f) = V\varphi_0(t,f)V^{-1}.$$

Then if $\varphi_0(t,f)$ is free, so is $\varphi(t,f)$.

# Proof of Haag's theorem part I (sketch)

1. First prove intertwiner relates vacua: $V\Omega_0 = \Omega$. Then follows

$$\begin{aligned}\langle\Omega_0|\varphi_0(t,f)\varphi_0(t,h)\Omega_0\rangle &= \langle\Omega_0|V^{-1}\varphi(t,f)VV^{-1}\varphi(t,h)V\Omega_0\rangle \\ &= \langle\Omega|\varphi(t,f)\varphi(t,h)\Omega\rangle.\end{aligned}$$

2. This implies

$$\langle\Omega_0|\varphi_0(t,\mathbf{x})\varphi_0(t,\mathbf{y})\Omega_0\rangle = \langle\Omega|\varphi(t,\mathbf{x})\varphi(t,\mathbf{y})\Omega\rangle$$

in the sense of distributions, ie

$$\langle\Omega|\varphi(t,\mathbf{x})\varphi(t,\mathbf{y})\Omega\rangle = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\frac{e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{\sqrt{\mathbf{p}^2+m^2}}.$$

# Proof of Haag's theorem part II (sketch)

3. Use Poincaré covariance to show

$$\langle\Omega|\varphi(x)\varphi(y)\Omega\rangle = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\frac{e^{ip\cdot(x-y)}}{\sqrt{\mathbf{p}^2+m^2}}.$$

   for spacelike distances $(x-y)^2 < 0$.
4. Technical (edge of the wedge theorem, ...): then for all $x - y \in M$.
5. Employ JOST-SCHROER THEOREM :
   $\varphi$ has free-field 2-point function
   $\implies \varphi$ has free-field $n$-point correlators.

# Countering canonical narrative

## Heisenberg picture field (interacting field)

$$\varphi(t,\mathbf{x}) = e^{iHt}\varphi(\mathbf{x})e^{-iHt}$$

## Interaction picture field (free field)

$$\varphi_0(t,\mathbf{x}) = e^{iH_0t}\varphi(\mathbf{x})e^{-iH_0t}$$

## Intertwining relation

$$\varphi(t,\mathbf{x}) = e^{iHt}e^{-iH_0t}\varphi_0(t,\mathbf{x})e^{iH_0t}e^{-iHt} = V_t^\dagger\varphi_0(t,\mathbf{x})V_t$$

# Interaction picture S-matrix & Gell-Mann-Low formula

## Interaction picture S-matrix

$$\mathrm{S} = \lim_{t\to\infty} \lim_{s\to-\infty} e^{iH_0 t} e^{-iH(t-s)} e^{-iH_0 s} = \lim_{t\to\infty} \lim_{s\to-\infty} V_t V_s^\dagger$$

as time-ordered exponentials:

$$S = T\{e^{-i\int \mathcal{H}_I}\} \qquad V_t = T\{e^{-i\int_0^t \mathcal{H}_I}\}$$

## Gell-Mann-Low formula

$$\langle\Omega|\mathrm{T}\{\varphi(x_1)...\varphi(x_n)\}\Omega\rangle = \frac{\langle\Omega_0|\mathrm{T}\{\mathrm{S}\varphi_0(x_1)...\varphi_0(x_n)\}\Omega_0\rangle}{\langle\Omega_0|\mathrm{S}\Omega_0\rangle}$$
$$\neq \langle\Omega_0|\mathrm{T}\{\varphi_0(x_1)...\varphi_0(x_n)\}\Omega_0\rangle$$

Contradiction to Haag's theorem? No.

# Nontrivial spin case

Axioms modified

**Poincaré covariance (Axiom II)**

$$U(a,\Lambda)\phi_s(x)U(a,\Lambda) = \sum_{r=1}^{n} S_{sr}(\Lambda^{-1})\phi_r(\Lambda x + a)$$

$S_{sr}$ finite-dimensional representation of Poincaré group

**Anticommutator (Axiom III)**

For half-integer spin

$$[\phi_s(f), \phi_r(g)^\dagger]_+ = 0$$

if $f, g$ have mutually spacelike-separated support.

Then: Higher spin fields also affected by Haag's theorem.

# Special species: gauge fields

**Theorem (Strocchi, 1967)**

$A_\mu$ Wightman field with Poincaré covariance (Axiom II)

$$U(a,\Lambda)A_\mu(f)U(a,\Lambda)^\dagger = \Lambda_\mu^{\ \sigma} A_\sigma(\{a,\Lambda\})f)$$

or commuting at spacelike distances (Axiom III).
Assume $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfies $\partial_\mu F^{\mu\nu} = 0$, then

$$\langle \Omega, F_{\mu\nu}(f) F_{\rho\sigma}(g)\Omega\rangle = 0.$$

If all axioms are fulfilled, then $F_{\mu\nu} = 0$ and thus $A_\mu = \partial_\mu g(x)$.

Expedient (free case): Gupta-Bleuler quantisation in Krein space.

# Change of story: renormalisation

Haag's theorem no big surprise: UV divergences in PT

**backpedal: so far, wrong Lagrangian**

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{g}{4!}\varphi^4$$

**new, renormalised Lagrangian**

$$\mathcal{L}_r = \frac{1}{2}(\partial\varphi_r)^2 - \frac{1}{2}m_r^2\varphi_r^2 - \frac{g_r}{4!}\varphi_r^4 + \mathcal{L}_{ct}$$

with counterterm Lagrangian

$$\mathcal{L}_{ct} = \frac{1}{2}(Z-1)(\partial\varphi_r)^2 - \frac{1}{2}m_r^2(Z_m - 1)\varphi_r^2 - \frac{g_r}{4!}(Z_g - 1)\varphi_r^4$$

# Counterterms induce mass shift

Canonical narrative continued

## New renormalised interaction part

replace old by new interaction part

$$\mathcal{L}^r_{int} = -\frac{g_r}{4!} Z_g \varphi_r^4 \underbrace{-\frac{1}{2}(Z-1)(\partial\varphi_r)^2 - \frac{1}{2} m_r^2 (Z_m - 1)\varphi_r^2}_{\text{induce mass shift}} .$$

and put it through 'Gell-Mann-Low machine' with

## new renormalised S-matrix

$$\mathsf{S}_r = \mathsf{T}\{e^{i\int \mathcal{L}^r_{int}}\}$$

$\Rightarrow$ finite results in perturbation theory, physically sensible.

# Mass shift in free theory

Consider mass shift $m_0^2 \to m^2 = m_0^2 + \delta m^2$ in free theory[3]:

**Theorem (Haag's theorem for free fields)**

*Let $\varphi$ and $\varphi_0$ be two free fields of masses $m$ and $m_0$, respectively. If at time $t$ there is a unitary map $V$ such that*

$$\varphi(t, \mathbf{x}) = V \varphi_0(t, \mathbf{x}) V^{-1},$$

*then $m = m_0$.*

Interacting theory unitarily equivalent to free theory?

[3] Reed, Simon (Academic Press, 1975)

# Mass shift, canonical treatment

## Mass shift as interaction

mass shift $m^2 = m_0^2 + \delta m^2$ implemented by

$$\mathcal{L}_m = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_0^2\varphi^2 - \frac{1}{2}\delta m^2\varphi^2$$

perturbative Gell-Mann-Low procedure

$$\langle\Omega|\mathrm{T}\{\varphi(x)\varphi(y)\}\Omega\rangle = \frac{\langle\Omega_0|\mathrm{T}\{\mathrm{S}\varphi_0(x)\varphi_0(y)\}\Omega_0\rangle}{\langle\Omega_0|\mathrm{S}\Omega_0\rangle},$$

with S-matrix $\mathrm{S} = \mathrm{T}\{e^{-i\frac{1}{2}\delta m^2\int\varphi^2}\}$ and formal intertwiner

$$V_t = T\{e^{-i\frac{1}{2}\delta m^2\int_0^t\varphi^2}\}$$

# Mass shift intertwiner not unitary

## Claim

$\varphi_0(t,\mathbf{x})$ free field of mass $m_0$, for

$$\varphi(t,\mathbf{x}) = T\{e^{i\frac{1}{2}\delta m^2 \int_0^t \varphi_0^2}\}\varphi_0(t,\mathbf{x})T\{e^{-i\frac{1}{2}\delta m^2 \int_0^t \varphi_0^2}\}$$

subjected to Gell-Mann-Low procedure with mass shift interaction one gets

$$\langle\Omega|\mathrm{T}\{\varphi(x)\varphi(y)\}\Omega\rangle = \frac{\langle\Omega_0|\mathrm{T}\{\mathrm{S}\varphi_0(x)\varphi_0(y)\}\Omega_0\rangle}{\langle\Omega_0|\mathrm{S}\Omega_0\rangle} = i\Delta_F(x-y;m^2),$$

with mass $m^2 = m_0^2 + \delta m^2$ and S-matrix $S = T\{e^{-i\frac{1}{2}\delta m^2 \int \varphi_0^2}\}$.

Therefore $V_t = T\{e^{-i\frac{1}{2}\delta m^2 \int_0^t \varphi_0^2}\}$ not unitary.

# Mass shift destroys unitary equivalence

## Renormalisation leads to mass shift

Full renormalised theory: self-energy $\Sigma_r(g_r, p, m_r)$

$$G_r(g_r, p) = \frac{i}{p^2 - m_r^2 - \Sigma_r(g_r, p, m_r) + i0^+}$$

incurs momentum (and scale) and coupling-dependent mass shift!

Theory **cannot possibly** be unitary equivalent to theory with

$$G_{0,r}(p) = \frac{i}{p^2 - m_r^2 + i0^+}.$$

# Conclusion

1. Axiomatic framework interesting but not the answer.
2. Some aspects of import, especially Haag's theorem to be taken seriously.
3. Haag's theorem most likely averted by renormalisation.
4. Thus renormalised interacting quantum fields not unitarily equivalent to free fields.

Further problems: renormalised S-matrix still unitary ... ?