CALCULATING HIGGS PRODUCTION AT N3LO

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• Why do we compute?
• What do we want to compute?
• How do we compute?
• What do we find?
Motivation

• Discovery marks the beginning of the experimental era of Higgs physics

• Determination of the properties of the Higgs will be a challenge for years to come

Amazing progress from the experiments
Motivation

• Higgs couplings are a gateway to possible BSM scenarios

• Measuring deviations from the standard model prediction can indicate new physics

• This requires highly accurate standard model predictions
The gluon fusion cross section

- The dominant Higgs production mode at the LHC is gluon fusion
  - Loop-induced process

- The Higgs boson is light compared to the top quark

- The top loop can be integrated out $\rightarrow$ effective theory
The gluon fusion cross section

- The tree-level coupling of the gluons to the Higgs is described by a dimension five operator

\[ \mathcal{L} = \mathcal{L}_{\text{QCD}} - \frac{1}{4v} C_1 H G_{\mu \nu}^a G_a^{\mu \nu} \]

- Operators with higher dimension can be included in the computation

- This leads to a systematic expansion of the gluon fusion cross section in the top mass

- Sub-leading corrections in the top-mass are known at NNLO

- In the following I will only talk about the leading term in the effective theory
The gluon fusion cross section

- The gluon fusion cross-section in perturbation theory is

\[ \sigma(pp \rightarrow H + X) = \tau \sum_{ij} \int_{\tau}^{1} dz L_{ij}(z) \hat{\sigma}_{ij} \left( \frac{\tau}{z} \right) \]

- We compute the inclusive partonic cross section

- The partonic cross section is a function of

\[ z = \frac{m^2_h}{\hat{s}} \quad \tau = \frac{m^2_h}{E^2_{cm}} \]

- In perturbation theory the partonic cross section can be expanded

\[ \hat{\sigma}(z) = \hat{\sigma}^{LO}(z) + \alpha_s \hat{\sigma}^{NLO}(z) + \alpha_s^2 \hat{\sigma}^{NNLO}(z) + \alpha_s^3 \hat{\sigma}^{N3LO}(z) + \ldots \]
The gluon fusion cross section

- Diagrammatic contributions at NNNLO

- triple virtual
- double virtual real
- real virtual squared
- double real virtual
- triple real
The triple virtual

- The triple virtual is directly related to the three loop QCD form factor

- The QCD form factor is well known
  - at one loop
  - at two loops [Gonsalves; Kramer, Lampe; Gehrmann, Huber, Maitre]
  - at three loops [Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]

- The pure loop contributions are not a problem in the calculation
The gluon fusion cross section

• All other contributions involve the real emission of additional particles into the final state

• Need to do phase space integrals
Unitarity

- Optical theorem:

\[
\text{Im} \quad \circ \quad = \quad \int d\Phi
\]

- Discontinuities of loop integrals are phase space integrals

- Discontinuities of loop integrals are given by Cutkosky’s rule:

\[
\frac{1}{p^2 - m^2 + i\epsilon} \rightarrow \delta^+(p^2 - m^2) = \delta(p^2 - m^2)\theta(p^0)
\]
Reverse unitarity

- Optical theorem:

\[ \text{Im} = \int d\Phi \]

- The optical theorem can be read ‘backwards’

- This way, phase space integrals can be expressed as unitarity cuts of loop integrals
  
  [Anastasiou, Melnikov; Anastasiou, Dixon, Melnikov, Petriello]

- We can compute loop integrals with cuts instead of phase space integrals

- This makes the rich technology developed for loop integrals available
IBPs and master integrals

- Loop integrals are in general not independent but related by Integration-by-parts identities (IBPs)

- The IBPs form a system of equations for a given class of loop integrals

- The system can be solved algorithmically expressing all integrals through a small basis set of integrals (master integrals)
IBPs and master integrals

- IBP reductions greatly reduce complexity

- Double-virtual real contributions:
  - 68273802 integrals before reduction
  - 72 integrals after reduction

\[
\frac{(\epsilon - 1)(2\epsilon - 1)(3\epsilon - 2)(3\epsilon - 1)(6\epsilon - 5)(6\epsilon - 1)}{\epsilon^4(\epsilon + 1)(2\epsilon - 3)}
\]
IBPs and differential equations

• Having access to IBP technology allows us to derive differential equations for master integrals

• The derivative of a master integral w.r.t. kinematic invariants can be expressed as a linear combination of master integrals

• Leads to a coupled system of linear differential equations for the master integrals

\[
\begin{bmatrix}
\frac{\partial}{\partial \bar{z}} - 3\epsilon \frac{1}{1 - \bar{z}} \\
\epsilon \frac{1}{1 - \bar{z}} \\
-3\epsilon \frac{1}{1 - \bar{z}}
\end{bmatrix}
\]

\[
\bar{z} = 1 - z = \frac{s - m_h^2}{s}
\]
Differential equations and boundaries

• Integrating the differential equations for the master integrals yields general solutions

• These general solutions need to be fixed using boundary conditions

• Natural boundary condition for the problem at

$$\bar{z} = 0 \iff \hat{s} = m_h^2$$

• This corresponds to the soft limit of the process
The threshold expansion

• It is possible to systematically expand the cross section at threshold

• This yields
  • the soft-virtual approximation for the cross-section
  • boundary conditions for the differential equation

• Around threshold the cross section can be approximated by a power series

\[ \hat{\sigma} = \hat{\sigma}_{-1} + \hat{\sigma}_0 + \bar{z}\hat{\sigma}_1 + \mathcal{O}(\bar{z})^2 \]
The soft-virtual approximation

- All required integrals can be computed analytically
  - 22 three-loop integrals
  - 3 double-virtual real integrals
  - 7 real-virtual squared integrals
  - 10 double-real virtual integrals
  - 8 triple real integrals

- Additionally
  - three-loop splitting functions
  - three-loop beta functions
  - three-loop Wilson coefficient

[Baikov, Chetyrkin, Smirnov, Smirnov, Steinhauser; Gehrmann, Glover, Huber, Ikizlerli, Studerus]
[Duhr, Gehrmann; Li, Zhu]
[Anastasiou, Duhr, FD, Herzog, Mistlberger; Kilgore]
[Anastasiou, Duhr, FD, Herzog, Mistlberger; Li, von Manteufel, Schabinger, Zhu]
[Anastasiou, Duhr, FD, Mistlberger]

[Moch, Vogt, Vermaseren]
[Tarasov, Vladimirov, Zharkov; Larin, Vermaseren]
[Chetyrkin, Kniehl, Steinhauser; Schroder, Steinhauser; Chetyrkin, Kuhn, Sturm]
The integrals

• We want to compute all the integrals analytically

• Every integral is individually divergent and gives rise to up to six poles in dimensional regularisation

• Many integrals are trivial to compute:

\[
\begin{align*}
\Gamma(4 - 4\epsilon)\Gamma(2 - 3\epsilon) &= \frac{\Gamma(4 - 4\epsilon)\Gamma(2 - 3\epsilon)}{\epsilon(1 - 2\epsilon)^2\Gamma(4 - 6\epsilon)\Gamma(1 - \epsilon)} \\
&= \frac{1}{\epsilon} + \frac{14}{3} + (24 - 6\zeta_2)\epsilon + \left(-28\zeta_2 - 42\zeta_3 + \frac{400}{3}\right)\epsilon^2 + (-144\zeta_2 - 196\zeta_3 - 195\zeta_4 + \frac{2320}{3})\epsilon^3 + (252\zeta_3\zeta_2 - 800\zeta_2 - 1008\zeta_3 - 910\zeta_4 - 1302\zeta_5 + 4576)\epsilon^4 \\
&\quad + \left(882\zeta_3^2 + 1176\zeta_2\zeta_3 - 5600\zeta_3 - 4640\zeta_2 - 4680\zeta_4 - 6076\zeta_5 - \frac{9219}{2}\zeta_6 + 81920\right)\epsilon^5 + O(\epsilon)^6
\end{align*}
\]
The integrals

- Other integrals not so much

\[ \mathcal{I}_{9,1}(\epsilon) = - \int_0^\infty dt_1 dt_2 \int_0^1 dx_1 dx_2 dx_3 t_1^{2-4\epsilon} (1 + t_1)^{\epsilon-1} t_2^{1-2\epsilon} \]
\[ \times x_1^{-\epsilon} (1 - x_1)^{2-4\epsilon} x_2^{1-3\epsilon} (1 - x_2)^{-\epsilon} x_3^{-\epsilon} (1 + t_2 x_3)^{1-3\epsilon} (1 + t_2 x_2 x_3)^\epsilon \]
\[ \times (t_1 t_2^2 x_1 x_2 x_3 + t_2^2 x_2 x_3 + t_1 t_2 x_1 x_2 + t_1 t_2 x_3 + t_2 x_2 x_3 + t_2 + t_1 + 1)^{3\epsilon-3} , \]

\[ \mathcal{I}_{9,2}(\epsilon) = \int_0^\infty dt_1 dt_2 \int_0^1 dx_1 dx_2 dx_3 t_1^{2-4\epsilon} (1 + t_1)^{\epsilon-1} t_2^{1-2\epsilon} \]
\[ \times x_1^{1-\epsilon} (1 - x_1)^{2-4\epsilon} x_2^{1-3\epsilon} (1 - x_2)^{-\epsilon} x_3^{-\epsilon} (1 + t_2 x_3)^{1-3\epsilon} (1 + t_2 x_2 x_3)^\epsilon \]
\[ \times (t_1 t_2^2 x_1 x_2 x_3 + t_2^2 x_1 x_2 x_3 + t_2 x_1 + t_1 t_2 x_1 x_2 + t_1 t_2 x_3 + t_2 x_1 x_2 x_3 + t_1 + x_1)^{3\epsilon-3} , \]
Two computational problems

• Two computational problems that need to be solved

• Phase space integrals for the boundary conditions need to be computed analytically

• Differential equations need to be solved in terms of some useful functions

• What connects these two problems?

• How do we solve them?
• Why do we compute? ✔
• What do we want to compute? ✔
• How do we compute?
• What do we find?
Multiple polylogarithms

- Large classes of loop integrals can be expressed in terms of multiple polylogarithms

\[ G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t) \quad \text{Li}_n(z) = \int_0^z \frac{dt}{t} \text{Li}_{n-1}(t) \]

- The classical polylogarithms, HPLs, 2dHPLs, cyclotomic polylogarithms, etc are special cases of the multiple polylogarithms

- The classical polylogarithms satisfy various complicated functional identities

\[ -\text{Li}_2(z) - \log(z) \log(1 - z) = \text{Li}_2(1 - z) - \frac{\pi^2}{6} \]

- For the multiple polylogarithms these identities are in general not known
Multiple polylogarithms

• Not knowing the functional identities is a problem

• Even if the physics of a result is very simple, the analytical expression might be very complicated
  • The simplicity of the answer might be hidden behind the various functional equations

• Famous example:
  • The two-loop hexagon remainder function in N=4 SYM as computed by Del Duca, Duhr and Smirnov is a 17 page expression
  • After Goncharov, Spradlin, Vergu and Volovich simplified it using functional identities it can be written in 4 lines
Multiple polylogarithms

• Not knowing the functional identities is a problem

• Too complicated results are not just a formal or aesthetic problem

• Without using functional identities there might be huge cancellation between divergent sub-pieces of the result even though the complete result is finite

• Too complicated results are not useable for phenomenology because numerical implementations are not feasible

• Need functional identities to express result in a simple basis
Multiple polylogarithms

• Not knowing the functional identities is a problem

• The integrand might not be in the right form to perform the integration

• Result can only be obtained if functional identities between polylogarithms are known
Multiple polylogarithms are a very active field of research in pure mathematics.

Mathematicians have discovered algebraic structures that underlie the polylogarithms.

When we usually think of functional identities we think of complicated functional equations that are obtained by performing intricate variable transformations of the integral representations

\[- \text{Li}_2(z) - \log(z) \log(1 - z) = \text{Li}_2(1 - z) - \frac{\pi^2}{6}\]
Number theory

• Mathematicians have conjectured that all functional equations between polylogarithms follow from a simple algebraic structure

• All functional equations between polylogarithms can be obtained from pure combinatorics

• The algebraic structure that governs the polylogarithms is called a Hopf algebra
Hopf algebras

• What is a **Hopf algebra**?

• It is an **algebra**: A vector space with an operation that allows us to combine two elements into one (multiplication)

• It is also a **coalgebra**: A vector space with an operation that allows us to break an element into two elements (comultiplication)

• Disclaimer: The following explanation is very handwaving and omits many mathematical details
Hopf algebras

- An example of the algebra part of a Hopf algebra is the **shuffle algebra** of the multiple polylogarithms.

- Shuffle product: Takes two sets and intersperses them in all possible ways while keeping the ordering of the elements of each set among themselves.

  \[
  ab \shuffle cd = abcd + acbd + acdb + cabd + cadb + adab
  \]

- Analogy: Riffle shuffling two stacks of cards.

  \[
  \log(x) \log(1 - x) = -G(0, 1, x) - G(1, 0, x)
  \]
Hopf algebras

- The shuffle algebra is not the only Hopf algebra that is carried by the multiple polylogarithms.

- In fact, the multiple polylogarithms carry three Hopf algebra structures:
  - **Shuffle algebra**: Shuffle product, Deconcatenation (from iterated integral representation).
  - **Goncharov’s Hopf algebra**: Multiplication, the "coproduct".
  - **Stuffle algebra**: Stuffle product, Deconcatenation (from nested sum representation).

These structures are compatible with each other.
Hopf algebras

• The comultiplication of the Hopf algebra for polylogarithms is the **coproduct**

• It splits a word in all possible ways

\[ \Delta(abcd) = abcd \otimes 1 + abc \otimes d + ab \otimes cd + a \otimes bcd + 1 \otimes abcd \]

• We can iterate this splitting until we have broken the word into products of single letters

[Goncharov; Duhr]
Functional equations

- The coproduct can be applied to polylogarithms [Goncharov; Duhr]

- The word is here the list of indices \( \{a_n\} \) of a polylogarithm

- Examples:

\[
\Delta(\log x) = 1 \otimes \log x + \log x \otimes 1
\]

\[
\Delta(\text{Li}_n(x)) = 1 \otimes \text{Li}_n(x) + \sum_{k=0}^{n-1} \text{Li}_{n-k}(x) \otimes \frac{\log^k(x)}{k!}
\]

\[
\Delta(\text{Li}_2(x)) = 1 \otimes \text{Li}_2(x) - \log(1 - x) \otimes \log(x) + 1 \otimes \text{Li}_2(x)
\]
Functional equations

• The coproduct can be used to derive functional equations for polylogarithms

• The coproduct is applied to the polylogarithm to split it into simpler pieces

• The functional identities for these simpler pieces might be known

• If not, the coproduct is repeatedly applied until only ordinary logarithms are left
Example

• Assume you want to calculate:
  \[ \int_0^1 dx \frac{\text{Li}_2 \left( \frac{ax}{1-x} \right)}{x(1-x)} \]

• Using the coproduct it is possible to derive the following functional identity:
  \[
  \text{Li}_2 \left( \frac{ax}{1-x} \right) = G \left( 0, 1; x \right) - G \left( 0, \frac{1}{1+a}; x \right) \\
  - G \left( 1, 1; x \right) + G \left( 1, \frac{1}{1+a}; x \right)
  \]

• Now all the integrations are trivial:
  \[
  G \left( a_1, \ldots, a_n; z \right) = \int_0^z \frac{dt}{t-a_1} G(a_2, \ldots, a_n; t)
  \]
The integrals

\[ I_{9,1}(\epsilon) = - \int_0^\infty dt_1 dt_2 \int_0^1 dx_1 dx_2 dx_3 t_1^{2-4\epsilon} (1 + t_1)^{\epsilon-1} t_2^{1-2\epsilon} \]
\[ \times x_1^{-\epsilon} (1 - x_1)^{2-4\epsilon} x_2^{1-3\epsilon} (1 - x_2)^{-\epsilon} x_3^{-\epsilon} (1 + t_2 x_3)^{1-3\epsilon} (1 + t_2 x_2 x_3)^{\epsilon} \]
\[ \times (t_1 t_2^2 x_1 x_2 x_3 + t_2^2 x_2 x_3 + t_1 t_2 x_1 x_2 + t_1 t_2 x_3 + t_2 x_2 x_3 + t_2 + t_1 + 1)^{3\epsilon-3} , \]

\[ I_{9,2}(\epsilon) = \int_0^\infty dt_1 dt_2 \int_0^1 dx_1 dx_2 dx_3 t_1^{2-4\epsilon} (1 + t_1)^{\epsilon-1} t_2^{1-2\epsilon} \]
\[ \times x_1^{1-\epsilon} (1 - x_1)^{2-4\epsilon} x_2^{1-3\epsilon} (1 - x_2)^{-\epsilon} x_3^{-\epsilon} (1 + t_2 x_3)^{1-3\epsilon} (1 + t_2 x_2 x_3)^{\epsilon} \]
\[ \times (t_1 t_2^2 x_1 x_2 x_3 + t_2^2 x_1 x_2 x_3 + t_2 x_1 + t_1 t_2 x_1 x_2 + t_1 t_2 x_3 + t_2 x_1 x_2 x_3 + t_1 + x_1)^{3\epsilon-3} , \]
Number theory

- Number theory helps us here
- The integral can be done one step at a time
- We use the coproduct to derive the needed functional identities at each step
- Integrate over one variable at a time using the basic definition of the multiple polylogarithms

\[ G(a_1, \ldots, a_n; z) = \int_{0}^{z} \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \]

- Number theory gives us a way to solve the integrals algorithmically

[Brown]
Number theory

• The previous integral can be computed one step at a time

• In the process one finds functional identities like:

• Such identities can not be found in the literature

• No one wants to derive them using integral transformations

• Number theory and the coproduct give you a simple way to obtain them on the fly
The integrals

- When the smoke clears, one finds:

\[
\frac{160}{\epsilon^5} - \frac{1712}{\epsilon^4} + \frac{1}{\epsilon^3}(-120\zeta_2 + 2784) + \frac{1}{\epsilon^2}(-120\zeta_3 + 1284\zeta_2 + 31968) \\
+ \frac{1}{\epsilon}(2520\zeta_4 + 1284\zeta_3 - 2088\zeta_2 - 216864) + 15720\zeta_5 + 1920\zeta_2\zeta_3 \\
- 26964\zeta_4 - 2088\zeta_3 - 23976\zeta_2 + 795744 + \epsilon\left(82520\zeta_6 + 9600\zeta_3^2\right) \\
- 168204\zeta_5 - 20544\zeta_2\zeta_3 + 43848\zeta_4 - 23976\zeta_3 + 162648\zeta_2 - 2449440 \\
+ \mathcal{O}(\epsilon^2).
\]

- Thanks to these modern techniques we were able to compute all boundary conditions analytically

- We obtain the soft-virtual approximation of the gluon fusion cross section at N3LO
Differential equations

• If we want to calculate the cross section for general kinematics we need more than just the first two terms in the expansion

• We want to solve the differential equations for the master integrals in a closed form

• General form of the system of differential equations

\[ \frac{\partial_x f_i}{x} = M_{ij}(x, \epsilon) f_j \]

• Can describe very general functions

• In simple cases it is possible to go to a less general, canonical, form

\[ \frac{\partial_x g_i}{x} = \epsilon \left( \frac{A^0_{ij}}{x} + \frac{A^1_{ij}}{1 - x} \right) g_j \]  

[Henn]
Differential equations

- In this form the solutions are of the simple form

\[ g_i(x) = g_i^0 + \epsilon A_{ij} \int dx \left( \frac{g_j(x)}{x} + \frac{g_j(x)}{1-x} \right) + \ldots \]

- Directly related to the integral representation of the multiple polylogarithms

\[ G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t) \]
• Why do we compute?

• What do we want to compute?

• How do we compute it?

• What do we find?
First N3LO approximation
Our method is working
13.11.2014 - Next-to-Soft

Method can be extended beyond the soft limit
Huge correction

LHC @ 13TeV
pp→h+X gluon fusion
MSTW08 68cl
μ=μ_R=μ_F=m_h
gg→h+X subchannel

“Soft expansion unreliable”
“Fixed order calculations unreliable”
LHC @ 13 TeV
$gg \rightarrow h + X$ subchannel
MSTW08 68% 
$\mu = \mu_R = \mu_F = m_h$

Truncation order

$\sigma_{gg}^{N^3LO}/pb$
LHC @ 13TeV
pp→h+X gluon fusion
MSTW08 68d
μ=μ_R=μ_F=f_h
gg→h+X subchannel
LHC @ 13TeV

$\mu = \mu_R = \mu_F$

$\sigma / \text{pb}$ vs $\mu / m_H$

LO, NLO, NNLO, NNNLO

LO ABM, NLO ABM, NNLO ABM, N3LO ABM

Preliminary
Comparison with ggF-Wg study

Preliminary

Run 1 HXSWG recommendation

mh/2 : 47.03
mh : 46.10
**ATLAS**  \( \sqrt{s} = 8 \text{ TeV}, \ 20.3 \text{ fb}^{-1} \)

\[ \sigma_{pp \to H}, \ m_H = 125.4 \text{ GeV} \]

- \( H \to \gamma \gamma \)
- \( H \to ZZ^* \to 4l \)
- comb. data
- syst. unc.

\[ \sigma_{ggF} + \sigma_{XH}, \ \sigma_{XH} = 3.0 \pm 0.1 \text{ pb} \]

\[ XH = \text{VBF} + \text{VH} + \text{ttH} + \text{bbH} \]

- QCD scale uncertainty
- Total uncertainty (scale + PDF + \( \alpha_s \))

Comparison with data:
- LHC-XS
- \( N^3LO \)
- \( \text{ADDGFHLM} \)
Conclusions

• We have finished the first ever complete calculation of a hadron collider process at N3LO in QCD

• We can provide the first reliable phenomenological predictions at N3LO from 30 orders in the threshold expansion

• We find a 2% correction compared to NNLO at $\mu = m_h/2$

• Dramatic reduction of the scale dependence

• We will have an updated prediction of the LHC soon

• In the future: Drell-Yan, Differential cross sections, etc.
THANK YOU