# Cuts and coproducts of Feynman integrals

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Based on:
1401.3546 with Samuel Abreu, Claude Duhr, Einan Gardi
1504.00206 with Samuel Abreu, Hanna Grönqvist
work in preparation with Samuel Abreu, Claude Duhr, Einan Gardi

# Motivation: loops and cuts

- Loop integrals are **necessary**

  ...for high precision at high energy

- Loop integrals are **hard**

- “Cuts” are **useful**

  ...both analytically and numerically

  Definition: Cuts put propagators on shell.

  **Since cuts have been useful, we would like to know how to understand their meaning, going deeper than pattern-matching with master integrals.**

# Cuts and Hopf algebra of

Cutkosky: Cuts are discontinuities across branch cuts

We claim:

- For integrals in the class of multiple polylogarithms (MPL), the discontinuities described by cuts are naturally found within the Hopf algebra of MPL.
- There is a graphical Hopf algebra that
  - involves cut diagrams, and
  - corresponds to the Hopf algebra of MPL.

# First claim

3 equivalent definitions of discontinuities: "Cut = Disc = $\delta$"

Familiar for the first cut; we extend it to sequences of cuts.

I will explain this claim and give some examples, and then comment on reconstruction of the full integral from its cuts.

# Outline

- Cuts (review)
- Coproducts of MPL in Hopf algebra (background)
- Sequential discontinuities: Cut, Disc, $\delta$
- Examples

  Reconstruction: from cuts to integrals (via coproducts)
- The graphical Hopf algebra

# Review: unitarity cuts

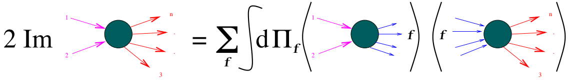
$$\text{“ Disc = Cut ”}$$

Discontinuities = Landau singularities = replace propagators by Dirac delta functions in integral.

$$S = 1 + iT$$

The unitarity condition: $S^\dagger S = 1$.

$$2\,\mathrm{Im}\, T = T^\dagger T$$

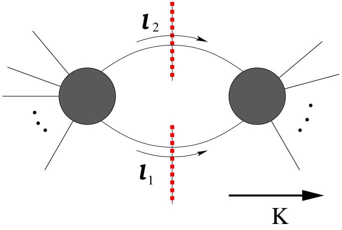



Cut across one channel, with any number of loops.

# Unitarity Cuts

$$\Delta A^{1-\text{loop}} = \int d^4\ell \; \delta^+(\ell_1^2) \; \delta^+(\ell_2^2) \; A^{\text{tree}}_{\text{Left}}(\ell_1, \ell_2) \; A^{\text{tree}}_{\text{Right}}(\ell_1, \ell_2)$$

Cut conditions: $\ell_1^2 = 0, \qquad \ell_2^2 = 0.$




By unitarity, this is the discontinuity of the amplitude across a branch cut.

# Review: 1-loop cut construction

- Expansion in master integrals, thanks to Passarino-Veltman reduction
- Theorem: "cut-constructibility"
- Cut both sides, solve for coefficients. Nontrivial work, but well established by now.

**Generalized** cuts are particularly useful.

Classic alternative: dispersion relations.

# Beyond one loop

Integrals get complicated. Various sophisticated techniques (e.g. MB, DE)

What is the set of master integrals? How do we evaluate them?

Analytic techniques are still tough.

# Multiple polylogarithms (MPL)

A large class of integrals are described by multiple polylogarithms:

$$I(a_0; a_1, \ldots, a_n; a_{n+1}) \equiv \int_{a_0}^{a_{n+1}} \frac{dt}{t - a_n} I(a_0; a_1, \ldots, a_{n-1}; t)$$

Examples:

$$I(0; 0; z) = \log z, \quad I(0; a; z) = \log\left(1 - \frac{z}{a}\right)$$
$$I(0; \vec{a}_n; z) = \frac{1}{n!} \log^n \left(1 - \frac{z}{a}\right), \quad I(0; \vec{0}_{n-1}, a; z) = -\mathrm{Li}_n\left(\frac{z}{a}\right)$$

*Harmonic* polylog if all $a_i \in \{-1, 0, 1\}$.

$n$ is the *transcendental weight*.

Observation: most known Feynman integrals can be written in terms of classical and harmonic polylogs.

# Hopf algebra

Product and coproduct:

$$\mu: \quad \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \qquad \Delta: \quad \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$$

Compatible with each other:

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)\,,$$

The algebra is graded by transcendental weight:

$$\mathcal{H}_n \xrightarrow{\Delta} \bigoplus_{k=0}^{n} \mathcal{H}_k \otimes \mathcal{H}_{n-k}\,,$$

and

$$\Delta_{n_1,\ldots,n_k} : \mathcal{H}_n \to \mathcal{H}_{n_1} \otimes \ldots \otimes \mathcal{H}_{n_k}\,.$$

# Hopf algebra of MPL

Goncharov's coproduct formula for MPL (modulo $\pi$):

$$\Delta I(a_0; a_1, \ldots, a_n; a_{n+1})$$
$$= \sum_{0=i_0<\cdots<i_k<i_{k+1}=n+1} I(a_0; a_{i_1}, \ldots, a_{i_k}; a_{m+1}) \otimes \prod_{p=0}^{k} I(a_{i_p}; a_{i_p+1}, \ldots, a_{i_{p+1}-1}; a_{i_{p+1}})$$

Examples:

$$\begin{aligned}
\Delta(a \cdot b) &= \Delta(a) \cdot \Delta(b) \\
\Delta(1) &= 1 \otimes 1 \\
\Delta(\log z) &= 1 \otimes \log z + \log z \otimes 1 \\
\Delta(\log x \log y) &= 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1 \\
\Delta(\mathrm{Li}_n(z)) &= 1 \otimes \mathrm{Li}_n(z) + \mathrm{Li}_n(z) \otimes 1 + \sum_{k=1}^{n-1} \mathrm{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!}
\end{aligned}$$

# Symbols of MPL

The "symbol" $\mathcal{S}$ is essentially the maximal iteration.

$$\mathcal{S}(F) \equiv \Delta_{1,\ldots,1}(F) \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_1 \,.$$

$$\begin{aligned}
\mathcal{S}\left(\frac{1}{n!}\log^n z\right) &= \underbrace{z \otimes \cdots \otimes z}_{n \text{ times}} \\
\mathcal{S}(\mathrm{Li}_n(z)) &= -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{(n-1) \text{ times}}
\end{aligned}$$

Functions are all weight $1 = \log$.

(symbol is familiar from remainder functions [Goncharov, Spradlin, Vergu, Volovich])

# Coproducts of Feynman integrals

Observation: without internal masses, coproduct can be written such that

$$\Delta_{1,n-1}F = \sum_i \log(-s_i) \otimes f_{s_i}$$

- first entries are Mandelstam invariants,
- and each second entry $f_{s_i}$ is the discontinuity of $F$ in the channel $s_i$.

[Gaiotto, Maldacena, Sever, Viera]

Thus: the coproduct captures standard cuts.

What about generalized cuts?

# Cut=Disc=$\delta$ for generalized cuts

- Need to define generalized cuts. In real kinematics, we do it as a sequence of traditional cuts.
- Need to specify kinematic regions.
- Need to identify the MPL variables and explain the correspondence.
- Limited by: number of channels, transcendental weight, and number of variables.

# Definition of Disc

The discontinuity across the branch cut.

$$\mathrm{Disc}_x \left[F(x \pm i0)\right] = F(x \pm i0) - F(x \mp i0),$$

Example:

$$\mathrm{Disc}_x \log(x + i0) = 2\pi i\, \theta(-x).$$

Sequential:

$$\mathrm{Disc}_{x_1,\ldots,x_k} F \quad \equiv \quad \mathrm{Disc}_{x_k} \left(\mathrm{Disc}_{x_1,\ldots,x_{k-1}} F\right).$$

# Definition of multiple cuts

$$\mathrm{Cut}_{s_1,\ldots,s_k}\, F$$

With *real* kinematics.

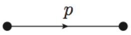
Defined by: cut propagators + consistent energy flow + corresponding kinematic region

Multiple cuts are taken simultaneously.

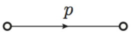
# Cutting Rules

Traditional [Veltman]:

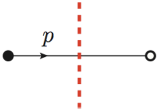
$$\bullet = i \qquad\qquad \circ = -i$$




$$= \frac{i}{p^2 + i\varepsilon}$$




$$= \frac{-i}{p^2 - i\varepsilon}$$



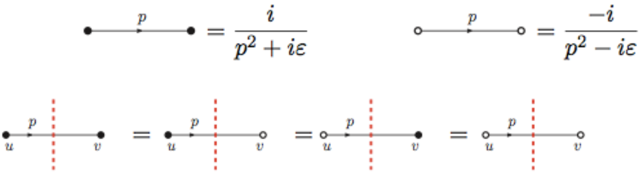

$$= 2\pi\,\delta\left(p^2\right)\theta\left(p_0\right)$$

for massless scalar theory.

# Cutting Rules

Generalized:

$$\bullet = i \qquad \circ = -i$$



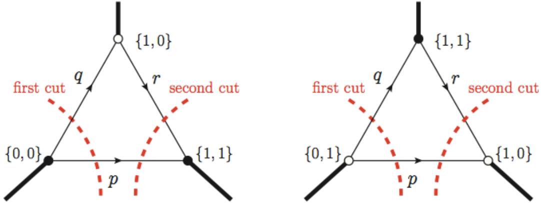

$$= 2\pi\,\delta\left(p^2\right) \prod_{i:c_i(u)\neq c_i(v)} \theta\left([c_i(v) - c_i(u)]p_0\right)$$

Colors are $c_i = 0, 1$ for each cut $i$: we overlay consistent energy flow across each cut.

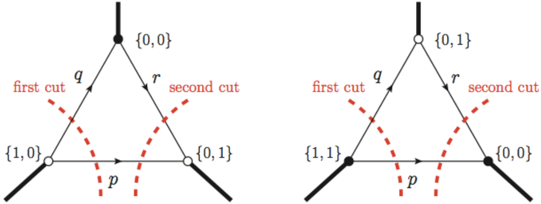
We *must* allow repeated cuts of same propagator or same loop.

# Cutting Rules

Example:




$\theta(p_0)\theta(q_0)\theta(r_0)$ $\qquad$ $\theta(p_0)\theta(-p_0) = 0$




$\theta(-p_0)\theta(p_0) = 0$ $\qquad$ $\theta(-p_0)\theta(-q_0)\theta(-r_0)$

# From Mandelstam invariants to MPL variables

From the Largest Time Equation [Veltman]:

$$F + F^* = -\sum_s \mathrm{Cut}_s\, F,$$

Hence:

$$\mathrm{Disc}_s\, F = -\,\mathrm{Cut}_s\, F.$$

Generalize to:

$$\boxed{\mathrm{Cut}_{s_1,\ldots,s_k}\, F = (-1)^k\, \mathrm{Disc}_{s_1,\ldots,s_k}\, F.}$$

Valid in a particular kinematic region: cut invariants $s_i$ positive, others negative.

# Definition of $\delta$

If

$$\Delta_{\underbrace{1,1,\ldots,1}_{k \text{ times}},n-k} F = \sum_{\{x_1,\ldots,x_k\}} \log x_1 \otimes \cdots \otimes \log x_k \otimes g_{x_1,\ldots,x_k} \,,$$

then

$$\delta_{x_1,\ldots,x_k} F \cong g_{x_1,\ldots,x_k} \,.$$

More precisely: match branch points. The "$\cong$" means modulo $\pi$.

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Motivated by coproduct identity : $\Delta \; \text{Disc} = (\text{Disc} \otimes 1) \, \Delta$ [Duhr]
and first-entry condition.

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More precisely: match branch points. The "$\cong$" means modulo $\pi$.

Motivated by coproduct identity : $\Delta$ Disc $=$ (Disc $\otimes 1$) $\Delta$ [Duhr]
and first-entry condition.

If $\delta_x F \cong g_x$, then $\text{Disc}_x F \cong (\text{Disc}_x \otimes 1)(\log x \otimes g_x) = \pm 2\pi i\, g_x$. Sign determined by $i\varepsilon$ prescription.

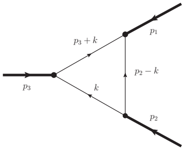
# Coproduct and discontinuities for Feynman integrals

$$\mathrm{Disc}_{s_1} F = (-2\pi i)\, \delta_{s_1} F.$$

$$\boxed{\mathrm{Disc}_{s_1,\ldots,s_k} F = \sum_{x_1,\ldots,x_k} \pm(2\pi i)^k \delta_{x_1,\ldots,x_k} F.}$$

- Assume prior knowledge of alphabet (e.g. from cuts)
- Underlying claim: kinematics put us on the branch cuts, so that it is correct to use our definition of Disc.

# Basic example: 3-mass triangle

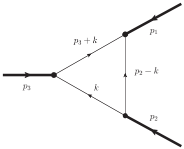



$$
\begin{aligned}
T &= -\frac{i}{p_1^2}\frac{2}{z-\bar z}\left(\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar z)+\frac{1}{2}\log(z\bar z)\log\left(\frac{1-z}{1-\bar z}\right)\right)\\
&\equiv -\frac{i}{p_1^2}\frac{2}{z-\bar z}\,\mathcal{P}_2
\end{aligned}
$$

where

$$
z\bar z=\frac{p_2^2}{p_1^2}, \qquad (1-z)(1-\bar z)=\frac{p_3^2}{p_1^2}
$$

# Basic example: 3-mass triangle




$$T = -\frac{i}{p_1^2}\frac{2}{z-\bar{z}}\left(\mathrm{Li}_2(z)-\mathrm{Li}_2(\bar{z})+\frac{1}{2}\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right)$$

$$\equiv -\frac{i}{p_1^2}\frac{2}{z-\bar{z}}\,\mathcal{P}_2$$

$$\mathrm{Disc}_{p_2^2}\,T = -\frac{2\pi}{p_1^2(z-\bar{z})}\log\frac{1-z}{1-\bar{z}}$$

# Coproduct of the 3-mass triangle

$$\begin{aligned}\Delta\mathcal{P}_2 &= \mathcal{P}_2\otimes 1 + 1\otimes\mathcal{P}_2 + \frac{1}{2}\log(z\bar{z})\otimes\log\frac{1-z}{1-\bar{z}} + \frac{1}{2}\log[(1-z)(1-\bar{z})]\otimes\log\frac{\bar{z}}{z}\\ &= \mathcal{P}_2\otimes 1 + 1\otimes\mathcal{P}_2 + \frac{1}{2}\log\left(-p_2^2\right)\otimes\log\frac{1-z}{1-\bar{z}} + \frac{1}{2}\log\left(-p_3^2\right)\otimes\log\frac{\bar{z}}{z}\\ &\quad +\frac{1}{2}\log(-p_1^2)\otimes\log\frac{1-1/\bar{z}}{1-1/z}\end{aligned}$$

Alphabet: $\{z, \bar{z}, 1-z, 1-\bar{z}\}$.

# Coproduct of the 3-mass triangle

$$\Delta \mathcal{P}_2 = \mathcal{P}_2 \otimes 1 + 1 \otimes \mathcal{P}_2 + \frac{1}{2}\log(z\bar{z}) \otimes \log\frac{1-z}{1-\bar{z}} + \frac{1}{2}\log[(1-z)(1-\bar{z})] \otimes \log\frac{\bar{z}}{z}$$
$$= \mathcal{P}_2 \otimes 1 + 1 \otimes \mathcal{P}_2 + \frac{1}{2}\log\left(-p_2^2\right) \otimes \log\frac{1-z}{1-\bar{z}} + \frac{1}{2}\log\left(-p_3^2\right) \otimes \log\frac{\bar{z}}{z}$$
$$+\frac{1}{2}\log(-p_1^2) \otimes \log\frac{1-1/\bar{z}}{1-1/z}$$

Alphabet: $\{z, \bar{z}, 1-z, 1-\bar{z}\}$.

Here

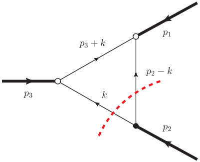
$$z = \frac{1}{p^2}\left(p_1^2 + p_2^2 - p_3^2 + \sqrt{\lambda}\right), \qquad \bar{z} = \frac{1}{p^2}\left(p_1^2 + p_2^2 - p_3^2 - \sqrt{\lambda}\right),$$

where $\lambda \equiv (p_1^2)^2 + (p_2^2)^2 + (p_3^2)^2 - 2p_1^2p_2^2 - 2p_1^2p_3^2 - 2p_2^2p_3^2$.
Hence

$$z\bar{z} = \frac{p_2^2}{p_1^2}, \qquad (1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}, \qquad \sqrt{\lambda} = z - \bar{z}$$

# First cut of the 3-mass triangle

Cut in the $p_2^2$ channel.



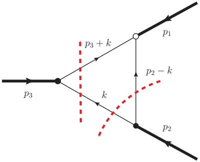

Kinematic region: $p_2^2 > 0$; $p_1^2, p_3^2 < 0$.

$$\begin{aligned} \mathrm{Cut}_{p_2^2}\, T &= \frac{2\pi}{p_1^2(z-\bar{z})} \log \frac{1-z}{1-\bar{z}} \\ &= -\,\mathrm{Disc}_{p_2^2}\, T \end{aligned}$$

$$\delta_{p_2^2} \mathcal{P}_2 = \frac{1}{2} \log \frac{1-z}{1-\bar{z}}$$

$$\mathrm{Disc}_{p_2^2}\, T = (-2\pi i)\delta_{p_2^2} T.$$

# Second cut of the 3-mass triangle




$$\text{Cut}_{p_3^2, p_2^2} T = \frac{4\pi^2 i}{p_1^2 (z - \bar{z})}$$

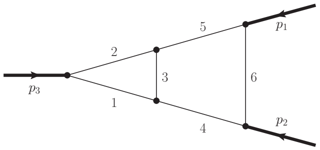
Kinematic region: $p_3^2, p_2^2 > 0; \quad p_1^2 < 0$
Equivalently: $\bar{z} < 0, z > 1$.

Now we have to match the alphabet with Mandelstam invariants:

$$\text{Disc}_{p_2^2, p_3^2} T = \text{Cut}_{p_2^2, p_3^2} T.$$

$$\text{Disc}_{p_2^2, p_3^2} T = 4\pi^2 \, \delta_{p_2^2, 1-z} T$$

# 2-loop example: 3-point ladder

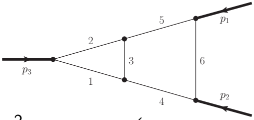



$$L = i\left(p_1^2\right)^{-2} \frac{1}{(1-z)(1-\bar{z})(z-\bar{z})} F$$

$$F = 6\left[\mathrm{Li}_4(z) - \mathrm{Li}_4(\bar{z})\right] - 3\log(z\bar{z})\left[\mathrm{Li}_3(z) - \mathrm{Li}_3(\bar{z})\right] + \frac{1}{2}\log^2(z\bar{z})\left[\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})\right].$$

$$z\bar{z} = \frac{p_2^2}{p_1^2}, \quad (1-z)(1-\bar{z}) = \frac{p_3^2}{p_1^2}$$

# Coproduct of the ladder

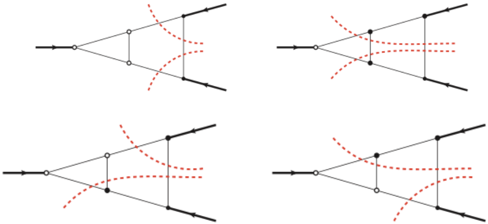



$$
\begin{aligned}
\Delta_{1,1,2}F \quad = \quad & \log \frac{p_3^2}{p_1^2} \otimes \log z \otimes \left( \log z \log \bar{z} - \frac{1}{2}\log^2 \bar{z} \right) \\
& - \log \frac{p_3^2}{p_1^2} \otimes \log \bar{z} \otimes \left( \log z \log \bar{z} - \frac{1}{2}\log^2 z \right) \\
& - \log \frac{p_2^2}{p_1^2} \otimes \log(1-z) \otimes \left( \log z \log \bar{z} - \frac{1}{2}\log^2 z \right) \\
& + \log \frac{p_2^2}{p_1^2} \otimes \log(1-\bar{z}) \otimes \left( \log z \log \bar{z} - \frac{1}{2}\log^2 \bar{z} \right) \\
& + \log \frac{p_2^2}{p_1^2} \otimes \log(z\bar{z}) \otimes [\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z})] \,,
\end{aligned}
$$

Individual cut diagrams diverge, but sums are finite.

Same alphabet as triangle. $\frac{p_2^2}{p_1^2} = z\bar{z}, \ \frac{p_3^2}{p_1^2} = (1-z)(1-\bar{z})$

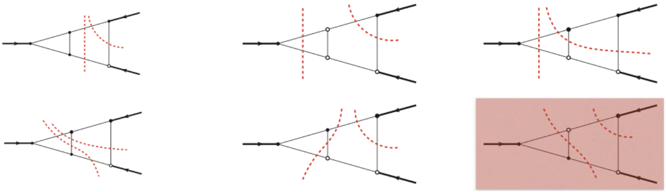
# Two cuts of the ladder



$$\mathrm{Cut}_{p_1^2,p_2^2}\, F = \mathrm{Disc}_{p_1^2,p_2^2}\ F = -(2\pi i)^2\, \delta_{p_1^2,\bar{z}} F\,,$$
$$\mathrm{Cut}_{p_2^2,p_1^2}\, F = \mathrm{Disc}_{p_2^2,p_1^2}\ F = (2\pi i)^2\, [\delta_{p_2^2,z} + \delta_{p_2^2,1-z}] F\,.$$

Variables matched within the correct kinematic region:
$p_1^2, p_2^2 > 0$ and $p_3^2 < 0$, or equivalently $0 < \bar{z} < 1 < z$.
Follow $i\varepsilon$ for signs.

# Two cuts of the ladder

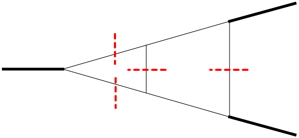


$$\mathrm{Cut}_{p_1^2,p_3^2}\, F \;=\; \mathrm{Disc}_{p_1^2,p_3^2}\, F \;=\; -(2\pi i)^2\, \delta_{p_1^2,1-z} F\,,$$
$$\mathrm{Cut}_{p_3^2,p_1^2}\, F \;=\; \mathrm{Disc}_{p_3^2,p_1^2}\, F \;=\; (2\pi i)^2\, [\delta_{p_3^2,\bar{z}} + \delta_{p_3^2,1-\bar{z}}] F\,.$$

Kinematic region: $p_1^2, p_3^2 > 0$ and $p_2^2 < 0$, or equivalently $\bar{z} < 0 < z < 1$.

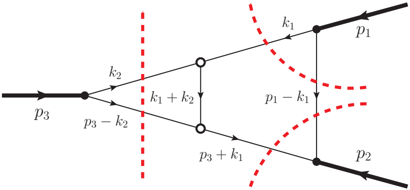
# Cuts live strictly within their own regions

Notice: the "1236" cut diagram is common to both double-cuts, but gives different values!



$\stackrel{?}{=}$ "Disc"

# Third cut of the ladder?




Cut $= 0$ in the kinematic region $p_1^2, p_2^2, p_3^2 > 0$.

Disc $= 0$ : no region detects the three cuts simultaneously.

It's consistent, but we should continue to more complicated examples. [in progress]

# Reconstruction: from cut to symbol

Integrability condition on symbols: for each $k$,

$$\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}\, d\log a_{i_k} \wedge d\log a_{i_{k+1}} \left[a_{i_1} \otimes \cdots \otimes a_{i_{k-1}} \otimes a_{i_{k+2}} \otimes \cdots \otimes a_{i_n}\right] = 0\,.$$

Combine with first entry condition (=Mandelstam invariant) and known cut(s).

Related to exchanging order of cuts.

# Reconstruction: from cut to symbol

Integrability condition on symbols: for each $k$,

$$\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}\, d\log a_{i_k} \wedge d\log a_{i_{k+1}} \left[a_{i_1} \otimes \cdots \otimes a_{i_{k-1}} \otimes a_{i_{k+2}} \otimes \cdots \otimes a_{i_n}\right] = 0\,.$$

Combine with first entry condition (=Mandelstam invariant) and known cut(s).
Example: $p_2^2$ cut of triangle.

$$\frac{1}{2}\,(z\bar{z}) \otimes \frac{1-z}{1-\bar{z}}$$

- For integrability, add

$$\frac{1}{2}\,(1-z) \otimes \bar{z} - \frac{1}{2}\,(1-\bar{z}) \otimes z$$

- For the first entry condition, add

$$\frac{1}{2}\,(1-\bar{z}) \otimes \bar{z} - \frac{1}{2}\,(1-z) \otimes z$$

Both conditions are satisfied.

Result:

$$\mathcal{S}(\mathcal{T}) = \frac{1}{2}\,z\bar{z} \otimes \frac{1-z}{1-\bar{z}} + \frac{1}{2}\,(1-z)(1-\bar{z}) \otimes \frac{\bar{z}}{z}$$

# Reconstruction: from cut to symbol

Integrability condition on symbols: for each $k$,

$$\sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n}\, d\log a_{i_k} \wedge d\log a_{i_{k+1}} \left[a_{i_1} \otimes \cdots \otimes a_{i_{k-1}} \otimes a_{i_{k+2}} \otimes \cdots \otimes a_{i_n}\right] = 0\,.$$

Combine with first entry condition (=Mandelstam invariant) and known cut(s).

Reconstruction of the symbol of the ladder is unique from any of its single *or double* cuts.

Knowledge of alphabet is crucial.

# Reconstruction: from symbol to full function

- In general, integrating a symbol is an unsolved problem.
- But in many cases we have enough information to constrain the function uniquely & algebraically.
- In the same example, from the $p_2^2$ cut of triangle, given that:

$$\mathcal{S}(\mathcal{T}) = \frac{1}{2}\, z\bar{z} \otimes \frac{1-z}{1-\bar{z}} + \frac{1}{2}\,(1-z)(1-\bar{z}) \otimes \frac{\bar{z}}{z}$$

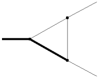
  and antisymmetry under $z \leftrightarrow \bar{z}$, the solution

$$\begin{aligned} \mathcal{T} &= \mathcal{P}_2(z) \\ &= \left(\mathrm{Li}_2(z) - \mathrm{Li}_2(\bar{z}) + \frac{1}{2}\log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right) \end{aligned}$$

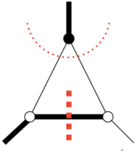
  is unique.
- Ladder & massive triangles are easy too. At most, fix a single free constant by numerical evaluation at a point.

# Generalizations for internal masses

- First entries adjusted for thresholds

$$\mathcal{S}\Bigg($$  $$\Bigg) = \frac{1}{\epsilon}\,\frac{m^2-p^2}{m^2} + m^2 \otimes \frac{m^2\left(m^2-p^2\right)}{p^2} + \left(m^2-p^2\right) \otimes \frac{p^2}{\left(m^2-p^2\right)^2} + \mathcal{O}\left(\epsilon\right)\ .$$

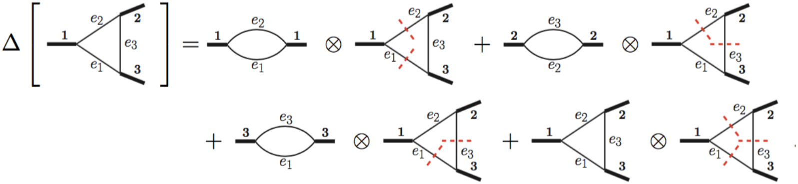
- Include cuts of massive propagators



 $=$  $= 2\pi\,\delta\left(p^2-m^2\right)$

# Coproducts as diagrams

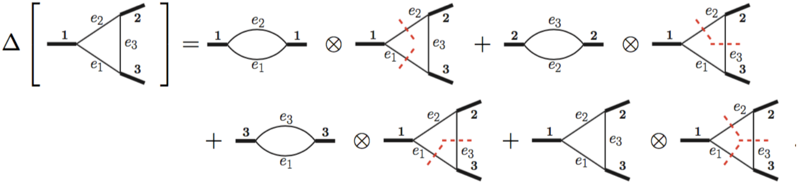
[in preparation with Abreu, Duhr, Gardi]



Second entries are discontinuities; first entries have discontinuities.

# Coproducts as diagrams

[in preparation with Abreu, Duhr, Gardi]




Second entries are discontinuities; first entries have discontinuities.

Motivated by the identity

$$\Delta\ \text{Disc} = (\text{Disc} \otimes 1)\,\Delta.$$

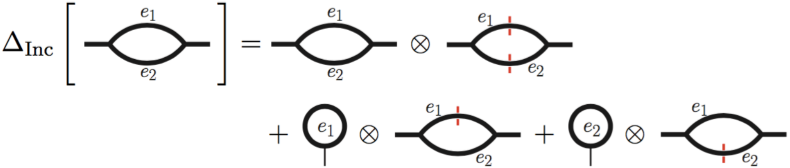
The companion relation

$$\Delta\,\partial = (1 \otimes \partial)\,\Delta$$

produces useful differential equations.

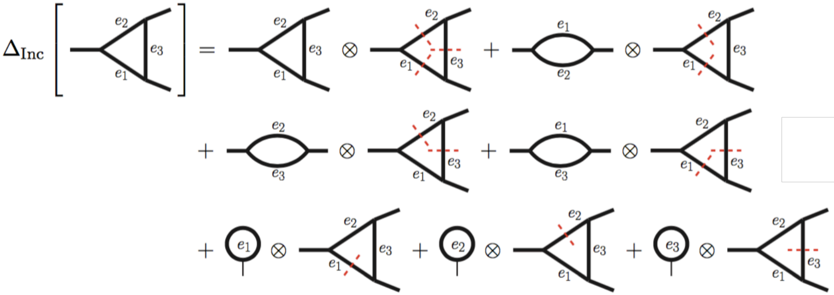
# Diagram operations: pinch $\otimes$ cut

Pinch and cut *complementary* subsets of edges:



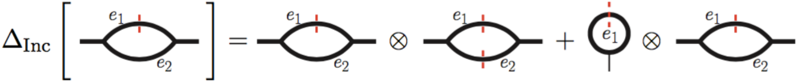
# Diagram operations: pinch ⊗ cut

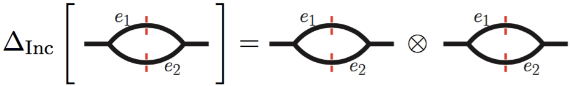
Pinch and cut *complementary* subsets of edges:



# Diagram operations: pinch $\otimes$ cut

Pinch and cut *complementary* subsets of edges:




Can also start with a cut diagram.

Operation is purely combinatorial and is the coproduct of a full Hopf algebra with product, unit, counit, antipode.

# 2 equivalent Hopf algebras

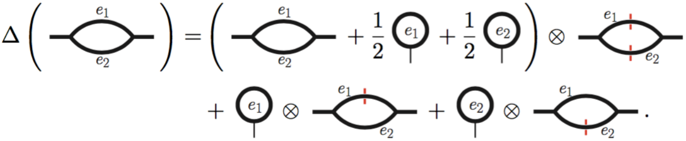
The combinatorial algebra agrees with the Hopf algebra on the MPL of evaluated diagrams!

But we have to make adjustments:

- Integrals live in different dimensions. With $n$ propagators, $D = n - 2\epsilon$ for $n$ even; $D = n + 1 - 2\epsilon$ for $n$ odd. Conjecture: these integrals form a basis of 1-loop integrals.

  E.g. box and triangle in $D = 4 - 2\epsilon$, bubble and tadpole in $D = 2 - 2\epsilon$.

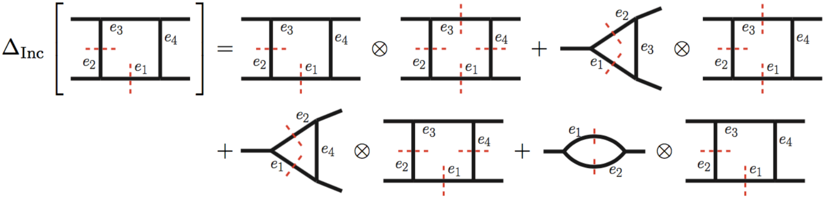
- Need to insert extra terms:




  Isomorphic to the more basic construction. (For any value of 1/2.)

# What are these generalized cuts?

Generalized cuts are sensitive to kinematic regions. Our graphical relations make no reference to kinematics, and we need complex cuts like 4-cuts of boxes. Here, we define cuts as residues.




At least at 1-loop, differences are proportional to $i\pi$, which drops out of the coproduct.

The residues can be written beautifully in terms of determinants.
(Gram and modified Cayley)

Not very clear how to generalize to higher loops, but the nature of coproducts suggests it should be possible. [related work: Brown; Bloch and Kreimer]

# Evidence for the graphical conjecture

- all tadpoles and bubbles
- triangles and boxes with several combinations of internal and external masses
- consistency checks for more complicated boxes, 0m pentagon, 0m hexagon

Checked to several orders in $\epsilon$.

# Summary & Outlook

- Cuts are used to compute Feynman integrals and explain simplicity of amplitudes.
- Cut diagrams are discontinuities of loop amplitudes.
- Hopf algebra structure identifies generalized discontinuities!
- Use Hopf algebra to interpret cut diagrams and reconstruct multi-loop integrals.
- The Hopf algebra can also be written diagrammatically. Diagram representations of cuts and differential equations.

- Make contact with maximal cuts/ complex residues of multiloop integrals?
- Expand cutting rules to capture more of the coproduct? e.g. repeated cuts in the same channel, crossed cuts, ...
- For diagrammatic algebra beyond one loop: what is a good basis? what are the exact operations? what about elliptic functions?
- Use diagrammatic operations to generate differential equations.

Aim to compute cuts & reconstruct amplitudes cleanly and efficiently.