Evaluating Feynman integrals by uniformly transcendental differential equations

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Based on collaboration with Johannes Henn, Bernhard Mistlberger and Alexander Smirnov
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- Introduction. The method of differential equations
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- Introduction. The method of differential equations
- Evaluating non-planar on-shell three-loop four-point massless integrals
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- MPL(1)
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- MPL(1)
- Conclusion
Evaluating Feynman integrals by uniformly transcendental differential equations

Introduction. The method of differential equations

[A.V. Kotikov’91, E. Remiddi’97, T. Gehrmann & E. Remiddi’00, J. Henn’13]
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Gehrmann & Remiddi: a method to evaluate *master integrals*. It is assumed that the problem of reduction to master integrals is solved.
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Henn: use uniform transcendental (UT) bases!
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Solve DE
Let $f = (f_1, \ldots, f_N)$ be primary master integrals (MI) for a given family of dimensionally regularized (with $D = 4 - 2\epsilon$) Feynman integrals.
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Let \( x = (x_1, \ldots, x_n) \) be kinematical variables and/or masses, or some new variables introduced to ‘get rid of square roots’.
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DE:

\[
\partial_i f(\epsilon, x) = A_i(\epsilon, x)f(\epsilon, x),
\]

where \( \partial_i = \frac{\partial}{\partial x_i} \), and each \( A_i \) is an \( N \times N \) matrix.
Henn (2013): turn to a new basis where DE take the form

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In the differential form,

$$d f(\epsilon, x) = \epsilon (d \tilde{A}(x)) f(x, \epsilon),$$

where

$$\tilde{A} = \sum_k \tilde{A}_{\alpha_k} \log(\alpha_k).$$
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and \( \tilde{A}_{\alpha_k} \) are constant matrices. The arguments of the logarithms \( \alpha_i \) (letters) are functions of \( x \). Elements of such basis turn out to be uniformly transcendental (UT).
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Let us call it \textit{epsilon form}.
The case of two scales, i.e. with one variable in the DE, i.e. $n = 1$. 
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One tries to achieve the following form of DE:

\[
 f'(\epsilon, x) = \epsilon \sum_{k} \frac{a_k}{x - x^{(k)}} f(\epsilon, x). 
\]

where \( x^{(k)} \) is the set of singular points of the DE and \( N \times N \) matrices \( a_k \) are independent of \( x \) and \( \epsilon \).
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For example, if \( x_k = 0, -1, 1 \) then results for elements of such a basis are expressed in terms of HPL.
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- In simple situations where integrals can be expressed in terms of gamma functions, just adjust indices properly.
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- Replace propagators by delta functions and analyze whether the resulting expression is UT.
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- Use Feynman parametrization
- Replace propagators by delta functions and analyze whether the resulting expression is UT.
- An approach using Magnus and Dyson series expansion
A part of the procedure is algorithmically described in [T. Gehrmann, A. von Manteuffel, L. Tancredi and E. Weihs’14]
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Constructing UT elements of the basis at the level of integrand [Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz and J. Trnka’14]
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- Constructing UT elements of the basis at the level of integrand [Z. Bern, E. Herrmann, S. Litsey, J. Stankowicz and J. Trnka’14]
- An algorithmical description in the case of one variable [R.N. Lee’14]
Evaluating Feynman integrals by uniformly transcendental differential equations

Evaluating non-planar on-shell three-loop four-point massless integrals
The kinematics: \( p_i^2 = 0, \ s = (p_1 + p_2)^2, \ t = (p_1 + p_3)^2, \ u = (p_2 + p_3)^2 = -s - t. \)
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A and E [J. Henn, A. & V. Smirnov’13]

B, C, D, F, G, H, I [J. Henn, B. Mistlberger and V. Smirnov’15]
in progress

\[
F_{a_1, \ldots, a_{15}}^D(s, t; D) = \frac{1}{(i\pi^{D/2})^3} \int \int \int \frac{d^D k_1 \ d^D k_2 \ d^D k_3}{(-k_1^2)^{a_1} \ [-(p_2 - k_1 + k_2)^2]^{a_2} \ [-(k_2^2)^{a_3}} \times \frac{[-(k_1 - k_3)^2]}{[-(p_1 + k_3)^2]^{a_4} \ [-(p_1 + k_2)^2]^{a_5} \ [-(p_1 + p_2 + p_3 + k_2 - k_3)^2]^{a_6}} \times \frac{[-(p_3 + k_1)^2]}{[-(p_3 + k_1)^2]^{a_7} \ [-(k_1 - k_2)^2]^{a_8} \ [-(k_2 - k_3)^2]^{a_9} \ [-(k_3 - p_3)^2]^{a_{10}}}.
\]
Partial results:
master integrals for D apart from the top sector [R.N. Lee’14]
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Results expressed in terms of HPL
$H_{a_1,a_2,...,a_n}(x)$, $a_i = 1, 0, -1$,
[E. Remiddi and J.A.M. Vermaseren’00]
IBP reduction by FIRE and by a private code by Bernhard Mistlberger.
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In all the cases, initial DE are transformed into

$$\partial_x f(x, \epsilon) = \epsilon \left[ \frac{a}{x} + \frac{b}{1 + x} \right] f(x, \epsilon).$$

where $a$ and $b$ are constant matrices.
Boundary conditions.
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Three singular points, at $x = 0$, $x = -1$, and $x = \infty$, corresponding to the limits $s \to 0$, $u \to 0$, and $t \to 0$, respectively.
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For planar diagrams A and E, the condition of the absence of singularities at $u = 0$ served as a very powerful boundary condition. As a result, only simple information about integrals expressed in terms of gamma functions fixed completely the solution of the DE.
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There is no this condition in the non-planar cases because non-planar diagrams have singularities in all the three channels.
Studying limits, $s \to 0$, $t \to 0$, $u \to 0$. 
Studying limits, $s \to 0, t \to 0, u \to 0$.

Typical contributions to the asymptotic expansion in the limit $x = t/x \to 0$:
- hard-hard-hard contribution,
- collinear-collinear-collinear contribution,
- ultrasoft-collinear-collinear contribution.
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- hard-hard-hard contribution,
- collinear-collinear-collinear contribution,
- ultrasoft-collinear-collinear contribution.

The code {	exttt{asy.m}}
[A. Pak and A. Smirnov’10, B. Jantzen, A.S. and V.S.’12]
(which is now included into FIESTA [A.S.’09-15])
→ expression of contributions of regions
Three last elements of the basis

\[-\epsilon^6 s(s + t)(2sF_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0} - sF_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,-1}) \]
\[-F_{1,1,0,0,0,1,1,1,1,1,1,0,0,0,0,0} + F_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,-1,0,-1}) \],

\[\epsilon^6 st(3F_{1,1,0,0,0,1,1,1,1,1,1,0,0,0,0,0} - 2F_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,0,-1}) \]
\[-F_{1,1,1,1,1,1,0,0,0,-1,0,-1}) \],

\[\epsilon^6 s\left(-\frac{3}{2}s^2 F_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0} + \frac{3}{2}s^2 F_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,-1} \right) \]
\[\left(-\frac{9}{4} sF_{1,1,0,1,1,1,1,1,1,1,0,0,0,0,0,-1} + \frac{5}{4} sF_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,0,-1} \right) \]
\[-2sF_{1,1,1,1,1,1,1,1,1,0,0,0,-1,0,-1} + \frac{3}{2} sF_{1,1,1,1,1,1,1,1,1,1,0,0,0,0,0,-2} \)
\[-5F_{1,1,1,1,1,1,0,0,0,0,0,0,-1} + 4F_{1,1,1,0,1,1,1,1,1,1,0,0,0,-1,0,-1} \]
\[+3F_{1,1,1,0,1,1,1,1,1,1,0,0,0,0,-2} - 2F_{1,1,1,1,1,1,1,1,1,1,0,0,0,-1,0,-2}) \]
Our analytical result for element 28 is

\[-(1/3) - (10 \, \text{ep}^2 \, \text{Pi}^2) / 2 + (10 \, \text{ep}^3 \, \text{Pi}^3) / 9 +
\]
\[23/24 \, \text{ep}^3 \, \text{Pi}^3 - (271 \, \text{ep}^4 \, \text{Pi}^4) / 4320 - (\]
\[10201 \, \text{ep}^5 \, \text{Pi}^5) / 2880 - (23819 \, \text{ep}^6 \, \text{Pi}^6) / 20160 +
\]
\[1/2 \, \text{ep} \, H[{-1}, \, x] - 7/24 \, \text{ep}^3 \, \text{Pi}^3 \, H[{-1}, \, x] -
\]
\[35/12 \, \text{ep}^4 \, \text{Pi}^3 \, H[{-1}, \, x] - 3809/960 \, \text{ep}^5 \, \text{Pi}^4 \, H[{-1}, \, x] -
\]
\[1157/72 \, \text{ep}^6 \, \text{Pi}^4 \, H[{-1}, \, x] + 1/2 \, \text{ep} \, H[{0}, \, x] +
\]
\[1/2 \, \text{ep}^2 \, \text{Pi} \, H[{0}, \, x] - 61/24 \, \text{ep}^3 \, \text{Pi}^2 \, H[{0}, \, x] +
\]
\[27/8 \, \text{ep}^4 \, \text{Pi}^3 \, H[{0}, \, x] - 56301/7!6 \, \text{ep}^5 \, \text{Pi}^4 \, H[{0}, \, x] + (\]
\[58537 \, \text{ep}^6 \, \text{Pi}^5 \, H[{0}, \, x]) / 2880 +
\]
\[9/2 \, \text{ep}^3 \, \text{Pi} \, H[{-1}, \, -1], \, x] -
\]
\[35/12 \, \text{ep}^4 \, \text{Pi}^2 \, H[{-1}, \, -1], \, x] -
\]
\[683/24 \, \text{ep}^5 \, \text{Pi}^3 \, H[{-1}, \, -1], \, x] +
\]
\[3361/240 \, \text{ep}^6 \, \text{Pi}^4 \, H[{-1}, \, -1], \, x] - 1/2 \, \text{ep}^2 \, H[{-1}, \, 0], \, x] -
\]
\[5/2 \, \text{ep} \, \text{Pi}^3 \, H[{-1}, \, 0], \, x] + 77/24 \, \text{ep}^4 \, \text{Pi}^2 \, H[{-1}, \, 0], \, x] +
\]
\[395/24 \, \text{ep}^5 \, \text{Pi}^3 \, H[{-1}, \, 0], \, x] + (\]
\[739 \, \text{ep}^6 \, \text{Pi}^4 \, H[{-1}, \, 0], \, x] / 2880 - 1/2 \, \text{ep}^2 \, H[{0}, \, -1], \, x] -
\]
\[97/24 \, \text{ep}^3 \, \text{Pi} \, H[{0}, \, -1], \, x] +
\]
\[77/4 \, \text{ep}^3 \, \text{Pi}^3 \, H[{0}, \, -1], \, x] + (1/2880)
\]
\[18691 \, \text{ep}^6 \, \text{Pi}^4 \, H[{0}, \, -1], \, x] - 5/2 \, \text{ep}^3 \, \text{Pi} \, H[{0}, \, 0], \, x] +
\]
\[79/12 \, \text{ep}^4 \, \text{Pi}^2 \, H[{0}, \, 0], \, x] -
\]
\[445/24 \, \text{ep}^5 \, \text{Pi}^3 \, H[{0}, \, 0], \, x] +
\]
\[73/240 \, \text{ep}^6 \, \text{Pi}^4 \, H[{0}, \, 0], \, x] - 9/2 \, \text{ep}^3 \, H[{-1}, \, -1], \, x] +...\]
Evaluating planar three-loop vertex integrals

[J. Henn, A. Smirnov and V. Smirnov’15]
[J. Henn, A. Smirnov, V. Smirnov & M. Steinhauser’16]
Evaluating planar three-loop vertex integrals
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Numerical evaluation of planar and non-planar three-loop
threshold integrals with FIESTA [P. Marquard, J.H. Piclum,
D. Seidel and M. Steinhauser’14]
(evaluating NRQCD/QCD matching coefficients)
Evaluating Feynman integrals by uniformly transcendental differential equations

\[ F_{a_1, \ldots, a_{12}} = \int \int \int \frac{d^D k_1 \, d^D k_2 \, d^D k_3}{[m^2 - (k_1 + p_1)^2]^{a_1}[m^2 - (k_2 + p_1)^2]^{a_2}} \]
\[ \times \frac{1}{[m^2 - (k_3 + p_1)^2]^{a_3}[m^2 - (k_3 + p_2)^2]^{a_4}[m^2 - (k_2 + p_2)^2]^{a_5}} \]
\[ \times \frac{1}{[m^2 - (k_1 + p_2)^2]^{a_6}[-k_1^2]^{a_7}[-(k_1 - k_2)^2]^{a_8}[-(k_2 - k_3)^2]^{a_9}} \]
\[ \times \frac{1}{[-(k_1 - k_3)^2]^{a_{10}}[-k_2^2]^{-a_{11}}[-k_3^2]^{-a_{12}}} \]

at \( p_1^2 = m^2 \), \( p_2^2 = m^2 \) at general \( s \equiv q^2 = (p_1 - p_2)^2 \)
or at threshold, \( s = 4m^2 \).
\[ F_{a_1, \ldots, a_{12}} = \int \int \int \frac{d^D k_1 \, d^D k_2 \, d^D k_3}{[m^2 - (k_1 + p_1)^2]^{a_1} [m^2 - (k_2 + p_1)^2]^{a_2}} \]
\times \frac{1}{[m^2 - (k_3 + p_1)^2]^{a_3} [m^2 - (k_3 + p_2)^2]^{a_4} [m^2 - (k_2 + p_2)^2]^{a_5}} \]
\times \frac{1}{[m^2 - (k_1 + p_2)^2]^{a_6} [-k_1^2]^{a_7} [-(k_1 - k_2)^2]^{a_8} [-(k_2 - k_3)^2]^{a_9}} \]
\times \frac{1}{[-(k_1 - k_3)^2]^{a_{10}} [-k_2^2]^{-a_{11}} [-k_3^2]^{-a_{12}}} \]

at \( p_1^2 = m^2, p_2^2 = m^2 \) at general \( s \equiv q^2 = (p_1 - p_2)^2 \)
or at threshold, \( s = 4m^2 \).
Each index can be positive but the total number of positive indices cannot be more than 9. This family of integrals can be represented as the union of eight subfamilies.
Evaluating Feynman integrals by uniformly transcendental differential equations

Evaluating planar three-loop vertex integrals
90 master integrals for general $q^2$ and 51 threshold master integrals

\[
F_{0,0,0,1,1,1,0,0,0,0,0,0}, F_{0,0,0,1,1,1,0,0,0,1,0,1}, F_{0,0,0,1,1,1,0,0,1,1,0,0}, F_{0,0,1,0,0,0,0,0,1,0,1},
\]

\[
F_{0,0,1,0,0,1,0,1,1,0,0,0}, F_{0,0,1,0,0,1,0,1,2,0,0,0}, F_{0,0,1,0,1,1,0,0,0,1,0,0}, F_{0,0,0,0,0,1,0,1,0,1,1,1},
\]

\[
F_{0,0,1,0,0,1,0,1,0,1,1,0}, F_{0,0,1,0,0,1,0,1,0,1,2,0}, F_{0,0,1,0,0,1,0,1,0,2,1,0}, F_{0,0,1,0,0,1,0,1,1,0,0,1},
\]

\[
F_{0,0,1,0,0,1,0,1,1,1,1,0}, F_{0,0,1,0,1,1,0,0,1,1,0,0,0}, F_{0,1,1,0,0,0,1,1,0,1,0,0}, F_{0,0,1,0,0,1,0,1,1,1,1,1},
\]

\[
F_{0,0,1,0,0,1,0,1,1,1,1,1}, F_{0,0,1,0,1,1,0,0,1,1,0,1,2,0}, F_{0,0,1,0,1,2,0,0,1,1,0,0,0}, F_{0,1,1,0,0,1,0,1,0,1,0,1},
\]

\[
F_{0,1,1,0,0,1,0,1,0,2,0,1,0,1,1,1,0,0,0,0}, F_{0,1,1,0,0,1,0,1,1,1,1,0,0,0,0}, F_{0,1,1,0,0,1,0,1,1,1,0,1,0,0,0},
\]

\[
F_{0,1,1,0,1,0,1,0,1,1,0,0,0,0}, F_{0,1,1,0,1,0,1,1,0,1,0,0,0,0}, F_{0,1,1,0,1,1,0,0,1,0,0,1,0,1},
\]

\[
F_{0,1,1,0,1,1,0,0,1,0,1,1,1,1,1,0,0,0,0}, F_{0,1,1,0,1,1,1,0,1,0,1,1,0,0,0,0}, F_{0,1,1,0,1,1,1,0,1,0,1,1,0,0,0},
\]

\[
F_{1,1,1,0,0,0,0,1,0,1,1,1,1}.
\]
Evaluating Feynman integrals by uniformly transcendental differential equations

Evaluating planar three-loop vertex integrals

(1) (2) (3) (4) (5)

(6a) (7) (8) (9) (10a)

(11b) (12) (13) (14a) (15b)
Evaluating Feynman integrals by uniformly transcendental differential equations

Evaluating planar three-loop vertex integrals

(16) (17) (18a) (19) (20) (21) (22) (23a) (24b) (25) (26)
Evaluating Feynman integrals by uniformly transcendental differential equations

Evaluating planar three-loop vertex integrals
Evaluating Feynman integrals by uniformly transcendental differential equations

Evaluating planar three-loop vertex integrals

(42)  (43a)  (44b)  (45c)  (46)

(47a)  (48)  (49)  (50)  (51)
It is convenient to introduce the variable

$$\frac{s}{m^2} = -\frac{(1 - x)^2}{x}$$
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\[ \frac{s}{m^2} = - \frac{(1 - x)^2}{x} \]

The values \( x = 1 \) and \( x = -1 \) correspond to \( s = 0 \) and \( s = 4m^2 \).
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DE

\[
f'(\epsilon, x) = \epsilon \tilde{A}'(x) f(x, \epsilon),
\]

where \( \tilde{A} = \sum_k \tilde{A}_{\alpha_k} \log(\alpha_k) \) and the letters \( \alpha_k \) are \( x, 1 + x, 1 - x, 1 + x + x^2 \).
90 elements of this basis $f(x)$ are

$$
\left\{ F_{0,0,0,3,3,0,0,0,0,0,0} , \quad \varepsilon \frac{x^2 - 1}{x} F_{0,0,2,1,3,3,0,0,0,0,0,0} , \quad \ldots \\
\varepsilon^6 \frac{(1 - x^2)^2}{x^2} F_{1,0,1,1,1,1,1,1,0,0,0} , \quad (1 - 2\varepsilon)\varepsilon^4 F_{1,2,1,0,0,0,1,1,1,0,0,1} \right\}
$$
90 elements of this basis $f(x)$ are

$$\{ F_{0,0,0,3,3,0,0,0,0,0,0} , \quad \varepsilon \frac{x^2 - 1}{x} F_{0,0,2,1,3,3,0,0,0,0,0,0}, \ldots \}$$

$$\varepsilon^6 \frac{(1 - x^2)^2}{x^2} F_{1,0,1,1,1,1,1,1,1,0,0,0} , \quad (1 - 2\varepsilon)\varepsilon^4 F_{1,2,1,0,0,0,1,1,1,0,0,1} \}$$

A solution in an epsilon-expansion with coefficients written in terms of multiple (Goncharov) polylogarithms (MPL)

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

with indices $a_i$ taken from the seven-letters alphabet

$$\{ 0, r_1, r_3, -1, r_4, r_2, 1 \}$$

with

$$r_{1,2} = \frac{1}{2} \left( 1 \pm \sqrt{3} i \right) , \quad r_{3,4} = \frac{1}{2} \left( -1 \pm \sqrt{3} i \right) .$$
A typical expression for analytical results for the elements of the basis

\[
\begin{align*}
\text{ep}^{-4} & \times (-24 \times G\{-1, 1\} \times G\{0, -1\}, x) \times G\{0, -1\}, 1) + \\
& 24 \times G\{0, -1\}, 1) \times G\{0, -1\}, x) - 23 \times G\{0, -1\}, 1) \times G\{0, 0\}, x) - \\
& 12 \times G\{-1, 1\} \times G\{0\}, x) \times G\{0, 1\}, 1) + 12 \times G\{0, -1\}, 1) \times G\{0, 1\}, 1) - \\
& (23 \times G\{0, 0\}, x) \times G\{0, 1\}, 1) )/2 + 12 \times G\{0, -1\}, 1) \times G\{0, 1\}, x) + \\
& 6 \times G\{0, 1\}, 1) \times G\{0, 1\}, x) + 12 \times G\{0, -1\}, 1) \times G\{0, 1\}, x) + \\
& 6 \times G\{0, 1\}, 1) \times G\{0, 1\}, 1) - 9 \times G\{0, -1\}, 1) \times G\{0, 1\}, x) - \\
& (9 \times G\{0, 1\}, 1) \times G\{r1, 0\}, x)) / 2 - 9 \times G\{0, -1\}, 1) \times G\{r2, 0\}, x) - \\
& (9 \times G\{0, 1\}, 1) \times G\{r2, 0\}, x)) / 2 + 24 \times G\{0\}, x) \times G\{-1, 0, -1\}, 1) + \\
& 12 \times G\{-1, 0, -1\}, 1) \times G\{0, 0, -1\}, 1) + 24 \times G\{0\}, x) \times G\{0, -1\}, 1) \times \\
& 48 \times G\{1\}, x) \times G\{0, 0, -1\}, 1) - 18 \times G\{r1, 0\}, x) \times G\{0, 0, -1\}, 1) - \\
& 18 \times G\{r2, 0\}, x) \times G\{0, 0, -1\}, 1) + 24 \times G\{-1, 0, 1\}, 1) - \\
& (57 \times G\{0\}, x) \times G\{0, 0, 1\}, 1) )/2 + 24 \times G\{1\}, x) \times G\{0, 0, 1\}, 1) - \\
& (21 \times G\{r1, 0\}, x) \times G\{0, 0, 1\}, 1) )/2 - (21 \times G\{r2, 0\}, x) \times G\{0, 0, 1\}, 1)) / 2 - \\
& 6 \times G\{0\}, x) \times G\{0, 1, 1\}, 1) - 24 \times G\{-1, 0, 0\}, x) + \\
& 36 \times G\{-1, 0, 0\}, 0\}, x) - 24 \times G\{-1, 1, 0\}, 0\}, x) + \\
& 24 \times G\{0, 0, -1\}, 0\}, x) + 2 \times G\{0, 0, -1\}, 0\}, x) + 12 \times G\{0, -1, 1, 0\}, x) - \\
& 23 \times G\{0, 0, 0\}, x) - (23 \times G\{0, 0, 1\}, 0\}, x) )/2 + \\
& 12 \times G\{0, 0, 1\}, 1\}, x) + (11 \times G\{0, 0, 1\}, 0\}, x) )/2 + \\
& 6 \times G\{0, 0, 1\}, x) - 24 \times G\{0, 0, 1\}, 1\}, x] + 12 \times G\{1, 0, -1\}, 0\}, x) + \\
& 15 \times G\{1\}, 0\}, 0\}, x) + 6 \times G\{1, 0, 1\}, 0\}, x) - 12 \times G\{1, 1\}, 0\}, x) - \\
& 9 \times G\{r1, 0\}, 0\}, 0\}, x] + 6 \times G\{r1, 0\}, 0\}, 0\}, x] - \\
& (9 \times G\{r1, 0\}, 1\}, 0\}, x)) / 2 + (3 \times G\{r1, 0\}, 1\}, 0\}, x)) / 2 - \\
& 9 \times G\{r2, 0\}, 0\}, 0\}, x] + 6 \times G\{r2, 0\}, 0\}, 0\}, x] - \\
& (9 \times G\{r2, 0\}, 1\}, 0\}, x)) / 2 + (3 \times G\{r2, 1\}, 0\}, 0\}, x)) / 2 + \\
& (3 \times G\{0\}, x) \times Zeta[3] / 2 - (3 \times G\{r1, 0\}, x) \times Zeta[3]) / 2 - \\
& (3 \times G\{r2, 0\}, x) \times Zeta[3]) / 2 - \\
& (3 \times (16 \times G\{0\}, -1\}, 1\}, 1) - 2 + 8 \times G\{0\}, -1\}, 1\}, 1\}, 1\}, 1\} + ...
\end{align*}
\]
To obtain analytical results for the 51 threshold master integrals use threshold expansion

\[ F(a_1, \ldots, a_{12}; q^2, m^2) \sim \sum_{n=n_0}^{\infty} \sum_{j=0}^{3} (4m^2 - q^2)^{n-j} F_{n,j}(a_1, \ldots, a_{12}; q^2). \]
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Threshold master integrals are one-scale integrals \(F_{0,0}(a_1, \ldots, a_{12}; m^2)\) defined with \(q^2\) set to \(4m^2\).
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Threshold master integrals are one-scale integrals \( F_{0,0}(a_1, \ldots, a_{12}; m^2) \) defined with \( q^2 \) set to \( 4m^2 \).

We cannot just set \( q^2 = 4m^2 \), i.e. \( x = -1 \) in our basis because of \( 1/(x + 1) \) and \( 1/(x + 1)^2 \) in some coefficients.
To obtain analytical results for the 51 threshold master integrals use threshold expansion

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Threshold master integrals are one-scale integrals \( F_{0,0}(a_1, \ldots, a_{12}; m^2) \) defined with \( q^2 \) set to \( 4m^2 \).

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Expand ‘naively’ in \( x + 1 \) the corresponding integrals. Introduce one more (13th) index for the order of this derivative in \( s \), i.e. deal with the family

\[ F'(a_1, \ldots, a_{12}, a_{13}) = \left( \frac{\partial}{\partial s} \right)^{-a_{13}} F(a_1, \ldots, a_{12}) \bigg|_{s=4m^2} \]
Using IBP relations for integrals at general $q$ and expanding all the terms naively in $q^2$ at $q^2 = 4m^2 \rightarrow 15$ IBP relations.
Using IBP relations for integrals at general $q$ and expanding all
the terms naively in $q^2$ at $q^2 = 4m^2$ $\rightarrow$ 15 IBP relations.

A naive differentiation in $s$ of all the terms of the naive
expansion [P.A. Baikov and V.A. Smirnov’2000] $\rightarrow$ one more
relation.
Using IBP relations for integrals at general $q$ and expanding all the terms naively in $q^2$ at $q^2 = 4m^2 \rightarrow 15$ IBP relations.

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Using IBP relations for integrals at general $q$ and expanding all the terms naively in $q^2$ at $q^2 = 4m^2$ → 15 IBP relations.

A naive differentiation in $s$ of all the terms of the naive expansion [P.A. Baikov and V.A. Smirnov’2000] → one more relation.

Then $F'(a_1, \ldots, a_{12}, a_{13})$ are reduced to master integrals (with FIRE).

They are all with $a_{13} = 0$, i.e. directly correspond to the 51 master threshold integrals.
Matching at threshold
Matching at threshold

\( x = y - 1, \ y \to 0: \)

\[
f'(\epsilon, y) = \epsilon \frac{\tilde{A}'(y)}{y} f(\epsilon, y),
\]

where \( \tilde{A}'(y) = A_0 + yA_1 + y^2A_2 + \ldots. \)
Matching at threshold

\[ x = y - 1, \ y \to 0: \]

\[ f'(\epsilon, y) = \epsilon \frac{\tilde{A}'(y)}{y} f(\epsilon, y), \]

where \( \tilde{A}'(y) = A_0 + yA_1 + y^2A_2 + \ldots \).

In the language of differential equations, the naive part of the expansion near \( y = 0 \) corresponds to zero eigenvalues of the matrix \( A_0 \) while eigenvalues proportional to \( \epsilon \) correspond to other contributions.
To obtain expansions near $y = 0$ of the elements of the basis in higher orders in $y$, we use a trick from the theory of differential equations (presented, e.g., in [Wasov’s book]).
To obtain expansions near \( y = 0 \) of the elements of the basis in higher orders in \( y \), we use a trick from the theory of differential equations (presented, e.g., in [Wasov’s book]).

Construct a polynomial \( P = 1 + \sum_{r=1} P_r y^r \) such that the DE for the function \( g \) defined by \( f = Pg \) takes the form \( yg'(y) = A_0 g(y) \) (with \( A_0 \) is independent of \( y \)).
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Then the solution of this equation is just \( g = y^{A_0} g_0 \) with a boundary value \( g_0 \).
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Then the solution of this equation is just \( g = y^{A_0} g_0 \) with a boundary value \( g_0 \).

We implemented this algorithm and constructed \( P_r \) up to \( r = 5 \).
Evaluating Feynman integrals by uniformly transcendental differential equations

Evaluating planar three-loop vertex integrals

Equating the part of our analytic results for the basis without $\log(x + 1)$ and the naive part of the threshold expansion expressed in terms of the 51 threshold MI.
Equating the part of our analytic results for the basis without \( \log(x + 1) \) and the naive part of the threshold expansion expressed in terms of the 51 threshold \( \text{Ml} \).

Solving these equations \( \rightarrow \) coefficients of the epsilon expansion of the \( \text{Ml} \) up to some order written in terms of MPL \( G(a_1, \ldots, a_n; 1) \) with \( a_1 \neq 1 \) and \( a_i \) taken from the alphabet \( \{0, r_1, r_3, -1, r_4, r_2, 1\} \).
Equating the part of our analytic results for the basis without \( \log(x + 1) \) and the naive part of the threshold expansion expressed in terms of the 51 threshold \( M_i \).

Solving these equations \( \rightarrow \) coefficients of the epsilon expansion of the \( M_i \) up to some order written in terms of MPL \( G(a_1, \ldots, a_n; 1) \) with \( a_1 \neq 1 \) and \( a_i \) taken from the alphabet \( \{0, r_1, r_3, -1, r_4, r_2, 1\} \).

Examples of our results

[J. Henn, A. Smirnov and V. Smirnov’15]
\[
F_{0,0,1,0,1,1,0,1,0,1,1,1} = -\frac{27}{2} \log(2) G_R(0, 0, r_2, -1) - \frac{181\zeta(5)}{32} - \frac{21}{2} \log^2(2) \zeta(3)
\]
\[
+ \frac{115\pi^2 \zeta(3)}{48} - 12 \text{Li}_5 \left( \frac{1}{2} \right) - 12 \log(2) \text{Li}_4 \left( \frac{1}{2} \right) - \frac{2 \log^5(2)}{5} + \frac{1}{6} \pi^2 \log^3(2)
\]
\[
- \frac{81}{8} G_R(0, 0, r_4, 1) \log(2) + \frac{277}{960} \pi^4 \log(2),
\]
\[
F_{0,0,1,1,1,1,0,1,0,1,1,0} = -\frac{27}{4} \log(2) G_R(0, 0, r_2, -1) - \frac{341\zeta(5)}{64} - \frac{21}{4} \log^2(2) \zeta(3)
\]
\[
+ \frac{211\pi^2 \zeta(3)}{96} - 6 \text{Li}_5 \left( \frac{1}{2} \right) - 6 \log(2) \text{Li}_4 \left( \frac{1}{2} \right) - \frac{\log^5(2)}{5}
\]
\[
+ \frac{1}{12} \pi^2 \log^3(2) - \frac{81}{16} G_R(0, 0, r_4, 1) \log(2) + \frac{277\pi^4 \log(2)}{1920}.
\]
\[ F_{0,0,1,0,1,1,0,1,0,1,1,1} = -\frac{27}{2} \log(2) G_R(0, 0, r_2, -1) - \frac{181 \zeta(5)}{32} - \frac{21}{2} \log^2(2) \zeta(3) \]
\[ + \frac{115 \pi^2 \zeta(3)}{48} - 12 \text{Li}_5 \left( \frac{1}{2} \right) - 12 \log(2) \text{Li}_4 \left( \frac{1}{2} \right) - \frac{2 \log^5(2)}{5} + \frac{1}{6} \pi^2 \log^3(2) \]
\[ - \frac{81}{8} G_R(0, 0, r_4, 1) \log(2) + \frac{277}{960} \pi^4 \log(2), \]
\[ F_{0,0,1,1,1,1,0,1,0,1,1,0} = -\frac{27}{4} \log(2) G_R(0, 0, r_2, -1) - \frac{341 \zeta(5)}{64} - \frac{21}{4} \log^2(2) \zeta(3) \]
\[ + \frac{211 \pi^2 \zeta(3)}{96} - 6 \text{Li}_5 \left( \frac{1}{2} \right) - 6 \log(2) \text{Li}_4 \left( \frac{1}{2} \right) - \frac{\log^5(2)}{5} \]
\[ + \frac{1}{12} \pi^2 \log^3(2) - \frac{81}{16} G_R(0, 0, r_4, 1) \log(2) + \frac{277 \pi^4 \log(2)}{1920}. \]

where

\[ G(a_1, \ldots, a_n; 1) = G_R(a_1, \ldots, a_n) + i \ G_I(a_1, \ldots, a_n) \]
Constructing bases for $G_R(a_1, \ldots, a_n)$ and $G_I(a_1, \ldots, a_n)$. 
Constructing bases for $G_R(a_1, \ldots, a_n)$ and $G_I(a_1, \ldots, a_n)$.

A linear basis in this set of constants up to weight 3 [D. Broadhurst’98] in terms of $Cl_2(\pi/3), \log(2), \log(3), \pi, \zeta(3)$ and $\text{Li}_n$ of some arguments.
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Bases for the alphabet with letters $0, 1, -1$ [D. Broadhurst’96]
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with letters $0, -1, r_2$ [D. Broadhurst’98]

Constants present in results for Feynman integrals up to
weight 5 were discussed in
[Fleischer and M. Kalmykov’99, Davydychev M. Kalmykov’00,
M. Kalmykov and B. Kniehl’10].
For example,

\[ G_I(r_2) = -\frac{\pi}{3} , \quad G_R(-1) = \log(2) , \]

\[ G_R(0, 0, 1) = -\zeta(3) , \quad G_R(0, 0, 0, 1) = -\frac{\pi^4}{90} , \]

\[ G_R(0, 0, 0, 0, 1) = -\zeta(5) , \]

\[ G_R(0, 0, 1, 1, -1) = -2\Li_5(\frac{1}{2}) - 2\Li_4(\frac{1}{2}) \log(2) - \frac{\pi^2\zeta(3)}{96} \]

\[ + \frac{151\zeta(5)}{64} - \frac{\log^5(2)}{15} + \frac{1}{18} \pi^2 \log^3(2) - \frac{1}{96} \pi^4 \log(2) . \]
Evaluating Feynman integrals by uniformly transcendental differential equations

\[ \text{MPL} \]

\[ G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \]
MPL

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In the special case where \(a_i = 0\) for all \(i\) one has by definition

\[ G(0, \ldots, 0; z) = \frac{1}{n!} \ln^n z . \]
MPL

\[ G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t) \]

In the special case where \( a_i = 0 \) for all \( i \) one has by definition

\[ G(0, \ldots, 0; z) = \frac{1}{n!} \ln^n z. \]

If \( a_w \neq 0 \), then

\[ G(\rho a_1, \ldots, \rho a_w; \rho z) = G(a_1, \ldots, a_w; z) \]

so that one can express them in terms of \( G(\ldots; 1) \).
The MPL can be represented as multiple nested sums

\[
\text{Li}_{m_1, \ldots, m_k}(x_1, \ldots, x_k) = \sum_{n_k=1}^{\infty} \sum_{n_{k-1}=1}^{n_k-1} \cdots \sum_{n_1=1}^{n_2-1} \frac{x_1^{n_1}}{n_1^{m_1}} \cdots \frac{x_k^{n_k}}{n_k^{m_k}}
\]

\[= Z_{m_k, \ldots, m_1}(\infty; x_k, \ldots, x_1) \]

\[= (-1)^k G \left( \underbrace{0, \ldots, 0}_{m_k-1}, \frac{1}{x_k}, \ldots, \underbrace{0, \ldots, 0}_{m_1-1}, \frac{1}{x_1 \ldots x_k}; 1 \right) \]
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\text{Li}_{m_1, \ldots, m_k}(x_1, \ldots, x_k) = \sum_{n_k=1}^{\infty} \sum_{n_{k-1}=1}^{n_k-1} \cdots \sum_{n_1=1}^{n_2-1} \frac{x_1^{n_1}}{n_1^{m_1}} \cdots \frac{x_k^{n_k}}{n_k^{m_k}}
\]

\[
= Z_{m_k, \ldots, m_1}(\infty; x_k, \ldots, x_1)
\]

\[
= (-1)^k G \left( \underbrace{0, \ldots, 0}_{m_k-1}, \frac{1}{x_k}, \ldots, \underbrace{0, \ldots, 0}_{m_1-1}, \frac{1}{x_1 \ldots x_k}; 1 \right).
\]

Since the arguments of the Li- and Z-functions involved have the form \( x_i = \lambda^{p_i} \) for \( p_i = 0, \ldots, 5 \) we introduce an auxiliary function

\[
L_{m_1, \ldots, m_k}(p_1, \ldots, p_k) = \text{Li}_{m_1, \ldots, m_k}(\lambda^{p_1}, \ldots, \lambda^{p_k})
\]
\[ G(a_1, \ldots, a_n; 1) \text{ satisfy various relations.} \]
$G(a_1, \ldots, a_n; 1)$ satisfy various relations.

We start with using shuffle, stuffle, regularization and distribution relations following Zhao [J. Zhao’07].
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Shuffle relations

$$G(a_1, \ldots, a_{w_1}; z) \ G(b_1, \ldots, b_{w_2}; z) = \sum_{c \in a \cup b} G(c_1, \ldots, c_{w_1+w_2}; z),$$
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Stuffle relations

$$L_{a_1, \ldots, a_{w_1}}(p_1, \ldots, p_{w_1}) \ L_{b_1, \ldots, b_{w_2}}(q_1, \ldots, q_{w_2})$$

$$= \sum_{c \in a \ast b} L_{c_1, \ldots, c_n}(r_1, \ldots, r_n)$$
$G(a_1, \ldots, a_n; 1)$ satisfy various relations. We start with using shuffle, stuffle, regularization and distribution relations following Zhao [J. Zhao’07].

Shuffle relations

$$G(a_1, \ldots, a_{w_1}; z) \ G(b_1, \ldots, b_{w_2}; z) = \sum_{c \in a \oplus b} G(c_1, \ldots, c_{w_1+w_2}; z) ,$$

Stuffle relations

$$L_{a_1,\ldots,a_{w_1}} (p_1, \ldots, p_{w_1}) \ L_{b_1,\ldots,b_{w_2}} (q_1, \ldots, q_{w_2}) = \sum_{c \in a * b} L_{c_1,\ldots,c_n} (r_1, \ldots, r_n)$$

They are written for $a_1 \geq 2, b_1 \geq 2$ for a given weight $w = \sum a_i + \sum b_i$ and then are translated into the language of $G(a_1, \ldots, a_w; 1)$.
Regularization relations

\[ \{ L_1(0)L_{a_1,\ldots,a_k}(q_1,\ldots,q_k) \}_{\text{stuffle}} = \{ L_1(0)L_{a_1,\ldots,a_k}(q_1,\ldots,q_k) \}_{\text{shuffle}} \]
Regularization relations

\[ \{ L_1(0) L_{a_1, \ldots, a_k}(q_1, \ldots, q_k) \}_{\text{shuffle}} = \{ L_1(0) L_{a_1, \ldots, a_k}(q_1, \ldots, q_k) \}_{\text{shuffle}} \]

The numbers \( L_1(0) \) correspond to the variable \( T \) introduced by Zagier in the case of MZV
[K. Ihara, M. Kaneko & D. Zagier’06] and used by Zhao [J. Zhao’07] in the case of MPL at \( n \)-th roots of unity.
Regularization relations

\[ \{ L_1(0)L_{a_1, \ldots, a_k}(q_1, \ldots, q_k) \}_{\text{shuffle}} = \{ L_1(0)L_{a_1, \ldots, a_k}(q_1, \ldots, q_k) \}_{\text{shuffle}} \]

The numbers \( L_1(0) \) correspond to the variable \( T \) introduced by Zagier in the case of MZV 
[K. Ihara, M. Kaneko & D. Zagier’06] 
and used by Zhao [J. Zhao’07] in the case of MPL at \( n \)-th roots of unity.

(Regularized double shuffle relations.)
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(Regularized double shuffle relations.)

Singular terms are cancelled in the difference.
Distribution relations

\[ \text{Li}_{a_1, \ldots, a_k}(x_1, \ldots, x_k) = d^{a_1 + \ldots + a_k - k} \]
\[ \times \sum_{(y_1, \ldots, y_k): y_j^d = x_j, j = 1, \ldots, k} \text{Li}_{a_1, \ldots, a_k}(y_1, \ldots, y_k) \]
Distribution relations

\[
\text{Li}_{a_1, \ldots, a_k}(x_1, \ldots, x_k) = d^{a_1 + \ldots + a_k - k} \times \sum_{(y_1, \ldots, y_k): y_j^d = x_j, j=1,\ldots,k} \text{Li}_{a_1, \ldots, a_k}(y_1, \ldots, y_k)
\]

for \( d = 2 \) and \( d = 3 \).
In our case, \( n = 6 \), we use these four types of relations, then turn to the real and imaginary parts and use also the complex conjugation relations

\[
G(a_1^*, \ldots, a_n^*; 1) = G(a_1, \ldots, a_n; 1)^*
\]

with \( r_1^* = r_2, r_3^* = r_4 \).
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We solved these relations up to weight 6 recursively with the respect to the weight.
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We solved these relations up to weight 6 recursively with the respect to the weight.

The total number of relations grows fast when the weight is increased. At weight 6, we have \( 654452 \) equations for the real parts and \( 654937 \) equations for the imaginary parts of \( G(a_1, \ldots, a_n; 1) \).
The relations for the $6 \times 7^{w-1}$ numbers $G_R(a_1, \ldots, a_w; 1)$ or $G_I(a_1, \ldots, a_w; 1)$ are linear equations. We solved them for $w = 1, 2, \ldots, 6$ with a code written in Mathematica.
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It turns out, however, that the resulting constants, independent in the sense of these relations, are still $Q$-linearly dependent, i.e. one can linearly express some of them in terms of a smaller set of the constants and products of constants of lower weights.
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We revealed additional relations using PSLQ algorithm [H.R.P. Ferguson, D.H. Bailey, and S. Arno] and ginac [C. Bauer, A. Frink and R. Kreckel] to evaluate MPLs with a big accuracy (up to 4000 digits).
### The dimensions of the bases

<table>
<thead>
<tr>
<th>w</th>
<th>$\tilde{D}_R(w)$</th>
<th>$D_R(w)$</th>
<th>$\tilde{D}_I(w)$</th>
<th>$D_I(w)$</th>
<th>$PSLQ_R$</th>
<th>$PSLQ_I$</th>
</tr>
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<tbody>
<tr>
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<td>2</td>
<td>1</td>
<td>1</td>
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<td>13</td>
<td>76</td>
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<td>68</td>
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<tr>
<td>6</td>
<td>25</td>
<td>195</td>
<td>25</td>
<td>182</td>
<td>39</td>
<td>49</td>
</tr>
</tbody>
</table>
Our basis for the real parts of $G(a_1, \ldots, a_4; 1)$ consists of 5 constants of weight 4

$$\{\text{GR}[0, 0, r2, -1], \text{GR}[0, 0, r4, 1], \text{GR}[r2, 1, 1, -1],$$
$$\text{GR}[r2, 1, 1, r3], \text{GR}[r2, 1, r2, -1]\}$$

and 25 products of constants of lower weights

$$\{\text{GR}[-1]^4, \text{GI}[r2]^2 \text{GR}[-1]^2, \text{GI}[r2]^4, \text{GR}[-1]^3 \text{GR}[r4],$$
$$\text{GI}[r2]^2 \text{GR}[-1] \text{GR}[r4], \text{GR}[-1]^2 \text{GR}[r4]^2, \text{GI}[r2]^2 \text{GR}[r4]^2,$$
$$\text{GR}[-1] \text{GR}[r4]^3, \text{GR}[r4]^4, \text{GI}[r2] \text{GI}[0, r2] \text{GR}[-1],$$
$$\text{GI}[r2] \text{GI}[0, r2] \text{GR}[r4], \text{GI}[0, r2]^2, \text{GR}[-1]^2 \text{GR}[r2, -1],$$
$$\text{GI}[r2]^2 \text{GR}[r2, -1], \text{GR}[-1] \text{GR}[r4] \text{GR}[r2, -1], \text{GR}[r4]^2 \text{GR}[r2, -1],$$
$$\text{GR}[r2, -1]^2, \text{GR}[-1] \text{GR}[0, 0, 1], \text{GR}[r4] \text{GR}[0, 0, 1],$$
$$\text{GI}[r2] \text{GI}[0, 1, r4], \text{GI}[r2] \text{GI}[0, r2, -1], \text{GR}[-1] \text{GR}[r2, 1, -1],$$
$$\text{GR}[r4] \text{GR}[r2, 1, -1], \text{GR}[-1] \text{GR}[r2, 1, r3], \text{GR}[r4] \text{GR}[r2, 1, r3]\}$$
Our basis for the imaginary parts of $G(a_1, \ldots, a_4; 1)$ consists of 5 constants of weight 4

\{GI[0, 0, 0, r2], GI[0, 1, 1, r4], GI[0, 1, r2, -1], GI[0, 1, r2, r3],
  GI[0, r2, 1, -1]\}

and 20 products of constants of lower weights

\{GI[r2] GR[-1]^3, GI[r2]^3 GR[-1], GI[r2] GR[-1]^2 GR[r4],
  GI[r2]^3 GR[r4], GI[r2] GR[-1] GR[r4]^2, GI[r2] GR[r4]^3,
  GI[0, r2] GR[-1]^2, GI[r2]^2 GI[0, r2], GI[0, r2] GR[-1] GR[r4],
  GI[0, r2] GR[r4]^2, GI[r2] GR[-1] GR[r2, -1],
  GI[r2] GR[r4] GR[r2, -1], GI[0, r2] GR[r2, -1], GI[r2] GR[0, 0, 1],
  GI[0, 1, r4] GR[-1], GI[0, 1, r4] GR[r4], GI[0, r2, -1] GR[-1],
  GI[0, r2, -1] GR[r4], GI[r2] GR[r2, 1, -1], GI[r2] GR[r2, 1, r3]\}
Our basis for the real parts of $G(a_1, \ldots, a_5; 1)$ consists of 13 constants of weight 5

\{GR[0, 0, 0, 1], GR[0, 0, 1, 1, -1], GR[0, 0, 1, 1, r4],
GR[0, 1, r2, r3], GR[0, 1, r2, r4],
GR[r2, 1, 1, -1], GR[r2, 1, 1, -1, r2], GR[r2, 1, 1, -1, -1],
GR[r2, 1, 1, r3], GR[r2, 1, 1, r2, r3],
GR[r2, 1, 1, r4, -1]\}

and 63 products of constants of lower weights

\{GR[-1]^5, GI[r2]^2 GR[-1]^3, GI[r2]^4 GR[-1], GR[-1]^4 GR[r4],
GR[r2]^2 GR[-1]^2 GR[r4], GI[r2]^4 GR[r4], GR[-1]^3 GR[r4]^2,
GR[-1] GR[r4]^4, GR[r4]^5, GI[r2] GI[0, r2] GR[-1]^2,
GI[r2]^3 GI[0, r2], GI[r2] GI[0, r2] GR[-1] GR[r4],
GI[r2] GI[0, r2] GR[r4]^2, GI[0, r2]^2 GR[-1], GI[0, r2]^2 GR[r4],
GR[-1]^3 GR[r2, -1], GI[r2]^2 GR[-1] GR[r2, -1],
GR[-1] GR[r4]^2 GR[r2, -1], GR[r4]^3 GR[r2, -1],
GI[r2] GI[0, r2] GR[r2, -1], GR[-1] GR[r2, -1]^2,
GR[r4] GI[r2, -1]^2, GR[-1]^2 GR[0, 0, 1], GI[r2]^2 GR[0, 0, 1],
GR[-1] GR[r4] GI[0, 0, 1], GR[r4]^2 GR[0, 0, 1],
GR[r2, -1] GI[0, 0, 1, r4] GR[-1],
GI[r2] GI[0, 1, r4] GR[r4], GI[0, r2] GI[0, 1, r4],
GI[r2] GI[0, r2, -1] GR[-1], GI[r2] GI[0, r2, -1] GR[r4],
GI[0, r2] GI[0, r2, -1], GR[-1]^2 GR[r2, 1, -1],
GI[r2]^2 GR[r2, 1, -1], GR[-1] GR[r4] GR[r2, 1, -1],
GR[r4]^2 GR[r2, 1, -1], GR[r2, -1] GR[r2, 1, -1],
GR[-1]^2 GR[r2, 1, r3], GI[r2]^2 GR[r2, 1, r3],
GR[-1] GR[r4] GR[r2, 1, r3], GR[r4]^2 GR[r2, 1, r3],
GR[r2, -1] GR[r2, 1, r3], GI[r2] GI[0, 0, 0, r2],
GR[-1] GR[0, 0, r2, -1], GR[r4] GI[0, 0, r2, -1],
GR[-1] GR[0, 0, r4, 1], GR[r4] GR[0, 0, r4, 1],
GI[r2] GI[0, 1, r4], GI[r2] GI[0, 1, r2, -1],
GI[r2] GI[0, 1, r2, r3], GI[r2] GI[0, r2, 1, -1],
GR[-1] GR[r2, 1, 1, -1], GR[r4] GR[r2, 1, 1, -1],
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GR[-1] GR[r2, 1, r2, -1], GR[r4] GR[r2, 1, r2, -1],
GR[-1] GR[r2, 1, r2, r3], GR[r4] GR[r2, 1, r2, r3]\}
Our basis for the imaginary parts of $G(a_1, \ldots, a_5; 1)$ consists of 11 constants of weight 5

\{
GI[0, 0, 0, 1, r2], GI[0, 0, 0, 1, r4], GI[0, 0, 0, r2, -1],
GI[0, 1, 1, -1, r2], GI[0, 1, 1, -1, r4], GI[0, 1, 1, r4],
GI[0, 1, r2, r3], GI[0, 1, r4, -1], GI[0, 1, r4, r1],
GI[0, r2, r3, r2], GI[0, r2, 1, 1, -1]\}

and 57 products of constants of lower weights

\{
GI[r2] GR[-1]^4, GI[r2]^3 GR[-1]^2, GI[r2]^5, GI[r2] GR[-1]^3 GR[r4],
GI[r2]^3 GR[-1] GR[r4], GI[r2] GR[-1]^2 GR[r4]^2, GI[r2]^3 GR[r4]^2, 
, GI[r2]^2 GR[r4]^3, GI[r2] GI[0, r2] GR[-1]^2 GR[r4],
GI[0, r2] GR[r4]^3, GI[r2] GI[0, r2] GR[-1]^2 GR[r2, -1],
GI[r2]^3 GR[r2, -1], GI[r2] GR[-1] GR[r4] GR[r2, -1],
GI[r2] GR[r4]^2 GR[r2, -1], GI[0, r2] GR[-1] GR[r2, -1],
GI[0, r2] GR[r4] GR[r2, -1], GI[r2] GR[r2, -1]^2, 
GI[r2] GR[-1] GR[0, 0, 1], GI[r2] GR[r4] GR[0, 0, 1],
GI[0, r2] GR[0, 0, 1], GI[0, 1, r4] GR[-1]^2, GI[r2]^2 GI[0, 1, r4],
GI[0, 1, r4] GR[-1] GR[r4], GI[0, 1, r4] GR[r4]^2, 
GI[0, 1, r4] GR[r2, -1], GI[0, r2, -1] GR[-1]^2, 
GI[r2]^2 GI[0, r2, -1], GI[0, r2, -1] GR[-1] GR[r4],
GI[0, r2, -1] GR[r4]^2, GI[0, r2, -1] GR[r2, -1],
GI[r2] GR[-1] GR[r2, 1, -1], GI[r2] GR[r4] GR[r2, 1, -1],
GI[0, r2] GR[r2, 1, -1], GI[r2] GR[-1] GR[r2, 1, r3],
GI[r2] GR[r4] GR[r2, 1, r3], GI[0, r2] GR[r2, 1, r3],
GI[0, 0, 0, r2] GR[-1], GI[0, 0, 0, r2] GR[r4],
GI[r2] GR[0, 0, r2, -1], GI[r2] GR[0, 0, r4, 1],
GI[0, 1, r4] GR[-1], GI[0, 1, 1, r4] GR[r4],
GI[0, 1, r2, -1] GR[-1], GI[0, 1, r2, -1] GR[r4],
GI[0, 1, r2, 1, -1] GR[-1], GI[0, 1, r2, 1, -1] GR[r4],
GI[0, 1, r2, 1, -1] GR[-1], GI[0, 1, r2, 1, -1] GR[r4],
GI[0, 1, r2, r3] GR[-1], GI[0, 1, r2, r3] GR[r4],
GI[0, r2, 1, -1] GR[-1], GI[0, r2, 1, -1] GR[r4],
GI[r2] GR[r2, 1, 1, -1], GI[r2] GR[r2, 1, 1, r3],
GI[r2] GR[r2, 1, r2], GI[r2] GR[r2, 1, r2]}

Evaluating Feynman integrals by uniformly transcendental differential equations
Our basis for the real parts of $G(a_1, \ldots, a_6; 1)$ consists of 25 constants of weight 6 and 170 products of constants of lower weights.
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Our basis for the imaginary parts of $G(a_1, \ldots, a_6; 1)$ consists of less than 74 constants of weight 6 and 157 products of constants of lower weights.
The dimensions of the spaces generated by real and imaginary parts of MPL of a given weight are, respectively, \((F_{2w+2} + F_{w+1})/2\), and \((F_{2w+2} - F_{w+1})/2\), where \(F(n)\) is a Fibonacci number [D. Broadhurst’14].
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Broadhurst explicitly described, at any \(w\), parts of conjectured bases consisting of \(G(a_1, \ldots, a_6; 1)\) using only letters 0, −1, \(r_4\).
The dimensions of the spaces generated by real and imaginary parts of MPL of a given weight are, respectively, \((F_{2w+2} + F_{w+1})/2\), and \((F_{2w+2} - F_{w+1})/2\), where \(F(n)\) is a Fibonacci number [D. Broadhurst’14].

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Zhao: if \(n\) is non-standard, i.e. has at least two prime factors, then the number of MPL at \(n\)th roots of unity coincides with the upper bounds of the motivic theory.
The dimensions of the spaces generated by real and imaginary parts of MPL of a given weight are, respectively, \((F_{2w+2} + F_{w+1})/2\), and \((F_{2w+2} - F_{w+1})/2\), where \(F(n)\) is a Fibonacci number [D. Broadhurst’14].

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Zhao: if \(n\) is non-standard, i.e. has at least two prime factors, then the number of MPL at \(n\)th roots of unity coincides with the upper bounds of the motivic theory. These upper bounds for the motivic fundamental group were computed by Deligne and Goncharov. The numbers of the corresponding generators coincide at \(n = 6\) with our numbers.
A lot of successful applications of the new strategy of the method of DE. A lot of pending projects.
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An explicit algorithmic description by Lee in the case of one variable.
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_to be continued_
Evaluating Feynman integrals by uniformly transcendental differential equations

Conclusion

BACKUP SLIDES
Lee [R.N. Lee’14]: transition to UT basis in three steps (in the case of one variable).
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- Reduction to a Fuchsian form where the singularities at all the points $x^{(k)}$ (including $x = \infty$) are simple poles.
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- Reduction to a Fuchsian form where the singularities at all the points $x^{(k)}$ (including $x = \infty$) are simple poles.
- Normalizing eigenvalues of the matrices which are coefficients the Fuchsian singularities when one tries to make them proportional to $\epsilon$. 

\[ \varepsilon \]
Lee [R.N. Lee’14]: transition to UT basis in three steps (in the case of one variable).

- Reduction to a Fuchsian form where the singularities at all the points $x^{(k)}$ (including $x = \infty$) are simple poles.
- Normalizing eigenvalues of the matrices which are coefficients the Fuchsian singularities when one tries to make them proportional to $\epsilon$.
- Providing a linear dependence on $\epsilon$. 
At each of the three steps, one is looking for a proper linear transformation of the current basis. The first two steps are based on the so-called balance transformation

$$\mathcal{B}(\mathbb{P}, x_1, x_2| x) = \overline{\mathbb{P}} + c \frac{x - x_2}{x - x_1} \mathbb{P},$$

where $c$ is a constant, $\mathbb{P}$, $\overline{\mathbb{P}}$ are the two complementary projectors, i.e. $\mathbb{P}^2 = \mathbb{P}$ and $\overline{\mathbb{P}} = \mathbb{I} - \mathbb{P}$. 
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\[ \mathcal{B}(\mathbb{P}, x_1, x_2 | x) = \mathbb{P} + c \frac{x - x_2}{x - x_1} \mathbb{P}, \]

where \( c \) is a constant, \( \mathbb{P}, \mathbb{P} \) are the two complementary projectors, i.e. \( \mathbb{P}^2 = \mathbb{P} \) and \( \mathbb{P} = I - \mathbb{P} \).

The idea of using a balance transformation is that with its help one can take care of one singular point \( x_1 \) (by providing a Fuchsian singularity or by normalizing eigenvalues corresponding to a given singular point) and not to spoil these properties at a second singular point, \( x_2 \).