The event GW150914: Gravitational waves and coalescing black-hole binaries

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Outline

- Some History on Gravitational Waves
- Details on GW150914
- Interferometric Detection of GWs
- Gravitational Waves
- Hamiltonian for General Relativity
- Binary Black Hole Spacetimes
- Higher order post-Newtonian Hamiltonians
- Orbital Motion (ISCO)
- EOB Formalism

Some History on Gravitational Waves

1916, 1918 Einstein

1923 Eddington

1941 Landau/Lifshitz

1957 H. Bondi

1960, 1969 J. Weber, bar detector

1969 K. Thorne, radiation reaction

1972 R. Weiss (MIT), interferometric detector

1974 H. Billing (MPI), interferometric detector

1974, 1978 Hulse-Taylor pulsar, radiation damping

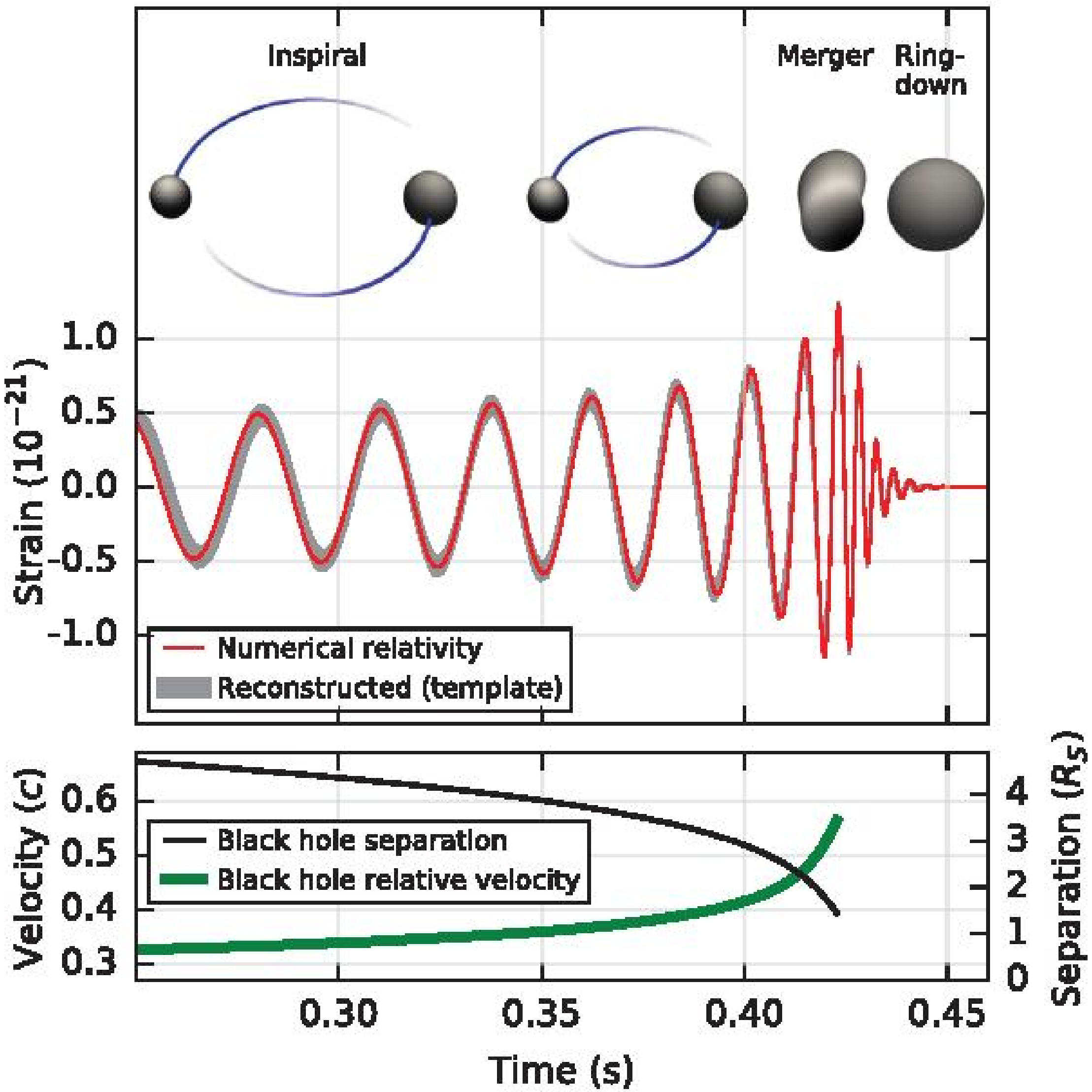
1989 LIGO (Caltech, MIT), Virgo (F-I), GEO600 (G-GB)

2002 LIGO, data aquisition

2015 aLIGO, data aquisition

Details on GW150914

B. P. Abbott *et al.*, Phys. Rev. Lett. **116**, 061102 (2016)



$$h(t)L = \Delta L(t) = \delta L_x - \delta L_y$$

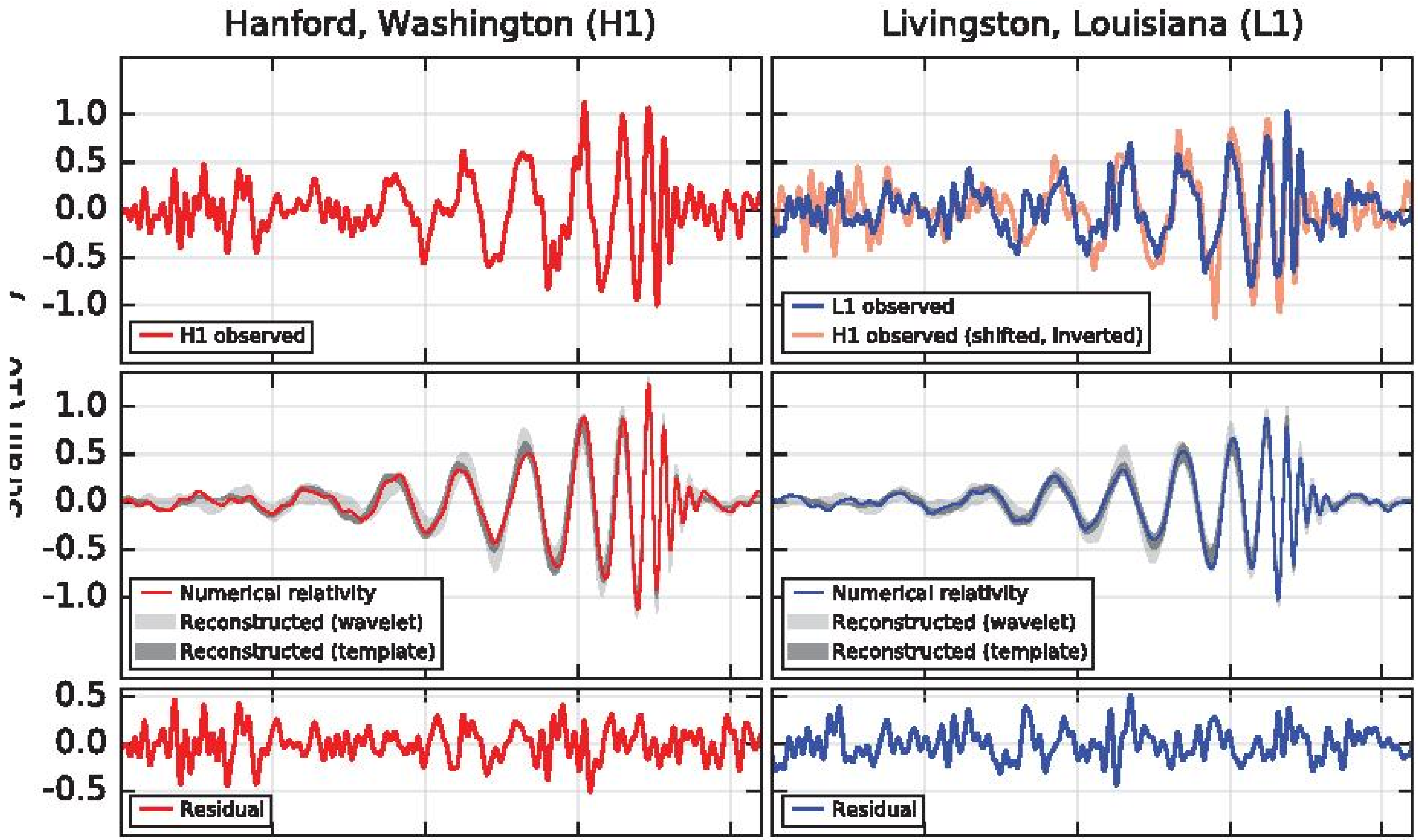

Hanford, Washington (H1)
Livingston, Louisiana (L1)
1.0
0.5
0.0
-0.5
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual

TABLE I. Source parameters for GW150914. We report median values with 90% credible intervals that include statistical errors, and systematic errors from averaging the results of different waveform models. Masses are given in the source frame; to convert to the detector frame multiply by $(1+z)$ [90]. The source redshift assumes standard cosmology [91].

Primary black hole mass	$36^{+5}_{-4} M_{\odot}$
Secondary black hole mass	$29^{+4}_{-4} M_{\odot}$
Final black hole mass	$62^{+4}_{-4} M_{\odot}$
Final black hole spin	$0.67^{+0.05}_{-0.07}$
Luminosity distance	410^{+160}_{-180} Mpc
Source redshift z	$0.09^{+0.03}_{-0.04}$

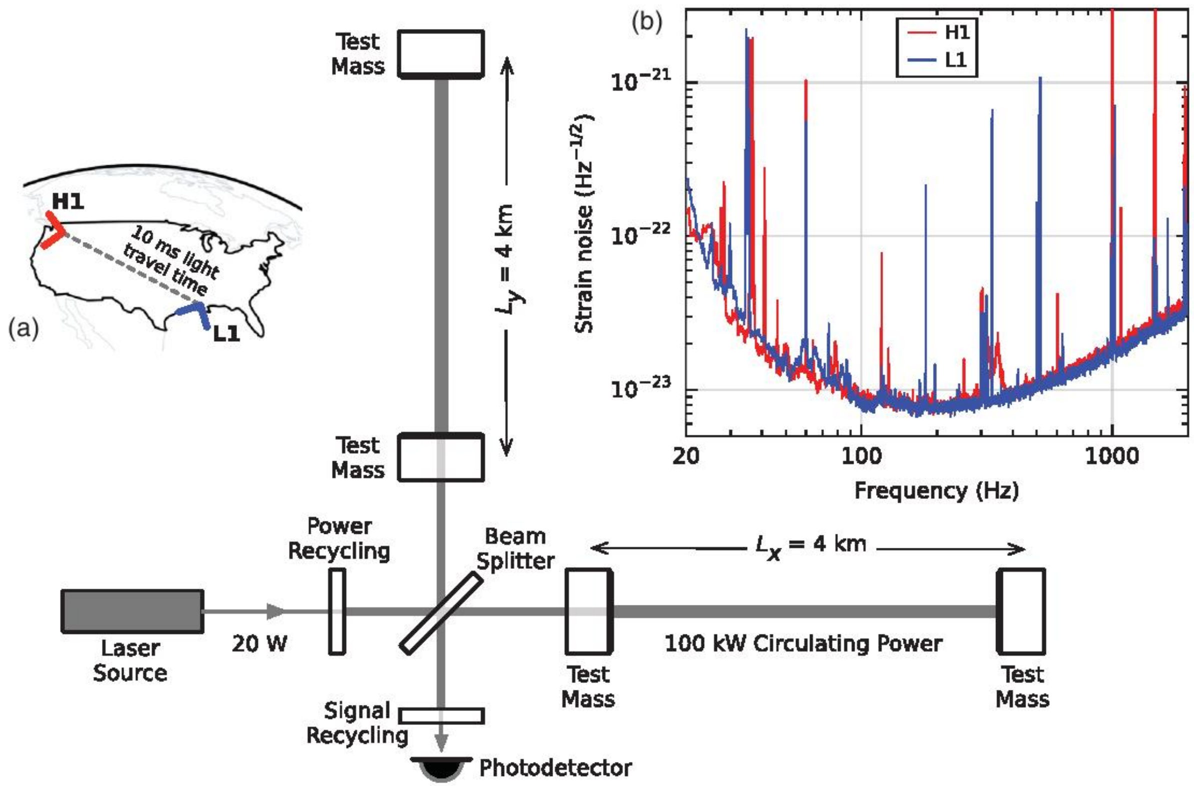
source power (3 solar masses radiated away in 0.2 seconds)

$$6 \times 10^{30}\mathrm{kg}\ 9 \times 10^{16} \left(\frac{\mathrm{m}}{\mathrm{sec}}\right)^2 /200\mathrm{ms} = 2.7 \times 10^{48}\mathrm{Watts} = 0.75 \times 10^{-4}\frac{c^5}{G}$$

maximum power of a single process

$$\frac{c^5}{G} = \frac{Mc^2}{(GM/c^2)/c} = 3.6 \times 10^{52}\mathrm{Watts}$$

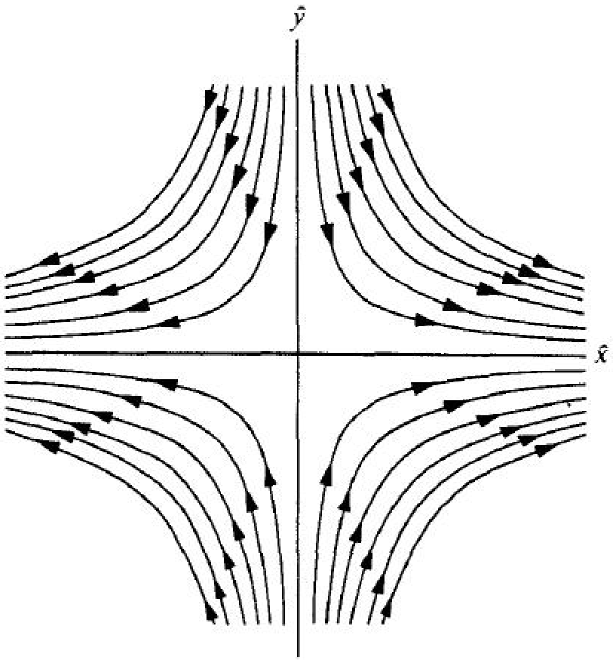
Interferometric Detection of GWs


(a)
H1
L1
10 ms light travel time
(b)
H1
L1
Strain noise (Hz$^{-1/2}$)
10^{-21}
10^{-22}
10^{-23}
20
100
1000
Frequency (Hz)
Test Mass
Test Mass
L_y = 4 km
L_x = 4 km
Power Recycling
Beam Splitter
Laser Source
20 W
100 kW Circulating Power
Test Mass
Test Mass
Signal Recycling
Photodetector

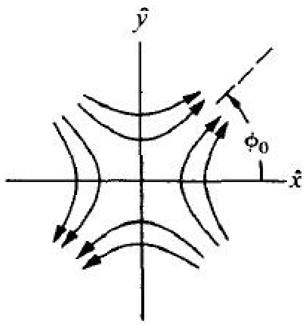
$$\begin{aligned}
\text{line element} \qquad ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
\text{gravitational wave} \qquad ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + h_{ij}^{\mathrm{TT}} dx^i dx^j \\
\text{electromagnetic wave} \qquad A &= A_\mu dx^\mu = A_i^{\mathrm{T}} dx^i
\end{aligned}$$

Fermi normal coordinates

$$ds^2 = -(1 - \frac{1}{2c^2} \ddot{h}_{ij}^{\mathrm{TT}} \hat{x}^i \hat{x}^j) c^2 d\hat{t}^2 + d\hat{x}^i d\hat{x}^i \qquad (|\hat{\mathbf{x}}| << c/f)$$



(a) Force lines for $\ddot{A}_\times = 0$, $\ddot{A}_+ > 0$



(b)

gravitational quadrupole wave

propagating in $\hat{z}$-direction

$$d^2\hat{x}/d\hat{t}^2 = \frac{1}{2}(\ddot{A}_+\hat{x} + \ddot{A}_\times\hat{y})$$

$$d^2\hat{y}/d\hat{t}^2 = \frac{1}{2}(-\ddot{A}_+\hat{y} + \ddot{A}_\times\hat{x})$$

$$d^2\hat{z}/d\hat{t}^2 = 0$$

MTW: Gravitation (1973)

chirp mass $$\mathcal{M} := \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} = \frac{c^3}{G}\left(\frac{5}{96}\frac{\dot{f}}{\pi^{8/3} f^{11/3}}\right)^{3/5}$$

velocity $$\frac{v}{c} = \left(\frac{GM\pi f}{c^3}\right)^{1/3}$$

gravitational quadrupole wave

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{2}{r}\frac{G}{c^4} P_{ijkm}(\mathbf{n}) \ddot{Q}_{km}\left(t - \frac{r}{c}\right)$$

$$Q_{km} = \sum_a m_a (x_k^a x_m^a - \frac{1}{3}\delta_{km} x_l^a x_l^a)$$

Gravitational Waves

Multipole expansion of far zone (FZ) field (e.g., Blanchet in LRR)

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{G}{c^4}\frac{P_{ijkm}(\mathbf{n})}{r}\sum_{l=2}^{\infty}\left\{\left(\frac{1}{c^2}\right)^{\frac{l-2}{2}}\frac{4}{l!}\,\mathrm{M}^{(l)}_{kmi_3\ldots i_l}(t-\frac{r_*}{c})\;N_{i_3\ldots i_l}\right.$$

$$\left. + \left(\frac{1}{c^2}\right)^{\frac{l-1}{2}}\frac{8l}{(l+1)!}\,\epsilon_{pq(k}\;\mathrm{S}^{(l)}_{m)pi_3\ldots i_l}(t-\frac{r_*}{c})\;n_q\;N_{i_3\ldots i_l}\right\}$$

$$\mathrm{M}_{ij}(t-\frac{r_*}{c}) = \widehat{\mathrm{M}}_{ij}\left(t-\frac{r_*}{c}\right)$$

$$+ \frac{2Gm}{c^3}\int_0^{\infty}dv\,\ln\left(\frac{v}{2b}\right)\widehat{\mathrm{M}}^{(2)}_{ij}(t-\frac{r_*}{c}-v)+O(1/c^4),$$

$$r_* = r + \frac{2Gm}{c^2}\ln\left(\frac{r}{cb}\right)+O(1/c^3)$$

Luminosity and energy loss

$$\mathcal{L}(t) = \frac{c^3}{32\pi G} \oint_{\mathrm{FZ}} (\partial_t h_{ij}^{\mathrm{TT}})^2 r^2 d\Omega$$

$$\begin{aligned}
\mathcal{L}(t) &= \frac{G}{5c^5} \sum_{n=0}^{\infty} \left(\frac{1}{c^2}\right)^n \hat{\mathcal{L}}_n(t) \\
&= \frac{G}{5c^5} \left\{ \mathrm{M}_{ij}^{(3)} \mathrm{M}_{ij}^{(3)} + \frac{1}{c^2} \left[\frac{5}{189} \mathrm{M}_{ijk}^{(4)} \mathrm{M}_{ijk}^{(4)} + \frac{16}{9} \mathrm{S}_{ij}^{(3)} \mathrm{S}_{ij}^{(3)} \right] \right. \\
&+ \left. \frac{1}{c^4} \left[\frac{5}{9072} \mathrm{M}_{ijkm}^{(5)} \mathrm{M}_{ijkm}^{(5)} + \frac{5}{84} \mathrm{S}_{ijk}^{(4)} \mathrm{S}_{ijk}^{(4)} \right] \right\}
\end{aligned}$$

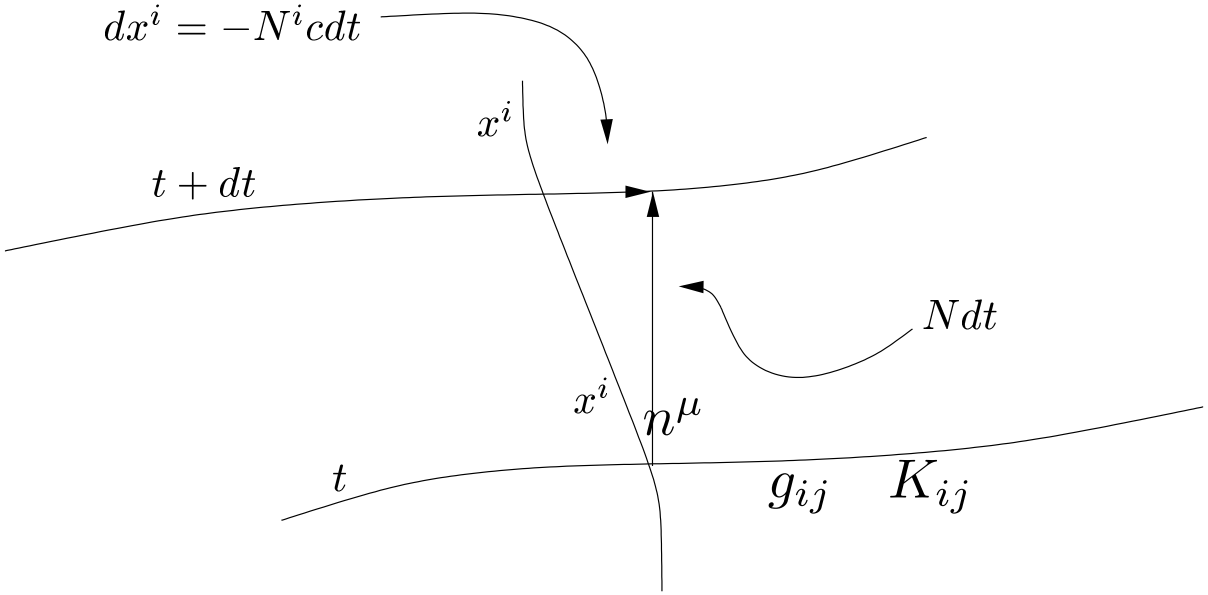
$$- < \frac{d\mathcal{E}(t_{\mathrm{ret}})}{dt} > \;=\; < \mathcal{L}(t) >$$

Hamiltonian for General Relativity

3+1 splitting of spacetime

$$n^\mu = (1, -N^i)/N$$

$$n_\mu = (-N, 0, 0, 0)$$



$$K_{ij} = -N\Gamma^0_{ij} = -Ng^{0\mu}(g_{i\mu,j} + g_{j\mu,i} - g_{ij,\mu})/2$$

$$\boxed{ds^2 = -(Ncdt)^2 + g_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)}$$

$$ds^2 = -(Ncdt)^2 + g_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt)$$

$$H = \int d^3x(N\mathcal{H} - N^i\mathcal{H}_i) + \frac{c^4}{16\pi G}\oint_{i^0} d^2s_i(g_{ij,j} - g_{jj,i})$$

$$N|_{i^0} = 1 + \mathcal{O}(1/r), \quad N^i|_{i^0} = \mathcal{O}(1/r)$$

If the constraints $\mathcal{H} = 0$ and $\mathcal{H}_i = 0$ are fulfilled and adapted coordinate conditions happen, then

$$H = \frac{c^4}{16\pi G}\oint_{i^0} d^2s_i(g_{ij,j} - g_{jj,i}) \equiv H_{\mathrm{ADM}}$$

Binary Black Hole Spacetimes

independent field variables

$$g_{ij} = \left(1 + \frac{1}{8}\phi\right)^4 \delta_{ij} + h_{ij}^{\mathrm{TT}}$$

$$\pi^{ii} = 0, \quad \pi^{ij} = -\gamma^{1/2}(K^{ij} - \gamma^{ij}K), \quad \pi_i^i = \pi^{ij}h_{ij}^{\mathrm{TT}},\ \gamma = \det(\gamma_{ij}),\ \gamma_{ij} = g_{ij}$$

unique decomposition: $\pi^{ij} = \tilde{\pi}^{ij} + \pi_{\mathrm{TT}}^{ij}$

$$\tilde{\pi}^{ij} = \partial_i \pi^j + \partial_j \pi^i - \frac{2}{3}\delta_{ij}\partial_k \pi^k$$

$\pi_{\mathrm{TT}}^{ij} c^3/16\pi G$: canonical conjugate to h_{ij}^{TT}

Hamilton and momentum constraints

$$\gamma^{1/2}\mathrm{R} = \frac{1}{\gamma^{1/2}}\left(\pi^i_j\pi^j_i - \frac{1}{2}\pi^i_i\pi^j_j\right) + \frac{16\pi G}{c^3}\sum_a \left(m_a^2c^2 + \gamma^{ij}p_{ai}p_{aj}\right)^{1/2}\delta_a$$

$G^{00} = \frac{8\pi G}{c^4}T^{00}$

$$-2\partial_j\pi^j_i + \pi^{kl}\partial_i\gamma_{kl} = \frac{16\pi G}{c^3}\sum_a p_{ai}\delta_a$$

$G^0_i = \frac{8\pi G}{c^4}T^0_i$

isolated BH

$$
\begin{aligned}
ds^2 &= -\left(\frac{1-\frac{Gm}{2rc^2}}{1+\frac{Gm}{2rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2rc^2}\right)^4 \delta_{ij}dx^idx^j \\
&= -\left(\frac{1-\frac{Gm}{2Rc^2}}{1+\frac{Gm}{2Rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2Rc^2}\right)^4 \delta_{ij}dX^idX^j
\end{aligned}
$$

symmetry transformation (inversion): $Rr = \left(\frac{Gm}{2c^2}\right)^2$

$R^2 = X^iX^i, \quad r^2 = x^ix^i$

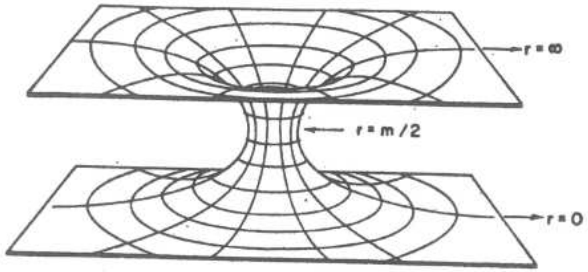


FIG. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2 = (1+m/2r)^4(dr^2+r^2d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold ($r\to 0$, $r\to\infty$).

Brill/Lindquist, JMP 1963

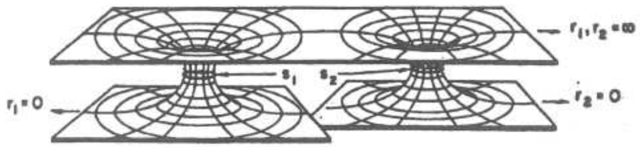
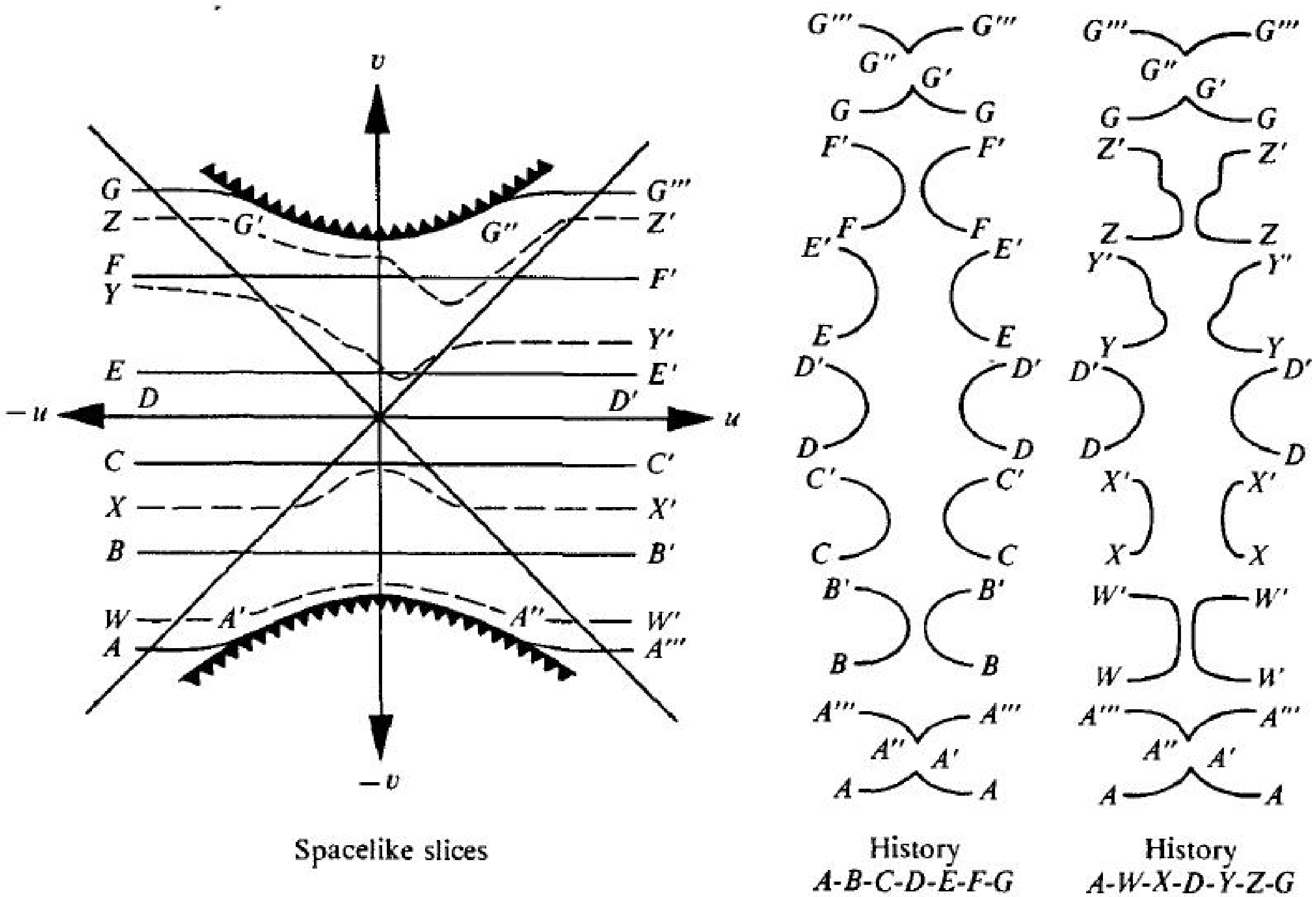


FIG. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

$$ds^2 = (1+m/2r_1+m/2r_2)^4 ds_F^2.$$



MTW: Gravitation (1973)

Brill-Lindquist BHs

maximally sliced

$$ds^2 = -\left(\frac{1 - \frac{\beta_1 G}{2r_1 c^2} - \frac{\beta_2 G}{2r_2 c^2}}{1 + \frac{\alpha_1 G}{2r_1 c^2} + \frac{\alpha_2 G}{2r_2 c^2}}\right)^2 c^2 dt^2 + \left(1 + \frac{\alpha_1 G}{2r_1 c^2} + \frac{\alpha_2 G}{2r_2 c^2}\right)^4 d\mathbf{x}^2$$

total energy: $$E_{ADM} = -\frac{c^4}{2\pi G}\oint_{i_0} ds_i \partial_i \Psi = (\alpha_1 + \alpha_2)c^2$$

$\Psi = 1 + \frac{\alpha_1 G}{2r_1 c^2} + \frac{\alpha_2 G}{2r_2 c^2}$

inversion map of the three-metric at the throat of black hole 1

$\mathbf{r}_1' = \mathbf{r}_1 \alpha_1^2 G^2 / 4c^4 r_1^2$

$\mathbf{r}_1' = \mathbf{x}' - \mathbf{x}_1, \quad \mathbf{r}_1 = \mathbf{x} - \mathbf{x}_1, \quad r_1 = |\mathbf{x} - \mathbf{x}_1|$

$$dl^2 = \Psi^4 d\mathbf{x}^2 = \left(1 + \frac{\alpha_1 G}{2r_1 c^2} + \frac{\alpha_2 G}{2r_2 c^2}\right)^4 d\mathbf{x}^2$$

$$dl^2 = \Psi'^4 d\mathbf{x}'^2 = \left(1 + \frac{\alpha_1 G}{2r'_1 c^2} + \frac{\alpha_1 \alpha_2 G^2}{4r_2 r'_1 c^4}\right)^4 d\mathbf{x}'^2$$

$$\mathbf{r}_2 = \frac{\alpha_1^2 G^2}{4c^4} \frac{\mathbf{r}'_1}{r_1'^2} + \mathbf{r}_{12}, \qquad \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$$

$$\boxed{m_1 = -\frac{c^2}{2\pi G} \oint_{i_0} ds'_i \partial'_i \Psi' = \alpha_1 + \frac{\alpha_1 \alpha_2 G}{2r_{12} c^2}}$$

$$\Psi' = 1 + \frac{\alpha_1 G}{2r'_1 c^2} + \frac{\alpha_1 \alpha_2 G^2}{4r_2 r'_1 c^4}$$

$$-\left(1+\frac{1}{8}\ \phi\right)\Delta\phi=\frac{16\pi G}{c^2}\sum_a m_a\delta_a \qquad (h_{ij}^{\mathrm{TT}}=0=p_{ai})$$

$$\phi=\frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right)$$

$$\alpha_a=m_a-\frac{m_a+m_b}{2}+\frac{c^2 r_{ab}}{G}\left(\sqrt{1+\frac{m_a+m_b}{c^2 r_{ab}/G}+\left(\frac{m_a-m_b}{2c^2 r_{ab}/G}\right)^2}-1\right)$$

$$\boxed{\mathrm{H}_{\mathrm{BL}}=(\alpha_1+\alpha_2)\ c^2=(m_1+m_2)\ c^2-G\ \frac{\alpha_1\alpha_2}{r_{12}}}$$

Metric in d-dimensional conformally flat space:

$$g_{ij} = \left(1 + \frac{1}{4}\frac{d-2}{d-1}\phi\right)^{\frac{4}{d-2}} \delta_{ij}$$

$$\phi = \frac{4G}{c^2}\frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d-2}{2}}}\left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right)$$

$$\Psi = 1 + \frac{1}{4}\frac{d-2}{d-1}\phi$$

$$-\Delta^{-1}\delta = \frac{\Gamma((d-2)/2)}{4\pi^{d/2}} r^{2-d}$$

$$\Psi = 1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right)$$

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}} \right) \right) \alpha_1 \delta_1 = m_1 \delta_1$$

$1 < d < 2$

$$\left(1 + \frac{G(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \frac{\alpha_2}{r_{12}^{d-2}} \right) \alpha_1 \delta_1 = m_1 \delta_1$$

Regularisation procedures for 4PN

Jaranowski/GS, PRD **92**, 124043 (2015)

3-dimensional Riesz-implemented Hadamard regularisation:

local poles

$$I^{\mathrm{RH}}(3,\epsilon_1,\epsilon_2) := \int i(\mathbf{x}) \left(\frac{r_1}{s_1}\right)^{\epsilon_1} \left(\frac{r_2}{s_2}\right)^{\epsilon_2} d^3x$$
$$= A + c_1\left(\frac{1}{\epsilon_1} + \ln\frac{r_{12}}{s_1}\right) + c_2\left(\frac{1}{\epsilon_2} + \ln\frac{r_{12}}{s_2}\right) + \dots$$

pole at infinity

$$I^{\mathrm{RH}}(3,a,b,\epsilon) := \int i(\mathbf{x}) \left(\frac{r_1}{r_0}\right)^{a\epsilon} \left(\frac{r_2}{r_0}\right)^{b\epsilon} d^3x$$
$$= A - c_\infty\left(\frac{1}{(a+b)\epsilon} + \ln\frac{r_{12}}{r_0}\right) + \dots$$

UV regularisation (small singularity-centered balls with radii l_a)

$$I^{\mathrm{RH}}(3, \epsilon_1, \epsilon_2) + \Delta I_1 + \Delta I_2$$

$$\Delta I_a := I_a(d) - I_a^{\mathrm{RH}}(3; \epsilon_a), \qquad a = 1, 2$$

$$I_a^{\mathrm{RH}}(3; \epsilon_a) = c_a \left(\frac{1}{\epsilon_a} + \ln \frac{l_a}{s_a} \right) + \ldots$$

$$I_a(d) = -\frac{c_a}{2\epsilon} - \frac{1}{2} c_a'(3) + c_a \, \ln \frac{l_a}{l_0} + \ldots, \qquad \epsilon := d - 3$$

$$c_a'(3) = c_{aa}'(3) + c_{ab}'(3) \, \ln \frac{r_{12}}{l_0}, \qquad a \neq b$$

$$G_D = G_N l_0^{d-3}, \qquad D = d + 1$$

$$
\begin{aligned}
(I^{\mathrm{RH}}(3,\epsilon_1,\epsilon_2) &+ \Delta I_1 + \Delta I_2)|_{\epsilon_a \to 0} \\
&= A - \frac{1}{2}(c'_{11}(3) + c'_{21}(3)) \\
&- \frac{c_1 + c_2}{2\epsilon} \\
&+ (c_1 + c_2 - \frac{1}{2}c'_{12}(3) - \frac{1}{2}c'_{22}(3))\ln\frac{r_{12}}{l_0} + \ldots
\end{aligned}
$$

at the end:

a unique total time derivative eliminates all poles and logarithms

IR regularisation 1 (outside large coordinates-origin-centered ball with radius R)

$$I^{\mathrm{RH}}(3,a,b,\epsilon) + \Delta I^1_\infty$$

$$\Delta I^1_\infty := \mathrm{FP} I^1_\infty - I^{\mathrm{RH}}_\infty(3,a,b,\epsilon)$$

$$I^{\mathrm{RH}}_\infty(3,a,b,\epsilon) = -c_\infty \left(\frac{1}{(a+b)\epsilon} + \ln \frac{R}{r_0} \right) + \ldots$$

$$\mathrm{FP} I^1_\infty = - \frac{\partial c^1_\infty}{\partial B}(3,0) - c^1_\infty(3,0) \ln \frac{R}{s}$$

$$(I^{\mathrm{RH}}(3,a,b,\epsilon) + \Delta I^1_\infty)|_{\epsilon \to 0} = A - \frac{\partial c^0_\infty}{\partial B}(0) - c_\infty \ln \frac{r_{12}}{s}$$

$$\Delta_d^{-1} \ddot{h}^{\mathrm{TT}}_{(4)ij} \to \Delta_d^{-1} \left[\left(\tfrac{r}{s}\right)^B \ddot{h}^{\mathrm{TT}}_{(4)ij} \right]^{\mathrm{TT}}$$

IR regularisation 2 (outside large coordinates-origin-centered ball with radius R)

$$\mathrm{FP}I^2_\infty = -c^2_\infty(3)\ln\frac{R}{s}$$

$$\left(\frac{r_1}{s}\right)^{\alpha_1}\left(\frac{r_2}{s}\right)^{\alpha_2}$$

$\beta = \alpha_1 + \alpha_2, \qquad \epsilon = d - 3$

$$I^2_\infty(d,\beta) = \frac{\eta(\epsilon,\beta)}{\beta - 2\epsilon}$$

$$\mathrm{FP}I^2_\infty = \mathrm{FP}_{\epsilon\to 0}\lim_{\beta\to 0}\frac{\eta(\epsilon,\beta) - \eta(\epsilon,2\epsilon)}{\beta - 2\epsilon}$$

“IR regularisation 2 - IR regularisation 1” shows a final ambiguity

Higher Order Post-Newtonian Hamiltonians

$$\Box^{-1}_{\rm sym} = \left(1 + \frac{1}{c^2}\Delta^{-1}\partial_t^2 + ...\right)\Delta^{-1}\delta(t-t')$$

$$G_{\rm ret} = \left(1 - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\partial_t + \frac{1}{2c^2}|\mathbf{r}-\mathbf{r}'|^2\partial_t^2 + ...\right)\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\delta(t-t')$$

binary black holes to 4PN order

$$\begin{aligned} H(t) &= m_1c^2 + m_2c^2 + H_N + \frac{1}{c^2}H_{[1PN]} \\ &+ \frac{1}{c^4}H_{[2PN]} + \frac{1}{c^6}H_{[3PN]} + \frac{1}{c^8}H_{[4PN]} + \dots \\ &+ \frac{1}{c^5}H_{[2.5PN]}(t) + \frac{1}{c^7}H_{[3.5PN]}(t) + \dots \end{aligned}$$

$\hat{H} = (H - Mc^2)/\mu, \qquad \mu = m_1 m_2/M, \qquad M = m_1 + m_2$

$\nu = \mu/M, \qquad 0 \leq \nu \leq 1/4$

test particles: $\nu = 0$, equal masses: $\nu = 1/4$

CMF: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \qquad \mathbf{p} = \mathbf{p}_1/\mu, \qquad r = r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|,$

$p_r = (\mathbf{n} \cdot \mathbf{p}), \qquad \mathbf{q} = (\mathbf{x}_1 - \mathbf{x}_2)/GM, \qquad \mathbf{n} = \mathbf{n}_{12} = \mathbf{q}/|\mathbf{q}|$

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$\hat{H}_{[1PN]} = \frac{1}{8}(-1 + 3\nu)p^4 - \frac{1}{2}[(3 + \nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned}
\hat{H}_{[2PN]} &= \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\
&+ \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\
&+ \frac{1}{2}[(5 + 8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1 + 3\nu)\frac{1}{q^3}
\end{aligned}$$

2.5PN dissipative binary dynamics

$$\hat{H}_{[2.5PN]}(\hat{t}) = \frac{2}{5}\frac{d^3Q_{ij}(\hat{t})}{d\hat{t}^3}\left(p_i p_j - \frac{n^i n^j}{q}\right)$$

$$Q_{ij}(\hat{t}) = \nu(q'^i q'^j - \frac{1}{3}q'^2\delta_{ij})$$

$\hat{t} = t/GM$

$$
\begin{aligned}
\hat{H}_{[3PN]} &= \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7+42\nu-53\nu^2-5\nu^3)p^6+(2-3\nu)\nu^2 p_r^2 p^4 \\
&+ 3(1-\nu)\nu^2 p_r^4 p^2 - 5\nu^3 p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27+136\nu+109\nu^2)p^4+\frac{1}{16}(17+30\nu)\nu p_r^2 p^2 \\
&+ \frac{1}{12}(5+43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3}+[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

$$
\begin{aligned}
\hat{H}_{[4PN]} \quad &= \quad \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right) p^{10} \\
&+ \quad \left\{\frac{45}{128}p^8 - \frac{45}{16}p^8\nu + \left(\frac{423}{64}p^8 - \frac{3}{32}p_r^2p^6 - \frac{9}{64}p_r^4p^4\right)\nu^2\right. \\
&+ \quad \left(-\frac{1013}{256}p^8 + \frac{23}{64}p_r^2p^6 + \frac{69}{128}p_r^4p^4 - \frac{5}{64}p_r^6 + \frac{35}{256}p_r^8\right)\nu^3 \\
&+ \quad \left.\left(-\frac{35}{128}p^8 - \frac{5}{32}p_r^2p^6 - \frac{9}{64}p_r^4p^4 - \frac{5}{32}p_r^6 - \frac{35}{128}p_r^8\right)\nu^4\right\}\frac{1}{q} \\
&+ \quad \left\{\frac{13}{8}p^6 + \left(-\frac{791}{64}p^6 + \frac{49}{16}p_r^2p^4 - \frac{889}{192}p_r^4 + \frac{369}{160}p_r^6\right)\nu\right. \\
&+ \quad \left(\frac{4857}{256}p^6 - \frac{545}{64}p_r^2p^4 + \frac{9475}{768}p_r^4 - \frac{1151}{128}p_r^6\right)\nu^2 \\
&+ \quad \left.\left(\frac{2335}{256}p^6 + \frac{1135}{256}p_r^2p^4 - \frac{1649}{768}p_r^4 + \frac{10353}{1280}p_r^6\right)\nu^3\right\}\frac{1}{q^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \left[\frac{105}{32}p^4 + C_{41}\,\nu + C_{42}\,\nu^2 + \left(-\frac{553}{128}p^4 - \frac{225}{64}p_r^2 - \frac{381}{128}p_r^4\right)\nu^3\right]\frac{1}{q^3} \\
&+ \quad \left\{\frac{105}{32} + C_{21}\,\nu + C_{22}\,\nu^2\right\}\frac{1}{q^4} \\
&+ \quad \left\{-\frac{1}{16} + c_{01}\,\nu + c_{02}\,\nu^2\right\}\frac{1}{q^5} \\
&- \quad \frac{1}{5}\hat{I}_{ij}^{(3)}(\hat{t}) \int_{-\infty}^{+\infty} dw \ln\left(\frac{|w|c}{2q}\right) \hat{I}_{ij}^{(4)}(\hat{t} - w)\,\nu
\end{aligned}
$$

$$
\begin{aligned}
C_{42} &= \left(-\frac{1189789}{28800}+\frac{18491}{16384}\pi^2\right)p^4+\left(-\frac{127}{3}-\frac{4035}{2048}\pi^2\right)p_r^2p^2 \\
&+ \left(\frac{57563}{1920}-\frac{38655}{16384}\pi^2\right)p_r^4
\end{aligned}
$$

$$
C_{22} = \left(\frac{672811}{19200}-\frac{158177}{49152}\pi^2\right)p^2+\left(-\frac{21827}{3840}+\frac{110099}{49152}\pi^2\right)p_r^2
$$

$$
c_{02} = -\frac{1256}{45}+\frac{7403}{3072}\pi^2
$$

$$C_{41} = \left(-\frac{589189}{19200} + \frac{2749}{8192}\pi^2\right) p^4 + \left(\frac{63347}{1600} - \frac{1059}{1024}\pi^2\right) p_r^2 p^2$$

$$+ \left(-\frac{23533}{1280} + \frac{375}{8192}\pi^2\right) p_r^4$$

$$C_{21} = \left(\frac{185761}{19200} - \frac{21837}{8192}\pi^2\right) p^2 + \left(\frac{3401779}{57600} - \frac{28691}{24576}\pi^2\right) p_r^2$$

$$c_{01} = -\frac{169199}{2400} + \frac{6237}{1024}\pi^2$$

4PN

Jaranowski/GS ('12,'13)[in part], Damour/Jaranowski/GS ('14)

$$
\begin{aligned}
H_{4\mathrm{PN}}^{\mathrm{near-zone\,(s)}}[\mathbf{x}_a, \mathbf{p}_a] &= H_{4\mathrm{PN}}^{\mathrm{loc}\,0}[\mathbf{x}_a, \mathbf{p}_a] \\
&+ \frac{2}{5}\,\frac{G^2 M}{c^8}\left(I_{ij}^{(3)}\right)^2\left(\ln\,\frac{r_{12}}{s} + C\right) \\
&+ \frac{d}{dt}\,G[\mathbf{x}_a, \mathbf{p}_a]
\end{aligned}
$$

$$
\begin{aligned}
H_{4\mathrm{PN}}^{\mathrm{tailsym\,(s)}}(t) &= -\frac{1}{5}\,\frac{G^2 M}{c^8}\,I_{ij}^{(3)}(t) \\
&\times \int_{-\infty}^{+\infty} dv \ln\left(\frac{|v|c}{2s}\right)\,I_{ij}^{(4)}(t-v)
\end{aligned}
$$

Matching to results by Bini/Damour for perturbed Schwarzschild metric from particle in circular motion yields $C = -\frac{1681}{1536}$.

$$H_{4\mathrm{PN}}^{\mathrm{tailsym}}(t) = -\frac{1}{5}\,\frac{G^2 M}{c^8}\, I_{ij}^{(3)}(t)\,\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}\,I_{ij}^{(3)}(t-v)$$

Classical gravitational “Lamb Shift” (orbital average):

$$\Delta E \quad = \quad -\frac{G}{5c^5}\,\frac{GM}{c^3 P}\int_0^P dt\left[I_{ij}^{(3)}(t)\,\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}\,I_{ij}^{(3)}(t-v)\right]$$

$$< \frac{dE}{dt} > \quad = \quad -\frac{G}{5c^5}\,\frac{1}{P}\int_0^P dt\left[I_{ij}^{(3)}(t) I_{ij}^{(3)}(t)\right] \quad \text{(Einstein's quad. formula)}$$

$$\Delta E_{\rm circOrb} = \frac{32}{5}\mu c^2 \left(\frac{\mu}{M j^2}\right)\left(\frac{1}{j^2}\right)^4 \left[\ln\left(\frac{1}{j^2}\right) + 2\ln 4 + 2\gamma_E\right]$$

$$j \equiv \frac{cJ}{GM\mu}, \qquad \frac{1}{j^2} = \frac{GM}{rc^2} < 1$$

Lamb Shift: $\delta E_{nS} = -\frac{8}{3\pi n^3} mc^2(\alpha)(Z\alpha)^4\left[\ln(Z\alpha) + \frac{1}{2}\ln K_n - \frac{11}{48}\right]$

$\alpha \equiv \frac{e^2}{\hbar c}$, $\quad Z\alpha < 1$, $\qquad$ L.S. Brown, QFT, CUP 1992 ["orb" only]

$\ln 4 \simeq 1.39$, $\quad \gamma_E \simeq 0.58$, $\quad \ln K_2 \simeq 2.81$

Orbital Motion (ISCO)

ISCO

$$H = H(\mathbf{p}, \mathbf{r}), \qquad p^2 = p_r^2 + j^2/r^2, \qquad p_r = (\mathbf{p}\cdot\mathbf{r})/r$$

circular orbits: $p_r = 0, \qquad p^2 = j^2/r^2, \qquad H = H(j, r)$

circular motion: $\frac{\partial}{\partial r} H(j,r) = 0 \rightarrow H(j)$

orbital frequency: $\omega = \frac{dH(j)}{dj} \rightarrow H(\omega)$

ISCO: $\boxed{\frac{dH(\omega)}{d\omega} = 0}$ or, alternatively $\frac{\partial^2}{\partial r^2} H(j,r) = 0$

$$\text{SBH:} \quad E(x) = \frac{1-2x}{(1-3x)^{1/2}} - 1$$

$$= -\frac{1}{2}x + \frac{3}{8}x^2 + \frac{27}{16}x^3 + \frac{675}{128}x^4 + \frac{3969}{256}x^5 + \ldots$$

$$E(x) \equiv \frac{H(x) - mc^2}{mc^2}, \qquad x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$$

circular orbits:

$$c^2 E_{4PN} \equiv \hat{H}_N + c^{-2}\hat{H}_{[1PN]} + c^{-4}\hat{H}_{[2PN]} + c^{-6}\hat{H}_{[3PN]} + c^{-8}\hat{H}_{[4PN]}$$

$$\begin{aligned} E_{4PN}(x) &= -\frac{x}{2} + \left(\frac{3}{8} + \frac{1}{24}\nu\right)x^2 + \left(\frac{27}{16} - \frac{19}{16}\nu + \frac{1}{48}\nu^2\right)x^3 \\ &+ \left(\frac{675}{128} + \left(-\frac{34445}{1152} + \frac{205}{192}\pi^2\right)\nu + \frac{155}{192}\nu^2 + \frac{35}{10368}\nu^3\right)x^4 \end{aligned}$$

$$-\tfrac{1}{2}\left(-\tfrac{3960}{128} + [c_1 + \tfrac{448}{15}\ln x]\nu + \left(-\tfrac{498449}{3456} + \tfrac{3157}{576}\pi^2\right)\nu^2 + \tfrac{301}{1728}\nu^3 + \tfrac{77}{31104}\nu^4\right)x^5$$

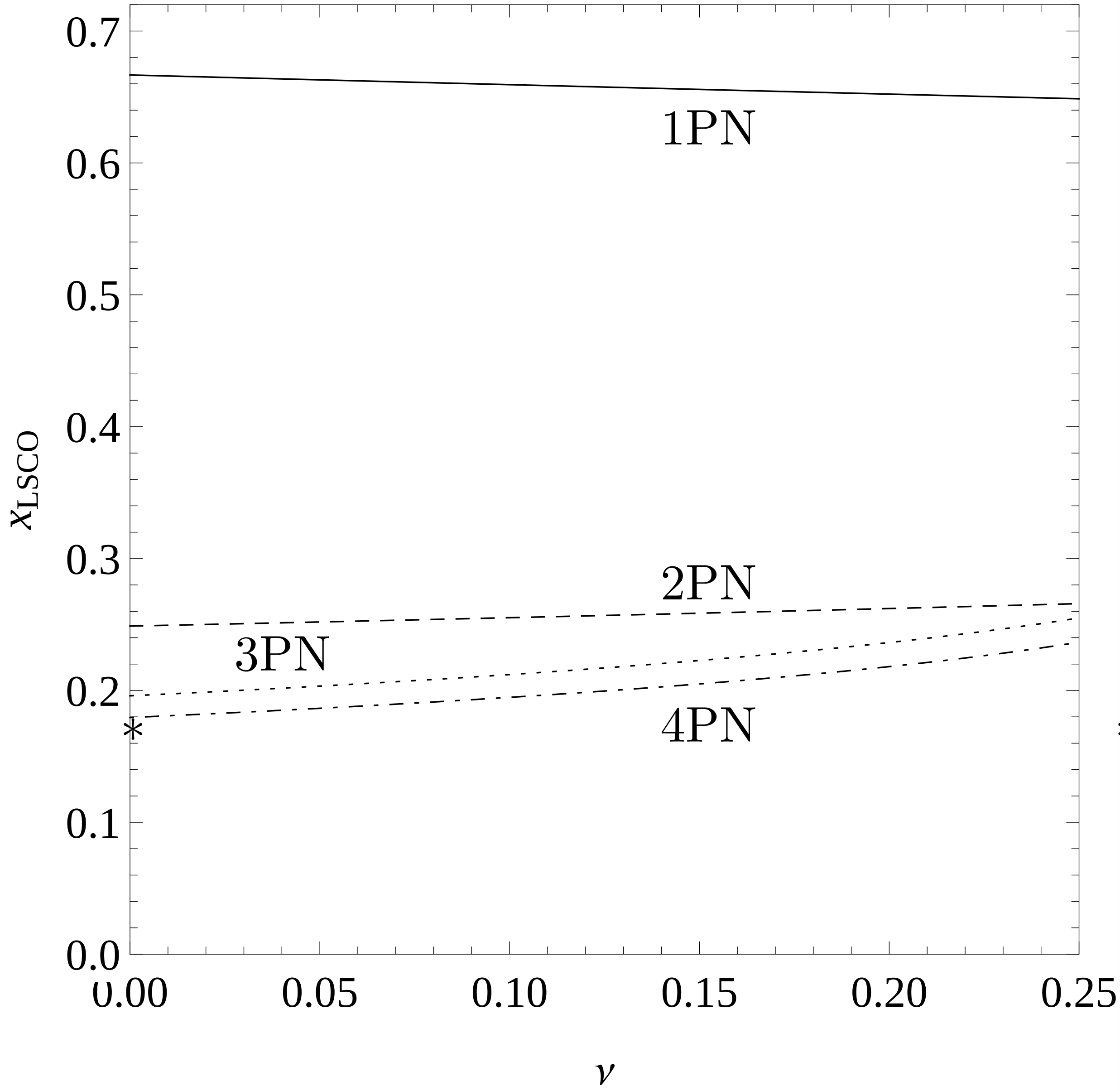
Damour ('10)[ln$x$], Blanchet/Detweiler/Le Tiec/Whiting ('10)[ln$x$]

Jaranowski/GS ('12)[ln$x$, ν^3, ν^4], ('13)[ν^2], Foffa/Sturani ('13) [ln$x$, ν^3, ν^4]

$$c_1 = -\tfrac{123671}{5760} + \tfrac{9037}{1536}\pi^2 + \tfrac{1792}{15}\ln 2 + \tfrac{896}{15}\gamma_E = 153.88...$$

Bini/Damour ('13), Le Tiec/Blanchet/Whiting ('12) [numerical value]

Jaranowski/GS ('13)



* test particle exact

spin-gravity interaction

leading order spin orbit

$$H_{\mathrm{SO}}^{\mathrm{LO}} = \frac{G}{c^2} \sum_a \sum_{b \neq a} \frac{1}{r_{ab}^2} (\mathbf{S}_a \times \mathbf{n}_{ab}) \cdot \left[\frac{3m_b}{2m_a} \mathbf{p}_a - 2\mathbf{p}_b \right]$$

leading order spin(1)-spin(2)

$$H_{\mathrm{S_1 S_2}}^{\mathrm{LO}} = \frac{G}{c^2} \sum_a \sum_{b \neq a} \frac{1}{2r_{ab}^3} \left[3(\mathbf{S}_a \cdot \mathbf{n}_{ab})(\mathbf{S}_b \cdot \mathbf{n}_{ab}) - (\mathbf{S}_a \cdot \mathbf{S}_b) \right]$$

leading order spin(1) spin(1)

$$H_{\mathrm{S_1 S_1}}^{\mathrm{LO}} = \frac{G}{c^2} \frac{1}{2r_{12}^3} \left[3(\mathbf{S}_1 \cdot \mathbf{n}_{12})(\mathbf{S}_1 \cdot \mathbf{n}_{12}) - (\mathbf{S}_1 \cdot \mathbf{S}_1) \right]$$

$$
\begin{aligned}
H_{\text{con}} &= H_N + H_{1PN} + H_{2PN} + H_{3PN} + H_{4PN} \\
&+ H_{SO}^{\text{LO}} + H_{S_1S_2}^{\text{LO}} + H_{S_1^2}^{\text{LO}} + H_{S_2^2}^{\text{LO}} \\
&+ H_{SO}^{\text{NLO}} + H_{S_1S_2}^{\text{NLO}} + H_{S_1^2}^{\text{NLO}} + H_{S_2^2}^{\text{NLO}} \\
&+ H_{SO}^{\text{NNLO}} + H_{S_1S_2}^{\text{NNLO}} \\
&+ H_{S_1^2}^{\text{LO}} + H_{S_2^2}^{\text{LO}} + H_{S_1^2}^{\text{NLO}} + H_{S_2^2}^{\text{NLO}} + H_{S_1^4}^{\text{LO}} + H_{S_1^4}^{\text{LO}} \\
&+ H_{p_1S_2^3}^{\text{LO}} + H_{p_2S_1^3}^{\text{LO}} + H_{p_1S_1S_2^2}^{\text{LO}} + H_{p_2S_2S_1^2}^{\text{LO}}
\end{aligned}
$$

$$
\begin{aligned}
H_{\text{diss}}(t) &= H_{2.5PN}(t) + H_{3.5PN}(t) \\
&+ H_{SO}^{\text{DLO}}(t) + H_{S_1S_2}^{\text{DLO}}(t)
\end{aligned}
$$

Steinhoff (2011), Hartung/Steinhoff/GS (2013), Levi/Steinhoff (2015)

EOB Formalism

Damour/Jaranowski/GS, arxiv:0803.0915

Damour, arXiv:1312.3505

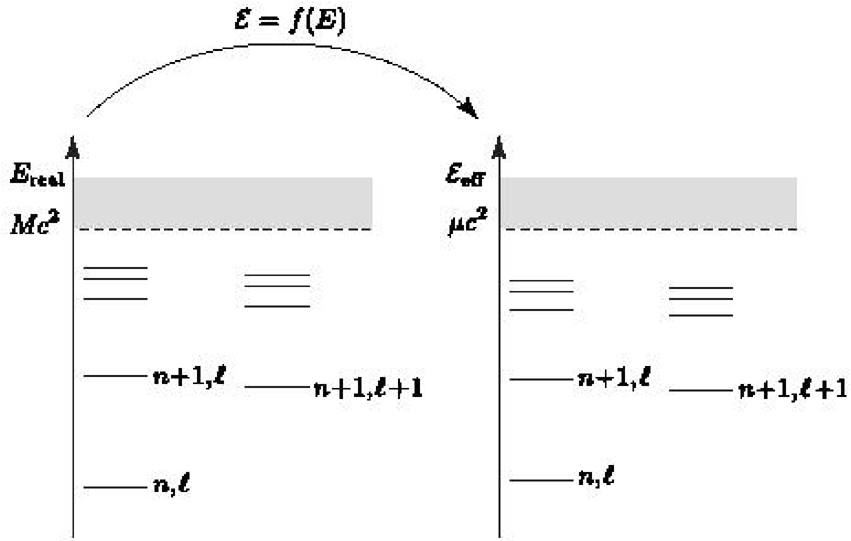


Figure 1: Sketch of the correspondence between the quantized energy levels of the real and effective conservative dynamics. n denotes the 'principal quantum number' ($n = n_r + \ell + 1$, with $n_r = 0, 1, \ldots$ denoting the number of nodes in the radial function), while ℓ denotes the (relative) orbital angular momentum ($\boldsymbol{L}^2 = \ell(\ell+1)\,\hbar^2$). Though the EOB method is purely classical, it is conceptually useful to think in terms of the underlying (Bohr-Sommerfeld) quantization conditions of the action variables I_R and J to motivate the identification between n and ℓ in the two dynamics.

Standard representation

$$\frac{H_{\text{eff}}}{\mu c^2} \equiv \frac{H^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4} = 1 + \frac{H_{\text{NR}}}{\mu c^2} + \frac{\nu}{2} \left(\frac{H_{\text{NR}}}{\mu c^2} \right)^2$$

$$H_{\text{NR}} \equiv H - M c^2$$

$$H = M c^2 \sqrt{1 + 2\nu \left(\frac{H_{\text{eff}}}{\mu c^2} - 1 \right)}$$

EOB representation

$$g_{\text{eff}}^{\mu\nu} P_\mu P_\nu + Q_4(P_i) = -\mu^2 c^2, \qquad H_{\text{eff}}^{\text{EOB}(1)} \equiv -P_0 c$$

$$H_{\text{eff}}^{\text{EOB}} = N_{\text{eff}}^i P_i c + N_{\text{eff}} c \sqrt{\mu^2 c^2 + \gamma_{\text{eff}}^{ij} P_i P_j + Q_4(P_i)}$$

$$H^{\text{EOB}} = M c^2 \sqrt{1 + 2\nu \left(\frac{H_{\text{eff}}^{\text{EOB}}}{\mu c^2} - 1 \right)}$$

Canonical transformation to connect:

$$\boxed{\mathrm{H}_{\rm eff}^{\rm EOB} \equiv H_{\rm eff}^{\rm EOB(1)}(X,P,S_0) + H_{\rm eff}^{\rm EOB(2)}(X,P,S_0,\sigma) = H_{\rm eff}(x,p,S_1,S_2)}$$

$$\begin{aligned} g_{\rm eff}^{\mu\nu} P_\mu P_\nu &= \frac{1}{R^2 + a^2 \cos^2\theta} \left(\Delta_R(R) P_R^2 + P_\theta^2 \right. \\ &+ \frac{1}{\sin^2\theta} \left(P_\phi + a \, \sin^2\theta \frac{P_t}{c} \right)^2 \\ &- \left. \frac{1}{\Delta_t(R)} \left((R^2 + a^2) \frac{P_t}{c} + a P_\phi \right)^2 \right) \end{aligned}$$

$$\Delta_t(R) = R^2 P_m^n \left[A(R) + \frac{a^2}{R^2} \right], \quad \Delta_R(R) = \Delta_t(R) D^{-1}(R)$$

$a = S_0/Mc$, $\quad P_m^n$: (n,m)-Padé approximation

$$A(R) = 1 - 2u + 2\nu u^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right) \nu u^4$$

$$D^{-1}(R) = 1 + 6\nu u^2 + 2(26 - 3\nu)\nu u^3$$

$u = \frac{GM}{Rc^2}$

possible improvement through (n,m)-Padé approximant:

$$1 + c_1 u + c_2 u^2 + ... + c_{n+m} u^{n+m} \sim \frac{1 + a_1 u + a_2 u^2 + ... + a_n u^n}{1 + b_1 u + b_2 u^2 + ... + b_m u^m}$$

Augmented through radiation reaction, the equations of motion for inspiral are easily solvable ordinary differential equations.

The connection to gravitational waveforms is given in, e.g. Damour, arXiv:1312.3505.

Therein, the generalisation to the merger of binary black holes and of the ringdown of the final black hole can be found too, including comparison with numerical relativity.

EOB played a crucial role in the identification of GW150914.