CALCULATING HIGGS PRODUCTION AT N3LO

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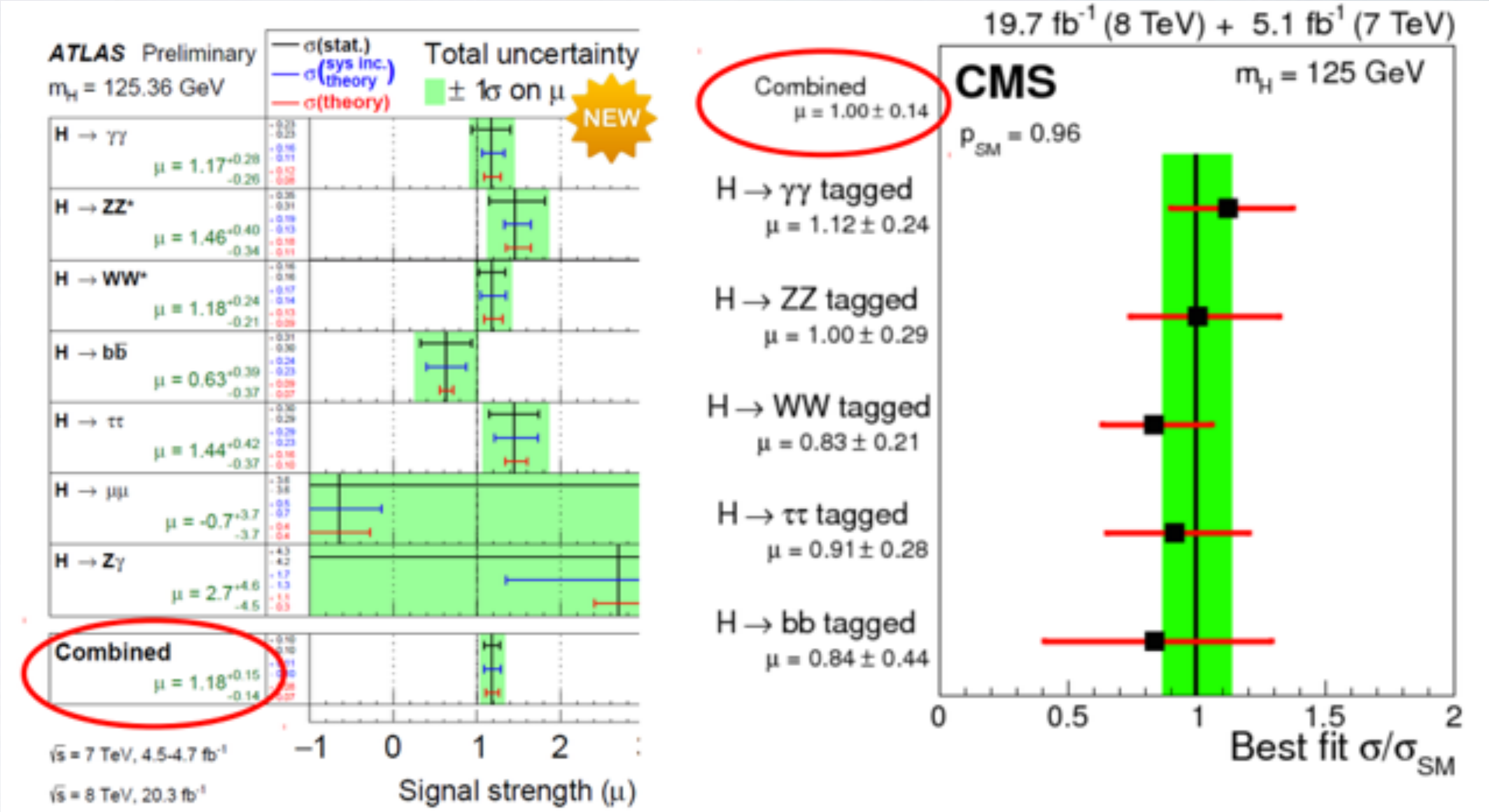
- Why do we compute?
- What do we want to compute?
- How do we compute?
- What do we find?

Motivation

- Discovery marks the beginning of the experimental era of Higgs physics

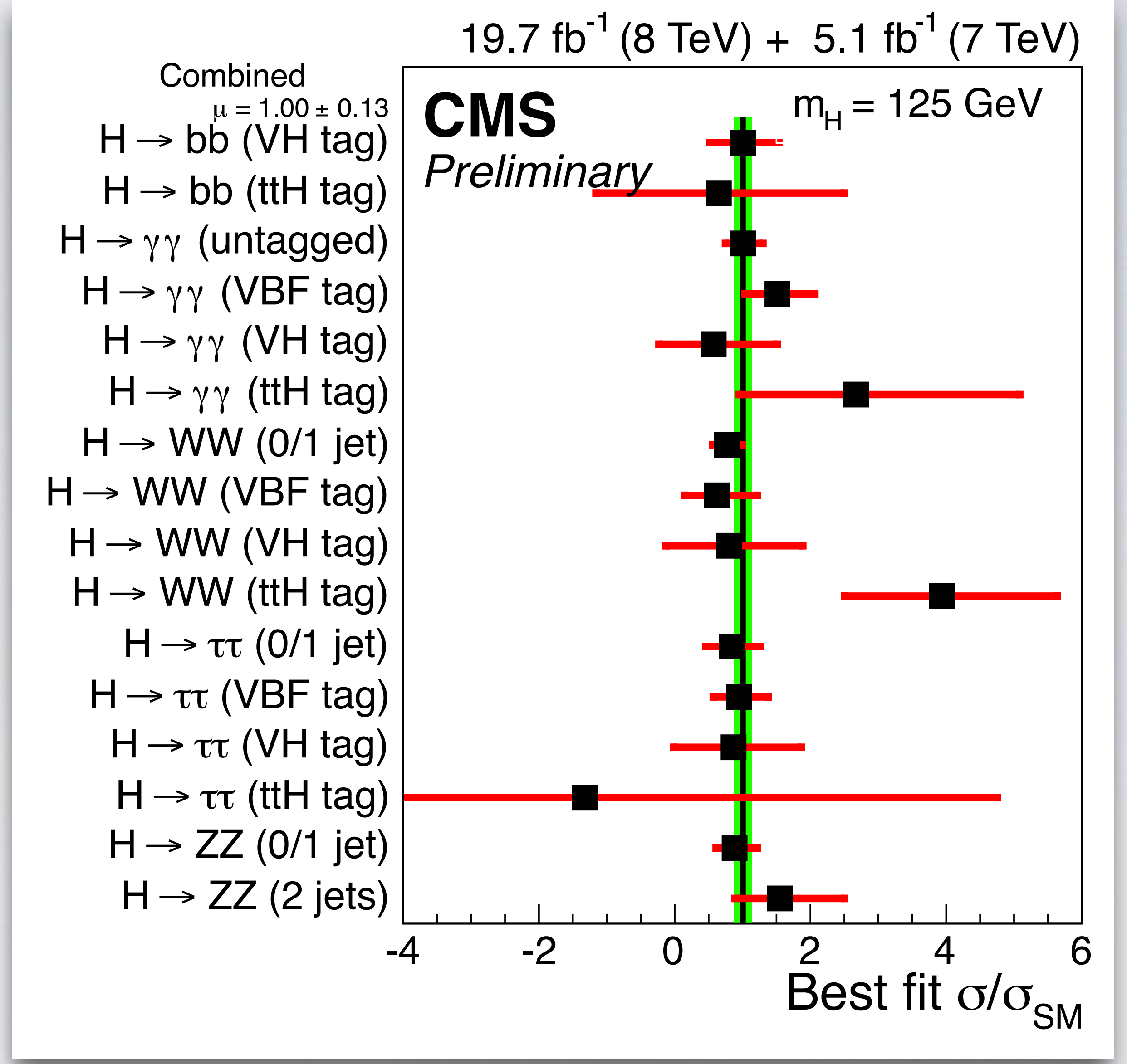
- Determination of the properties of the Higgs will be a challenge for years to come



Amazing progress from the experiments

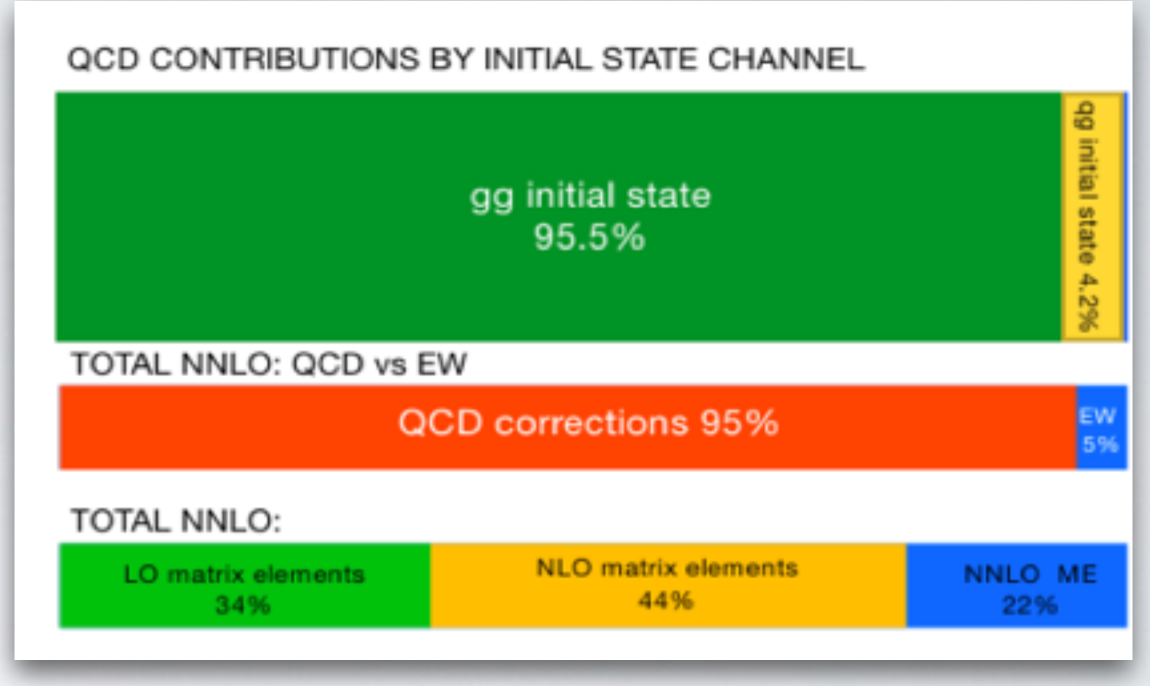
Motivation

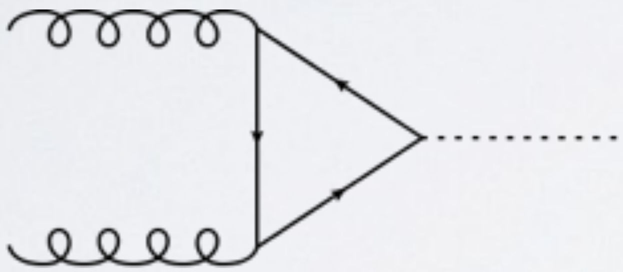
- Higgs couplings are a gateway to possible BSM scenarios
- Measuring deviations from the standard model prediction can indicate new physics
- This requires highly accurate standard model predictions



The gluon fusion cross section

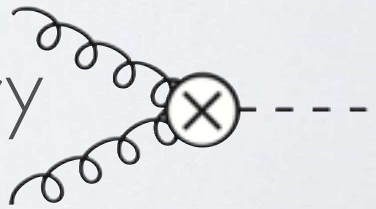
- The dominant Higgs production mode at the LHC is gluon fusion
  - Loop-induced process





- The Higgs boson is light compared to the top quark

- The top loop can be integrated out → effective theory



The gluon fusion cross section

- The tree-level coupling of the gluons to the Higgs is described by a dimension five operator

$$\mathcal{L} = \mathcal{L}_{\rm QCD} - \frac{1}{4v} C_1 H G^a_{\mu\nu} G_a^{\mu\nu}$$

- Operators with higher dimension can be included in the computation

- This leads to a systematic expansion of the gluon fusion cross section in the top mass

- Sub-leading corrections in the top-mass are known at NNLO
  [Harlander, Ozeren; Pak, Rogal, Steinhauser; Ball, Del Duca, Marzani, Forte, Vicini; Harlander, Mantler, Marzani, Ozeren]

- In the following I will only talk about the leading term in the effective theory

The gluon fusion cross section

- The gluon fusion cross-section in perturbation theory is

$$\sigma\left(pp \to H + X\right) = \tau \sum_{ij} \int_{\tau}^{1} dz \mathcal{L}_{ij}(z) \hat{\sigma}_{ij}\left(\frac{\tau}{z}\right)$$

- We compute the inclusive partonic cross section

- The partonic cross section is a function of

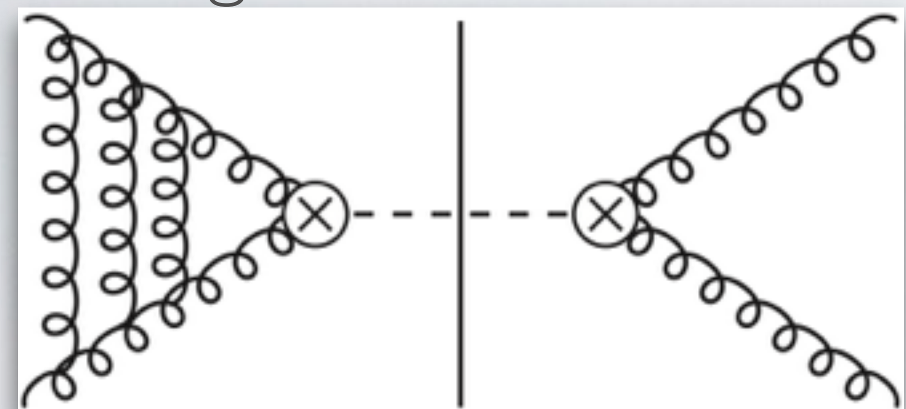
$$z = \frac{m_h^2}{\hat{s}} \qquad \tau = \frac{m_h^2}{E_{cm}^2}$$

- In perturbation theory the partonic cross section can be expanded

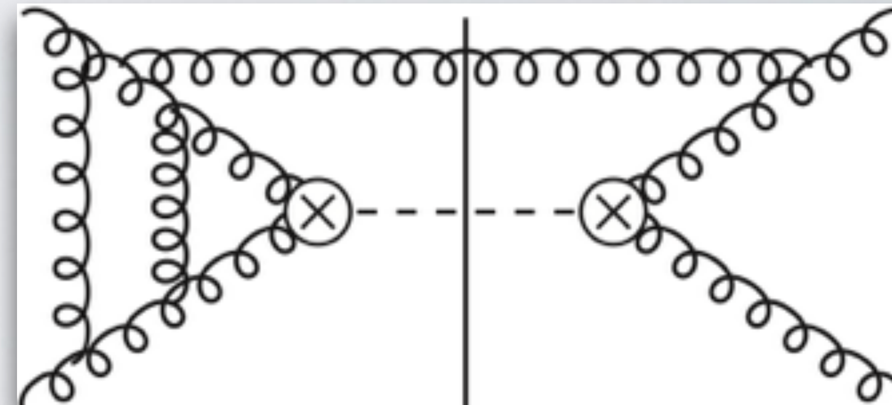
$$\hat{\sigma}(z) = \hat{\sigma}^{\mathrm{LO}}(z) + \alpha_s \hat{\sigma}^{\mathrm{NLO}}(z) + \alpha_s^2 \hat{\sigma}^{\mathrm{NNLO}}(z) + \alpha_s^3 \hat{\sigma}^{\mathrm{N3LO}}(z) + \ldots$$

The gluon fusion cross section

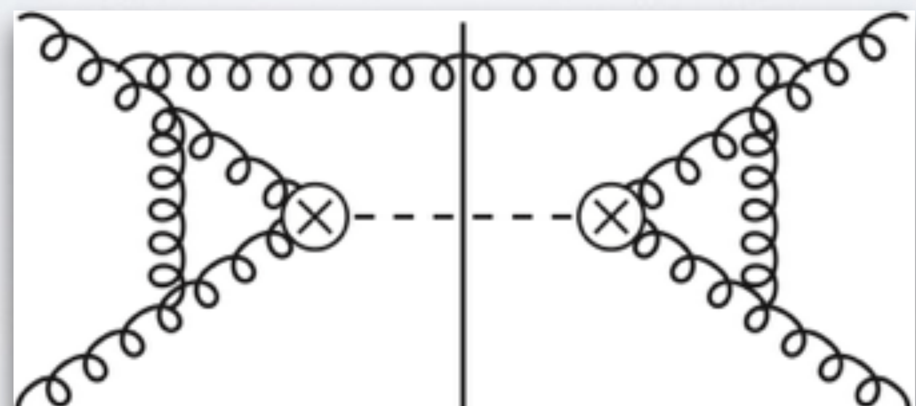
- Diagrammatic contributions at NNNLO



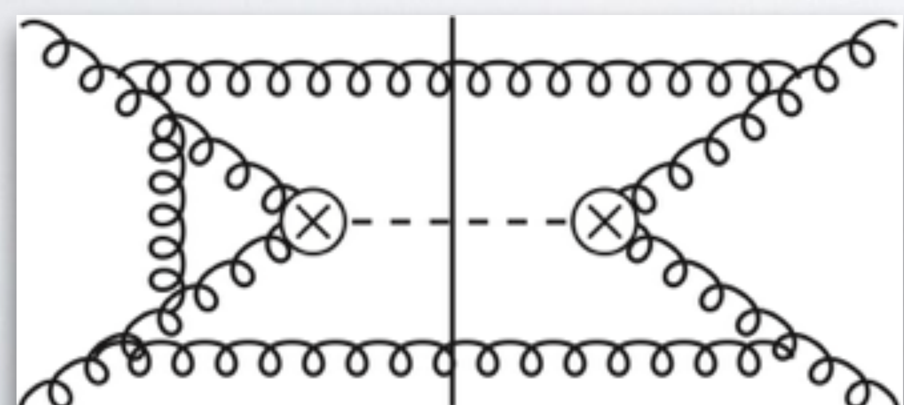
triple virtual



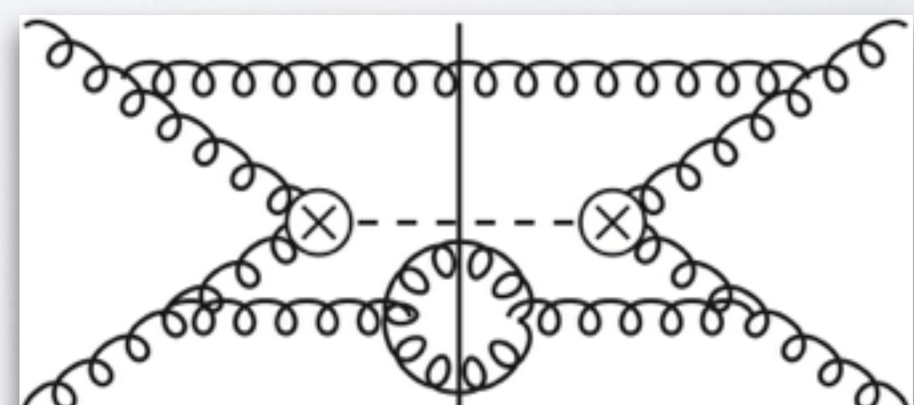
double virtual real



real virtual squared



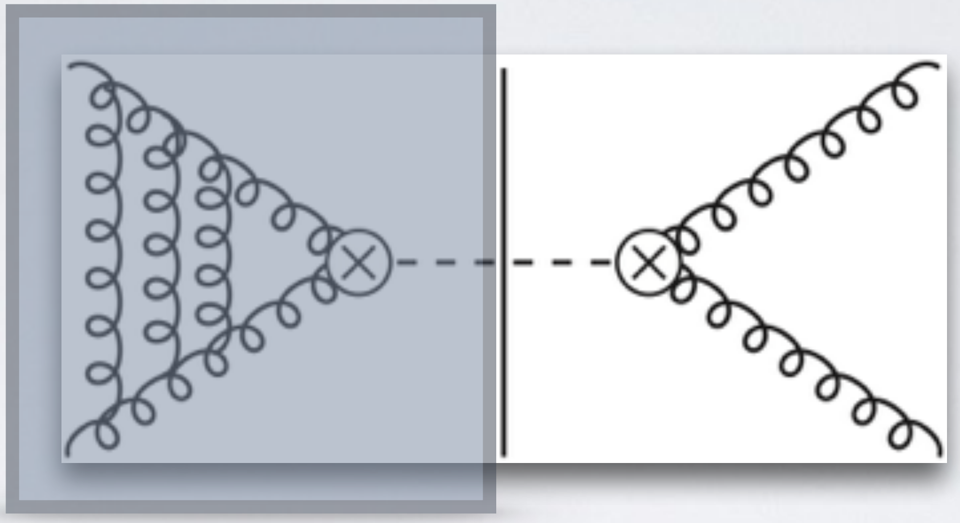
double real virtual



triple real

The triple virtual

- The triple virtual is directly related to the three loop QCD form factor

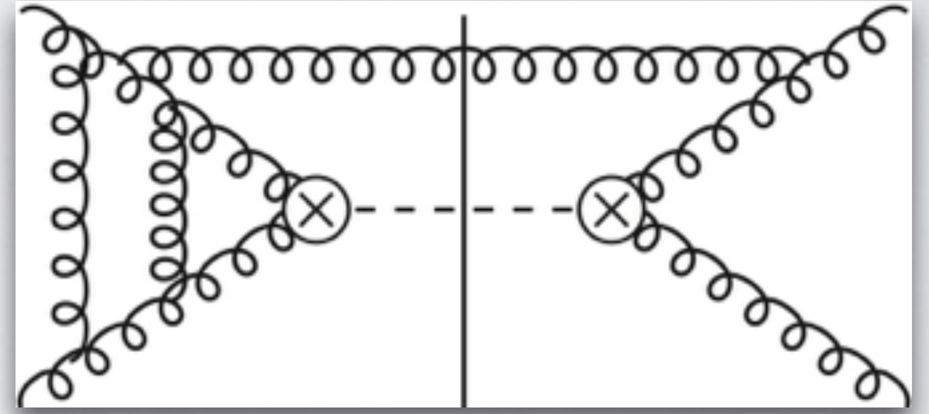


- The QCD form factor is well known
  - at one loop
  - at two loops [Gonsalves; Kramer, Lampe; Gehrmann, Huber, Maitre]
  - at three loops [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]
- The pure loop contributions are not a problem in the calculation

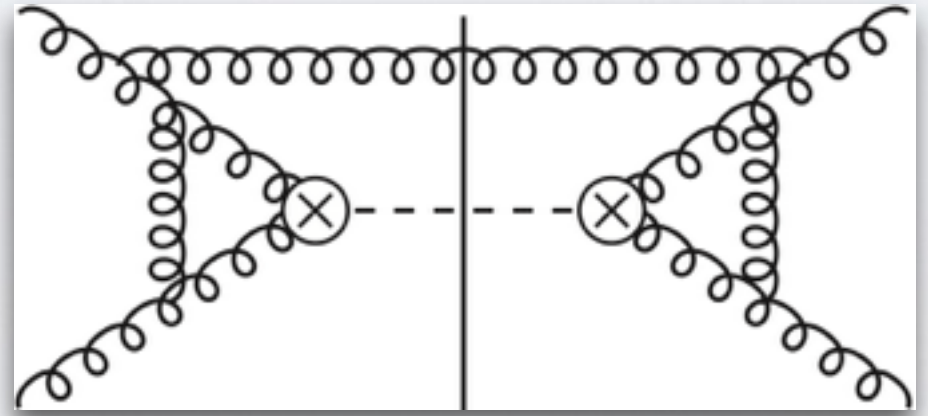
The gluon fusion cross section

- All other contributions involve the real emission of additional particles into the final state

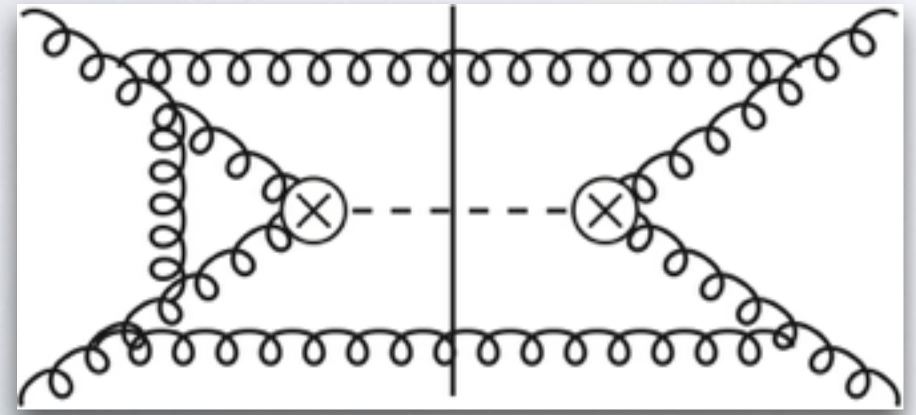
- Need to do phase space integrals



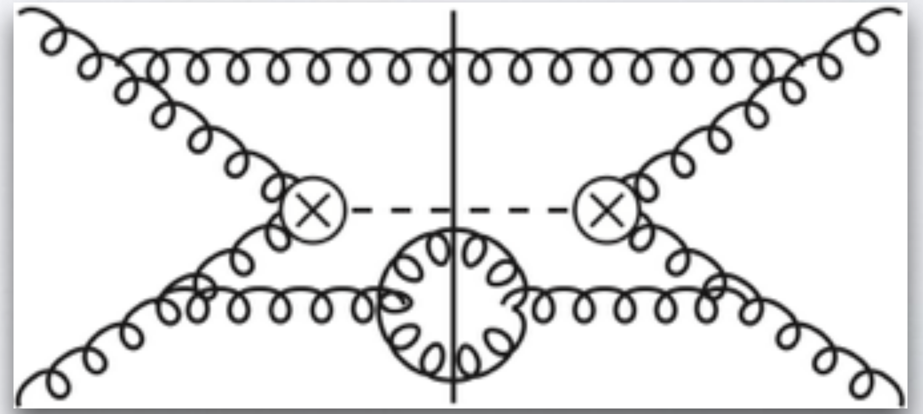
double virtual real



real virtual squared



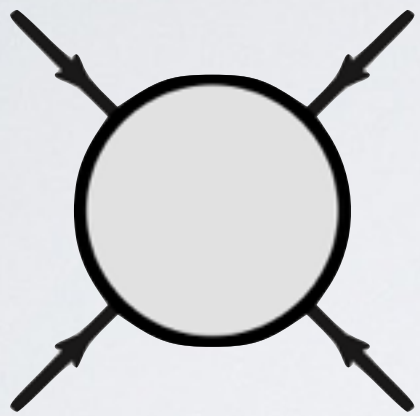
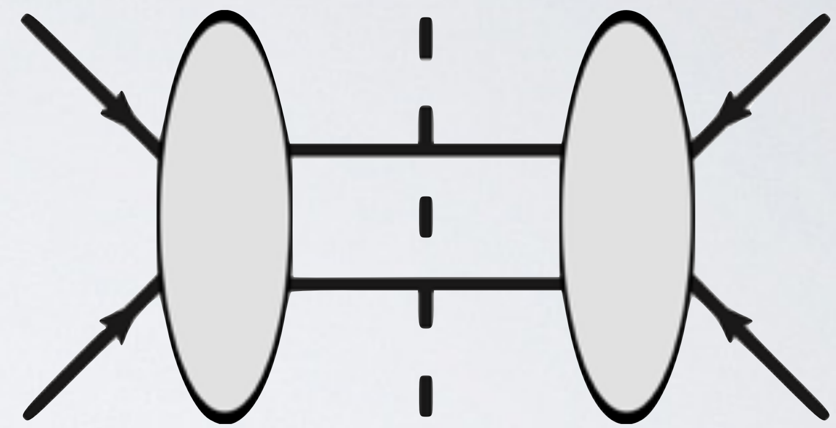
double real virtual



triple real

Unitarity

- Optical theorem:

Im  $= \int d\Phi$ 

  - Discontinuities of loop integrals are phase space integrals

- Discontinuities of loop integrals are given by Cutkosky's rule:

$$\frac{1}{p^2 - m^2 + i\epsilon} \to \delta^+(p^2 - m^2) = \delta(p^2 - m^2)\theta(p^0)$$

Reverse unitarity

- Optical theorem:

- The optical theorem can be read 'backwards'

- This way, phase space integrals can be expressed as unitarity cuts of loop integrals

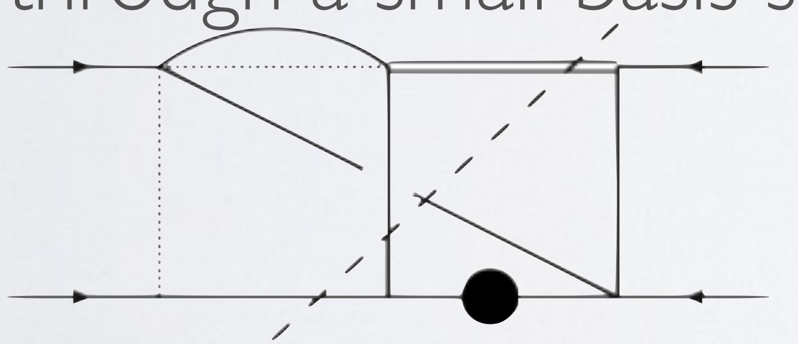
  [Anastasiou, Melnikov; Anastasiou, Dixon, Melnikov, Petriello]

- We can compute loop integrals with cuts instead of phase space integrals

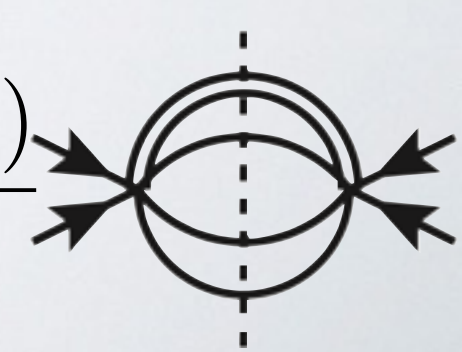
- This makes the rich technology developed for loop integrals available

IBPs and master integrals

- Loop integrals are in general not independent but related by Integration-by-parts identities (IBPs)
- The IBPs form a system of equations for a given class of loop integrals
- The system can be solved algorithmically expressing all integrals through a small basis set of integrals (master integrals)

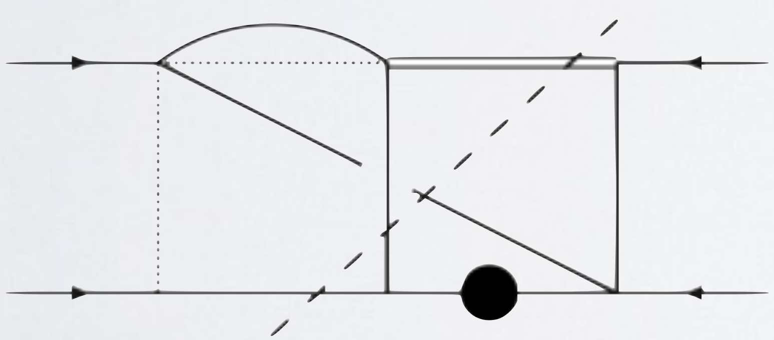


$$= -\frac{(\epsilon-1)(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(6\epsilon-5)(6\epsilon-1)}{\epsilon^4(\epsilon+1)(2\epsilon-3)}$$

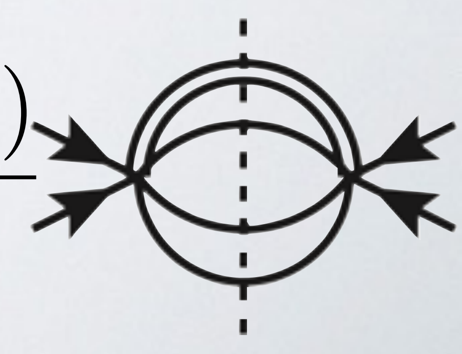


IBPs and master integrals

- IBP reductions greatly reduce complexity
- Double-virtual real contributions:
  - 68273802 integrals before reduction
  - 72 integrals after reduction



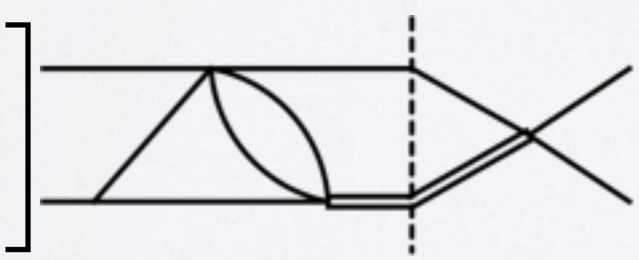
$$= -\frac{(\epsilon-1)(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)(6\epsilon-5)(6\epsilon-1)}{\epsilon^4(\epsilon+1)(2\epsilon-3)}$$



IBPs and differential equations

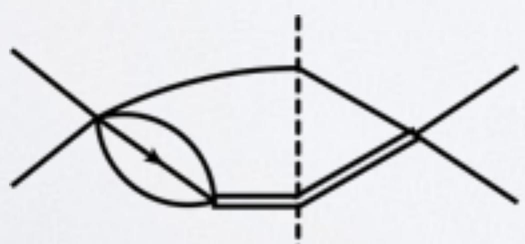
- Having access to IBP technology allows us to derive differential equations for master integrals
- The derivative of a master integral w.r.t. kinematic invariants can be expressed as a linear combination of master integrals
- Leads to a coupled system of linear differential equations for the master integrals

$$\left[\partial_{\bar{z}} - 3\epsilon \frac{1}{1-\bar{z}}\right]$$

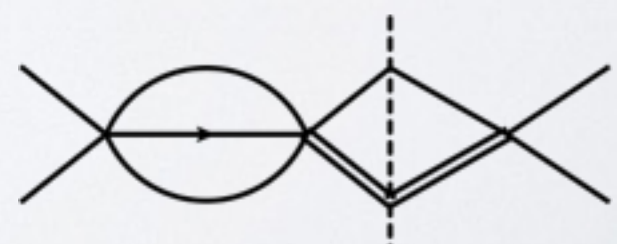


$$\bar{z} = 1 - z = \frac{s - m_h^2}{s}$$

$$= \epsilon \frac{1}{1-\bar{z}}$$



$$-3\epsilon \frac{1}{1-\bar{z}}$$



Differential equations and boundaries

- Integrating the differential equations for the master integrals yields general solutions
- These general solutions need to be fixed using boundary conditions
- Natural boundary condition for the problem at

$$\bar{z} = 0 \Longleftrightarrow \hat{s} = m_h^2$$

- This corresponds to the soft limit of the process

The threshold expansion

- It is possible to systematically expand the cross section at threshold
- This yields
  - the soft-virtual approximation for the cross-section
  - boundary conditions for the differential equation
- Around threshold the cross section can be approximated by a power series

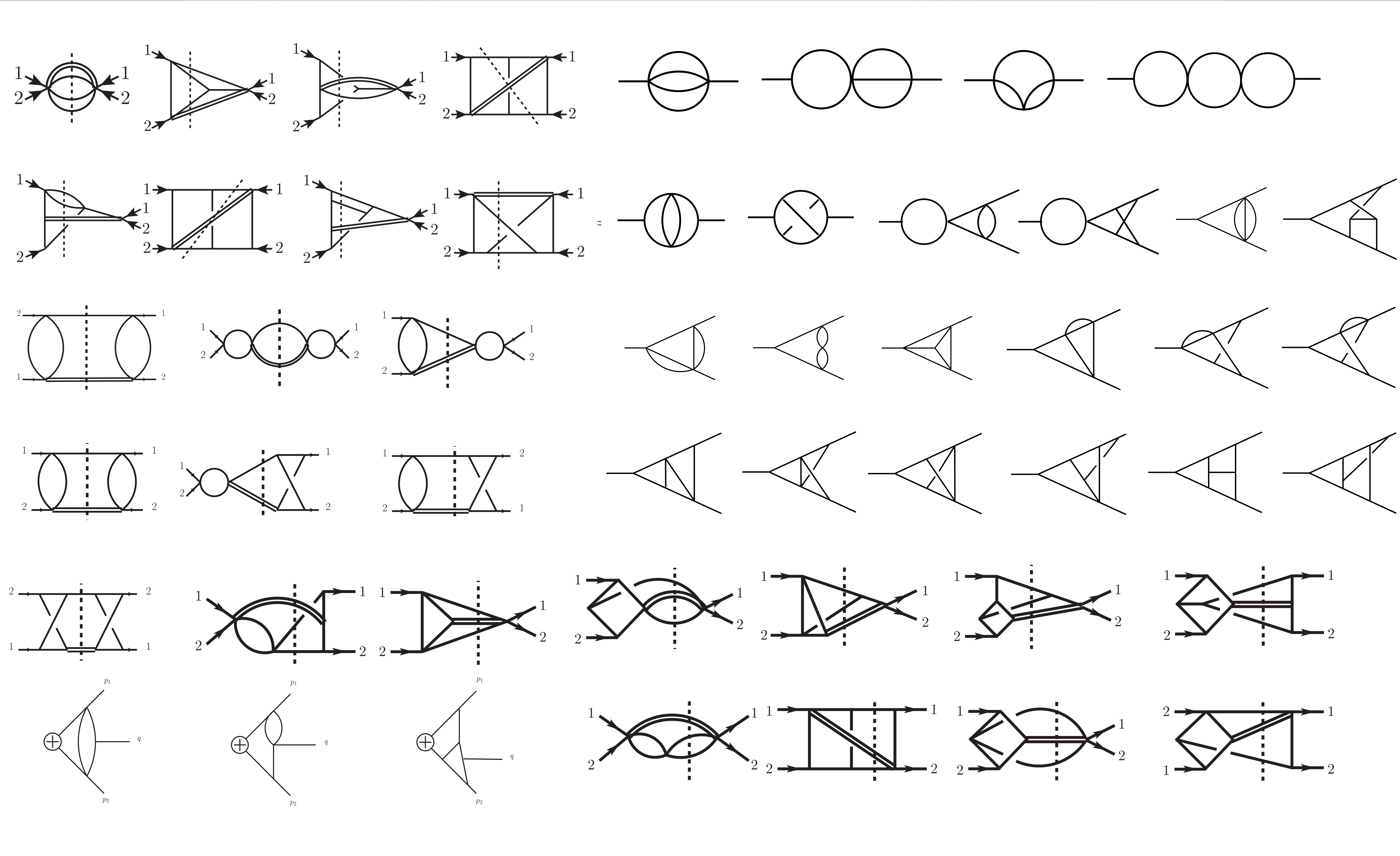
$$\hat{\sigma} = \hat{\sigma}_{-1} + \hat{\sigma}_0 + \bar{z}\hat{\sigma}_1 + \mathcal{O}(\bar{z})^2$$

The soft-virtual approximation

- All required integrals can be computed analytically
  - 22 three-loop integrals [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]
  - 3 double-virtual real integrals [Duhr, Gehrmann; Li, Zhu]
  - 7 real-virtual squared integrals [Anastasiou, Duhr, FD, Herzog, Mistlberger; Kilgore]
  - 10 double-real virtual integrals [Anastasiou, Duhr, FD, Herzog, Mistlberger; Li, von Manteufel, Schabinger, Zhu]
  - 8 triple real integrals [Anastasiou, Duhr, FD, Mistlberger]

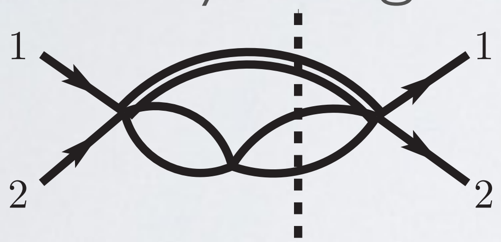
- Additionally
  - three-loop splitting functions [Moch, Vogt, Vermaseren]
  - three-loop beta functions [Tarasov, Vladimirov, Zharkov; Larin, Vermaseren]
  - three-loop Wilson coefficient [Chetyrkin, Kniehl, Steinhauser; Schroder, Steinhauser; Chetyrkin, Kuhn, Sturm]

The master integrals



The integrals

- We want to compute all the integrals analytically
- Every integral is individually divergent and gives rise to up to six poles in dimensional regularisation
- Many integrals are trivial to compute:



$$
= \frac{\Gamma(4-4\epsilon)\Gamma(2-3\epsilon)}{\epsilon(1-2\epsilon)^2\Gamma(4-6\epsilon)\Gamma(1-\epsilon)}
$$

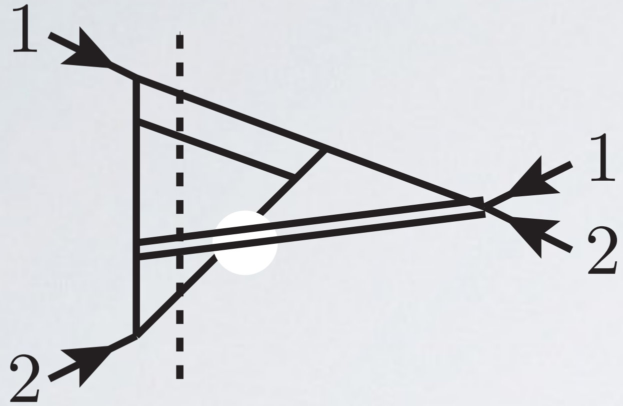
$$
= \frac{1}{\epsilon} + \frac{14}{3} + (24 - 6\zeta_2)\epsilon + \left(-28\zeta_2 - 42\zeta_3 + \frac{400}{3}\right)\epsilon^2 + \left(-144\zeta_2 - 196\zeta_3 \right.
$$

$$
\left. -195\zeta_4 + \frac{2320}{3}\right)\epsilon^3 + \left(252\zeta_3\zeta_2 - 800\zeta_2 - 1008\zeta_3 - 910\zeta_4 - 1302\zeta_5 + 4576\right)\epsilon^4
$$

$$
+ \left(882\zeta_3^2 + 1176\zeta_2\zeta_3 - 5600\zeta_3 - 4640\zeta_2 - 4680\zeta_4 - 6076\zeta_5 - \frac{9219}{2}\zeta_6 + 81920\right)\epsilon^5 + \mathcal{O}(\epsilon)^6
$$

The integrals

- Other integrals not so much



$$= \frac{\Gamma(12-6\epsilon)\Gamma(3-3\epsilon)\Gamma(1-\epsilon)}{\Gamma(5-6\epsilon)\Gamma(2-\epsilon)^4}\Big[\mathcal{I}_{9,1}(\epsilon)+\mathcal{I}_{9,2}(\epsilon)\Big]$$

$$\begin{aligned}
\mathcal{I}_{9,1}(\epsilon) = -\int_0^\infty dt_1\, dt_2 \int_0^1 dx_1\, dx_2\, dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\, t_2^{1-2\epsilon} \\
\times\, x_1^{-\epsilon}\,(1-x_1)^{2-4\epsilon}\, x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\, x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon} \\
\times\, \left(t_1t_2^2x_1x_2x_3 + t_2^2x_2x_3 + t_1t_2x_1x_2 + t_1t_2x_3 + t_2x_2x_3 + t_2 + t_1 + 1\right)^{3\epsilon-3}\,,
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_{9,2}(\epsilon) = \int_0^\infty dt_1\, dt_2 \int_0^1 dx_1\, dx_2\, dx_3\, t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\, t_2^{1-2\epsilon} \\
\times\, x_1^{1-\epsilon}\,(1-x_1)^{2-4\epsilon}\, x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\, x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon} \\
\times\, \left(t_1t_2^2x_1x_2x_3 + t_2^2x_1x_2x_3 + t_2x_1 + t_1t_2x_1x_2 + t_1t_2x_3 + t_2x_1x_2x_3 + t_1 + x_1\right)^{3\epsilon-3}\,,
\end{aligned}$$

Two computational problems

- Two computational problems that need to be solved
    - Phase space integrals for the boundary conditions need to be computed analytically
    - Differential equations need to be solved in terms of some useful functions
- What connects these two problems?
- How do we solve them?

- Why do we compute? ✔
- What do we want to compute? ✔
- How do we compute?
- What do we find?

Multiple polylogarithms

- Large classes of loop integrals can be expressed in terms of multiple polylogarithms

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \;\Big|\; \mathrm{Li}_n(z) = \int_0^z \frac{dt}{t} \mathrm{Li}_{n-1}(t)$$

- The classical polylogarithms, HPLs, 2dHPLs, cyclotomic polylogarithms, etc are special cases of the multiple polylogarithms

- The classical polylogarithms satisfy various complicated functional identities

$$-\mathrm{Li}_2(z) - \log(z)\log(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

- For the multiple polylogarithms these identities are in general not known

Multiple polylogarithms

- Not knowing the functional identities is a problem
- Even if the physics of a result is very simple, the analytical expression might be very complicated
  - The simplicity of the answer might be hidden behind the various functional equations
  - Famous example:
    - The two-loop hexagon remainder function in N=4 SYM as computed by Del Duca, Duhr and Smirnov is a 17 page expression
    - After Goncharov, Spradlin, Vergu and Volovich simplified it using functional identities it can be written in 4 lines

Multiple polylogarithms

- Not knowing the functional identities is a problem
- Too complicated results are not just a formal or aesthetic problem
- Without using functional identities there might be huge cancellation between divergent sub-pieces of the result even though the complete result is finite
- Too complicated results are not useable for phenomenology because numerical implementations are not feasible
- Need functional identities to express result in a simple basis

Multiple polylogarithms

- Not knowing the functional identities is a problem
- The integrand might not be in the right form to perform the integration
- Result can only be obtained if functional identities between polylogarithms are known

Number theory

- Multiple polylogarithms are a very active field of research in pure mathematics
- Mathematicians have discovered algebraic structures that underly the polylogarithms
- When we usually think of functional identities we think of complicated functional equations that are obtained by performing intricate variable transformations of the integral representations

$$-\mathrm{Li}_2(z) - \log(z)\log(1-z) = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$$

Number theory

- Mathematicians have conjectured that all functional equations between polylogarithms follow from a simple algebraic structure
- All functional equations between polylogarithms can be obtained from pure combinatorics
- The algebraic structure that governs the polylogarithms is called a **Hopf algebra**

Hopf algebras

- What is a **Hopf algebra**?
- It is an **algebra**: A vector space with an operation that allows us to combine two elements into one (multiplication)
- It is also a **coalgebra**: A vector space with an operation that allows us to break an element into two elements (comultiplication)
- Disclaimer: The following explanation is very handwaving and omits many mathematical details

Hopf algebras

- An example of the algebra part of a Hopf algebra is the **shuffle algebra** of the multiple polylogarithms

- Shuffle product: Takes two sets and intersperses them in all possible ways while keeping the ordering of the elements of each set among themselves

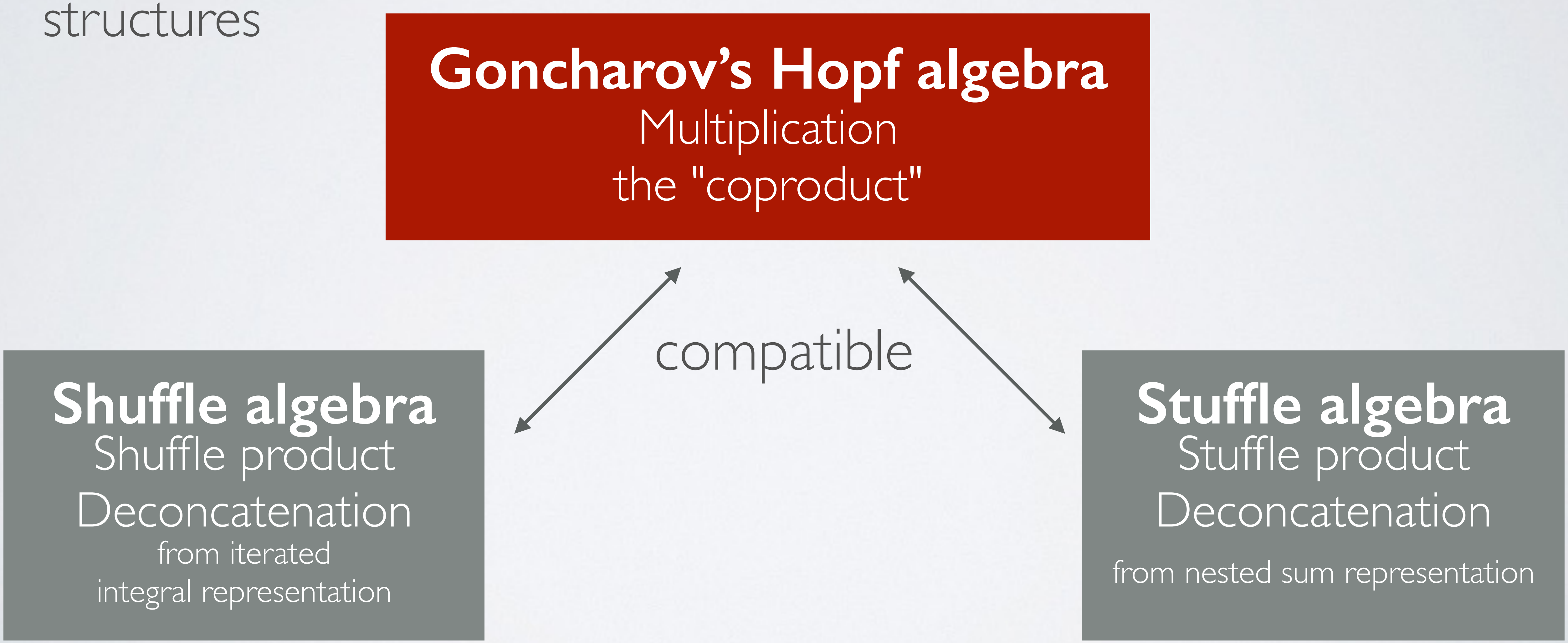
$$ab \sqcup\!\sqcup\, cd = abcd + acbd + acdb + cabd + cadb + adab$$

- Analogy: Riffle shuffling two stacks of cards.

$$\log(x)\log(1-x)$$
$$= -G(0,1,x) - G(1,0,x)$$

Hopf algebras

- The shuffle algebra is not the only Hopf algebra that is carried by the multiple polylogarithms
- In fact the multiple polylogarithms carry **three** Hopf algebra structures



Hopf algebras

- The comultiplication of the Hopf algebra for polylogarithms is the **coproduct**
- It splits a word in all possible ways

$$\Delta(abcd) = abcd \otimes 1 + abc \otimes d + ab \otimes cd + a \otimes bcd + 1 \otimes abcd$$

- We can iterate this splitting until we have broken the word into products of single letters

[Goncharov; Duhr]

Functional equations

- The coproduct can be applied to polylogarithms [Goncharov; Duhr]
- The word is here the list of indices $\{a_n\}$ of a polylogarithm

$$G(a_1, \ldots, a_n; z)$$

- Examples:

$$\Delta(\log x) = 1 \otimes \log x + \log x \otimes 1$$

$$\Delta(\text{Li}_n(x)) = 1 \otimes \text{Li}_n(x) + \sum_{k=0}^{n-1} \text{Li}_{n-k}(x) \otimes \frac{\log^k(x)}{k!}$$

$$\Delta(\text{Li}_2(x)) = 1 \otimes \text{Li}_2(x) - \log(1-x) \otimes \log(x) + 1 \otimes \text{Li}_2(x)$$

Functional equations

- The coproduct can be used to derive functional equations for polylogarithms
- The coproduct is applied to the polylogarithm to split it into simpler pieces
- The functional identities for these simpler pieces might be known
- If not, the coproduct is repeatedly applied until only ordinary logarithms are left

Example

- Assume you want to calculate: $\int_0^1 dx \frac{\mathrm{Li}_2\left(\frac{ax}{(1-x)}\right)}{x(1-x)}$

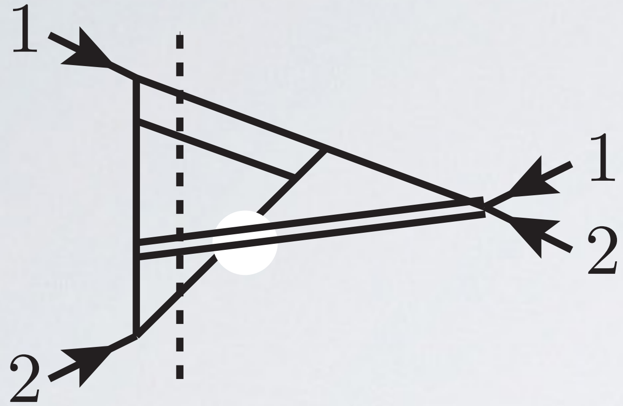
- Using the coproduct it is possible to derive the following functional identity:

$$\mathrm{Li}_2\left(\frac{ax}{(1-x)}\right) = G(0,1;x) - G\left(0,\frac{1}{1+a};x\right) - G(1,1;x) + G\left(1,\frac{1}{1+a};x\right)$$

- Now all the integrations are trivial:

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1} G(a_2,\ldots,a_n;t)$$

The integrals



$$= \frac{\Gamma(12-6\epsilon)\Gamma(3-3\epsilon)\Gamma(1-\epsilon)}{\Gamma(5-6\epsilon)\Gamma(2-\epsilon)^4}\Big[\mathcal{I}_{9,1}(\epsilon)+\mathcal{I}_{9,2}(\epsilon)\Big]$$

$$\begin{aligned}\mathcal{I}_{9,1}(\epsilon) = -\int_0^\infty dt_1\,dt_2\int_0^1 dx_1\,dx_2\,dx_3\,t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\,t_2^{1-2\epsilon}\\ \times x_1^{-\epsilon}\,(1-x_1)^{2-4\epsilon}\,x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\,x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon}\\ \times\left(t_1t_2^2x_1x_2x_3+t_2^2x_2x_3+t_1t_2x_1x_2+t_1t_2x_3+t_2x_2x_3+t_2+t_1+1\right)^{3\epsilon-3},\end{aligned}$$

$$\begin{aligned}\mathcal{I}_{9,2}(\epsilon) = \int_0^\infty dt_1\,dt_2\int_0^1 dx_1\,dx_2\,dx_3\,t_1^{2-4\epsilon}\,(1+t_1)^{\epsilon-1}\,t_2^{1-2\epsilon}\\ \times x_1^{1-\epsilon}\,(1-x_1)^{2-4\epsilon}\,x_2^{1-3\epsilon}\,(1-x_2)^{-\epsilon}\,x_3^{-\epsilon}\,(1+t_2x_3)^{1-3\epsilon}\,(1+t_2x_2x_3)^{\epsilon}\\ \times\left(t_1t_2^2x_1x_2x_3+t_2^2x_1x_2x_3+t_2x_1+t_1t_2x_1x_2+t_1t_2x_3+t_2x_1x_2x_3+t_1+x_1\right)^{3\epsilon-3},\end{aligned}$$

Number theory

- Number theory helps us here
- The integral can be done one step at a time
- We use the coproduct to derive the needed functional identities at each step
- Integrate over one variable at a time using the basic definition of the multiple polylogarithms

$$G\left(a_1, \ldots, a_n; z\right) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t)$$

- Number theory gives us a way to solve the integrals algorithmically

[Brown]

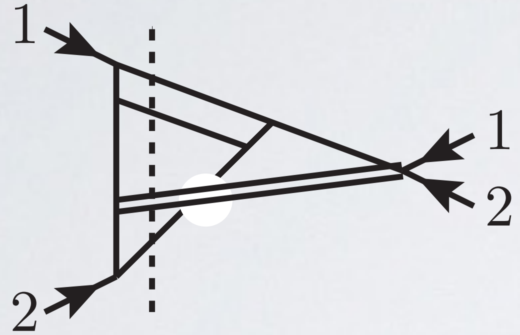
Number theory

- The previous integral can be computed one step at a time
- In the process one finds functional identities like:
- Such identities can not be found in the literature
- No one wants to derive them using integral transformations
- Number theory and the coproduct give you a simple way to obtain them on the fly

$$
\begin{aligned}
&G\left(\frac{t u x+u x-u+1}{x(t u+1)},\frac{t u x+u x-u+1}{x(t u+1)},-\frac{1}{t x},1,1\right)=\\
&-G\left(0,1,1,-\frac{1}{t},x\right)-G\left(0,1,-\frac{1}{t u},1,x\right)+G\left(0,1,-\frac{1}{t u},-\frac{1}{t},x\right)+G\left(0,1,-\frac{1}{t u},-\frac{1-u}{t u+u},x\right)-G\left(0,-\frac{1}{t u},1,1,x\right)+\\
&G\left(0,-\frac{1}{t u},1,-\frac{1}{t},x\right)+G\left(0,-\frac{1}{t u},1,-\frac{1-u}{t u+u},x\right)+G\left(0,-\frac{1}{t u},-\frac{1}{t u},1,x\right)-G\left(0,-\frac{1}{t u},-\frac{1}{t u},-\frac{1}{t},x\right)-\\
&G\left(0,-\frac{1}{t u},-\frac{1}{t u},-\frac{1-u}{t u+u},x\right)+G\left(0,-\frac{1}{t u},-\frac{1-u}{t u+u},1,x\right)-G\left(0,-\frac{1}{t u},-\frac{1-u}{t u+u},-\frac{1-u}{t u+u},x\right)-G\left(1,0,1,-\frac{1}{t},x\right)-\\
&G\left(1,0,-\frac{1}{t u},1,x\right)+G\left(1,0,-\frac{1}{t u},-\frac{1}{t},x\right)+G\left(1,0,-\frac{1}{t u},-\frac{1-u}{t u+u},x\right)-G\left(1,1,0,-\frac{1}{t},x\right)+G\left(1,1,-\frac{1}{t},-\frac{1}{t},x\right)+\\
&G\left(1,-\frac{1}{t},0,1,x\right)-G\left(1,-\frac{1}{t},0,-\frac{1-u}{t u+u},x\right)-G\left(1,-\frac{1}{t},1,1,x\right)+G\left(1,-\frac{1}{t},1,-\frac{1}{t},x\right)+G\left(1,-\frac{1}{t},1,-\frac{1-u}{t u+u},x\right)+\\
&G\left(1,-\frac{1}{t},-\frac{1}{t u},1,x\right)-G\left(1,-\frac{1}{t},-\frac{1}{t u},-\frac{1}{t},x\right)-G\left(1,-\frac{1}{t},-\frac{1}{t u},-\frac{1-u}{t u+u},x\right)-G\left(1,-\frac{1}{t u},0,1,x\right)+G\left(1,-\frac{1}{t u},0,-\frac{1}{t},x\right)+\\
&G\left(1,-\frac{1}{t u},0,-\frac{1-u}{t u+u},x\right)+G\left(1,-\frac{1}{t u},1,1,x\right)-G\left(1,-\frac{1}{t u},1,-\frac{1-u}{t u+u},x\right)-G\left(1,-\frac{1}{t u},-\frac{1}{t},-\frac{1}{t},x\right)+G\left(-\frac{1}{t},0,1,1,x\right)-\\
&G\left(-\frac{1}{t},0,1,-\frac{1-u}{t u+u},x\right)-G\left(-\frac{1}{t},0,-\frac{1-u}{t u+u},1,x\right)+G\left(-\frac{1}{t},0,-\frac{1-u}{t u+u},-\frac{1-u}{t u+u},x\right)-G\left(-\frac{1}{t},1,1,1,x\right)+G\left(-\frac{1}{t},1,1,-\frac{1}{t},x\right)+\\
&G\left(-\frac{1}{t},1,1,-\frac{1-u}{t u+u},x\right)+G\left(-\frac{1}{t},1,-\frac{1}{t u},1,x\right)-G\left(-\frac{1}{t},1,-\frac{1}{t u},-\frac{1}{t},x\right)-G\left(-\frac{1}{t},1,-\frac{1}{t u},-\frac{1-u}{t u+u},x\right)+G\left(-\frac{1}{t},1,-\frac{1-u}{t u+u},1,x\right)-\\
&G\left(-\frac{1}{t},1,-\frac{1-u}{t u+u},-\frac{1-u}{t u+u},x\right)+G\left(-\frac{1}{t},-\frac{1}{t u},1,1,x\right)-G\left(-\frac{1}{t},-\frac{1}{t u},1,-\frac{1}{t},x\right)-G\left(-\frac{1}{t},-\frac{1}{t u},1,-\frac{1-u}{t u+u},x\right)-\\
&G\left(-\frac{1}{t},-\frac{1}{t u},-\frac{1}{t u},1,x\right)+G\left(-\frac{1}{t},-\frac{1}{t u},-\frac{1}{t u},-\frac{1}{t},x\right)+G\left(-\frac{1}{t},-\frac{1}{t u},-\frac{1}{t u},-\frac{1-u}{t u+u},x\right)-G\left(-\frac{1}{t},-\frac{1}{t u},-\frac{1-u}{t u+u},1,x\right)+\\
&G\left(-\frac{1}{t},-\frac{1}{t u},-\frac{1-u}{t u+u},-\frac{1-u}{t u+u},x\right)-G\left(-\frac{1}{t u},0,1,1,x\right)+G\left(-\frac{1}{t u},0,1,-\frac{1}{t},x\right)+G\left(-\frac{1}{t u},0,1,-\frac{1-u}{t u+u},x\right)+\\
&G\left(-\frac{1}{t u},0,-\frac{1}{t u},1,x\right)-G\left(-\frac{1}{t u},0,-\frac{1}{t u},-\frac{1}{t},x\right)-G\left(-\frac{1}{t u},0,-\frac{1}{t u},-\frac{1-u}{t u+u},x\right)+G\left(-\frac{1}{t u},0,-\frac{1-u}{t u+u},1,x\right)-\\
&G\left(-\frac{1}{t u},0,-\frac{1-u}{t u+u},-\frac{1-u}{t u+u},x\right)+G\left(-\frac{1}{t u},1,0,-\frac{1}{t},x\right)+G\left(-\frac{1}{t u},1,1,1,x\right)-G\left(-\frac{1}{t u},1,1,-\frac{1-u}{t u+u},x\right)-G\left(-\frac{1}{t u},1,-\frac{1}{t},-\frac{1}{t},x\right)-\\
&G\left(-\frac{1}{t u},1,-\frac{1-u}{t u+u},1,x\right)+G\left(-\frac{1}{t u},1,-\frac{1-u}{t u+u},-\frac{1-u}{t u+u},x\right)-G\left(-\frac{1}{t u},-\frac{1}{t},0,1,x\right)+G\left(-\frac{1}{t u},-\frac{1}{t},0,-\frac{1-u}{t u+u},x\right)+\\
&G\left(-\frac{1}{t u},-\frac{1}{t},1,1,x\right)-G\left(-\frac{1}{t u},-\frac{1}{t},1,-\frac{1}{t},x\right)-G\left(-\frac{1}{t u},-\frac{1}{t},1,-\frac{1-u}{t u+u},x\right)-G\left(-\frac{1}{t u},-\frac{1}{t},-\frac{1}{t u},1,x\right)+\\
&G\left(-\frac{1}{t u},-\frac{1}{t},-\frac{1}{t u},-\frac{1}{t},x\right)+G\left(-\frac{1}{t u},-\frac{1}{t},-\frac{1}{t u},-\frac{1-u}{t u+u},x\right)+G\left(-\frac{1}{t u},-\frac{1}{t u},0,1,x\right)-G\left(-\frac{1}{t u},-\frac{1}{t u},0,-\frac{1}{t},x\right)-\\
&G\left(-\frac{1}{t u},-\frac{1}{t u},0,-\frac{1-u}{t u+u},x\right)-G\left(-\frac{1}{t u},-\frac{1}{t u},1,1,x\right)+G\left(-\frac{1}{t u},-\frac{1}{t u},1,-\frac{1-u}{t u+u},x\right)+G\left(-\frac{1}{t u},-\frac{1}{t u},-\frac{1}{t},-\frac{1}{t},x\right)
\end{aligned}
$$

The integrals

- When the smoke clears, one finds:



$$
\begin{aligned}
&= \frac{160}{\epsilon^5} - \frac{1712}{\epsilon^4} + \frac{1}{\epsilon^3}\Big(- 120\,\zeta_2 + 2784\Big) + \frac{1}{\epsilon^2}\Big(- 120\,\zeta_3 + 1284\,\zeta_2 + 31968\Big) \\
&+ \frac{1}{\epsilon}\Big(2520\,\zeta_4 + 1284\,\zeta_3 - 2088\,\zeta_2 - 216864\Big) + 15720\,\zeta_5 + 1920\,\zeta_2\,\zeta_3 \\
&- 26964\,\zeta_4 - 2088\,\zeta_3 - 23976\,\zeta_2 + 795744 + \epsilon\Big(82520\,\zeta_6 + 9600\,\zeta_3^2 \\
&- 168204\,\zeta_5 - 20544\,\zeta_2\,\zeta_3 + 43848\,\zeta_4 - 23976\,\zeta_3 + 162648\,\zeta_2 - 2449440\Big) \\
&+ \mathcal{O}(\epsilon^2)\,.
\end{aligned}
$$

- Thanks to these modern techniques we were able to compute all boundary conditions analytically

- We obtain the soft-virtual approximation of the gluon fusion cross section at N3LO

Differential equations

- If we want to calculate the cross section for general kinematics we need more than just the first two terms in the expansion

- We want to solve the differential equations for the master integrals in a closed form

- General form of the system of differential equations

$$\partial_x f_i = M_{ij}(x, \epsilon) f_j$$

- Can describe very general functions

- In simple cases it is possible to go to a less general, canonical, form

$$\partial_x g_i = \epsilon \left(\frac{A^0_{ij}}{x} + \frac{A^1_{ij}}{1-x} \right) g_j$$

[Henn]

Differential equations

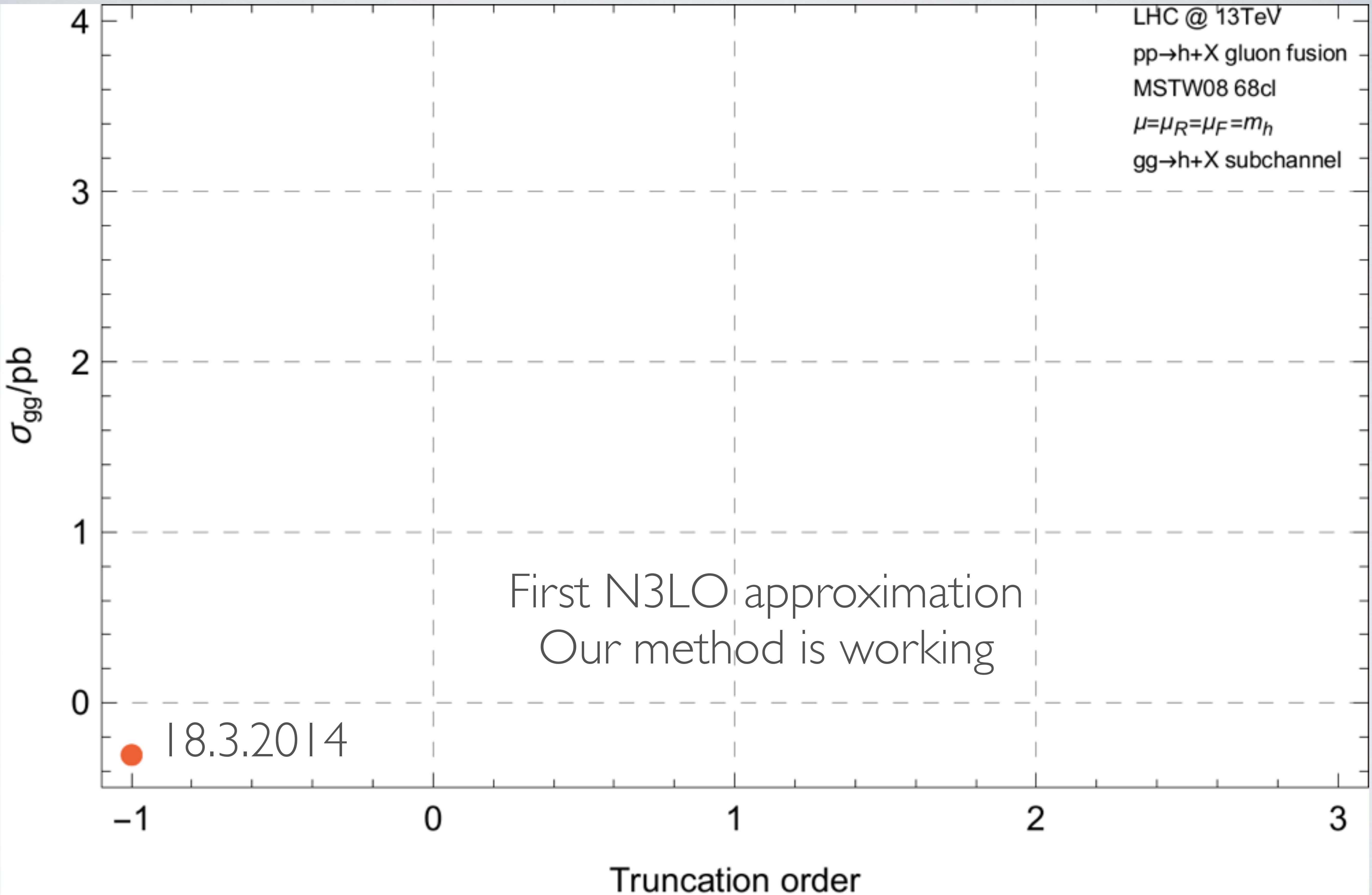
- In this form the solutions are of the simple form

$$g_i(x) = g_i^0 + \epsilon A_{ij} \int dx \left(\frac{g_j(x)}{x} + \frac{g_j(x)}{1-x} \right) + \dots$$

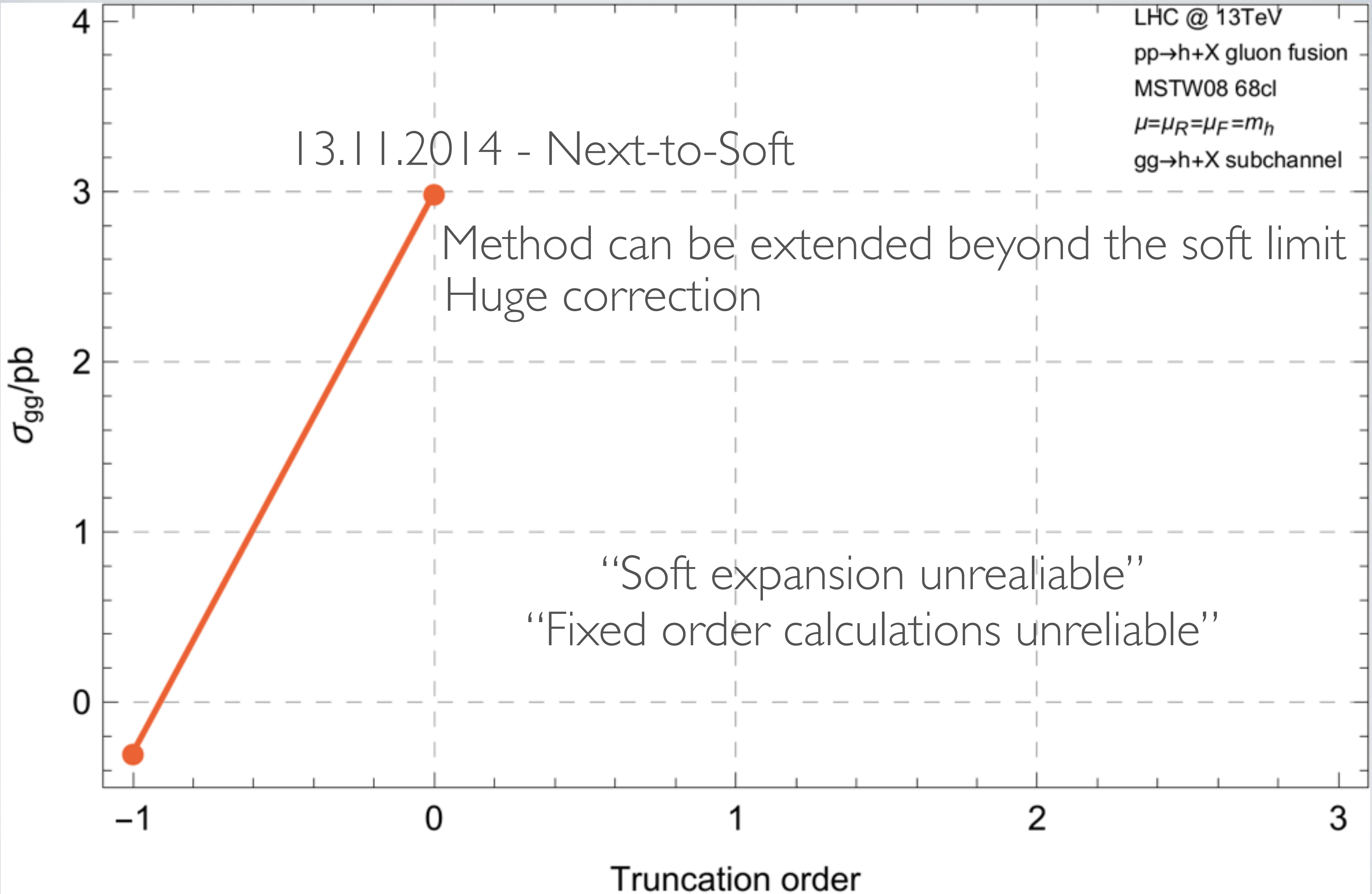
- Directly related to the integral representation of the multiple polylogarithms

$$G\left(a_1, \dots, a_n; z\right) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t)$$

- Why do we compute? ✔
- What do we want to compute? ✔
- How do we compute it? ✔
- What do we find?



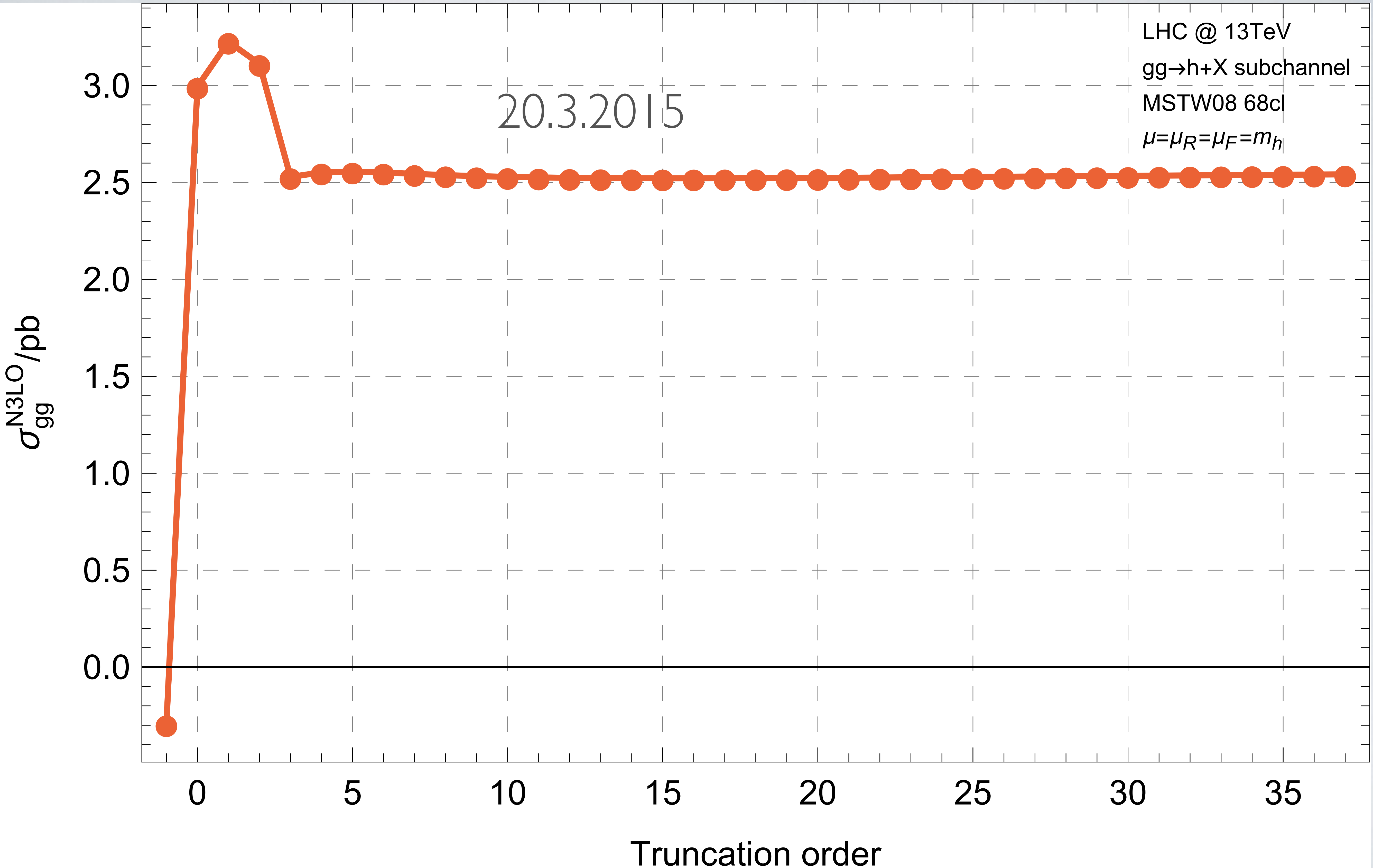
LHC @ 13TeV
pp→h+X gluon fusion
MSTW08 68cl
$\mu=\mu_R=\mu_F=m_h$
gg→h+X subchannel
σ_{gg}/pb
4
3
2
1
0
−1
0
1
2
3
Truncation order
First N3LO approximation
Our method is working
18.3.2014



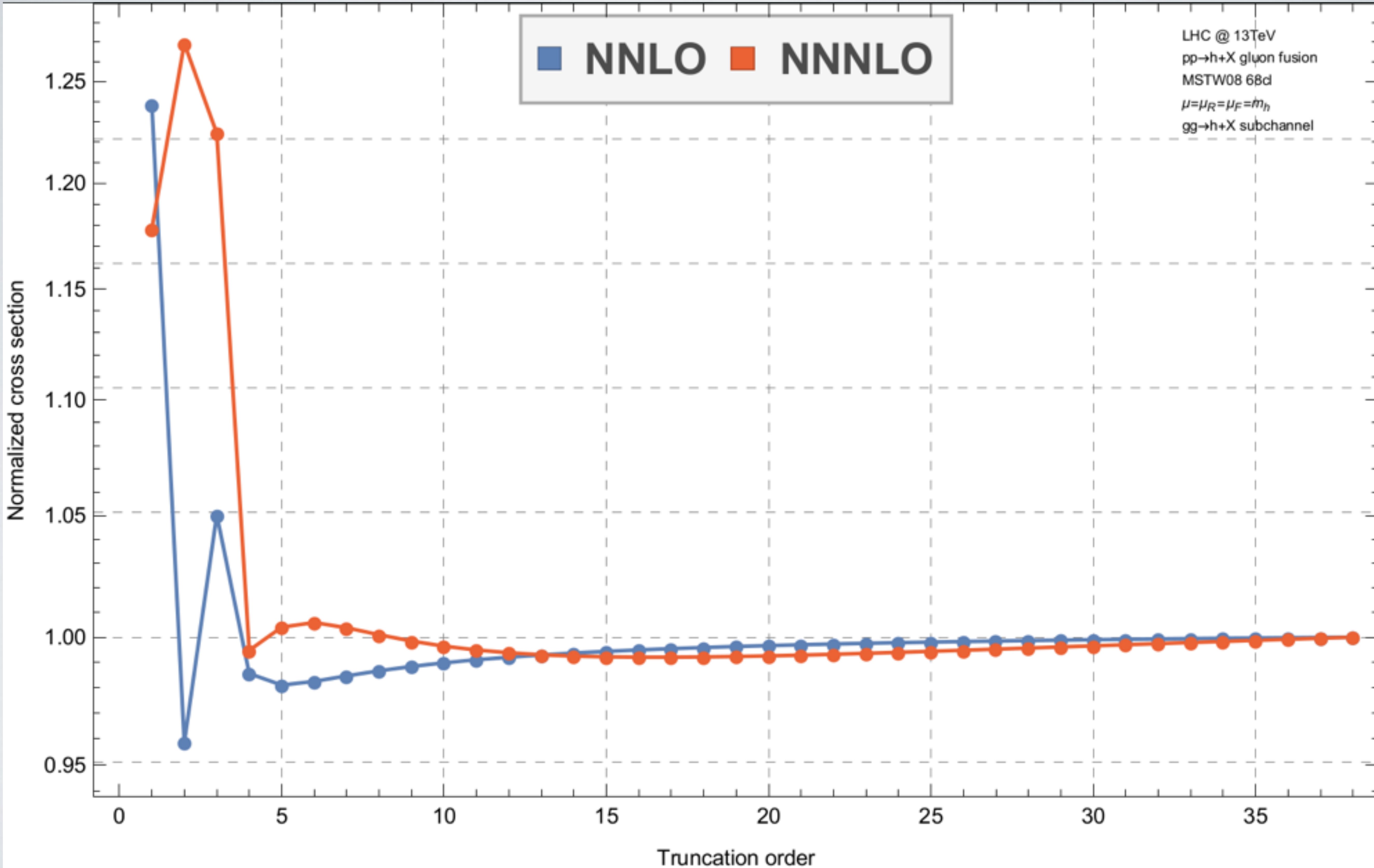
13.11.2014 - Next-to-Soft

Method can be extended beyond the soft limit
Huge correction

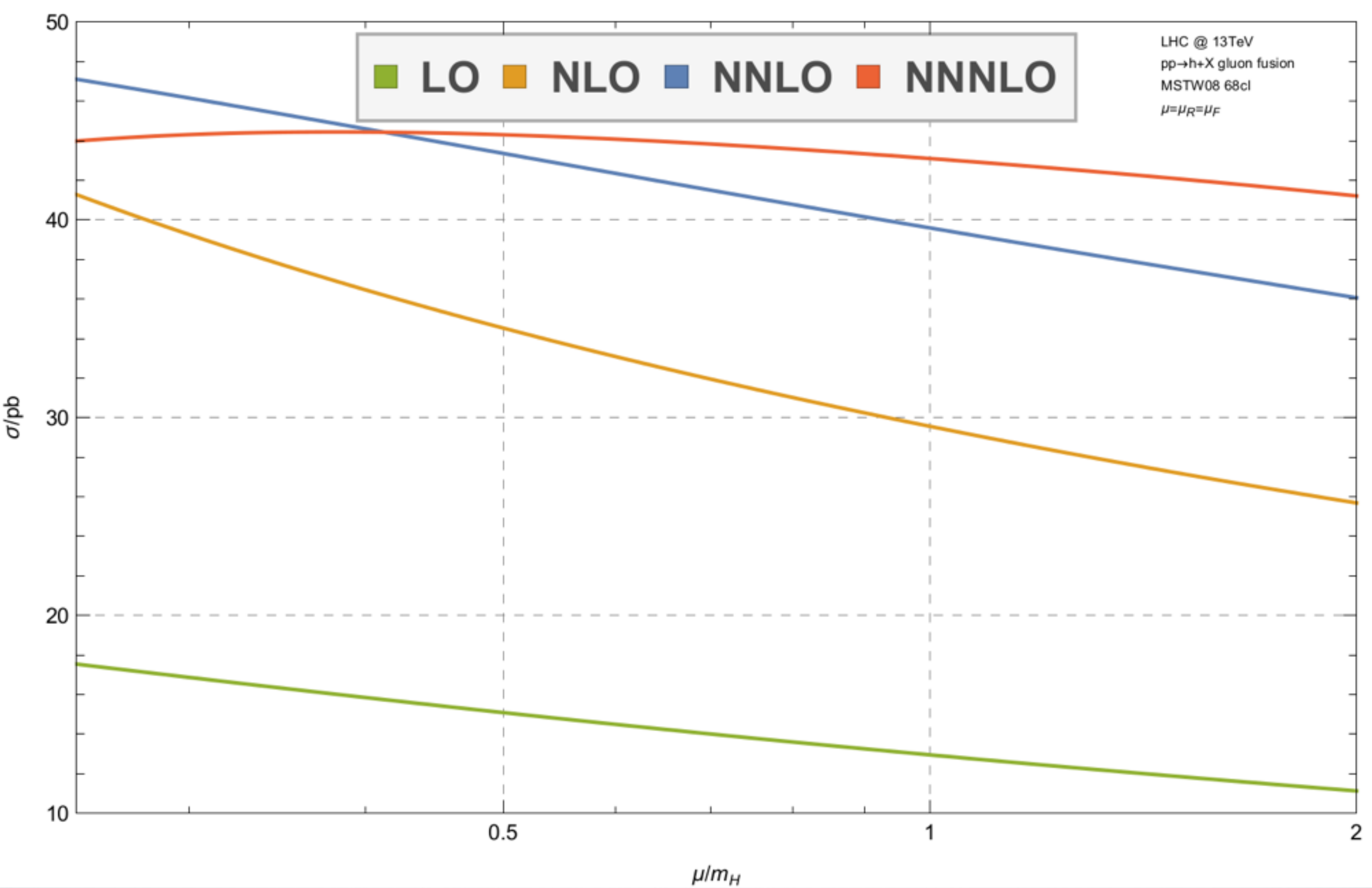
“Soft expansion unrealiable”
“Fixed order calculations unreliable”



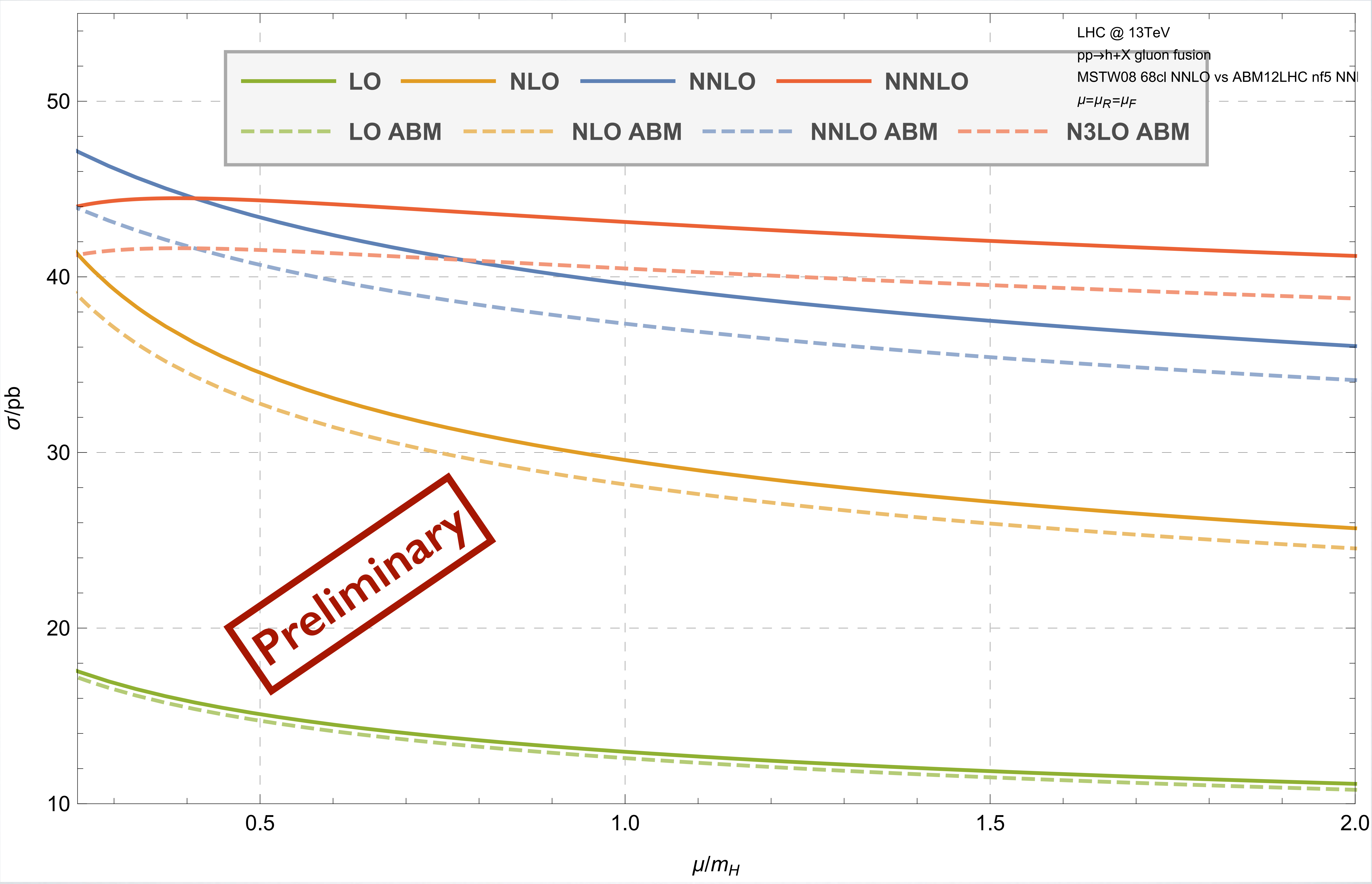
LHC @ 13TeV
gg→h+X subchannel
MSTW08 68cl
$\mu=\mu_R=\mu_F=m_h$
20.3.2015
σ_{gg}^{N3LO}/pb
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
25
30
35
Truncation order



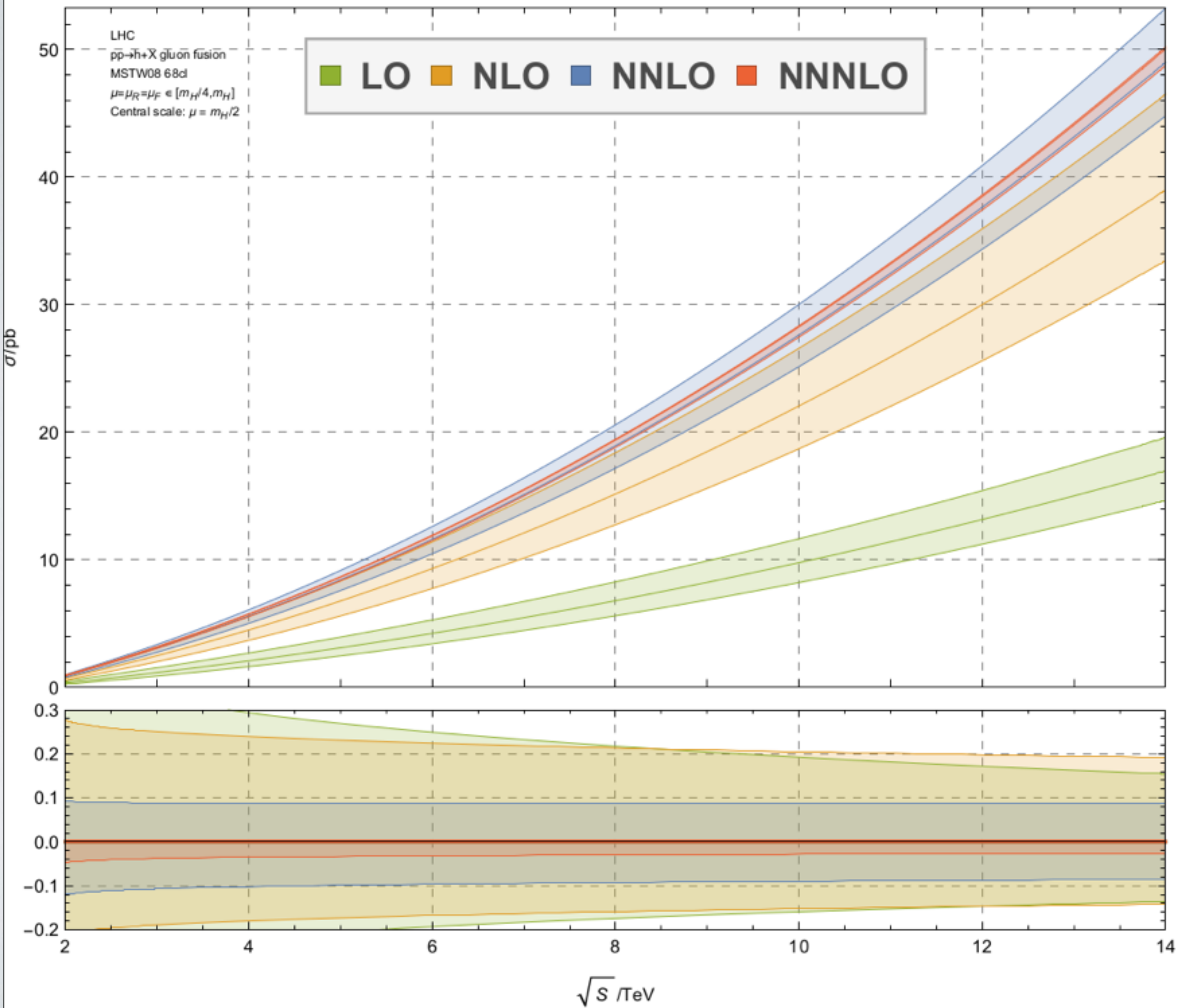
NNLO
NNNLO
LHC @ 13TeV
pp→h+X gluon fusion
MSTW08 68cl
$\mu=\mu_R=\mu_F=m_h$
gg→h+X subchannel
1.25
1.20
1.15
1.10
1.05
1.00
0.95
0
5
10
15
20
25
30
35
Normalized cross section
Truncation order



LO
NLO
NNLO
NNNLO
LHC @ 13TeV
pp→h+X gluon fusion
MSTW08 68cl
$\mu=\mu_R=\mu_F$
50
40
30
20
10
0.5
1
2
σ/pb
μ/m_H

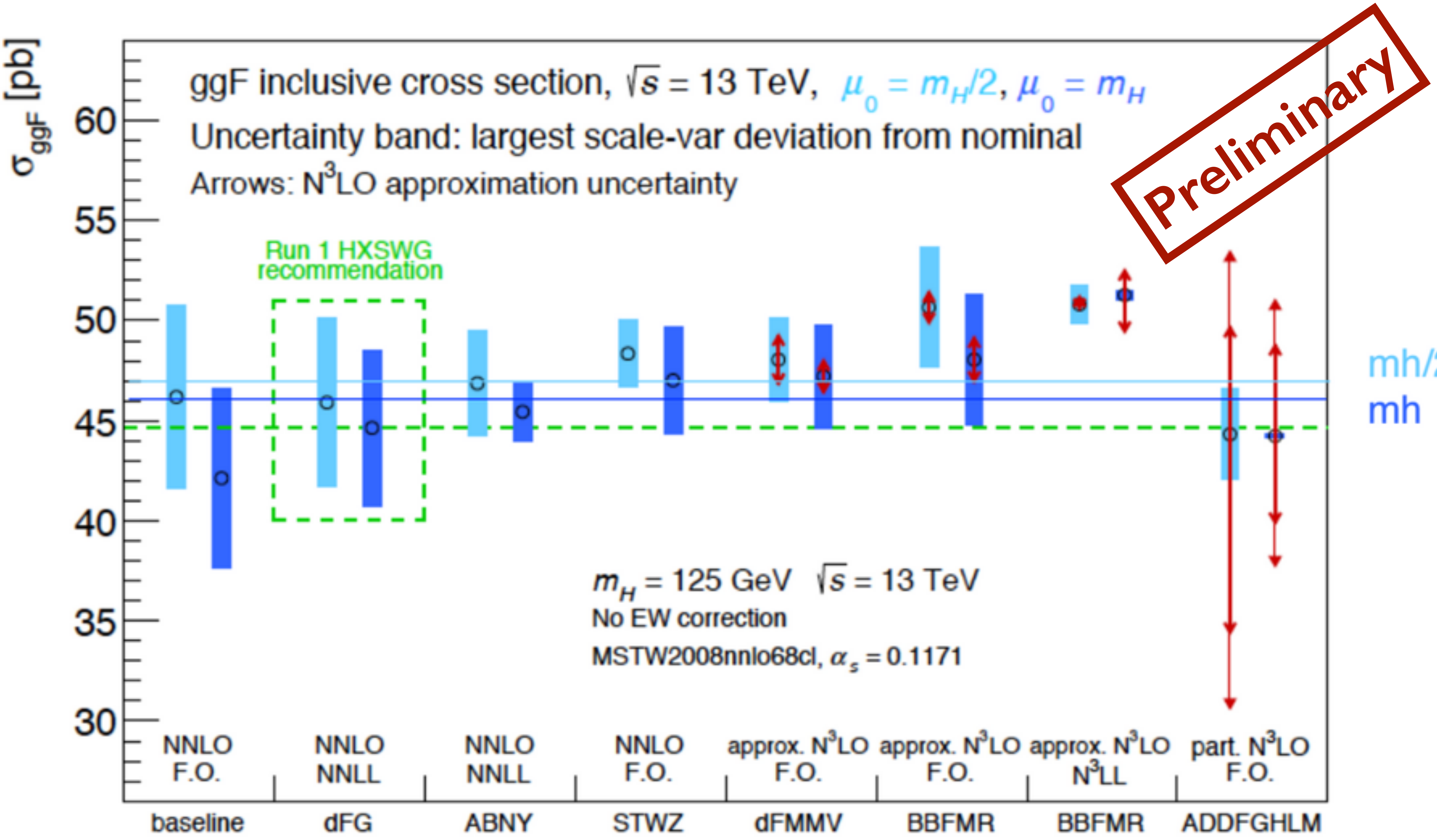


LHC @ 13TeV
pp→h+X gluon fusion
MSTW08 68cl NNLO vs ABM12LHC nf5 NNl
μ=μR=μF
LO
NLO
NNLO
NNNLO
LO ABM
NLO ABM
NNLO ABM
N3LO ABM
Preliminary
σ/pb
50
40
30
20
10
0.5
1.0
1.5
2.0
μ/mH



LHC
pp→h+X gluon fusion
MSTW08 68cl
$\mu=\mu_R=\mu_F \in [m_H/4, m_H]$
Central scale: $\mu = m_H/2$
LO
NLO
NNLO
NNNLO
σ/pb
$\sqrt{S}$ /TeV

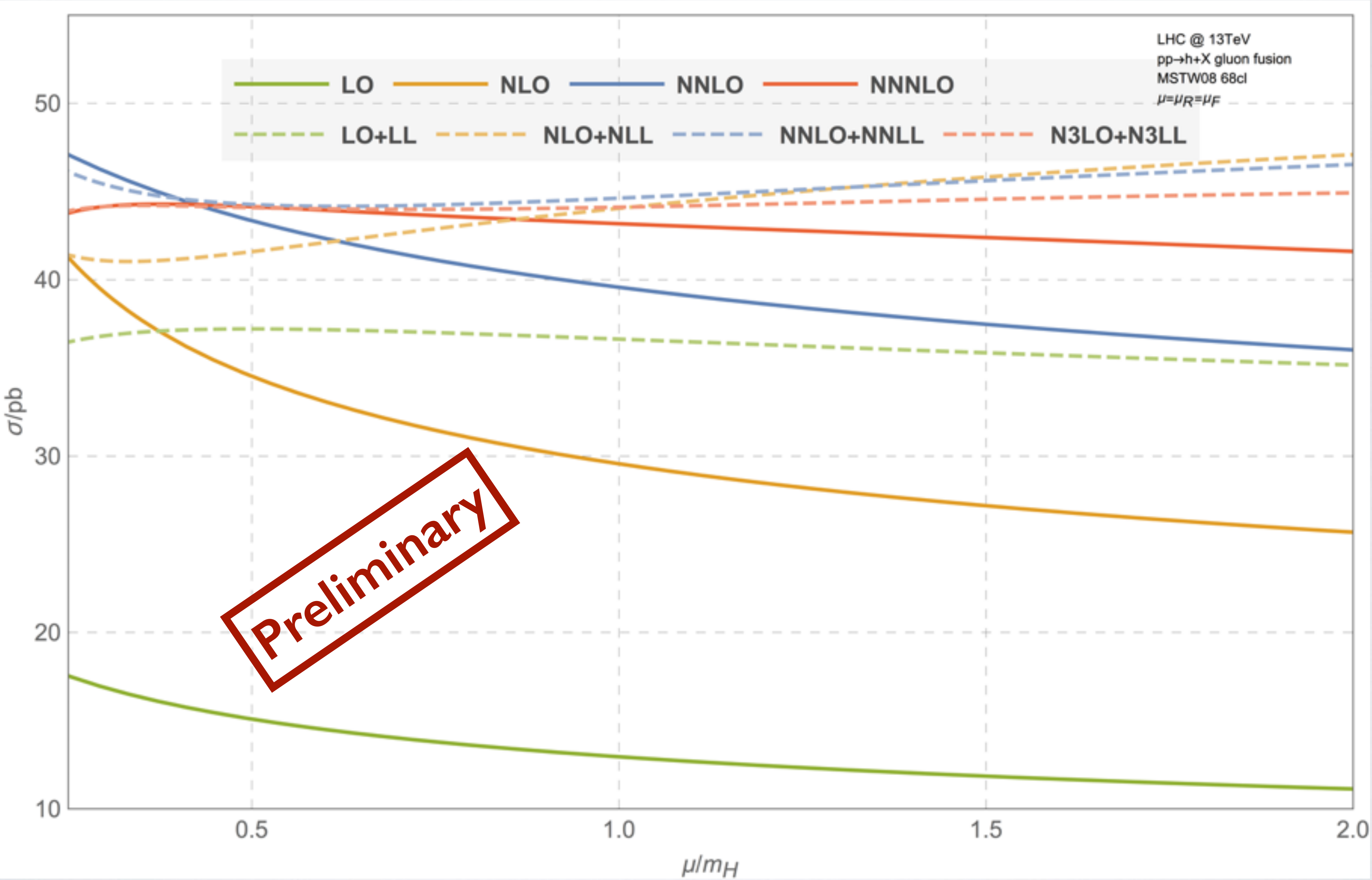
Comparison with ggF-Wg study

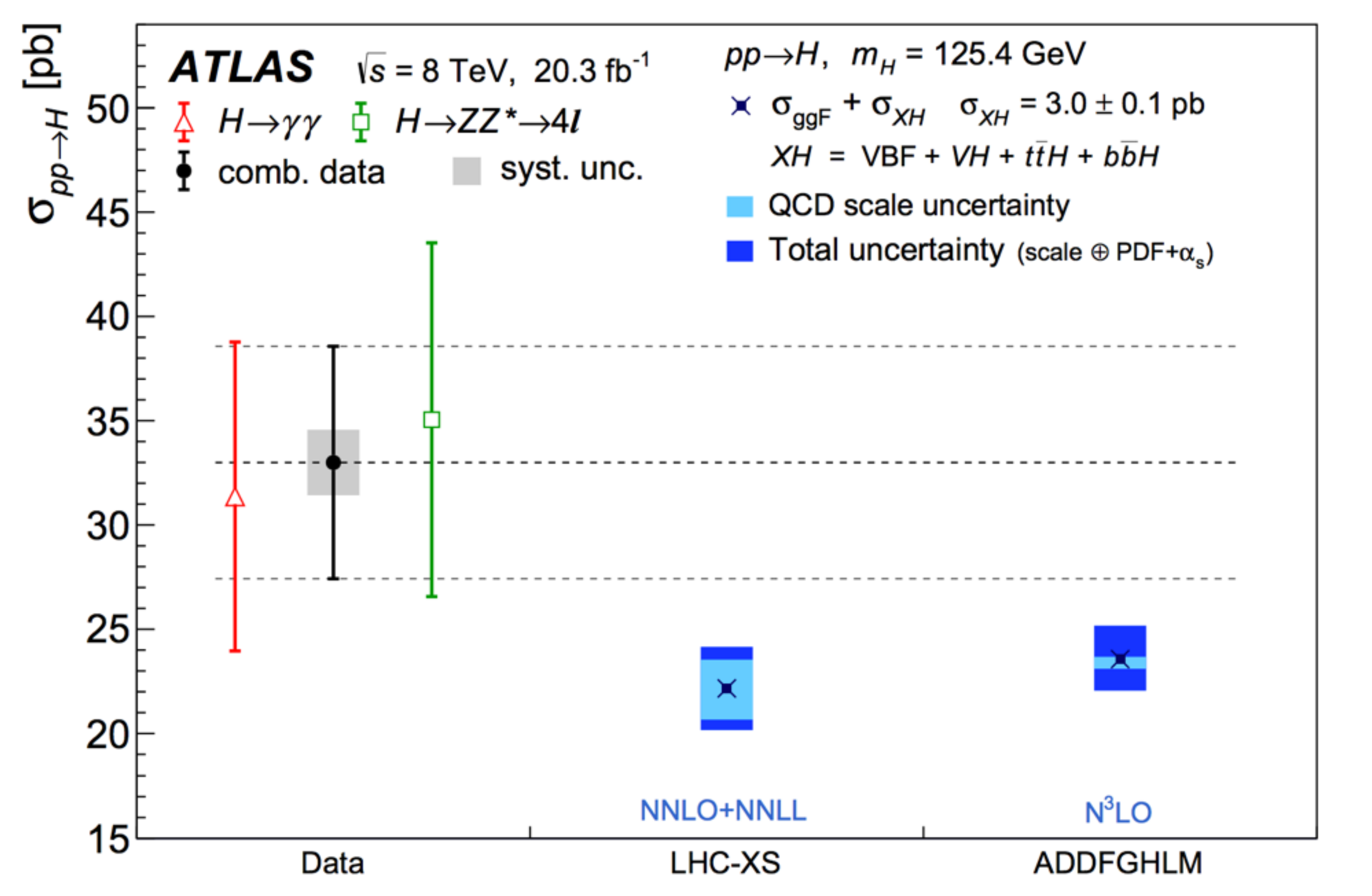


mh/2 : 47.03

mh : 46.10

Threshold resummation





ATLAS √s = 8 TeV, 20.3 fb⁻¹
H→γγ
H→ZZ*→4l
comb. data
syst. unc.
pp→H, m_H = 125.4 GeV
σ_ggF + σ_XH σ_XH = 3.0 ± 0.1 pb
XH = VBF + VH + tt̄H + bb̄H
QCD scale uncertainty
Total uncertainty (scale ⊕ PDF+α_s)
σ_pp→H [pb]
50
45
40
35
30
25
20
15
NNLO+NNLL
N3LO
Data
LHC-XS
ADDFGHLM

Conclusions

- We have finished the first ever complete calculation of a hadron collider process at N3LO in QCD
- We can provide the first reliable phenomenological predictions at N3LO from 30 orders in the threshold expansion
- We find a 2% correction compared to NNLO at $\mu = m_h/2$
- Dramatic reduction of the scale dependence
- We will have an updated prediction of the LHC soon
- In the future: Drell-Yan, Differential cross sections, etc.

THANK YOU