The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

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partly based on collaboration with E. Remiddi

[arXiv:1311.3342], [arXiv:1509.03330]
INTRODUCTION

- Our understanding of high energy physics is (mainly) based on perturbative calculations in the Standard Model.

- At LHC discovery of the Higgs boson and (for now) no real sign of any new physics effect.

- High energy physics is becoming more and more precision physics and in order to perform precision calculations we need to be able to compute more and more complicated Feynman diagrams.
Any Feynman Diagrams is *(after some tedious but elementary algebra!)* nothing but a collection of scalar **Feynman Integrals**

\[ I(p_1, p_2, q_1) = \begin{array}{c}
\begin{array}{c}
p_1 \\
p_2
\end{array}
\end{array} \begin{array}{c}
\begin{array}{c}
q_1 \\
q_2
\end{array}
\end{array} \quad \text{with} \quad q_2 = p_1 + p_2 - q_1 \]

A *(possible)* representation in **momentum space** *(massless case!)*

\[ I(p_1, p_2, q_1) = \int \frac{\mathcal{D}^d k \mathcal{D}^d l}{k^2 l^2 (k - l)^2 (k - p_1)^2 (k - p_{12})^2 (l - p_{12})^2 (l - q_1)^2} \]

**Typical 2-loop** Feynman Integral required for the computation of a \(2 \to 2\) scattering process.
How do we (tentatively!) compute **analytically** such integrals?

1. Integrals are **ill-defined** in $d = 4 \rightarrow$ need a **regularization procedure**!

2. Use of **dimensional regularization** to regulate **UV** and **IR divergences**.

3. Dimensional regularisation turned out to be **much MORE** than just a **regularization scheme**!

   \[\downarrow\]

   Dimensionally regularized Feynman integrals **always converge**!

   This allows to derive a large number of **unexpected relations**...

   - **Integration by Parts**, Lorentz invariance identities, Schouten Identities,...
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- *Integration by Parts*, Lorentz invariance identities, Schouten Identities,...
This large set of identities makes it *simpler* to compute **Feynman integrals** in $d$ continuous dimensions than in $d = 4$!

A general scalar Feynman Integral ($l$-loops) can be written as

$$ I(\sigma_1, ..., \sigma_s; \alpha_1, ..., \alpha_n) = \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{S_1^{\sigma_1} \cdots S_s^{\sigma_s}}{D_1^{\alpha_1} \cdots D_n^{\alpha_n}} $$

where

$$ D_n = (q_n^2 + m_n^2), \quad \text{are the \textbf{propagators}} $$

$$ S_n = k_i \cdot p_j, \quad \text{are \textbf{scalar products} among internal and external momenta} $$

This introduces the concept of **Topology**
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This introduces the concept of **Topology**
The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

1. Integration By Parts Identities (IBPs)

\[
\int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \left( \frac{\partial}{\partial k_j^\mu} v_\mu \frac{S_{1}^{\sigma_1} \ldots S_{s}^{\sigma_s}}{D_{1}^{\alpha_1} \ldots D_{n}^{\alpha_n}} \right) = 0, \quad v_\mu = k_j^\mu, \ p_k^\mu
\]

2. They generate huge systems of linear equations which relate integrals with different powers of numerators and denominators.

3. The integrals always belong to the same topology, as defined above.

The IBPs can be solved using computer algebra (Reduze2, AIR, FIRE5...)

As a result, all integrals are expressed as linear combination of a small subset of Master Integrals

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Dimensionally regularised Feynman Integrals fulfil differential equations!

[Kotikov '90, Remiddi '97, Gehrmann-Remiddi '00,...]

Let us take a topology of integrals which depend on two external invariants

\[ s, m^2 \rightarrow x = \frac{s}{m^2}. \]

\[ \mathcal{I}(s, m^2; \alpha_1, ..., \alpha_n) = \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{1}{D_1^{\alpha_1} ... D_n^{\alpha_n}}, \quad \text{(same with scalar products)} \]

Assume IBPs reduce all integrals into this topology to \( N \) Master Integrals

\[ m_i(s; d), \text{ with } i = 1, ..., N. \]
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Assume IBPs reduce all integrals into this topology to \textbf{N Master Integrals} \( m_i(s; \ d) \), with \( i = 1, ..., N \).
1) All integrals depend on $x = s/m^2$ only.

2) **Differentiation** w.r.t to an external invariant:

$$s = p_\mu p^\mu \quad \rightarrow \quad \frac{\partial}{\partial s} = \frac{1}{2s} \left( p_\mu \frac{\partial}{\partial p^\mu} \right)$$

$$\frac{\partial}{\partial s} \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{1}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}} = \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^d} \frac{1}{2s} \left( p_\mu \frac{\partial}{\partial p^\mu} \right) \frac{1}{D_1^{\alpha_1} \ldots D_n^{\alpha_n}}$$

3) With IBPs integrals on r.h.s. can be **reduced** again to the MIs

$$\frac{\partial}{\partial s} m_i(d; s) = \sum_{j=1}^{N} c_{ij}(d; s) m_j(d; s).$$

System of $N$ coupled differential equations for $m_i(d; s)$
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System of $N$ coupled differential equations for $m_i(d; s)$
How does this help in practice?
What if \( N = 1 \)? \( ( \text{There is only 1 MI!} ) \)

If there is only 1 master integral the situation is \textit{in principle} trivial:

\[
\frac{\partial}{\partial s} m(d; s) = c(d; s) m(d; s)
\]

\textbf{First order} linear equation, can be solved by \textit{quadrature}

\[
m(d; s) = C_0 \exp \left( \int_0^s dt \ c(d; t) \right)
\]

\textbf{Note}

Differential equations provide a \textit{natural integral representation} for \( m(d; s) \) in terms of the “correct variables”, i.e. the \textbf{Mandelstam variable} \( s \).
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What if $N > 1$? *(Life is not that easy anymore!)*

If the system is coupled, it corresponds to a **N-th order** differential equation for any of the MIs. **No general strategy for a solution is known.**

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**Observations**

1. We are **free** of choosing our **basis of MIs**
2. We are interested in the **expansion** for $d \to 4$

3. The **physical case** $d \to 4$ can be recovered from $d \to 2n$ with $n \in \mathbb{N}$
    
    [Tarasov '96; Lee '09]

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Changing the basis can **simplify** the structure of the differential equations!

A simplification for $d \to 2n$ equivalent to a simplification for $d \to 4$!!!
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The freedom of considering all possible even numbers of dimensions can help simplify substantially the problem!
Shifting differential equations of even numbers of dimensions

a) We start with a system of deq for N masters $m_1, \ldots, m_N$

$$\frac{\partial}{\partial x_{ij}} \vec{m}(d; x_{ij}) = A(d; x_{ij}) \vec{m}(d; x_{ij}),$$

b) Using Tarasov’s relations

$$\vec{m}(d-2; x_{ij}) = \Delta_{m_k} \vec{m}(d; x_{ij}) = C(d; x_{ij}) \vec{m}(d; x_{ij})$$

we can define a new basis

$$\vec{I}(d; x_{ij}) = \vec{m}(d-2; x_{ij})$$

Recall that $\Delta_{m_k}$ is a differential operator w.r.t. the internal masses. Its form depends only on the topology of the graph and can be computed algorithmically.

c) By construction, the new basis fulfils system with $d \to d - 2$

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a) Suppose that we can find a **canonical basis** in \( d = 2 \)

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\frac{\partial}{\partial x_{ij}} \vec{m}(d; x_{ij}) = (d - 2) A(x_{ij}) \vec{m}(d; x)
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b) Use **Tarasov’s operator** to define a new basis

\[
\vec{I}(d; x_{ij}) = \vec{m}(d - 2; x_{ij}) = \Delta_{mk} \vec{m}(d; x_{ij})
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Strong indication that **number of coupled MIs** as \( d \to 4 \) determines **complexity** of the problem.

1) If all masters decouple differential equations become **triangular** as \( d \to 4 \)
   a) In all these cases a **canonical basis** can be found
      [Kotikov '10; Henn '13]
      
      \[
      \frac{\partial}{\partial x_{ij}} \tilde{I}(d; x_{ij}) = (d - 4) A(x_{ij}) \tilde{I}(d; x)
      \]
      
      where \( A(x_{ij}) \) is in **d-log form**

   b) Results in terms of **multiple polylogarithms**
      
      \[
      G(0; x) = \ln x, \quad G(a; x) = \ln \left(1 - \frac{x}{a}\right)
      \]
      \[
      G(\vec{0}_n; x) = \frac{1}{n!} \ln^n x, \quad G(a, \vec{n}; x) = \int_0^x \frac{dt}{t - a} G(\vec{n}; t)
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2) If all but two masters decouple → a $2 \times 2$ block remains coupled.

a) Not many “classified” examples. The most famous ones

\[ S(d; p^2) = \quad , \quad T(d; q^2) = \]

b) $S(d; p^2)$ is reduced to 4 master integrals
$T(d; q^2)$ is reduced to 3 master integrals

c) In both cases 2 masters cannot be decoupled in the limit $d \rightarrow 2n$.
Both are known to contain elliptic functions

This stimulated current attempt to generalise multiple polylogarithms to elliptic (multiple?) polylogarithms
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\begin{array}{c}
\text{p} \\
\text{m}_1 \\
\text{m}_2 \\
\text{m}_3 \\
\end{array}
\end{array}, \quad T(d; q^2) = \begin{array}{c}
\begin{array}{c}
\text{q} \\
\text{m} \\
\text{m} \\
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Of course there are more complicated examples where three or more master integrals cannot be decoupled in the limit $d \rightarrow 2n$, but the latter are at the moment still poorly studied and it is not clear what kind of new functions one could expect to appear in such calculations.
To me one central question becomes:

How many MIs remain coupled in $d \to 2n$?

▶ Can we imagine a criterion to determine whether some MIs can be decoupled in the limit $d \to 2n$?

▶ Given a graph with $N$ MIs, they fulfil $N$ coupled differential equations in $d$ dimensions because the $N$ MIs are linearly independent in $d$!

Conjecture

The differential equations can be decoupled in $d = 2n$ if the IBPs degenerate in the limit $d \to 2n$, allowing some of the MIs to become linearly dependent.

If this happens we would expect them to bring no new information as $d \to 2n$ and therefore it should be possible to decouple them from the differential equations.
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▶ Can we imagine a criterion to determine whether some MIs can be decoupled in the limit $d \to 2n$?

▶ Given a graph with $N$ MIs, they fulfil $N$ coupled differential equations in $d$ dimensions because the $N$ MIs are linearly independent in $d$!

Conjecture

The differential equations can be decoupled in $d = 2n$ if the IBPs degenerate in the limit $d \to 2n$, allowing some of the MIs to become linearly dependent.

If this happens we would expect them to bring no new information as $d \to 2n$ and therefore it should be possible to decouple them from the differential equations.
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In order to verify this we need to read the IBPs in $d = 2n$ dimensions

1) Start with a topology with 2 masters $I_1(d; x), I_2(d; x)$ in $d$ dimensions and which satisfy

$$\frac{\partial}{\partial x} I_1(d; x) = c_{11}(d; x) I_1(d; x) + c_{12}(d; x) I_2(d; x)$$

$$\frac{\partial}{\partial x} I_2(d; x) = c_{21}(d; x) I_1(d; x) + c_{22}(d; x) I_2(d; x)$$

2) Now generate IBPs in $d$ dimensions, then fix $d = 2n$, and finally try to solve them

NB

The MIs are in general divergent BUT in the IBPs there can be NO factors $1/(d - 2n)!$. One can more formally expand IBPs in Laurent series in $(d - 2n)$, and solve the chained systems of IBPs order by order. The homogeneous part is always the same. This is what determines the degeneracy on the MIs in $d = 2n$. 
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3) Assume that by solving the $d = 2n$ IBPs we find

$$I_2(2n; x) = b(x) I_1(2n; x),$$

i.e. in $d = 2n$ there is only 1 master integral. It must fulfill a first order differential equation in the limit $d \to 2n$.

4) Where does such a relation come from?
If in the $d$-dimensional IBPs there is something like

$$K(d, x) = \frac{1}{d - 2n} \left( b_1(d; x) I_1(d; x) + b_2(d; x) I_2(d; x) \right),$$

5) This implies that, as far as the IBPs are concerned,

$$b_1(d; x) I_1(d; x) + b_2(d; x) I_2(d; x) = O(d - 2n),$$

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with \( \lim_{d \to 2n} b_i(d; x) = b_i(x) \).

The system of deq under this rotation becomes

\[
\frac{\partial}{\partial x} \mathcal{J}_1(d; x) = \left( c_{11}(d; x) + \frac{b_2(x)c_{21}(d; x) + b_1'(x)}{b_1(x)} \right) \mathcal{J}_1(d; x)
+ \left( b_1(x)c_{12}(d; x) + b_2(x)(c_{22}(d; x) - c_{11}(d; x)) + b_2'(x) - \frac{b_2(x)}{b_1(x)} \left[ b_2(x)c_{21}(d; x) + b_1'(x) \right] \right) \mathcal{J}_2(d; x)
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\]
We expect the diff. equation for the newly introduced master to **decouple**, i.e.

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\left( b_1(x)c_{12}(d; x) + b_2(x) (c_{22}(d; x) - c_{11}(d; x)) + b'_2(x) \\
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**NB**

No mathematical proof. Decoupling expected due to *linear dependence* of the two master integrals in the limit \( d \to 2n \). The rotation introduced above makes **manifest** the *first order differential equation* satisfied by one of the two original masters (in this case \( J_2 = I_2 \)).

This can be easily generalised to \( N \) master integrals.
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The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

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This can be easily generalised to \( N \) master integrals.
Explicit example, an easy two-loop sunrise

\[ p \]

\[ \mathcal{I}(s, m^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \]

\[ = \int \mathcal{D}^d k \mathcal{D}^d l \frac{(k \cdot p)^{\alpha_4} (l \cdot p)^{\alpha_5}}{(k^2)^{\alpha_1} (l^2)^{\alpha_2} ((k - l + p)^2 - m^2)^{\alpha_3}} \]

1. Using IBPs we can reduce all these integrals to only 2 Master Integrals.

\[ S_1 = \mathcal{I}(s, m^2; 1, 1, 1, 0, 0), \quad S_2 = \mathcal{I}(s, m^2; 1, 1, 2, 0, 0). \]

2. Derive now DE for these two integrals, we find:

\[ \frac{d}{ds} S_1 = \frac{(d - 3)}{s} S_1 - \frac{m^2}{s} S_2 \]

\[ \frac{d}{ds} S_2 = \frac{(d - 3)(3d - 8)}{2s(s - m^2)} S_1 + \left( \frac{2(d - 3)}{s - m^2} - \frac{3d - 8}{2s} \right) S_2 \]
Explicit example, an easy two-loop sunrise

\[ p = \mathcal{I}(s, m^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \]

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&= \int \mathcal{D}^d k \mathcal{D}^d l \frac{(k \cdot p)^{\alpha_4} (l \cdot p)^{\alpha_5}}{(k^2)^{\alpha_1} (l^2)^{\alpha_2} ((k - l + p)^2 - m^2)^{\alpha_3}}
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Decoupling of the diff. equations in $d = 4$

Following argument above, let us study the IBPs in $d = 4$. First of all note that both masters have a (UV) double pole

$$S_1(d; s) = \frac{1}{(d - 4)^2} S_1^{(-2)}(4; s) + \frac{1}{(d - 4)} S_1^{(-1)}(4; s) + O(1)$$

$$S_2(d; s) = \frac{1}{(d - 4)^2} S_2^{(-2)}(4; s) + \frac{1}{(d - 4)} S_2^{(-1)}(4; s) + O(1).$$

This is also highest pole for any other integral in this topology as $d \to 4!$

Expand IBPs in $(d - 4)$ and solve them order by order. As expected first order (double pole) gives degeneracy

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Note that

In this case solving the IBPs in $d = 4$ provides a real relation among the highest poles of the two master integrals! The graph does not contain any sub-topology and we have not neglected anything.

An (easy) explicit calculation gives

$$S_1(d; s) = \frac{1}{(d - 4)^2} \left( \frac{m^2}{2} \right) + \mathcal{O}\left( \frac{1}{(d - 4)} \right)$$

$$S_2(d; s) = \frac{1}{(d - 4)^2} \left( \frac{1}{2} \right) + \mathcal{O}\left( \frac{1}{(d - 4)} \right),$$

Confirming the relation above

$$S_2^{(-2)}(4; s) = \frac{1}{m^2} S_1^{(-2)}(4; s).$$

which comes for free if we properly read the IBPs in $d = 4$. 
As we discussed, this degeneracy can be found in the $d$-dimensional IBPs

a) Scan the result of the IBPs reduction in $d$ dimensional, look for relations which contain an overall $1/(d - 4)$. We find, for example

$$I(s, m^2; 2, 1, 1, 0, 0) = \left( \frac{1}{d - 4} \right) \frac{(d - 3)}{s - m^2} \left( 4m^2 S_2(d; s) - (3d - 8)S_1(d; s) \right)$$

And many other similar relations, all equivalent in the limit $d \to 4$!

b) Taking the $d \to 4$ limit of these relation we find of course

$$m^2 S_2(d; s) - S_1(d; s) = O(d - 4)$$

in the sense introduced above, i.e. this combination is zero when we put $d = 4$ in the IBPs.
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We can now finally simplify the differential equations.

1) Using results above, we perform the rotation

$$J_1(d; s) = S_1(d; s) - m^2 S_2(d; s), \quad J_2(d; s) = S_1(d; s).$$

2) As expected the differential equations decouple

$$\frac{d J_1}{d s} = \left[ \frac{2}{s-m^2} - \frac{1}{s} + (d-4) \left( \frac{2}{s-m^2} - \frac{3}{2s} \right) \right] J_1$$

$$+ (d-4) \left[ \frac{3}{2(s-m^2)} - \frac{1}{s} + \frac{3}{2} (d-4) \left( \frac{1}{s-m^2} - \frac{1}{s} \right) \right] J_2$$

$$\frac{d J_2}{d s} = \frac{1}{s} J_1 + \frac{(d-4)}{s} J_2.$$ 

The decoupling follows always this pattern, namely the new differential equation for $J_1$ contains an explicit $(d-4)$ which multiplies the master that has not been rotated, in this case $J_2$. 
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Decoupling of the diff. equations in $d = 2$

1) Let us check what happens in $d = 2$. In $d = 2$ the two masters have a double pole of IR origin.

2) In order to check whether there is any degeneracy we solve IBPs in $d = 2$, and find

$$S_2^{(-2)}(2; s) = \frac{1}{s - m^2} S_1^{(-2)}(2; s).$$

3) The corresponding $d$-dimensional IBP read

$$I(s, m^2; 1, 1, 1, 1, 0) = \left( \frac{1}{d - 2} \right) \frac{m^2}{3} \left( \left[ (2 - d) \frac{s}{m^2} + (3 - d) \right] S_1(d; s) - (s - m^2) S_2(d; s) \right)$$
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\[
\frac{dJ_1}{ds} = (d - 2) \left[ \frac{2}{s - m^2} - \frac{3}{2s} \right] J_1 + (d - 2) \left[ \frac{3(d - 2)}{2s} - \frac{2}{s - m^2} \right] J_2
\]

\[
\frac{dJ_2}{ds} = \left[ \frac{1}{s - m^2} - \frac{1}{s} \right] J_1 + \left[ \frac{(d - 2)}{s} - \frac{1}{s - m^2} \right] J_2.
\]

Again the decoupling follows the same pattern as before.
We can proceed and perform a change of basis as before

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\[
\begin{align*}
\frac{dJ_1}{ds} &= (d - 2) \left[ \frac{2}{s - m^2} - \frac{3}{2s} \right] J_1 + (d - 2) \left[ \frac{3(d - 2)}{2s} - \frac{2}{s - m^2} \right] J_2 \\
\frac{dJ_2}{ds} &= \left[ \frac{1}{s - m^2} - \frac{1}{s} \right] J_1 + \left[ \frac{(d - 2)}{s} - \frac{1}{s - m^2} \right] J_2.
\end{align*}
\]

Again the decoupling follows the same pattern as before.
Also in $d = 2$, we found a **real relation** between the two highest poles of the two masters! They become **linearly dependent in $d = 2$**!

Writing

$$S_1(d; s) = \frac{1}{(d - 2)^2} S_1^{(-2)}(2; s) + \frac{1}{(d - 2)} S_1^{(-1)}(2; s) + O(1)$$

$$S_2(d; s) = \frac{1}{(d - 2)^2} S_2^{(-2)}(2; s) + \frac{1}{(d - 2)} S_2^{(-1)}(2; s) + O(1).$$

The relation

$$S_2^{(-2)}(2; s) = \frac{1}{s - m^2} S_1^{(-2)}(2; s).$$

can be easily verified analytically or numerically.

**Question**

Is it always the case? Is it enough to find a relation between the highest poles of the two master integrals in order to decouple the differential equations?
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**Question**

Is it always the case? Is it enough to find a relation between the highest poles of the two master integrals in order to decouple the differential equations?
The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

The answer is, of course **NO**. Example **two-loop massive sunrise**

\[
\begin{align*}
\mathcal{I}(s, m^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) &= \mathcal{I}(s, m^2; \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\
&= \int \mathcal{D}^d k \mathcal{D}^d l \frac{(k \cdot p)^\alpha_4 (l \cdot p)^\alpha_5}{(k^2 - m^2)^\alpha_1 (l^2 - m^2)^\alpha_2 ((k - l + p)^2 - m^2)^\alpha_3}
\end{align*}
\]

1. Using **IBPs** we find again only 2 **Master Integrals**

\[
S_1 = \mathcal{I}(s, m^2; 1, 1, 1, 0, 0), \quad S_2 = \mathcal{I}(s, m^2; 1, 1, 2, 0, 0).
\]

2. If we neglect sub-topologies and solve IBPs in \( d = 2 \) or \( d = 4 \) we do not find any further relations. **Equations cannot be decoupled!** **Elliptic polylogaritms!**
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2. If we neglect sub-topologies and solve IBPs in \(d = 2\) or \(d = 4\) we do not find any further relations. Equations cannot be decoupled! **Elliptic polylogarithms!**
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The answer is, of course NO. Example two-loop massive sunrise

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\[ = \int D^d k D^d l \frac{(k \cdot p)^{\alpha_4} (l \cdot p)^{\alpha_5}}{(k^2 - m^2)^{\alpha_1} (l^2 - m^2)^{\alpha_2} ((k - l + p)^2 - m^2)^{\alpha_3}} \]

1. Using IBPs we we find again only 2 Master Integrals

\[ S_1 = \mathcal{I}(s, m^2; 1, 1, 1, 0, 0), \quad S_2 = \mathcal{I}(s, m^2; 1, 1, 2, 0, 0). \]

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Elliptic polylogarithms!
Both masters are finite in $d = 2$. On the other hand in $d \approx 4$

\[
S_1(d; s, m^2) = \frac{1}{(d - 4)^2} \left( \frac{3m^2}{8} \right) + \mathcal{O}\left( \frac{1}{(d - 4)} \right)
\]
\[
S_2(d; s, m^2) = \frac{1}{(d - 4)^2} \left( \frac{1}{8} \right) + \mathcal{O}\left( \frac{1}{(d - 4)} \right)
\]

Such that of course we have

\[
S_1^{(-2)}(s, m^2) - 3m^2 S_2^{(-2)}(s, m^2) = 0 .
\]

What happens if we use as new masters:

\[
\mathcal{I}_1 = S_1 , \quad \mathcal{I}_2 = S_1 - 3m^2 S_2 \quad ?
\]

The differential equations do not decouple in $d = 4$ ! What’s going wrong?
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To understand what is going on, let’s study IBPs for massive sunrise in $d = 4$

a) This time, **do not neglect** sub-topologies and expand all IBPs in Laurent series starting from the highest pole $1/(d - 4)^2$

b) For the highest pole we find **two relations** which can inverted giving

$$S_1^{(-2)}(d \to 4; s) = \frac{3}{2m^2} \, T^{(-2)}(d \to 4)$$
$$S_2^{(-2)}(d \to 4; s) = \frac{1}{2m^4} \, T^{(-2)}(d \to 4),$$

which just give the **poles** in terms of the **tadpole** (sub-topology)!

$$T(d) = \int \frac{\mathcal{D}^d k \mathcal{D}^d l}{(k^2 - m^2)(l^2 - m^2)} = \frac{(m^2)^{(d-2)}}{(d-2)^2(d-4)^2}. $$

Poles of MIs are **fake**! In other words, there exists a completely **finite** basis in $d = 2, 4, 6, \ldots$ etc, such that all poles are entirely determined by sub-topologies!
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The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

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Poles of MIs are **fake**! In other words, there exists a completely **finite** basis in $d = 2, 4, 6, \ldots$ etc, such that all poles are entirely determined by sub-topologies!
Method can be easily generalised to more complicated topologies

a) In **different numbers of dimensions** $d = n \in \mathbb{N}$

b) For any **other topology** irrespective of number of loops or masses!

As a (**natural**) extension the three-loop banana graph

\[
I_4(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) = \int \mathcal{D}^d k_1 \mathcal{D}^d k_2 \mathcal{D}^d k_3 \frac{(k_1 \cdot p)^{n_5}(k_2 \cdot p)^{n_6}(k_3 \cdot p)^{n_7}(k_1 \cdot k_2)^{n_8}(k_1 \cdot k_3)^{n_9}}{(k_1^2 - m_1^2)^{n_1}(k_2^2 - m_2^2)^{n_2}(k_3^2 - m_3^2)^{n_3}((k_1 + k_2 + k_3 - p)^2 - m_4^2)^{n_4}}
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\]
The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

The case of **all equal masses**

All masses equal \( m_4 = m_3 = m_2 = m_1 = m \) \( \rightarrow \) 3 MIs in \( d \) dimensions

\[
\mathcal{I}_1(d; s) = l_1(d; 1, 1, 1, 1, 0, 0, 0, 0), \quad \mathcal{I}_2(d; s) = l_1(d; 2, 1, 1, 1, 0, 0, 0, 0), \quad \mathcal{I}_3(d; s) = l_1(d; 3, 1, 1, 1, 0, 0, 0, 0).
\]

They remain independent if \( d = 2 \) \( \rightarrow \) The scalar master integrals fulfil a **third-order differential equation**!

\[
D_d^{(3)} \mathcal{I}_1(d; s) = 0,
\]

\[
D_d^{(3)} = \frac{d^3}{d \ s^3} + \frac{3 \ (64m^4 + 10(d - 5)m^2s - (d - 4)s^2)}{s(s - 4m^2)(s - 16m^2)} \frac{d^2}{d \ s^2}
\]

\[
+ \frac{(d - 4)(11d - 36)s^2 - 64(d - 4)d \ m^4 - 4 (216 + d(7d - 88)) m^2 s}{4 \ s^2(s - 4m^2)(s - 16m^2)} \frac{d}{d \ s}
\]

\[
+ \frac{(3 - d)(3d - 8) \ (2(d + 2)m^2 + (d - 4)s)}{4 \ s^2 (s - 4m^2)(s - 16m^2)}
\]
The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

The case of two different masses - two different configurations

\[ I^A_2(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) = I_4(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) \mid_{m_3 = m_2 = m_1 = m_a, m_4 = m_b} \]

\[ I^B_2(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) = I_4(d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) \mid_{m_2 = m_1 = m_a, m_4 = m_3 = m_b} \]

A) In configuration A 5 independent MIs in \( d \) dimensions

\[ I^A_1(d; s) = I^A_2(d; 1,1,1,1,0,0,0,0,0), \quad I^A_2(d; s) = I^A_2(d; 2,1,1,1,0,0,0,0,0), \]
\[ I^A_3(d; s) = I^A_2(d; 1,1,1,2,0,0,0,0,0), \quad I^A_4(d; s) = I^A_2(d; 3,1,1,1,0,0,0,0,0), \]
\[ I^A_5(d; s) = I^A_2(d; 2,2,1,1,0,0,0,0,0) \]

B) In configuration B 6 independent MIs in \( d \) dimensions

\[ I^B_1(d; s) = I^B_2(d; 1,1,1,1,0,0,0,0,0), \quad I^B_2(d; s) = I^B_2(d; 2,1,1,1,0,0,0,0,0), \]
\[ I^B_3(d; s) = I^B_2(d; 1,1,2,1,0,0,0,0,0), \quad I^B_4(d; s) = I^B_2(d; 3,1,1,1,0,0,0,0,0), \]
\[ I^B_5(d; s) = I^B_2(d; 2,2,1,1,0,0,0,0,0), \quad I^B_6(d; s) = I^B_2(d; 2,2,1,1,0,0,0,0,0) \].
The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

The case of **two different masses** - two different configurations

\[ I_A^2 (d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) = I_4 (d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) \mid m_3=m_2=m_1=m_a, m_4=m_b \]

\[ I_B^2 (d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) = I_4 (d; n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9) \mid m_2=m_1=m_a, m_4=m_3=m_b \]

A) In configuration A *5 independent* MIs in \( d \) dimensions

\[ I_A^1 (d; s) = I_A^2 (d; 1, 1, 1, 1, 0, 0, 0, 0, 0) \]
\[ I_A^2 (d; s) = I_A^2 (d; 2, 1, 1, 1, 0, 0, 0, 0, 0) \]
\[ I_A^3 (d; s) = I_A^2 (d; 1, 1, 1, 2, 0, 0, 0, 0, 0) \]
\[ I_A^4 (d; s) = I_A^2 (d; 3, 1, 1, 1, 0, 0, 0, 0, 0) \]
\[ I_A^5 (d; s) = I_A^2 (d; 2, 2, 1, 1, 0, 0, 0, 0, 0) \]

B) In configuration B *6 independent* MIs in \( d \) dimensions

\[ I_B^1 (d; s) = I_B^2 (d; 1, 1, 1, 1, 0, 0, 0, 0, 0) \]
\[ I_B^2 (d; s) = I_B^2 (d; 2, 1, 1, 1, 0, 0, 0, 0, 0) \]
\[ I_B^3 (d; s) = I_B^2 (d; 1, 1, 2, 1, 0, 0, 0, 0, 0) \]
\[ I_B^4 (d; s) = I_B^2 (d; 3, 1, 1, 1, 0, 0, 0, 0, 0) \]
\[ I_B^5 (d; s) = I_B^2 (d; 2, 2, 1, 1, 0, 0, 0, 0, 0) \]
\[ I_B^6 (d; s) = I_B^2 (d; 2, 1, 2, 1, 0, 0, 0, 0, 0) \]
Simplifying the system of differential equations in $d = 2$

A) Solving IBPs in $d = 2$ we find that 4 MIs remain independent

$$m_a^2(s - 5m_a^2 + m_b^2) \mathcal{I}_5^A(2; s) =$$

$$+ \frac{3m_a^2 + m_b^2 - s}{12 m_a^2} \mathcal{I}_1^A(2; s) + \frac{51m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + 2s)}{12 m_a^2} \mathcal{I}_2^A(2; s)$$

$$+ \frac{m_b^2(m_b^2 - s)}{6 m_a^2} \mathcal{I}_3^A(2; s) + \frac{21m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + s)}{6} \mathcal{I}_4^A(2; s).$$

B) Similarly we find here two relations such that again only 4 MIs remain independent

$$\mathcal{I}_5^B(s; 2) = a_1 \mathcal{I}_1(2; s) + a_2 \mathcal{I}_2(2, s) + a_3 \mathcal{I}_3(2, s) + a_4 \mathcal{I}_4(2, s)$$

$$\mathcal{I}_6^B(s; 2) = b_1 \mathcal{I}_1(2; s) + b_2 \mathcal{I}_2(2, s) + b_3 \mathcal{I}_3(2, s) + b_4 \mathcal{I}_4(2, s).$$

As for the two-loop sunrise, these relations can be used to decouple 1 (2) MIs from the system of differential equations.

4 differential equations remain coupled, corresponding to a fourth-order differential equation for the scalar amplitude.
Simplifying the system of differential equations in $d = 2$

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$$+ \frac{m_b^2 (m_b^2 - s)}{6 m_a^2} \mathcal{I}_3^A (2; s) + \frac{21m_a^4 + (m_b^2 - s)^2 - 6m_a^2 (m_b^2 + s)}{6} \mathcal{I}_4^A (2; s).$$

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+ \frac{m_b^2(m_b^2 - s)}{6 m_b^2} I_3^A(2; s) + \frac{21m_a^4 + (m_b^2 - s)^2 - 6m_a^2(m_b^2 + s)}{6} I_4^A(2; s).
\]

B) Similarly we find here two relations such that again only 4 MIs remain independent

\[
I_5^B(s; 2) = a_1 I_1(2; s) + a_2 I_2(2, s) + a_3 I_3(2, s) + a_4 I_4(2, s) \\
I_6^B(s; 2) = b_1 I_1(2; s) + b_2 I_2(2, s) + b_3 I_3(2, s) + b_4 I_4(2, s)
\]

As for the two-loop sunrise, these relations can be used to decouple 1 (2) MIs from the system of differential equations.

4 differential equations remain coupled, corresponding to a fourth-order differential equation for the scalar amplitude.
The general case with 4 different masses

In the most general case one finds 11 different master integrals in $d$ dimensions

\[ I_1(d; s) = l_4(d; 1, 1, 1, 0, 0, 0, 0, 0), \quad I_2(d; s) = l_4(d; 2, 1, 1, 0, 0, 0, 0, 0), \]
\[ I_3(d; s) = l_4(d; 1, 2, 1, 0, 0, 0, 0, 0), \quad I_4(d; s) = l_4(d; 1, 1, 2, 1, 0, 0, 0, 0), \]
\[ I_5(d; s) = l_4(d; 1, 1, 1, 2, 0, 0, 0, 0), \quad I_6(d; s) = l_4(d; 1, 1, 1, 3, 0, 0, 0, 0) \]
\[ I_7(d; s) = l_4(d; 2, 2, 1, 0, 0, 0, 0, 0), \quad I_8(d; s) = l_4(d; 2, 1, 2, 1, 0, 0, 0, 0) \]
\[ I_9(d; s) = l_4(d; 2, 1, 1, 2, 0, 0, 0, 0), \quad I_{10}(d; s) = l_4(d; 1, 2, 2, 1, 0, 0, 0, 0) \]
\[ I_{11}(d; s) = l_4(d; 1, 2, 1, 2, 0, 0, 0, 0). \]

Repeating the exercise of solving the IBPs in $d = 2$ we find 5 independent relations such that 6 master integrals remain independent!

→ corresponds to a sixth-order differential equation for the scalar amplitude
OPEN QUESTIONS

1. Is the criterion above necessary, together with sufficient?

2. Looking for relations in different even numbers of dimensions can give apparently different and complementary results. In what do these relations differ?

3. How do we know when we should stop looking? $d = 2, 4, 6, ...$
CONCLUSIONS

a) **Differential equations** are a very powerful tool for evaluating complicated *multi-loop and multi-scale* Feynman integrals.

b) If system of differential equations can be **decoupled** in the limit $d \to 4$ their solution as **Laurent series** in $(d - 4)$ becomes much easier.

c) We shouldn’t limit to study the system in the limit $d \to 4$. *A simplification in $d \to 2n$ is equivalent!*

d) I showed that **relations** useful to **decouple** systems of differential equations in the limit $d \to n \in \mathbb{N}$ can be found easily by studying the IBPs in the considered limit.
The role of the space-time dimensions in simplifying systems of differential equations for Feynman Integrals

THANKS!