Evaluating Feynman integrals by uniformly transcendental differential equations

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Based on collaboration with Johannes Henn, Bernhard Mistlberger and Alexander Smirnov

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- Introduction. The method of differential equations

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- Evaluating non-planar on-shell three-loop four-point massless integrals

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- MPL(1)
- Conclusion

[A.V. Kotikov'91, E. Remiddi'97, T. Gehrmann & E. Remiddi'00, J. Henn'13]

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Henn: use uniform transcendental (UT) bases!

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A lot of applications [J.M. Henn, A.V. Smirnov, V.A. Smirnov, K. Melnikov, F. Caola, R. Bonciani, V. Del Duca, H. Frellesvig, F. Moriello, M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi, T. Gehrmann, A. von Manteuffel, E. Weihs, F. Dulat, B. Mistlberger, R. N. Lee,...]

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- Solve DE

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DE:

$$\partial_i f(\epsilon, x) = A_i(\epsilon, x) f(\epsilon, x)\,,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and each A_i is an $N \times N$ matrix.

Henn (2013): turn to a new basis where DE take the form

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In the differential form,

$$d\, f(\epsilon, x) = \epsilon\, (d\, \tilde{A}(x))\, f(x, \epsilon) ,$$

where

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Let us call it *epsilon form*.

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where $x^{(k)}$ is the set of singular points of the DE and $N \times N$ matrices a_k are independent of x and ϵ.

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For example, if $x_k = 0, -1, 1$ then results for elements of such a basis are expressed in terms of HPL.

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- An approach using Magnus and Dyson series expansion [M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, U. Schubert, L. Tancredi'14]

- A part of the procedure is algorithmically described in [T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs'14]

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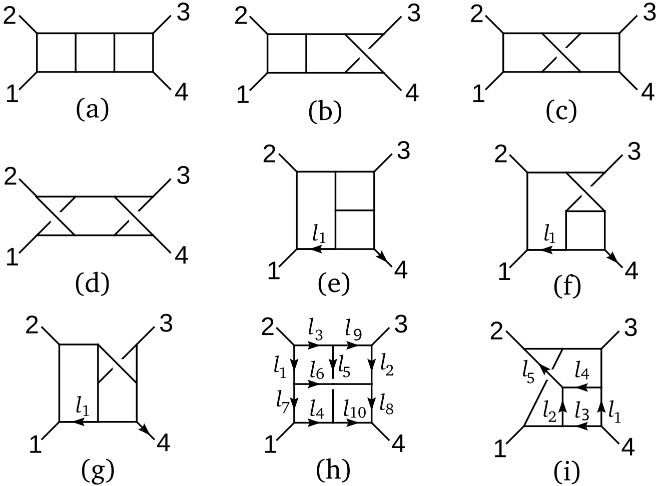
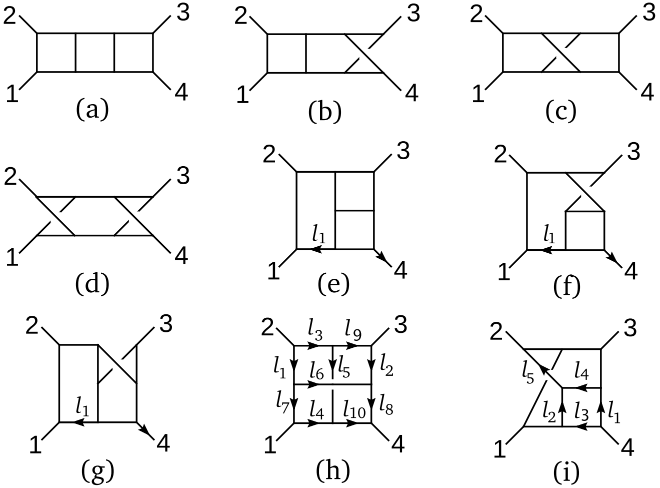
- An algorithmical description in the case of one variable [R.N. Lee'14]


2
3
1
4
(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
(i)
l_1
l_2
l_3
l_4
l_5
l_6
l_7
l_8
l_9
l_{10}



Motivation: three-loop amplitudes of $N = 8$ supergravity and $N = 4$ super-Yang-Mills theory [Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban'07]

The kinematics: $p_i^2 = 0$, $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_2 + p_3)^2 = -s - t$.

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$$F^D_{a_1,\ldots,a_{15}}(s,t;D) = \frac{1}{(i\pi^{D/2})^3} \int\int\int \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{(-k_1^2)^{a_1}[-(p_2-k_1+k_2)^2]^{a_2}[-k_2^2]^{a_3}}$$
$$\times \frac{[-(k_1-k_3)^2]^{-a_{11}}[-(p_1+k_3)^2]^{-a_{12}}[-(p_1+k_2)^2]^{-a_{13}}}{[-(p_1+p_2+k_2)^2]^{a_4}[-k_3^2]^{a_5}[-(p_1+p_2+p_3+k_2-k_3)^2]^{a_6}}$$
$$\times \frac{[-(p_3+k_1)^2]^{-a_{14}}[-(p_3+k_2)^2]^{-a_{15}}}{(-(p_1+k_1)^2)^{a_7}(-(k_1-k_2)^2)^{a_8}[-(k_2-k_3)^2]^{a_9}[-(k_3-p_3)^2]^{a_{10}}}\,.$$

Partial results:
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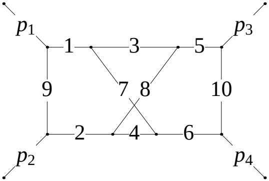
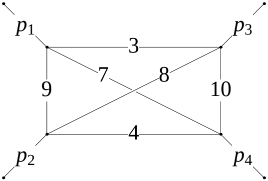
K_4 as a part of C [J. Henn, A.&V. Smirnov'13]





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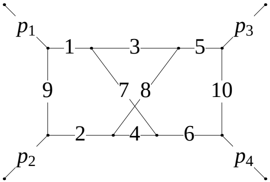

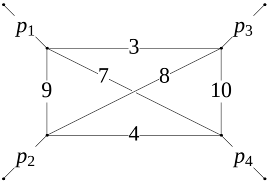

Results expressed in terms of HPL
$H_{a_1,a_2,\ldots,a_n}(x)$, $a_i = 1, 0, -1$,
[E. Remiddi and J.A.M. Vermaseren'00]

IBP reduction by `FIRE` and by a private code by Bernhard Mistlberger.

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In all the cases, initial DE are transformed into

$$\partial_x f(x,\epsilon) = \epsilon \left[\frac{a}{x} + \frac{b}{1+x}\right] f(x,\epsilon)\,.$$

where a and b are constant matrices.

Boundary conditions.

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Three singular points, at $x = 0$, $x = -1$, and $x = \infty$, corresponding to the limits $s \to 0$, $u \to 0$, and $t \to 0$, respectively.

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There is no this condition in the non-planar cases because non-planar diagrams have singularities in all the three channels.

Studying limits, $s \to 0, t \to 0, u \to 0$.

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Typical contributions to the asymptotic expansion in the limit $x = t/x \to 0$:
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The code `asy.m`
[A. Pak and A. Smirnov'10, B. Jantzen, A.S. and V.S.'12]
(which is now included into `FIESTA` [A.S.'09-15])
$\to$ expression of contributions of regions
[M. Beneke & V. S.'12] in terms of parametric integrals.

Three last elements of the basis

$$
\begin{aligned}
&-\epsilon^6 s(s+t)(2sF_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0}) - sF_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1})\\
&\qquad -F_{1,1,0,0,1,1,1,1,1,1,0,0,0,0,0}) + F_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1}))\,,\\
&\epsilon^6 st(3F_{1,1,0,0,1,1,1,1,1,1,0,0,0,0,0}) - 2F_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-1})\\
&\qquad -F_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1}))\,,\\
&\epsilon^6 s(-\frac{3}{2}s^2F_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0}) + \frac{3}{2}s^2F_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-1})\\
&\qquad -\frac{9}{4}sF_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,-1}) + \frac{5}{4}sF_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-1})\\
&\qquad -2sF_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-1}) + \frac{3}{2}sF_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,-2})\\
&\qquad -5F_{1,1,1,-1,1,1,1,1,1,1,0,0,0,0,-1}) + 4F_{1,1,1,0,1,1,1,1,1,1,0,0,-1,0,-1})\\
&\qquad +3F_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-2}) - 2F_{1,1,1,1,1,1,1,1,1,1,0,0,-1,0,-2}))\,.
\end{aligned}
$$

Our analytical result for element 28 is

```
-(1/3) - (I ep \[Pi])/2 + (10 ep^2 \[Pi]^2)/9 +
 23/24 I ep^3 \[Pi]^3 - (271 ep^4 \[Pi]^4)/4320 - (
 10201 I ep^5 \[Pi]^5)/2880 - (23819 ep^6 \[Pi]^6)/20160 +
 1/2 ep H[{-1}, x] - 7/24 ep^3 \[Pi]^2 H[{-1}, x] -
 35/12 I ep^4 \[Pi]^3 H[{-1}, x] - 3809/960 ep^5 \[Pi]^4 H[{-1}, x] -
 1157/72 I ep^6 \[Pi]^5 H[{-1}, x] + 1/2 ep H[{0}, x] +
 1/2 I ep^2 \[Pi] H[{0}, x] - 61/24 ep^3 \[Pi]^2 H[{0}, x] +
 27/8 I ep^4 \[Pi]^3 H[{0}, x] - 103/576 ep^5 \[Pi]^4 H[{0}, x] + (
 58537 I ep^6 \[Pi]^5 H[{0}, x])/2880 +
 9/2 I ep^3 \[Pi] H[{-1, -1}, x] -
 35/12 ep^4 \[Pi]^2 H[{-1, -1}, x] -
 683/24 I ep^5 \[Pi]^3 H[{-1, -1}, x] +
 3361/240 ep^6 \[Pi]^4 H[{-1, -1}, x] - 1/2 ep^2 H[{-1, 0}, x] -
 5/2 I ep^3 \[Pi] H[{-1, 0}, x] + 77/24 ep^4 \[Pi]^2 H[{-1, 0}, x] +
 395/24 I ep^5 \[Pi]^3 H[{-1, 0}, x] + (
 739 ep^6 \[Pi]^4 H[{-1, 0}, x])/2880 - 1/2 ep^2 H[{0, -1}, x] -
 97/24 ep^4 \[Pi]^2 H[{0, -1}, x] +
 77/4 I ep^5 \[Pi]^3 H[{0, -1}, x] + (1/2880)
 18691 ep^6 \[Pi]^4 H[{0, -1}, x] - 5/2 I ep^3 \[Pi] H[{0, 0}, x] +
 79/12 ep^4 \[Pi]^2 H[{0, 0}, x] -
 445/24 I ep^5 \[Pi]^3 H[{0, 0}, x] +
 73/240 ep^6 \[Pi]^4 H[{0, 0}, x] - 9/2 ep^3 H[{-1, -1, -1}, x] +...
```

Evaluating planar three-loop vertex integrals
[J. Henn, A. Smirnov and V. Smirnov'15]
[J. Henn, A. Smirnov,V. Smirnov & M. Steinhauser'16]

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Numerical evaluation of planar and non-planar three-loop threshold integrals with `FIESTA` [P. Marquard, J.H. Piclum, D. Seidel and M. Steinhauser'14]
(evaluating NRQCD/QCD matching coefficients)

$$
\begin{aligned}
F_{a_1,\ldots,a_{12}} = \int\int\int & \frac{\mathrm{d}^D k_1\, \mathrm{d}^D k_2\, \mathrm{d}^D k_3}{[m^2-(k_1+p_1)^2]^{a_1}[m^2-(k_2+p_1)^2]^{a_2}} \\
\times & \frac{1}{[m^2-(k_3+p_1)^2]^{a_3}[m^2-(k_3+p_2)^2]^{a_4}[m^2-(k_2+p_2)^2]^{a_5}} \\
\times & \frac{1}{[m^2-(k_1+p_2)^2]^{a_6}[-k_1^2]^{a_7}[-(k_1-k_2)^2]^{a_8}[-(k_2-k_3)^2]^{a_9}} \\
\times & \frac{1}{[-(k_1-k_3)^2]^{a_{10}}[-k_2^2]^{-a_{11}}[-k_3^2]^{-a_{12}}}
\end{aligned}
$$

at $p_1^2 = m^2, p_2^2 = m^2$ at general $s \equiv q^2 = (p_1 - p_2)^2$
or at threshold, $s = 4m^2$.

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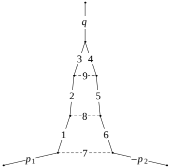
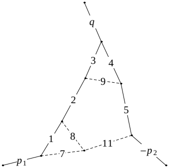
$$\times \frac{1}{[m^2-(k_3+p_1)^2]^{a_3}[m^2-(k_3+p_2)^2]^{a_4}[m^2-(k_2+p_2)^2]^{a_5}}$$
$$\times \frac{1}{[m^2-(k_1+p_2)^2]^{a_6}[-k_1^2]^{a_7}[-(k_1-k_2)^2]^{a_8}[-(k_2-k_3)^2]^{a_9}}$$
$$\times \frac{1}{[-(k_1-k_3)^2]^{a_{10}}[-k_2^2]^{-a_{11}}[-k_3^2]^{-a_{12}}}$$

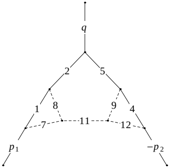
at $p_1^2 = m^2, p_2^2 = m^2$ at general $s \equiv q^2 = (p_1-p_2)^2$
or at threshold, $s = 4m^2$.
Each index can be positive but the total number of positive indices cannot be more than 9. This family of integrals can be represented as the union of eight subfamilies.


q
3 4
9
2 5
8
1 6
7
p1
−p2

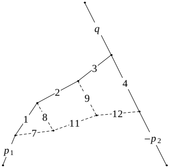
(1)


q
3 4
9
2
5
1 8
7
11
p1
−p2

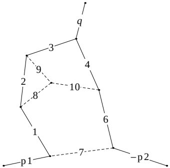
(2)


q
2 5
1 8 9 4
7 11 12
p1
−p2

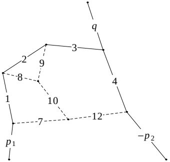
(3)


q
3
4
2
9
1 8
7 11 12
p1
−p2

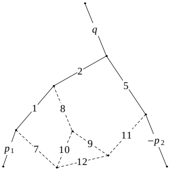
(4)


q
3
9
4
2
8
10
6
1
7
p1
−p2

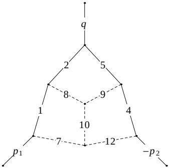
(5)


q
2
3
9
8
1
10
4
7
12
p1
−p2

(6)


q
2
5
1
8
11
9
7
10
12
p1
−p2

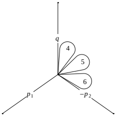
(7)

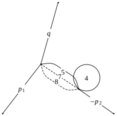

q
2 5
8 9
1
4
10
7 12
p1
−p2

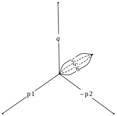
(8)

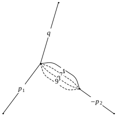
90 master integrals for general q^2 and 51 threshold master integrals

$$
\begin{aligned}
&F_{0,0,0,1,1,1,0,0,0,0,0,0},\ F_{0,0,0,0,1,1,0,0,0,1,0,1},\ F_{0,0,0,0,1,1,0,0,1,1,0,0},\ F_{0,0,1,0,0,0,0,1,0,1,1,0},\\
&F_{0,0,1,0,0,1,0,1,1,0,0,0},\ F_{0,0,1,0,0,1,0,1,2,0,0,0},\ F_{0,0,1,0,1,1,0,0,0,1,0,0},\ F_{0,0,0,0,0,1,0,1,0,1,1,1},\\
&F_{0,0,1,0,0,1,0,1,0,1,1,0},\ F_{0,0,1,0,0,1,0,1,0,1,2,0},\ F_{0,0,1,0,0,1,0,1,0,2,1,0},\ F_{0,0,1,0,0,1,0,1,1,0,0,1},\\
&F_{0,0,1,0,0,1,0,1,1,0,1,0},\ F_{0,0,1,0,0,1,0,1,1,0,2,0},\ F_{0,0,1,0,0,1,0,2,1,0,1,0},\ F_{0,0,1,0,1,1,0,0,0,1,0,1},\\
&F_{0,0,1,0,1,1,0,0,1,1,0,0},\ F_{0,0,1,0,1,1,0,0,1,2,0,0},\ F_{0,1,1,0,0,0,1,1,0,1,0,0},\ F_{0,0,1,0,0,1,0,1,0,1,1,1},\\
&F_{0,0,1,0,0,1,0,1,1,1,1,0},\ F_{0,0,1,0,1,1,0,1,0,1,1,0},\ F_{0,0,1,0,1,1,0,1,0,1,2,0},\ F_{0,0,1,0,1,2,0,1,0,1,1,0},\\
&F_{0,0,1,0,1,2,0,0,1,1,1,0},\ F_{0,0,1,1,0,1,0,1,1,0,2,0},\ F_{0,0,1,1,0,1,1,1,1,0,0,0},\ F_{0,1,1,0,0,1,0,1,0,1,0,1},\\
&F_{0,1,1,0,0,1,0,1,0,2,0,1},\ F_{0,1,1,0,0,1,0,2,0,1,0,1},\ F_{0,1,1,0,0,2,0,1,0,1,0,1},\ F_{0,1,1,0,0,1,1,1,0,1,0,0},\\
&F_{0,1,1,0,1,0,1,0,1,1,0,0},\ F_{0,1,1,0,1,0,1,1,0,1,0,0},\ F_{0,1,1,0,1,1,0,0,1,0,0,1},\ F_{0,1,1,0,1,1,0,0,1,1,0,0},\\
&F_{0,0,1,0,1,1,0,1,0,1,1,1},\ F_{0,0,2,0,1,1,0,1,0,1,1,1},\ F_{0,0,1,1,1,1,0,1,0,1,1,0},\ F_{0,0,1,1,1,1,0,1,0,1,2,0},\\
&F_{0,1,1,0,1,0,1,1,1,0,0,1},\ F_{0,1,1,0,1,1,0,1,0,1,0,1},\ F_{0,1,1,0,1,1,0,1,0,1,0,2},\ F_{0,1,1,0,1,1,0,1,0,2,0,1},\\
&F_{0,1,1,0,1,1,0,2,0,1,0,1},\ F_{0,1,1,0,1,1,1,1,0,1,0,0},\ F_{0,1,1,0,1,1,1,1,0,2,0,0},\ F_{0,1,1,1,1,1,0,1,0,1,0,0},\\
&F_{1,1,1,0,0,0,0,1,0,1,1,1},\ F_{0,1,1,0,1,1,1,1,1,0,0,1},\ F_{0,1,1,0,1,1,1,1,1,1,0,1}\ .
\end{aligned}
$$


(1)


(2)


(3)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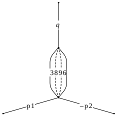
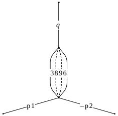
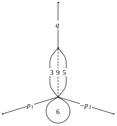
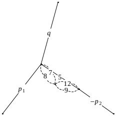
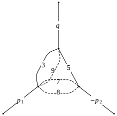
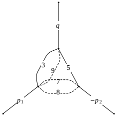

(5)

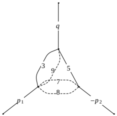

(6a)

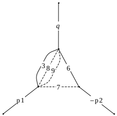

(7)

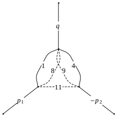

(8)

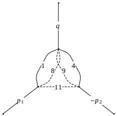

(9)

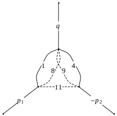

(10a)

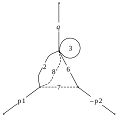

(11b)


(12)

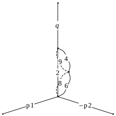

(13)


(14a)

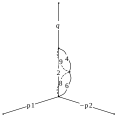

(15b)



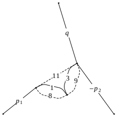
(16)



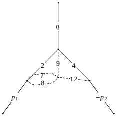
(17)



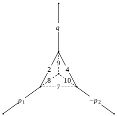
(18a)



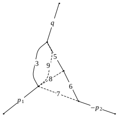
(19)



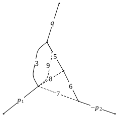
(20)



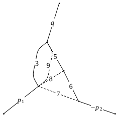
(21)



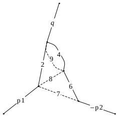
(22)



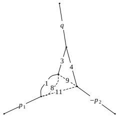
(23a)



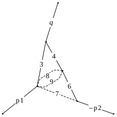
(24b)

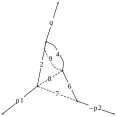


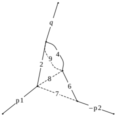
(25)

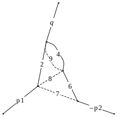


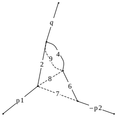
(26)

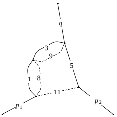

(27)

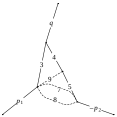

(28)


(29a)


(30b)


(31c)


(32)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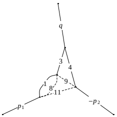
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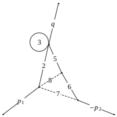
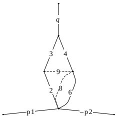
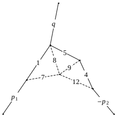
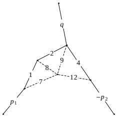
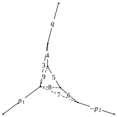
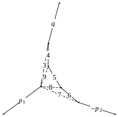

(35)


(36)


(37)


(38a)


(39)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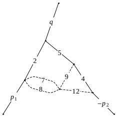
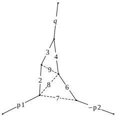
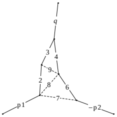
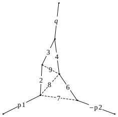
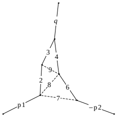
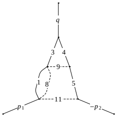

(41)


(42)


(43a)


(44b)


(45c)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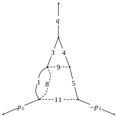
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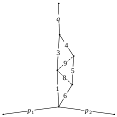
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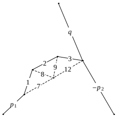
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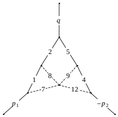
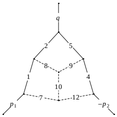

(50)


(51)

It is convenient to introduce the variable

$$\frac{s}{m^2} = -\frac{(1-x)^2}{x}$$

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The values $x = 1$ and $x = -1$ correspond to $s = 0$ and $s = 4m^2$.

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DE

$$f'(\epsilon, x) = \epsilon\, \tilde{A}'(x)\, f(x, \epsilon)\,,$$

where $\tilde{A} = \sum_k \tilde{A}_{\alpha_k} \log(\alpha_k)$ and the letters α_k are $x, 1+x, 1-x, 1+x+x^2$.

90 elements of this basis $f(x)$ are

$$\{F_{0,0,0,3,3,3,0,0,0,0,0,0}\,, \quad \varepsilon\frac{x^2-1}{x}F_{0,0,2,1,3,3,0,0,0,0,0,0}, \ldots$$
$$\varepsilon^6\frac{(1-x^2)^2}{x^2}F_{1,0,1,1,1,1,1,1,1,0,0,0}\,, \quad (1-2\varepsilon)\varepsilon^4 F_{1,2,1,0,0,0,1,1,1,0,0,1}\}$$

90 elements of this basis $f(x)$ are

$$\{F_{0,0,0,3,3,3,0,0,0,0,0,0}\,, \quad \varepsilon\frac{x^2-1}{x}F_{0,0,2,1,3,3,0,0,0,0,0,0}, \ldots$$
$$\varepsilon^6\frac{(1-x^2)^2}{x^2}F_{1,0,1,1,1,1,1,1,1,0,0,0}\,, \quad (1-2\varepsilon)\varepsilon^4 F_{1,2,1,0,0,0,1,1,1,0,0,1}\}$$

A solution in an epsilon-expansion with coefficients written in terms of multiple (Goncharov) polylogarithms (MPL)

$$G(a_1,\ldots,a_n;z)=\int_0^z \frac{\mathrm{d}t}{t-a_1}\,G(a_2,\ldots,a_n;t)$$

with indices a_i taken from the seven-letters alphabet $\{0, r_1, r_3, -1, r_4, r_2, 1\}$ with

$$r_{1,2}=\frac{1}{2}\left(1\pm\sqrt{3}\,\mathrm{i}\right)\,, \quad r_{3,4}=\frac{1}{2}\left(-1\pm\sqrt{3}\,\mathrm{i}\right)\,.$$

A typical expression for analytical results for the elements of the basis

```
ep^4*(-24*G[{-1}, 1]*G[{0}, x]*G[{0, -1}, 1] +
   24*G[{0, -1}, 1]*G[{0, -1}, x] - 23*G[{0, -1}, 1]*G[{0, 0}, x] -
   12*G[{-1}, 1]*G[{0}, x]*G[{0, 1}, 1] + 12*G[{0, -1}, x]*G[{0, 1}, 1] -
   (23*G[{0, 0}, x]*G[{0, 1}, 1])/2 + 12*G[{0, -1}, 1]*G[{0, 1}, x] +
   6*G[{0, 1}, 1]*G[{0, 1}, x] + 12*G[{0, -1}, 1]*G[{1, 0}, x] +
   6*G[{0, 1}, 1]*G[{1, 0}, x] - 9*G[{0, -1}, 1]*G[{r1, 0}, x] -
   (9*G[{0, 1}, 1]*G[{r1, 0}, x])/2 - 9*G[{0, -1}, 1]*G[{r2, 0}, x] -
   (9*G[{0, 1}, 1]*G[{r2, 0}, x])/2 + 24*G[{0}, x]*G[{-1, 0, -1}, 1] +
   12*G[{0}, x]*G[{-1, 0, 1}, 1] + 24*G[{0}, x]*G[{0, -1, -1}, 1] +
   24*G[{-1}, x]*G[{0, 0, -1}, 1] - 48*G[{0}, x]*G[{0, 0, -1}, 1] +
   48*G[{1}, x]*G[{0, 0, -1}, 1] - 18*G[{r1}, x]*G[{0, 0, -1}, 1] -
   18*G[{r2}, x]*G[{0, 0, -1}, 1] + 24*G[{-1}, x]*G[{0, 0, 1}, 1] -
   (57*G[{0}, x]*G[{0, 0, 1}, 1])/2 + 24*G[{1}, x]*G[{0, 0, 1}, 1] -
   (21*G[{r1}, x]*G[{0, 0, 1}, 1])/2 - (21*G[{r2}, x]*G[{0, 0, 1}, 1])/2 -
   6*G[{0}, x]*G[{0, 1, 1}, 1] - 24*G[{-1, -1, 0, 0}, x] +
   36*G[{-1, 0, 0, 0}, x] - 24*G[{-1, 1, 0, 0}, x] +
   24*G[{0, -1, -1, 0}, x] + 2*G[{0, -1, 0, 0}, x] + 12*G[{0, -1, 1, 0}, x] -
   23*G[{0, 0, -1, 0}, x] - (23*G[{0, 0, 1, 0}, x])/2 +
   12*G[{0, 1, -1, 0}, x] + (11*G[{0, 1, 0, 0}, x])/2 +
   6*G[{0, 1, 1, 0}, x] - 24*G[{1, -1, 0, 0}, x] + 12*G[{1, 0, -1, 0}, x] +
   15*G[{1, 0, 0, 0}, x] + 6*G[{1, 0, 1, 0}, x] - 12*G[{1, 1, 0, 0}, x] -
   9*G[{r1, 0, -1, 0}, x] + 6*G[{r1, 0, 0, 0}, x] -
   (9*G[{r1, 0, 1, 0}, x])/2 + (3*G[{r1, 1, 0, 0}, x])/2 -
   9*G[{r2, 0, -1, 0}, x] + 6*G[{r2, 0, 0, 0}, x] -
   (9*G[{r2, 0, 1, 0}, x])/2 + (3*G[{r2, 1, 0, 0}, x])/2 +
   (3*G[{0}, x]*Zeta[3])/2 - (3*G[{r1}, x]*Zeta[3])/2 -
   (3*G[{r2}, x]*Zeta[3])/2 -
   (3*(16*G[{0, -1}, 1]^2 + 8*G[{0, -1}, 1]*G[{0, 1}, 1] +...
```

To obtain analytical results for the 51 threshold master integrals use threshold expansion

$$F(a_1,\ldots,a_{12};q^2,m^2) \sim \sum_{n=n_0}^{\infty}\sum_{j=0}^{3}(4m^2-q^2)^{n-j\epsilon}F_{n,j}(a_1,\ldots,a_{12};q^2)\,.$$

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Threshold master integrals are one-scale integrals $F_{0,0}(a_1,\ldots,a_{12};m^2)$ defined with q^2 set to $4m^2$.

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Expand 'naively' in $x+1$ the corresponding integrals. Introduce one more (13th) index for the order of this derivative in s, i.e. deal with the family

$$F'(a_1,\ldots,a_{12},a_{13}) = \left(\frac{\partial}{\partial s}\right)^{-a_{13}} F(a_1,\ldots,a_{12})\Bigg|_{s=4m^2}$$

Using IBP relations for integrals at general q and expanding all the terms naively in q^2 at $q^2 = 4m^2 \rightarrow 15$ IBP relations.

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They are all with $a_{13} = 0$, i.e directly correspond to the 51 master threshold integrals.

Matching at threshold

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$x = y - 1$, $y \to 0$:

$$f'(\epsilon, y) = \epsilon \frac{\tilde{A}'(y)}{y} f(\epsilon, y)\,,$$

where $\tilde{A}'(y) = A_0 + yA_1 + y^2 A_2 + \ldots$.

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In the language of differential equations, the naive part of the expansion near $y = 0$ corresponds to zero eigenvalues of the matrix A_0 while eigenvalues proportional to ε correspond to other contributions.

To obtain expansions near $y = 0$ of the elements of the basis in higher orders in y, we use a trick from the theory of differential equations (presented, e.g., in [Wasov's book]).

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Construct a polynomial $P = 1 + \sum_{r=1} P_r y^r$ such that the DE for the function g defined by $f = Pg$ takes the form $yg'(y) = A_0 g(y)$ (with A_0 is independent of y).

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We implemented this algorithm and constructed P_r up to $r = 5$.

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Solving these equations $\rightarrow$ coefficients of the epsilon expansion of the MI up to some order written in terms of MPL $G(a_1,\ldots,a_n;1)$ with $a_1 \neq 1$ and a_i taken from the alphabet $\{0, r_1, r_3, -1, r_4, r_2, 1\}$.

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Examples of our results
[J. Henn, A. Smirnov and V. Smirnov'15]

$$
\begin{aligned}
F_{0,0,1,0,1,1,0,1,0,1,1,1} = & -\frac{27}{2}\log(2)G_R(0,0,r_2,-1) - \frac{181\zeta(5)}{32} - \frac{21}{2}\log^2(2)\zeta(3) \\
& + \frac{115\pi^2\zeta(3)}{48} - 12\mathrm{Li}_5\left(\frac{1}{2}\right) - 12\log(2)\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{2\log^5(2)}{5} + \frac{1}{6}\pi^2\log^3(2) \\
& - \frac{81}{8}G_R(0,0,r_4,1)\log(2) + \frac{277}{960}\pi^4\log(2)\,, \\
F_{0,0,1,1,1,1,0,1,0,1,1,0} = & -\frac{27}{4}\log(2)G_R(0,0,r_2,-1) - \frac{341\zeta(5)}{64} - \frac{21}{4}\log^2(2)\zeta(3) \\
& + \frac{211\pi^2\zeta(3)}{96} - 6\mathrm{Li}_5\left(\frac{1}{2}\right) - 6\log(2)\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{\log^5(2)}{5} \\
& + \frac{1}{12}\pi^2\log^3(2) - \frac{81}{16}G_R(0,0,r_4,1)\log(2) + \frac{277\pi^4\log(2)}{1920}\,.
\end{aligned}
$$

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$$+\frac{1}{12}\pi^2\log^3(2) - \frac{81}{16}G_R(0,0,r_4,1)\log(2) + \frac{277\pi^4\log(2)}{1920}\,.$$

where

$$G(a_1,\ldots,a_n;1) = G_R(a_1,\ldots,a_n) + \mathrm{i}\,G_I(a_1,\ldots,a_n)$$

Constructing bases for $G_R(a_1, \ldots, a_n)$ and $G_I(a_1, \ldots, a_n)$.

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Constants present in results for Feynman integrals up to weight 5 were discussed in [Fleischer and M. Kalmykov'99, Davydychev M. Kalmykov'00, M. Kalmykov and B. Kniehl'10].

For example,

$$
\begin{aligned}
G_I(r_2) &= -\frac{\pi}{3}\,, \quad G_R(-1) = \log(2)\,, \\
G_R(0,0,1) &= -\zeta(3)\,, \quad G_R(0,0,0,1) = -\frac{\pi^4}{90}\,, \\
G_R(0,0,0,0,1) &= -\zeta(5)\,, \\
G_R(0,0,1,1,-1) &= -2\mathrm{Li}_5\left(\frac{1}{2}\right) - 2\mathrm{Li}_4\left(\frac{1}{2}\right)\log(2) - \frac{\pi^2\zeta(3)}{96} \\
&\quad + \frac{151\zeta(5)}{64} - \frac{\log^5(2)}{15} + \frac{1}{18}\pi^2\log^3(2) - \frac{1}{96}\pi^4\log(2)\,.
\end{aligned}
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MPL

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If $a_w \neq 0$, then

$$G(\rho a_1, \ldots, \rho a_w; \rho z) = G(a_1, \ldots, a_w; z)$$

so that one can express them in terms of $G(\ldots; 1)$.

The MPL can be represented as multiple nested sums

$$
\begin{aligned}
\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) &= \sum_{n_k=1}^{\infty}\sum_{n_{k-1}=1}^{n_k-1}\cdots\sum_{n_1=1}^{n_2-1}\frac{x_1^{n_1}}{n_1^{m_1}}\cdots\frac{x_k^{n_k}}{n_k^{m_k}} \\
&= Z_{m_k,\ldots,m_1}(\infty;x_k,\ldots,x_1) \\
&= (-1)^k\, G\left(\underbrace{0,\ldots,0}_{m_k-1},\frac{1}{x_k},\ldots,\underbrace{0,\ldots,0}_{m_1-1},\frac{1}{x_1\ldots x_k};1\right)\,.
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Since the arguments of the Li- and Z-functions involved have the form $x_i=\lambda^{p_i}$ for $p_i=0,\ldots,5$ we introduce an auxiliary function

$$L_{m_1,\ldots,m_k}(p_1,\ldots,p_k)=\mathrm{Li}_{m_1,\ldots,m_k}\left(\lambda^{p_1},\ldots,\lambda^{p_k}\right)\ .$$

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Shuffle relations

$$G(a_1,\ldots,a_{w_1};z)\,G(b_1,\ldots,b_{w_2};z)=\sum_{c\in a\uplus b}G(c_1,\ldots,c_{w_1+w_2};z)\,,$$

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Stuffle relations

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They are written for $a_1\geq 2, b_1\geq 2$ for a given weight $w=\sum a_i+\sum b_i$ and then are translated into the language of $G(a_1,\ldots,a_w;1)$.

Regularization relations

$$\{L_1(0)L_{a_1,\ldots,a_k}(q_1,\ldots,q_k)\}_{\mathrm{stuffle}} = \{L_1(0)L_{a_1,\ldots,a_k}(q_1,\ldots,q_k)\}_{\mathrm{shuffle}}$$

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The numbers $L_1(0)$ correspond to the variable T introduced by Zagier in the case of MZV
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(Regularized double shuffle relations.)

Singular terms are cancelled in the difference.

Distribution relations

$$\mathrm{Li}_{a_1,\ldots,a_k}(x_1,\ldots,x_k) = d^{a_1+\ldots+a_k-k} \times \sum_{(y_1,\ldots,y_k):\, y_j^d=x_j, j=1,\ldots,k} \mathrm{Li}_{a_1,\ldots,a_k}(y_1,\ldots,y_k)$$

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for $d=2$ and $d=3$.

In our case, $n = 6$, we use these four types of relations, then turn to the real and imaginary parts and use also the complex conjugation relations

$$G(a_1^*, \dots, a_n^*; 1) = G(a_1, \dots, a_n; 1)^*$$

with $r_1^* = r_2, r_3^* = r_4$.

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The total number of relations grows fast when the weight is increased. At weight 6, we have 654452 equations for the real parts and 654937 equations for the imaginary parts of $G(a_1, \ldots, a_n; 1)$.

The relations for the $6 \times 7^{w-1}$ numbers $G_R(a_1, \ldots, a_w; 1)$ or $G_I(a_1, \ldots, a_w; 1)$ are linear equations. We solved them for $w = 1, 2, \ldots, 6$ with a code written in `Mathematica`.

The relations for the $6 \times 7^{w-1}$ numbers $G_R(a_1, \ldots, a_w; 1)$ or $G_I(a_1, \ldots, a_w; 1)$ are linear equations. We solved them for $w = 1, 2, \ldots, 6$ with a code written in `Mathematica`.

It turns out, however, that the resulting constants, independent in the sense of these relations, are still Q-linearly dependent, i.e. one can linearly express some of them in terms of a smaller set of the constants and products of constants of lower weights.

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We revealed additional relations using `PSLQ` algorithm [H.R.P. Ferguson, D.H. Bailey, and S. Arno] and `ginac` [C. Bauer, A. Frink and R. Kreckel] to evaluate MPLs with a big accuracy (up to 4000 digits).

The dimensions of the bases

w	$\bar{D}_R(w)$	$D_R(w)$	$\bar{D}_I(w)$	$D_I(w)$	$PSLQ_R$	$PSLQ_I$
1	2	2	1	1		
2	1	5	1	3		
3	3	12	2	9		2
4	5	30	5	25	2	6
5	13	76	11	68	11	17
6	25	195	25	182	39	49

Our basis for the real parts of $G(a_1, \ldots, a_4; 1)$ consists of 5 constants of weight 4

```
{GR[0, 0, r2, -1], GR[0, 0, r4, 1], GR[r2, 1, 1, -1],
 GR[r2, 1, 1, r3], GR[r2, 1, r2, -1]}
```

and 25 products of constants of lower weights

```
{GR[-1]^4, GI[r2]^2 GR[-1]^2, GI[r2]^4, GR[-1]^3 GR[r4],
 GI[r2]^2 GR[-1] GR[r4], GR[-1]^2 GR[r4]^2, GI[r2]^2 GR[r4]^2,
 GR[-1] GR[r4]^3, GR[r4]^4, GI[r2] GI[0, r2] GR[-1],
 GI[r2] GI[0, r2] GR[r4], GI[0, r2]^2, GR[-1]^2 GR[r2, -1],
 GI[r2]^2 GR[r2, -1], GR[-1] GR[r4] GR[r2, -1], GR[r4]^2 GR[r2, -1],
 GR[r2, -1]^2, GR[-1] GR[0, 0, 1], GR[r4] GR[0, 0, 1],
 GI[r2] GI[0, 1, r4], GI[r2] GI[0, r2, -1], GR[-1] GR[r2, 1, -1],
 GR[r4] GR[r2, 1, -1], GR[-1] GR[r2, 1, r3], GR[r4] GR[r2, 1, r3]}
```

Our basis for the imaginary parts of $G(a_1, \ldots, a_4; 1)$ consists of 5 constants of weight 4

```
{GI[0, 0, 0, r2], GI[0, 1, 1, r4], GI[0, 1, r2, -1], GI[0, 1, r2, r3],
  GI[0, r2, 1, -1]}
```

and 20 products of constants of lower weights

```
{GI[r2] GR[-1]^3, GI[r2]^3 GR[-1], GI[r2] GR[-1]^2 GR[r4],
 GI[r2]^3 GR[r4], GI[r2] GR[-1] GR[r4]^2, GI[r2] GR[r4]^3,
 GI[0, r2] GR[-1]^2, GI[r2]^2 GI[0, r2], GI[0, r2] GR[-1] GR[r4],
 GI[0, r2] GR[r4]^2, GI[r2] GR[-1] GR[r2, -1],
 GI[r2] GR[r4] GR[r2, -1], GI[0, r2] GR[r2, -1], GI[r2] GR[0, 0, 1],
 GI[0, 1, r4] GR[-1], GI[0, 1, r4] GR[r4], GI[0, r2, -1] GR[-1],
 GI[0, r2, -1] GR[r4], GI[r2] GR[r2, 1, -1], GI[r2] GR[r2, 1, r3]}
```

Our basis for the real parts of $G(a_1, \ldots, a_5; 1)$ consists of 13 constants of weight 5

```
{GR[0, 0, 0, 0, 1], GR[0, 0, 1, 1, -1], GR[0, 0, 1, 1, r4],
 GR[0, 0, 1, r2, -1], GR[0, 0, 1, r2, r3], GR[0, 0, 1, r2, r4],
 GR[0, 0, r2, 1, -1], GR[r2, 1, 1, -1, r2], GR[r2, 1, 1, 1, -1],
 GR[r2, 1, 1, 1, r3], GR[r2, 1, 1, r2, -1], GR[r2, 1, 1, r2, r3],
 GR[r2, 1, 1, r4, -1]}
```

and 63 products of constants of lower weights

```
{GR[-1]^5, GI[r2]^2 GR[-1]^3, GI[r2]^4 GR[-1], GR[-1]^4 GR[r4],
 GI[r2]^2 GR[-1]^2 GR[r4], GI[r2]^4 GR[r4], GR[-1]^3 GR[r4]^2,
 GI[r2]^2 GR[-1] GR[r4]^2, GR[-1]^2 GR[r4]^3, GI[r2]^2 GR[r4]^3,
 GR[-1] GR[r4]^4, GR[r4]^5, GI[r2] GI[0, r2] GR[-1]^2,
 GI[r2]^3 GI[0, r2], GI[r2] GI[0, r2] GR[-1] GR[r4],
 GI[r2] GI[0, r2] GR[r4]^2, GI[0, r2]^2 GR[-1], GI[0, r2]^2 GR[r4],
 GR[-1]^3 GR[r2, -1], GI[r2]^2 GR[-1] GR[r2, -1],
 GR[-1]^2 GR[r4] GR[r2, -1], GI[r2]^2 GR[r4] GR[r2, -1],
 GR[-1] GR[r4]^2 GR[r2, -1], GR[r4]^3 GR[r2, -1],
 GI[r2] GI[0, r2] GR[r2, -1], GR[-1] GR[r2, -1]^2,
 GR[r4] GR[r2, -1]^2, GR[-1]^2 GR[0, 0, 1], GI[r2]^2 GR[0, 0, 1],
 GR[-1] GR[r4] GR[0, 0, 1], GR[r4]^2 GR[0, 0, 1],
 GR[r2, -1] GR[0, 0, 1], GI[r2] GI[0, 1, r4] GR[-1],
 GI[r2] GI[0, 1, r4] GR[r4], GI[0, r2] GI[0, 1, r4],
 GI[r2] GI[0, r2, -1] GR[-1], GI[r2] GI[0, r2, -1] GR[r4],
 GI[0, r2] GI[0, r2, -1], GR[-1]^2 GR[r2, 1, -1],
 GI[r2]^2 GR[r2, 1, -1], GR[-1] GR[r4] GR[r2, 1, -1],
 GR[r4]^2 GR[r2, 1, -1], GR[r2, -1] GR[r2, 1, -1],
 GR[-1]^2 GR[r2, 1, r3], GI[r2]^2 GR[r2, 1, r3],
 GR[-1] GR[r4] GR[r2, 1, r3], GR[r4]^2 GR[r2, 1, r3],
 GR[r2, -1] GR[r2, 1, r3], GI[r2] GI[0, 0, 0, r2],
 GR[-1] GR[0, 0, r2, -1], GR[r4] GR[0, 0, r2, -1],
 GR[-1] GR[0, 0, r4, 1], GR[r4] GR[0, 0, r4, 1],
 GI[r2] GI[0, 1, 1, r4], GI[r2] GI[0, 1, r2, -1],
 GI[r2] GI[0, 1, r2, r3], GI[r2] GI[0, r2, 1, -1],
 GR[-1] GR[r2, 1, 1, -1], GR[r4] GR[r2, 1, 1, -1],
 GR[-1] GR[r2, 1, 1, r3], GR[r4] GR[r2, 1, 1, r3],
 GR[-1] GR[r2, 1, r2, -1], GR[r4] GR[r2, 1, r2, -1]}
```

Our basis for the imaginary parts of $G(a_1, \ldots, a_5; 1)$ consists of 11 constants of weight 5

```
{GI[0, 0, 0, 1, r2], GI[0, 0, 0, 1, r4], GI[0, 0, 0, r2, -1],
GI[0, 1, 1, -1, r2], GI[0, 1, 1, -1, r4], GI[0, 1, 1, 1, r4],
GI[0, 1, 1, r2, r3], GI[0, 1, 1, r4, -1], GI[0, 1, 1, r4, r1],
GI[0, 1, r2, r3, r2], GI[0, r2, 1, 1, -1]}
```

and 57 products of constants of lower weights

```
{GI[r2] GR[-1]^4, GI[r2]^3 GR[-1]^2, GI[r2]^5, GI[r2] GR[-1]^3 GR[r4],
 GI[r2]^3 GR[-1] GR[r4], GI[r2] GR[-1]^2 GR[r4]^2, GI[r2]^3 GR[r4]^2,
 GI[r2] GR[-1] GR[r4]^3, GI[r2] GR[r4]^4, GI[0, r2] GR[-1]^3,
GI[r2]^2 GI[0, r2] GR[-1], GI[0, r2] GR[-1]^2 GR[r4],
GI[r2]^2 GI[0, r2] GR[r4], GI[0, r2] GR[-1] GR[r4]^2,
GI[0, r2] GR[r4]^3, GI[r2] GI[0, r2]^2, GI[r2] GR[-1]^2 GR[r2, -1],
GI[r2]^3 GR[r2, -1], GI[r2] GR[-1] GR[r4] GR[r2, -1],
GI[r2] GR[r4]^2 GR[r2, -1], GI[0, r2] GR[-1] GR[r2, -1],
GI[0, r2] GR[r4] GR[r2, -1], GI[r2] GR[r2, -1]^2,
GI[r2] GR[-1] GR[0, 0, 1], GI[r2] GR[r4] GR[0, 0, 1],
GI[0, r2] GR[0, 0, 1], GI[0, 1, r4] GR[-1]^2, GI[r2]^2 GI[0, 1, r4],
GI[0, 1, r4] GR[-1] GR[r4], GI[0, 1, r4] GR[r4]^2,
GI[0, 1, r4] GR[r2, -1], GI[0, r2, -1] GR[-1]^2,
GI[r2]^2 GI[0, r2, -1], GI[0, r2, -1] GR[-1] GR[r4],
GI[0, r2, -1] GR[r4]^2, GI[0, r2, -1] GR[r2, -1],
GI[r2] GR[-1] GR[r2, 1, -1], GI[r2] GR[r4] GR[r2, 1, -1],
GI[0, r2] GR[r2, 1, -1], GI[r2] GR[-1] GR[r2, 1, r3],
GI[r2] GR[r4] GR[r2, 1, r3], GI[0, r2] GR[r2, 1, r3],
GI[0, 0, 0, r2] GR[-1], GI[0, 0, 0, r2] GR[r4],
GI[r2] GR[0, 0, r2, -1], GI[r2] GR[0, 0, r4, 1],
GI[0, 1, 1, r4] GR[-1], GI[0, 1, 1, r4] GR[r4],
GI[0, 1, r2, -1] GR[-1], GI[0, 1, r2, -1] GR[r4],
GI[0, 1, r2, r3] GR[-1], GI[0, 1, r2, r3] GR[r4],
GI[0, r2, 1, -1] GR[-1], GI[0, r2, 1, -1] GR[r4],
GI[r2] GR[r2, 1, 1, -1], GI[r2] GR[r2, 1, 1, r3],
GI[r2] GR[r2, 1, r2, -1]}
```

Our basis for the real parts of $G(a_1, \ldots, a_6; 1)$ consists of 25 constants of weight 6 and 170 products of constants of lower weights

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Our basis for the imaginary parts of $G(a_1, \ldots, a_6; 1)$ consists of less than 74 constants of weight 6 and 157 products of constants of lower weights

- The dimensions of the spaces generated by real and imaginary parts of MPL of a given weight are, respectively, $(F_{2w+2} + F_{w+1})/2$, and $(F_{2w+2} - F_{w+1})/2$, where $F(n)$ is a Fibonacci number [D. Broadhurst'14].

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- Broadhurst explicitly described, at any w, parts of conjectured bases consisting of $G(a_1, \ldots, a_6; 1)$ using only letters $0, -1, r_4$.
- Zhao: if n is *non-standard*, i.e. has at least two prime factors, then the number of MPL at nth roots of unity coincides with the upper bounds of the motivic theory. These upper bounds for the motivic fundamental group were computed by Deligne and Goncharov. The numbers of the corresponding generators coincide at $n = 6$ with our numbers.

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to be continued

BACKUP SLIDES

Lee [R.N. Lee'14]: transition to UT basis in three steps (in the case of one variable).

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- Reduction to a Fuchsian form where the singularities at all the points $x^{(k)}$ (including $x = \infty$) are simple poles.

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- Reduction to a Fuchsian form where the singularities at all the points $x^{(k)}$ (including $x = \infty$) are simple poles.
- Normalizing eigenvalues of the matrices which are coefficients the Fuchsian singularities when one tries to make them proportional to ϵ.
- Providing a linear dependence on ϵ.

At each of the three steps, one is looking for a proper linear transformation of the current basis. The first two steps are based on the so-called balance transformation

$$\mathcal{B}(\mathbb{P}, x_1, x_2|x) = \overline{\mathbb{P}} + c\frac{x - x_2}{x - x_1}\mathbb{P},$$

where c is a constant, $\mathbb{P}$, $\overline{\mathbb{P}}$ are the two complementary projectors, i.e. $\mathbb{P}^2 = \mathbb{P}$ and $\overline{\mathbb{P}} = \mathbb{I} - \mathbb{P}$.

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The idea of using a balance transformation is that with its help one can take care of one singular point x_1 (by providing a Fuchsian singularity or by normalizing eigenvalues corresponding to a given singular point) and not to spoil these properties at a second singular point, x_2.