# Non-leptonic $B$-decays at NNLO in QCD: Techniques and phenomenology

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Based on

- Bell, TH: JHEP **1412** (2014) 129 [arXiv:1410.2804]
- Kränkl, TH: [arXiv:1503.00735], accepted by JHEP
- Bell, Beneke, Li, TH: in preparation

Theory seminar, Zeuthen, April 9th, 2015

# Introduction to non-leptonic $B$ decays

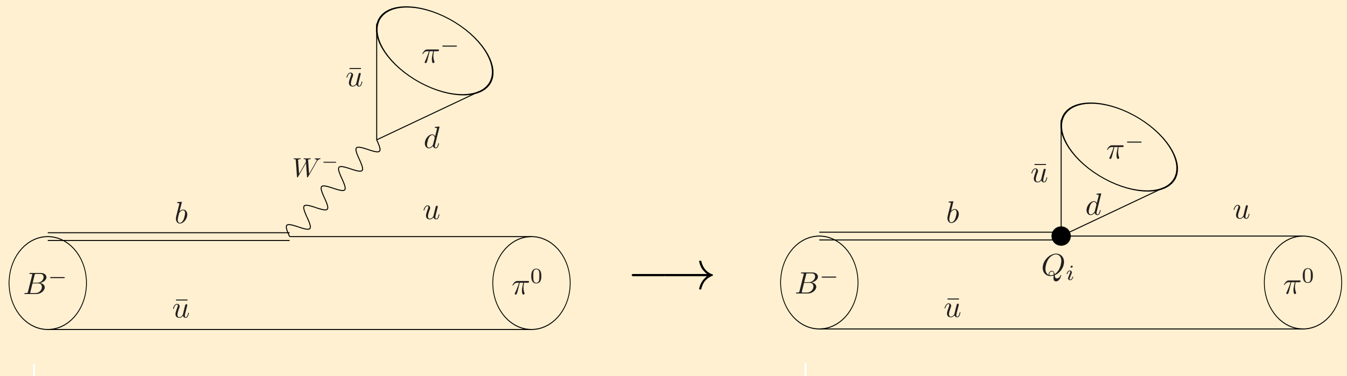
- Non-leptonic $B$ decays offer a rich and interesting phenomenology
  - Large data sets from $B$-factories, Tevatron, LHCb, future Super-flavour factory
  - $\mathcal{O}(100)$ final states
  - Numerous observables:
    * branching ratios
    * CP asymmetries
    * Polarisations
    * Dalitz plot analyses
    * Combinations thereof
- Test of CKM mechanism (CP violation)
- Indirect search for New Physics
  - Not as sensitive as rare or radiative $B$ decays, but large data sets

# Introduction to non-leptonic $B$ decays

- Theoretical description complicated by purely hadronic initial and final state
  - QCD effects from many different scales

- Theory approaches for two-body decays
  - Factorisation approaches: PQCD *[Keum,Li,Sanda'00]*, QCDF *[Beneke,Buchalla,Neubert,Sachrajda'99-'01]*
    - Disentangle long and short distances
  - QCD Factorisation
    - Systematic framework to all orders in $\alpha_s$ and leading power in $\Lambda/m_b$
    - Problems with factorisation of power suppressed and annihilation contributions. Endpoint divergences.
  - Flavour symmetries: Isospin, U-Spin ($d \leftrightarrow s$), V-Spin ($u \leftrightarrow s$), Flavour SU(3)
    - Only few a priori assumptions about scales needed
    - Implementation of symmetry breaking difficult

# Effective theory for $B$ decays




- $M_W$, $M_Z$, $m_t \gg m_b$: integrate out heavy gauge bosons and $t$-quark

- Effective Hamiltonian: *[Buras,Buchalla,Lautenbacher'96; Chetyrkin,Misiak,Münz'98]*

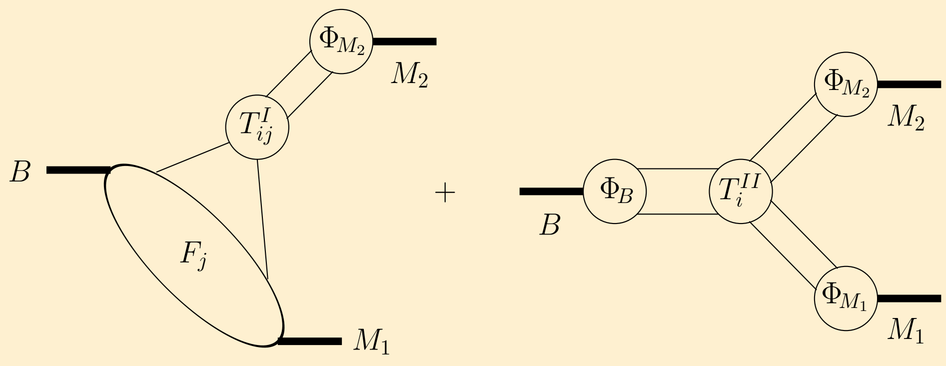
$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} \lambda_p \left[ C_1 Q_1^p + C_2 Q_2^p + \sum_{k=3}^{6} C_k Q_k + C_8 Q_8 \right] + \text{h.c.}$$

$$Q_1^p = (\bar{d}_L \gamma^\mu T^a p_L)(\bar{p}_L \gamma_\mu T^a b_L) \qquad Q_4 = (\bar{d}_L \gamma^\mu T^a b_L) \textstyle\sum_q (\bar{q} \gamma_\mu T^a q) \qquad Q_8 = -\frac{g_s}{16\pi^2}\, m_b\, \bar{d}_L\ \sigma_{\mu\nu} G^{\mu\nu} b_R$$

$$Q_2^p = (\bar{d}_L \gamma^\mu p_L)(\bar{p}_L \gamma_\mu b_L) \qquad Q_5 = (\bar{d}_L \gamma^\mu \gamma^\nu \gamma^\rho b_L) \textstyle\sum_q (\bar{q} \gamma_\mu \gamma_\nu \gamma_\rho q)$$

$$Q_3 = (\bar{d}_L \gamma^\mu b_L) \textstyle\sum_q (\bar{q} \gamma_\mu q) \qquad Q_6 = (\bar{d}_L \gamma^\mu \gamma^\nu \gamma^\rho T^a b_L) \textstyle\sum_q (\bar{q} \gamma_\mu \gamma_\nu \gamma_\rho T^a q) \qquad \lambda_p = V_{pb} V_{pd}^*$$

# QCD factorisation




- Amplitude in the limit $m_b \gg \Lambda_{\rm QCD}$ *[Beneke,Buchalla,Neubert,Sachrajda'99-'04]*

$$\langle M_1 M_2 | Q_i | \bar{B} \rangle \simeq m_B^2 \, F_+^{B\to M_1}(0) \, f_{M_2} \int_0^1 du \;\; T_i^I(u) \, \phi_{M_2}(u)$$

$$+ f_B \, f_{M_1} \, f_{M_2} \int_0^1 d\omega dv du \;\; T_i^{II}(\omega, v, u) \, \phi_B(\omega) \, \phi_{M_1}(v) \, \phi_{M_2}(u)$$

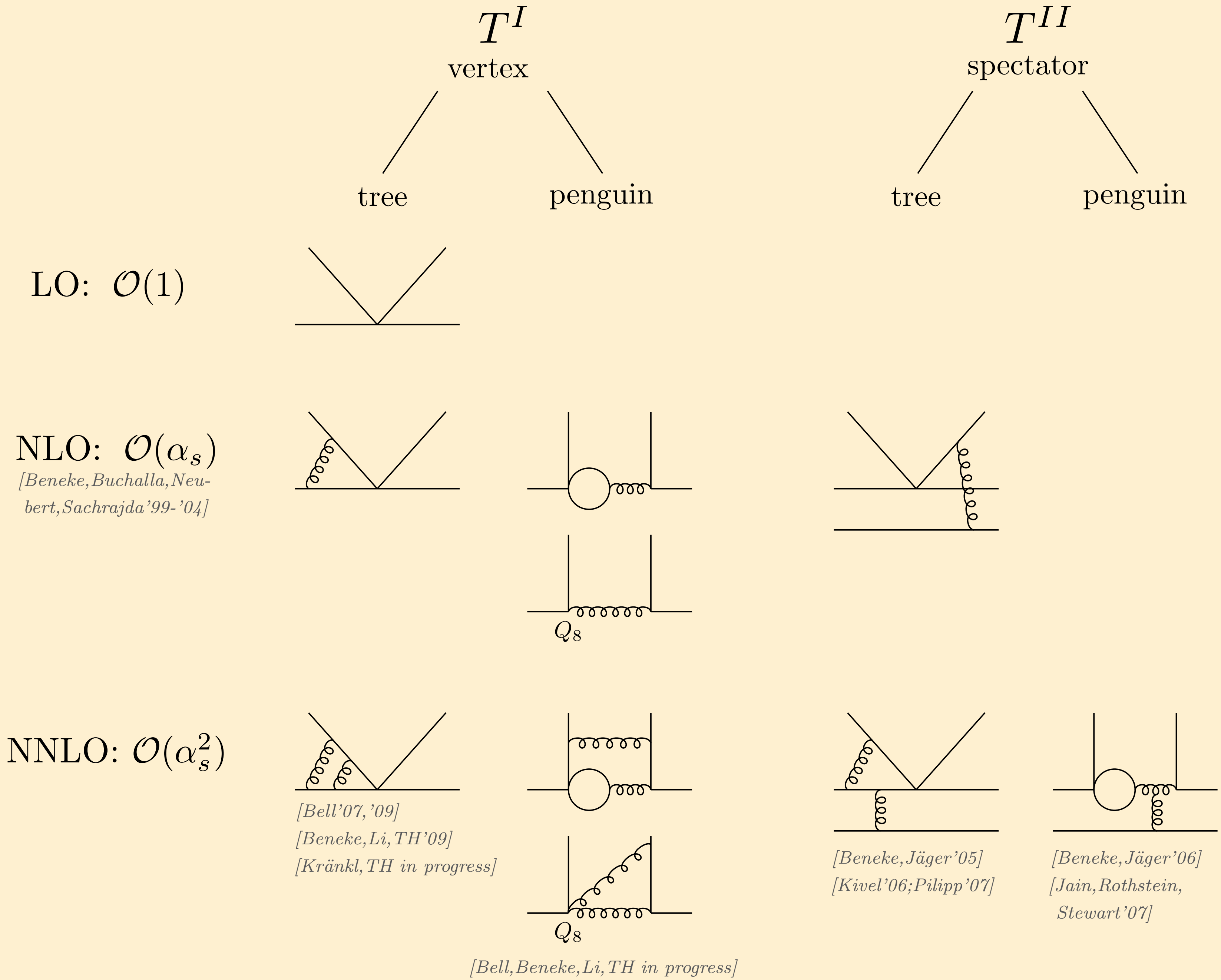
- $T^{I,II}$ : Hard scattering kernels, perturbatively calculable

- $F_+$ : $B \to M$ form factor
  $f_i$ : decay constants
  $\phi_i$ : light-cone distribution amplitudes

  } Universal. From Sum Rules, Lattice

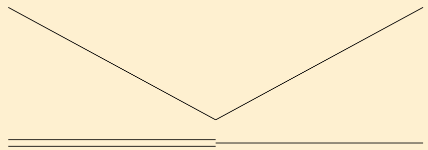
- Strong phases are $\mathcal{O}(\alpha_s)$ and/or $\mathcal{O}(\Lambda_{\rm QCD}/m_b)$

# Anatomy of QCD factorisation

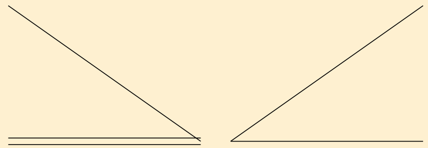


# Classification of amplitudes

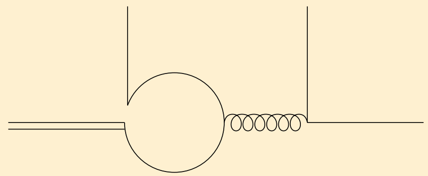
- $\alpha_1$ : colour-allowed tree amplitude



- $\alpha_2$ : colour-suppressed tree amplitude



- $\alpha_4^{u,c}$ : QCD penguin amplitudes



$$
\begin{aligned}
\sqrt{2}\,\langle \pi^-\pi^0|\,\mathcal{H}_{eff}\,|B^-\rangle &= A_{\pi\pi}\;\lambda_u\big[\alpha_1(\pi\pi)+\alpha_2(\pi\pi)\big]\\
\langle \pi^+\pi^-|\,\mathcal{H}_{eff}\,|\bar{B}^0\rangle &= A_{\pi\pi}\;\big\{\lambda_u\big[\alpha_1(\pi\pi)+\alpha_4^u(\pi\pi)\big]+\lambda_c\,\alpha_4^c(\pi\pi)\big\}\\
-\,\langle \pi^0\pi^0|\,\mathcal{H}_{eff}\,|\bar{B}^0\rangle &= A_{\pi\pi}\;\big\{\lambda_u\big[\alpha_2(\pi\pi)-\alpha_4^u(\pi\pi)\big]-\lambda_c\,\alpha_4^c(\pi\pi)\big\}
\end{aligned}
$$

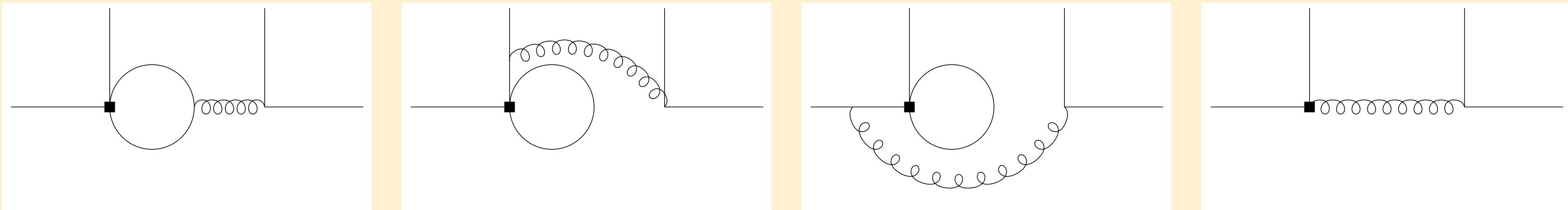
$$
\begin{aligned}
\langle \pi^-\bar{K}^0|\,\mathcal{H}_{eff}\,|B^-\rangle &= A_{\pi\bar{K}}\;\Big[\lambda_u^{(s)}\,\alpha_4^u+\lambda_c^{(s)}\,\alpha_4^c\Big]\\
\langle \pi^+K^-|\,\mathcal{H}_{eff}\,|\bar{B}^0\rangle &= A_{\pi\bar{K}}\Big[\lambda_u^{(s)}\,(\alpha_1+\alpha_4^u)+\lambda_c^{(s)}\,\alpha_4^c\Big]
\end{aligned}
$$

*[Beneke,Neubert'03]*

- Tree amplitudes $\alpha_1$ and $\alpha_2$ known analytically to NNLO *[Bell'07'09; Beneke,Li,TH'09]*

# Penguin amplitudes $a_4^u$ and $a_4^c$ to NLO

- NLO:



$$\alpha_4^u(\pi\pi) = -0.029 - [0.002 + 0.001i]_V + [0.003 - 0.013i]_P + [?? + ??\ i]_{\mathcal{O}(\alpha_s^2)}$$

$$+ \left[\frac{r_{\rm sp}}{0.485}\right] \left\{[0.001]_{\rm LO} + [0.001 + 0.000i]_{HV+HP} + [0.001]_{\rm tw3}\right\}$$

$$= (-0.024^{+0.004}_{-0.002}) + (-0.012^{+0.003}_{-0.002})i$$

$$\alpha_4^c(\pi\pi) = -0.029 - [0.002 + 0.001i]_V - [0.001 + 0.007i]_P + [?? + ??\ i]_{\mathcal{O}(\alpha_s^2)}$$

$$+ \left[\frac{r_{\rm sp}}{0.485}\right] \left\{[0.001]_{\rm LO} + [0.001 + 0.001i]_{HV+HP} + [0.001]_{\rm tw3}\right\}$$

$$= (-0.028^{+0.005}_{-0.003}) + (-0.006^{+0.003}_{-0.002})i$$

# Motivation for NNLO

- Direct CP asymmetries start at $\mathcal{O}(\alpha_s)$
  - Large (scale) uncertainties
  - NNLO is only first perturbative correction
  - NNLO is NLO for direct CP asymmetries!
- NLO results for tree amplitudes

$$\alpha_1(\pi\pi) = 1.009 + [0.023 + 0.010\,i]_{\rm NLO} - \left[\frac{r_{\rm sp}}{0.445}\right] \Big\{[0.014]_{\rm LOsp} + [0.008]_{\rm tw3}\Big\} = 1.010 + 0.010i$$

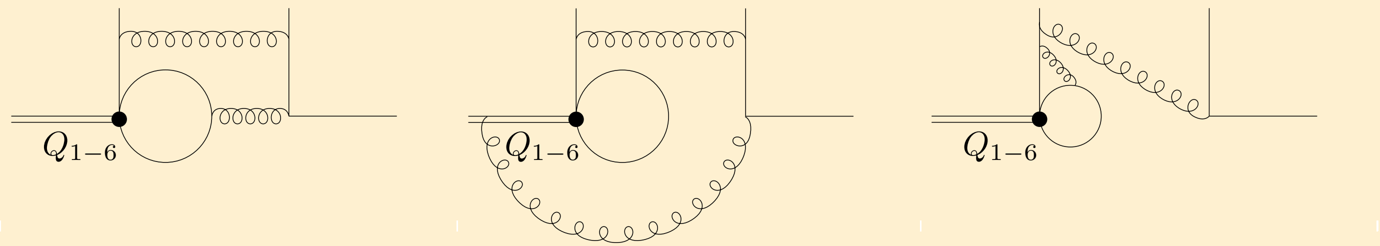
$$\alpha_2(\pi\pi) = 0.220 - [0.179 + 0.077\,i]_{\rm NLO} + \left[\frac{r_{\rm sp}}{0.445}\right] \Big\{[0.114]_{\rm LOsp} + [0.067]_{\rm tw3}\Big\} = 0.222 - 0.077i$$

  - Large cancellation in LO + NLO in $\alpha_2$. Particularly sensitive to NNLO
- Problems with colour-suppressed, tree-dominated decays (e.g. $\bar{B}^0 \to \pi^0\pi^0$)
  - However: New preliminary result by Belle: $\mathcal{B}(\bar{B}^0 \to \pi^0\pi^0) = (0.90 \pm 0.16) \cdot 10^{-6}$
- Does factorisation hold at NNLO?

# Penguin amplitudes at two loops

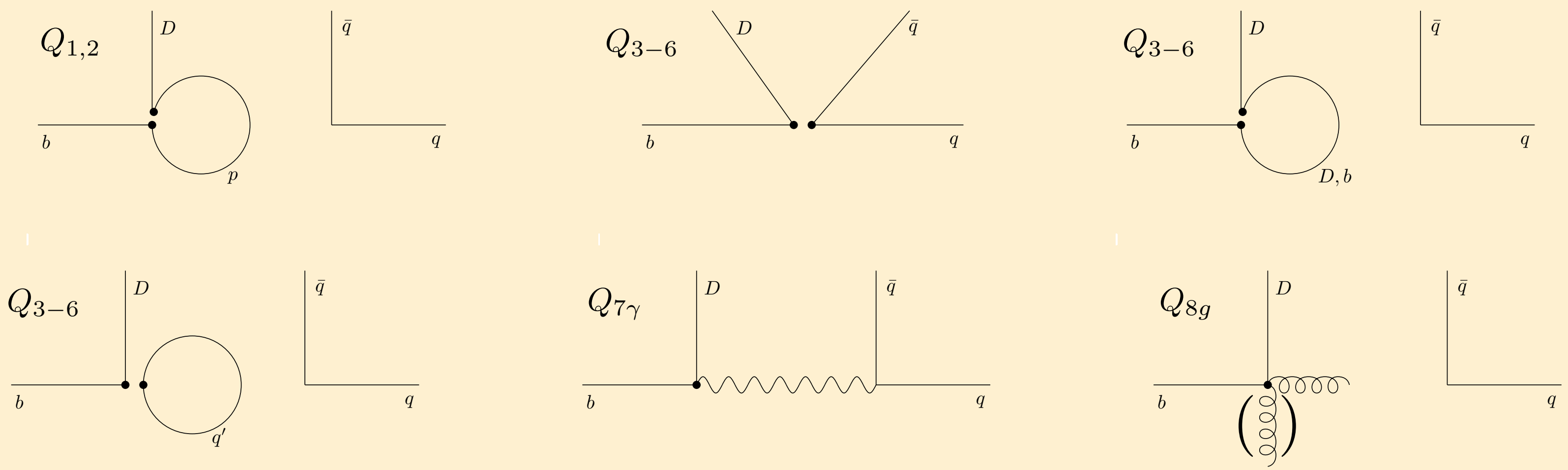
[Bell,Beneke,Li,TH, in preparation]

- $\mathcal{O}(70)$ diagrams at NNLO.




[Kim,Yoon'11]

- Quite some book-keeping due to various insertions

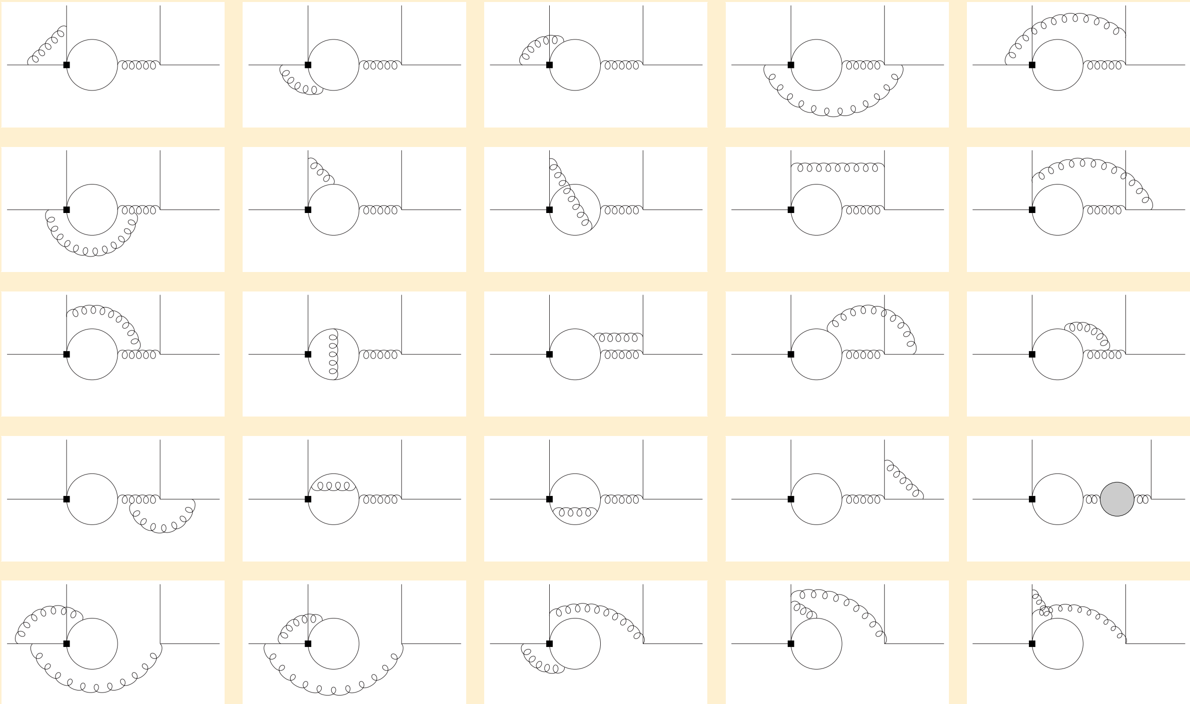



- !!! Focus on $Q_1^{u,c}$ and $Q_2^{u,c}$ insertions !!!

# Penguin amplitudes at two loops

*[Bell,Beneke,Li,TH, in preparation]*

- For $Q_{1,2}^{u,c}$ only a subset of $\sim 25$ diagrams contributes

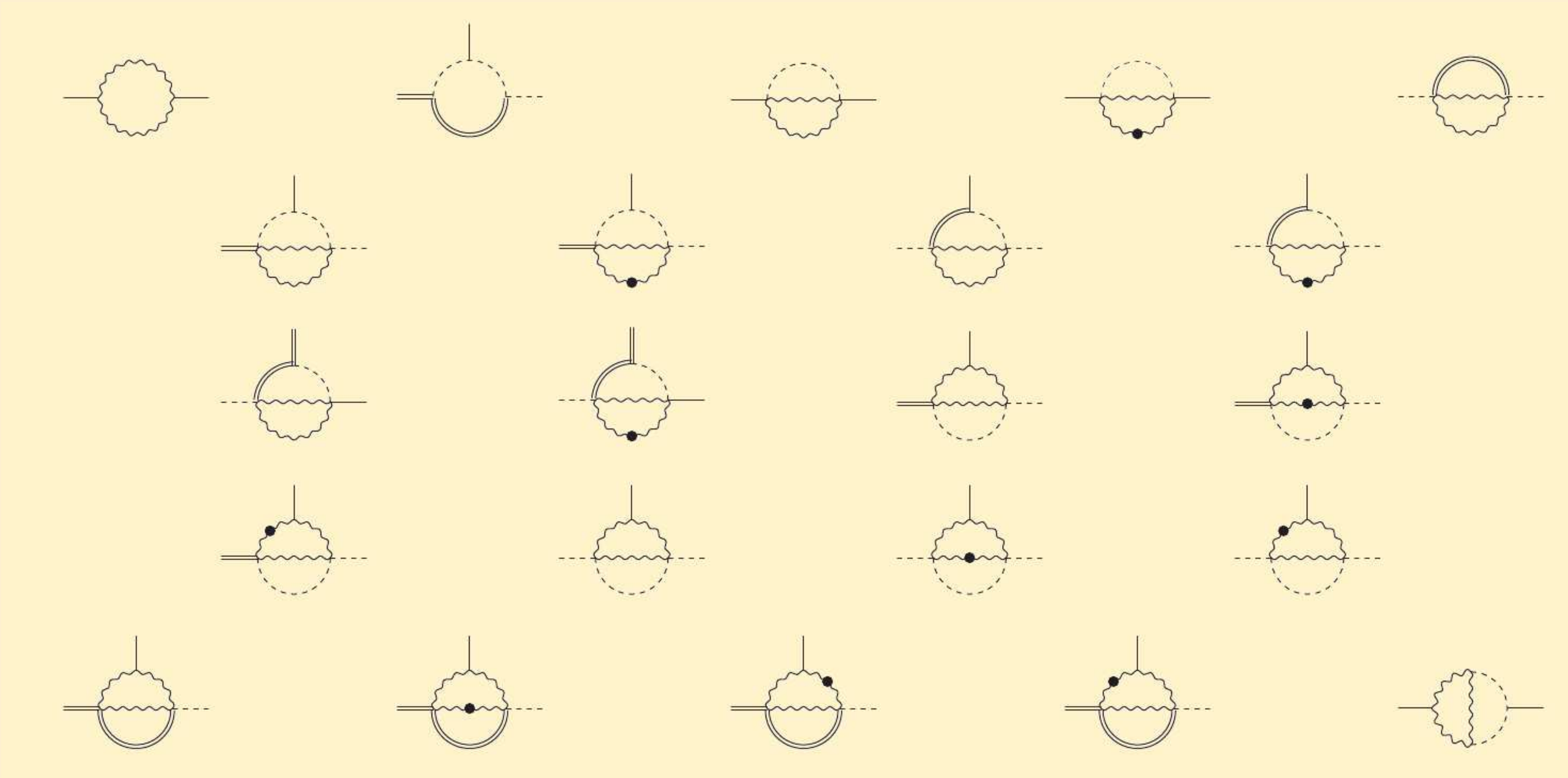


- Regularize UV and IR divergences dimensionally. Poles up to $1/\epsilon^3$
- Reduction: Integration-by-parts relations, Laporta algorithm

  *[Tkachov'81; Chetyrkin,Tkachov'81] [Laporta'01; Anastasiou,Lazopoulos'04; Smirnov'08; Studerus,von Manteuffel'10,'12]*

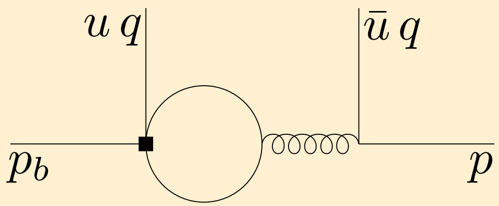
  – Obtain a set of 29 master integrals

# Master integrals



- Double: $m_b^2$ ,   wavy: $m_c^2$ ,   solid: $\bar{u}\, m_b^2$ ,   dashed: $0$ .
- Genuine two-scale problem: $u$ , $z_c \equiv m_c^2/m_b^2$
- Most integrals are four-liners with three external legs
- Only one five-liner: Two-point function, one-scale integral

# Kinematics




$$p^2 = q^2 = 0$$

$$p_b^2 = m_b^2$$

- Fermion loop with either $m = 0$ or $m = m_c$.
- Genuine two-scale problem: $\bar{u}$, $m_c^2/m_b^2$
- Threshold at $\bar{u} = 4m_c^2/m_b^2$
- Choice of suitable kinematic variables crucial

$$s = \sqrt{1 - 4z_c/\bar{u}}\,,\ r = \sqrt{1 - 4z_c} \quad \longleftrightarrow \quad \bar{u}\,,\ z_c = \frac{m_c^2}{m_b^2} \quad \longleftrightarrow \quad s_1 = \sqrt{1 - 4/\bar{u}}\,,\ r$$

$$\updownarrow$$

$$p = \frac{1 - \sqrt{u^2 + 4\bar{u}z_c}}{\bar{u}}\,,\ r$$

# Computing the masters

- Use differential equations in canonical form *[Henn'13]*

$$\frac{\partial}{\partial x_m} \vec{M}(\epsilon, x_n) = \epsilon\, A_m(x_n)\, \vec{M}(\epsilon, x_n)$$

  - Can write this as a total differential

$$d\, \vec{M}(\epsilon, x_n) = \epsilon\; d\tilde{A}(x_n)\; \vec{M}(\epsilon, x_n)$$

    * Together with boundary conditions, $\tilde{A}(x_n)$ completely fixes the solution

- Benefits of canonical basis
  - System disentangles order by order in $\epsilon$
  - Homogeneous (pure) functions to all orders in $\epsilon$
  - No fake higher weights
  - QCD amplitude much simpler, especially denominators of pre-factors of masters
  - Suitable for convolution with LCDA

# Computing the masters

- Found canonical basis for all masters *[Bell,TH'14]*
  - Finding of canonical basis mostly trial and error
  - Compute lowest orders in $\epsilon$ from Feynman or MB parameters to find candidate
  - Plug into differential equation to verify
- First example of canonical basis in case of 2 different internal masses
- Boundary conditions
  - All but two integrals vanish in $u = 0$, $u = 1$ or $r = 0$
  - Remaining two coincide with known integrals in $u = 1$
- Found analytical solution in terms of iterated integrals
- Checks
  - ${}_pF_q$, MB representations, sector decomposition

*[see e.g. Maitre,TH'05,'07; Smirnov'99; Tausk'99; Czakon'05; Borowka,Carter,Heinrich'12]*

# Iterated integrals

- Harmonic polylogarithms [Remiddi,Vermaseren'99]

$$H_{a_1,a_2,\ldots,a_n}(x) = \int_0^x dt\, f_{a_1}(t)\, H_{a_2,\ldots,a_n}(t)$$

- Integer weights

$$f_1(x) = \frac{1}{1-x}\,, \qquad f_0(x) = \frac{1}{x}\,, \qquad f_{-1}(x) = \frac{1}{1+x}$$

- "+" and "−"-weights

$$f_+(x) = f_1(x) + f_{-1}(x) = \frac{2}{1-x^2}\,, \qquad f_-(x) = \quad f_1(x) - f_{-1}(x) = \frac{2x}{1-x^2}$$

- Generalisation to arbitrary weights

$$f_w(x) = \frac{1}{w-x}\,, \qquad f_{-w}(x) = \frac{1}{w+x}$$

# Iterated integrals

- Arbitrary "+" and "−"-weights

$$f_{w^+}(x) = f_w(x) + f_{-w}(x) = \frac{2w}{w^2 - x^2}, \qquad f_{w^-}(x) = f_w(x) - f_{-w}(x) = \frac{2x}{w^2 - x^2}$$

- Occurring weights

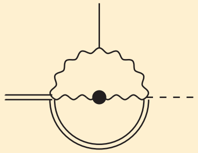
$$w_0 = 0 \qquad w_3 = \frac{r^2+1}{2}$$
$$w_1 = 1 \qquad w_4 = 1 + \sqrt{1-r^2}$$
$$w_2 = r \qquad w_5 = 1 - \sqrt{1-r^2}$$

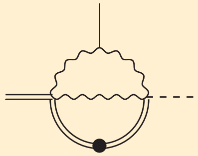
- Close relation to Goncharov polylogarithms

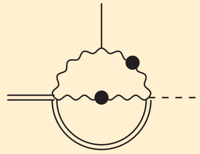
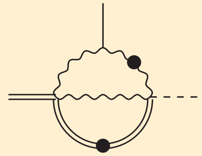
$$G_{a_1,a_2,\ldots,a_n}(x) = \int_0^x \frac{dt}{t-a_1} G_{a_2,\ldots,a_n}(t)$$

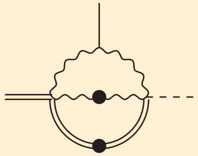
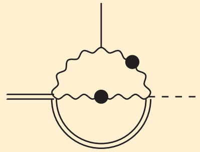
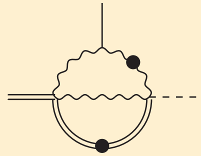
$$H_{w_2^+}(x) = G_{-r}(x) - G_r(x)$$

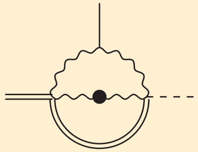
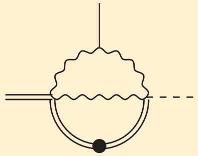
# Canonical basis for master integrals I

$\frac{M_{18}}{u\epsilon^3} =$ 

$\frac{M_{19}}{u\epsilon^3} =$ 

$-\frac{2\,M_{20}}{u\bar{u}s\epsilon^2} =$  $+$ 

$\frac{M_{21}}{\epsilon^2} = \frac{2[(1+\bar{u})^2 z_c - \bar{u}^2]}{\bar{u}}$  $- \bar{u}s^2(1+\bar{u})\Big[$  $+$  $\Big]$

$+\frac{2\epsilon u}{m_b^2}\Big[$  $+$  $\Big]$

- Differential equation (sample)

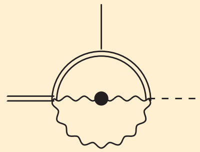
$$\frac{dM_{19}}{ds} = \frac{4\epsilon M_{18}\, r\, \left(r^2+1\right)}{((r^2+1)^2-4s^2)} - \frac{2\epsilon M_{19}\, r\, \left(r^2+s^2-2\right)}{(1-r^2)(r^2-s^2)} + \frac{4\epsilon M_{20}\, r\, s}{\left((r^2+1)^2-4s^2\right)} - \frac{\epsilon M_{21}\, r\, \left(r^2+1\right)}{((r^2+1)^2-4s^2)}$$

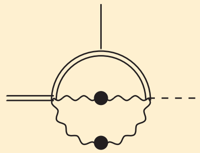
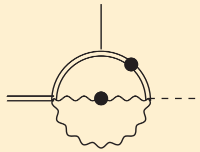
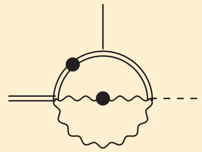
- Boundary conditions
  - $M_{18}$ and $M_{19}$ vanish in $s = r$ (i.e. in $u = 0$)
  - $M_{20}$ and $M_{21}$ vanish in $s = +i\infty$ (i.e. in $u = 1$)

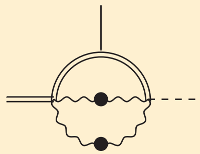
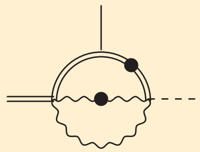
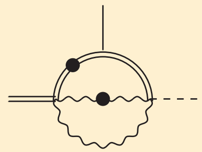
# Solution to $M_{18}$

$$
\begin{aligned}
M_{18}(r,s) =&\epsilon^3\,[-\frac{\pi^2}{6}\,H_{w_1^-}(r)+\frac{\pi^2}{6}\,H_{w_1^-}(s)-\frac{\pi^2}{12}\,H_{w_3^-}(r)+\frac{\pi^2}{12}\,H_{w_3^-}(s)-i\pi\,H_{w_1^-}(r)\,H_{w_3^+}(s)\\
&+H_{w_1^-}(s)\,H_{w_1^-,w_1^-}(r)-H_{w_3^-}(s)\,H_{w_1^-,w_1^-}(r)+i\pi\,H_{w_1^-,w_3^+}(r)-i\pi\,H_{w_1^+,w_1^-}(r)\\
&+i\pi\,H_{w_1^+,w_1^-}(s)-\frac{1}{2}\,i\pi\,H_{w_3^-,w_1^+}(r)+\frac{1}{2}\,i\pi\,H_{w_3^-,w_1^+}(s)+\frac{1}{2}\,i\pi\,H_{w_3^+,w_1^-}(r)\\
&+\frac{1}{2}\,i\pi\,H_{w_3^+,w_1^-}(s)+H_{w_1^-}(r)\,H_{w_3^+,w_1^+}(s)-3\,H_{w_1^-,w_1^-,w_1^-}(r)+H_{w_1^-,w_1^-,w_3^-}(r)\\
&+H_{w_1^-,w_3^-,w_1^-}(r)-H_{w_1^-,w_3^+,w_1^+}(r)+H_{w_1^+,w_1^-,w_1^+}(r)-H_{w_1^+,w_1^-,w_1^+}(s)\\
&+H_{w_3^-,w_1^-,w_1^-}(r)+\frac{1}{2}\,H_{w_3^-,w_1^+,w_1^+}(r)-\frac{1}{2}\,H_{w_3^-,w_1^+,w_1^+}(s)-\frac{1}{2}\,H_{w_3^+,w_1^-,w_1^+}(r)\\
&-\frac{1}{2}\,H_{w_3^+,w_1^-,w_1^+}(s)-H_{w_3^+,w_1^+,w_1^-}(r)+2\,H_{w_1^-}(r)\,H_{w_1^-}(s)\,\ln(2)-2\,i\pi\,H_{w_1^+}(r)\,\ln(2)\\
&+2\,i\pi\,H_{w_1^+}(s)\,\ln(2)-2\,H_{w_1^-}(r)\,H_{w_3^-}(s)\,\ln(2)+i\pi\,H_{w_3^+}(r)\,\ln(2)\\
&-i\pi\,H_{w_3^+}(s)\,\ln(2)-4\,H_{w_1^-,w_1^-}(r)\,\ln(2)+2\,H_{w_1^-,w_3^-}(r)\,\ln(2)+2\,H_{w_3^-,w_1^-}(r)\,\ln(2)\\
&-2\,H_{w_3^+,w_1^+}(r)\,\ln(2)+2\,H_{w_3^+,w_1^+}(s)\,\ln(2)-2\,H_{w_1^-}(r)\,\ln^2(2)+2\,H_{w_1^-}(s)\,\ln^2(2)\\
&+2\,H_{w_3^-}(r)\,\ln^2(2)-2\,H_{w_3^-}(s)\,\ln^2(2)-H_{w_1^-}(r)\,\mathrm{Li}_2(1-z_c)+H_{w_1^-}(s)\,\mathrm{Li}_2(1-z_c)\\
&+H_{w_3^-}(r)\,\mathrm{Li}_2(1-z_c)-H_{w_3^-}(s)\,\mathrm{Li}_2(1-z_c)]+\mathcal{O}(\epsilon^4)
\end{aligned}
$$

# Canonical basis for master integrals II

$$\frac{M_{23}}{u\epsilon^3} =$$ 

$$\frac{M_{24}}{\epsilon^2} = \frac{2(1+s_1)\sqrt{1+\frac{8z_c(1-s_1)}{(1+s_1)^2}}}{1-s_1}\Bigg[$$  $$+\,2$$  $$-\,\frac{2(1+s_1)}{1-s_1}$$  $$\Bigg]$$

$$\frac{M_{25}}{\epsilon^2} = \frac{2(1-s_1)\sqrt{1+\frac{8z_c(1+s_1)}{(1-s_1)^2}}}{1+s_1}\Bigg[$$  $$+\,2$$  $$-\,\frac{2(1-s_1)}{1+s_1}$$  $$\Bigg]$$

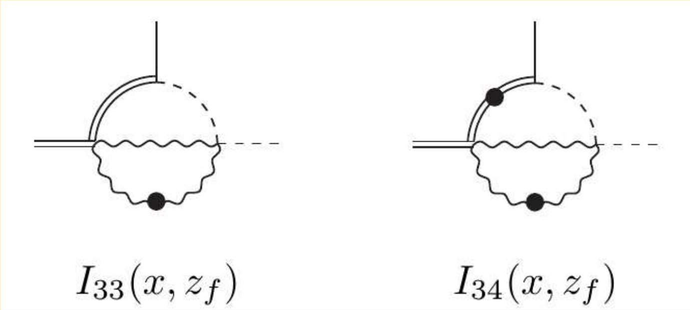
- Differential equation

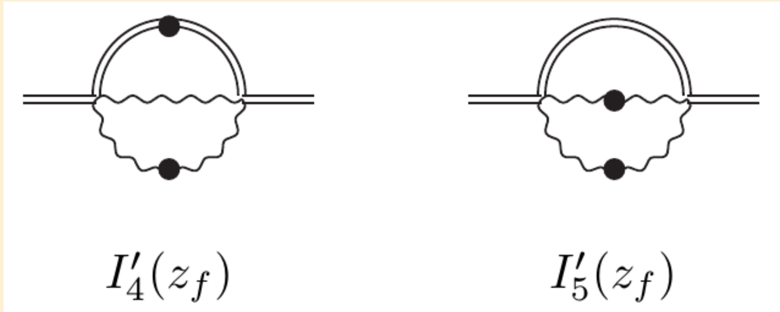
$$\frac{dM_{23}}{ds_1} = \frac{2\,\epsilon\,M_{23}\,s_1\left(5-s_1^2\right)}{(1-s_1^2)\,(3+s_1^2)} - \frac{\epsilon\,M_{24}\,(3-s_1)}{4\,(1-s_1^2)\,\sqrt{1+\frac{8\,z_c(1-s_1)}{(1+s_1)^2}}} + \frac{\epsilon\,M_{25}\,(3+s_1)}{4\,(1-s_1^2)\,\sqrt{1+\frac{8\,z_c(1+s_1)}{(1-s_1)^2}}}$$

- Variable transformation to rationalize irrational factors:

$$t = \frac{1-s_1}{2} + \frac{1+s_1}{2}\sqrt{1+\frac{2(1-r^2)(1-s_1)}{(1+s_1)^2}} \qquad v = \frac{1+s_1}{2} + \frac{1-s_1}{2}\sqrt{1+\frac{2(1-r^2)(1+s_1)}{(1-s_1)^2}}$$

# Canonical basis for master integrals III




$$M_{28} = \epsilon^3 u\, I_{33}$$

$$M_{29} = \frac{\epsilon^2}{2m_b^2}\Big\{2u\big(1-\bar{u}p\big)m_b^2 I_{34} - \big(\bar{u}p - 1 + 2\sqrt{z_c}\big)\big(I_5' + 2I_4'\big)\Big\}$$

- Differential equation

$$\frac{dM_{28}}{dp} = \frac{2\epsilon M_{28}\left(p^2+2p-r^2-2\right)\left(p^2-pr^2-p+1\right)}{\left(p^2-1\right)\left(2p-r^2-1\right)\left(p^2-2p+r^2\right)} - \frac{2\epsilon M_{29}\left(p^2+2p-r^2-2\right)}{\left(p^2-1\right)\left(2p-r^2-1\right)}$$

- Boundary conditions
  - $M_{28,29}$ vanish in $s = r$ (i.e. in $u = 0$)

# Results: Penguin Amplitudes

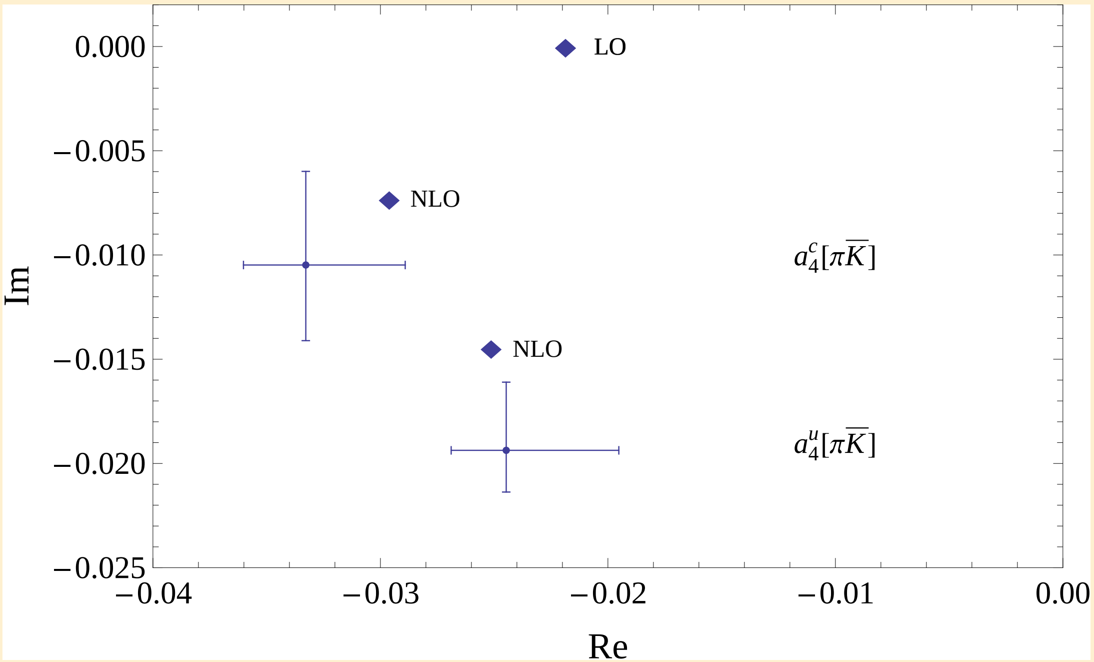
- PRELIMINARY numbers: Only $Q_{1,2}$ contribution. Inputs from [Beneke,Li,TH'09]

$$\alpha_4^u(\pi\bar{K})/10^{-2} = -2.87 - [0.09 + 0.09i]_{\rm V} + [0.49 - 1.32i]_{\rm P_1} - [0.32 + 0.71i]_{\rm P_2}$$

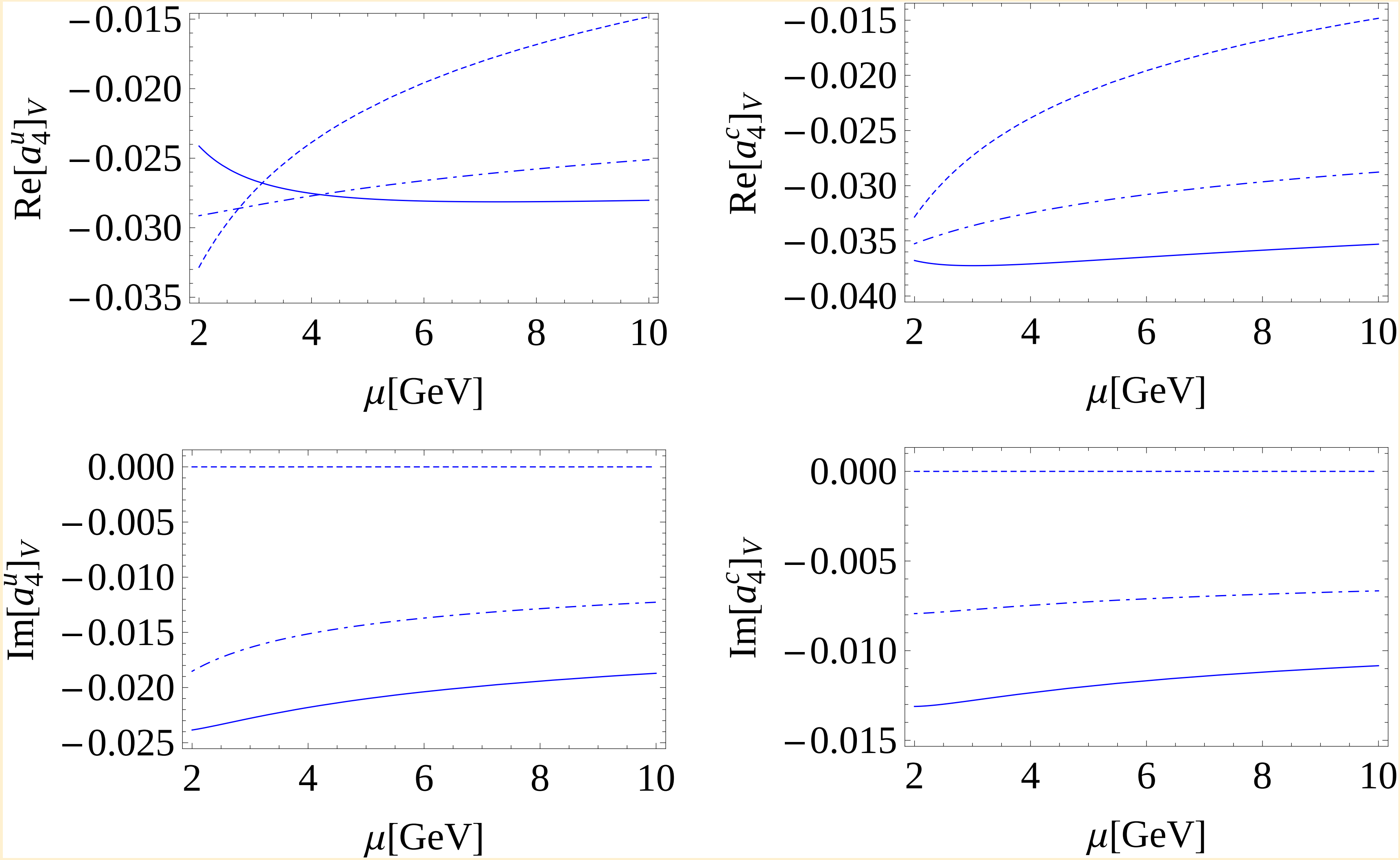
$$+ \left[\frac{r_{\rm sp}}{0.445}\right]\left\{[0.13]_{\rm LO} + [0.15 + 0.12i]_{\rm HV} - [0.01 - 0.06i]_{\rm HP} + [0.07]_{\rm tw3}\right\} = (-2.45^{+0.49}_{-0.24}) - (1.94^{+0.32}_{-0.20})i$$

$$\alpha_4^c(\pi\bar{K})/10^{-2} = -2.87 - [0.09 + 0.09i]_{\rm V} + [0.05 - 0.62i]_{\rm P_1} - [0.77 + 0.50i]_{\rm P_2}$$

$$+ \left[\frac{r_{\rm sp}}{0.445}\right]\left\{[0.13]_{\rm LO} + [0.15 + 0.12i]_{\rm HV} - [0.01 - 0.03i]_{\rm HP} + [0.07]_{\rm tw3}\right\} = (-3.33^{+0.44}_{-0.27}) - (1.05^{+0.45}_{-0.36})i$$



# Scale dependence



# Results: Amplitude ratios

| Ratio | NLO | NNLO |
|---|---|---|
| $\frac{P_{\pi\pi}}{T_{\pi\pi}}$ | $-0.121 - 0.021i$ | $-0.124^{+0.031}_{-0.060} + (-0.026^{+0.045}_{-0.046})i$ |
| $\frac{P_{\rho\rho}}{T_{\rho\rho}}$ | $-0.035 - 0.009i$ | $-0.041^{+0.020}_{-0.016} + (-0.014^{+0.019}_{-0.018})i$ |
| $\frac{P_{\pi\rho}}{T_{\pi\rho}}$ | $-0.038 - 0.005i$ | $-0.040^{+0.016}_{-0.030} + (-0.009^{+0.026}_{-0.026})i$ |
| $\frac{P_{\rho\pi}}{T_{\rho\pi}}$ | $0.040 + 0.002i$ | $0.036^{+0.042}_{-0.023} + (-0.001^{+0.033}_{-0.033})i$ |
| $\frac{C_{\pi\pi}}{T_{\pi\pi}}$ | $0.317 - 0.040i$ | $0.320^{+0.255}_{-0.142} + (-0.030^{+0.150}_{-0.091})i$ |
| $\frac{C_{\rho\rho}}{T_{\rho\rho}}$ | $0.165 - 0.064i$ | $0.176^{+0.187}_{-0.133} + (-0.054^{+0.142}_{-0.104})i$ |
| $\frac{C_{\pi\rho}}{T_{\pi\rho}}$ | $0.219 - 0.064i$ | $0.212^{+0.197}_{-0.112} + (-0.062^{+0.114}_{-0.079})i$ |
| $\frac{C_{\rho\pi}}{T_{\rho\pi}}$ | $0.092 - 0.080i$ | $0.112^{+0.189}_{-0.144} + (-0.065^{+0.152}_{-0.115})i$ |
| $\frac{T_{\rho\pi}}{T_{\pi\rho}}$ | $0.821 + 0.016i$ | $0.810^{+0.262}_{-0.200} + (\ 0.010^{+0.062}_{-0.062})i$ |
| $\frac{\alpha_4^c(\pi K)}{\alpha_1(\pi\pi) + \alpha_2(\pi\pi)}$ | $-0.085 - 0.019i$ | $-0.087^{+0.022}_{-0.036} + (-0.021^{+0.029}_{-0.029})i$ |
| $\frac{\alpha_4^c(\pi K^*)}{\alpha_1(\pi\pi) + \alpha_2(\pi\pi)}$ | $-0.029 - 0.005i$ | $-0.030^{+0.015}_{-0.026} + (-0.007^{+0.023}_{-0.023})i$ |
| $\frac{\alpha_4^c(\rho K)}{\alpha_1(\rho\rho) + \alpha_2(\rho\rho)}$ | $0.037 + 0.004i$ | $0.034^{+0.039}_{-0.021} + (\ 0.001^{+0.030}_{-0.030})i$ |
| $\frac{\alpha_4^c(\rho K^*)}{\alpha_1(\rho\rho) + \alpha_2(\rho\rho)}$ | $-0.023 - 0.010i$ | $-0.027^{+0.027}_{-0.016} + (-0.012^{+0.024}_{-0.023})i$ |

- PRELIMINARY numbers: Only $Q_{1,2}$ contribution. Inputs from *[Beneke,Li,TH'09]*

# Results: BR and $A_{\rm CP}$

- PRELIMINARY numbers: Only $Q_{1,2}$ contribution. Inputs from [Beneke,Li,TH'09]
- Branching ratios (1st line, in $10^{-6}$) and direct CP asymmetries (2nd line, in $10^{-2}$).

| | NNLO | NLO | Experiment |
|---|---|---|---|
| $B^- \to \pi^- \pi^0$ | $5.43\,^{+2.66\,+2.05\,+1.27\,+0.52}_{-2.14\,-1.73\,-0.57\,-0.50}$ | $5.33$ | $5.48\,^{+0.35}_{-0.34}$ |
| | $-0.18\,^{+0.03\,+0.08\,+0.03\,+0.01}_{-0.05\,-0.07\,-0.02\,-0.01}$ | $-0.09$ | $2.6\,^{+3.9}_{-3.9}$ |
| $\bar{B}_d^0 \to \pi^+ \pi^-$ | $7.47\,^{+3.15\,+3.36\,+0.30\,+1.18}_{-2.61\,-2.76\,-0.60\,-0.66}$ | $7.30$ | $5.10\,^{+0.19}_{-0.19}$ |
| | $-9.31\,^{+1.82\,+3.08\,+0.40\,+15.74}_{-2.06\,-3.39\,-1.06\,-15.01}$ | $-7.67$ | $-31\,^{+5}_{-5}$ |
| $\bar{B}_d^0 \to \pi^0 \pi^0$ | $0.35\,^{+0.14\,+0.19\,+0.33\,+0.20}_{-0.11\,-0.11\,-0.09\,-0.10}$ | $0.33$ | $1.33\,^{+0.46}_{-0.46}$ |
| | $44.2\,^{+7.9\,+18.3\,+17.2\,+47.7}_{-7.3\,-39.3\,-25.8\,-65.8}$ | $43.7$ | $43\,^{+24}_{-24}$ |
| $B^- \to \pi^- \bar{K}^0$ | $16.03\,^{+0.79\,+9.66\,+0.87\,+13.51}_{-0.77\,-6.68\,-1.28\,-\ \ 5.61}$ | $14.94$ | $23.79\,^{+0.75}_{-0.75}$ |
| | $0.67\,^{+0.15\,+0.25\,+0.04\,+0.76}_{-0.16\,-0.24\,-0.04\,-0.64}$ | $0.62$ | $-1.5\,^{+1.9}_{-1.9}$ |
| $B^- \to \pi^0 K^-$ | $9.57\,^{+0.79\,+5.00\,+0.18\,+7.15}_{-0.74\,-3.50\,-0.39\,-3.01}$ | $8.97$ | $12.94\,^{+0.52}_{-0.51}$ |
| | $9.58\,^{+2.26\,+2.18\,+0.09\,+10.50}_{-2.25\,-3.25\,-0.11\,-11.61}$ | $8.86$ | $4.0\,^{+2.1}_{-2.1}$ |
| $\bar{B}_d^0 \to \pi^+ K^-$ | $14.01\,^{+1.09\,+8.43\,+0.12\,+11.92}_{-1.03\,-5.76\,-0.26\,-\ \ 4.92}$ | $12.88$ | $19.57\,^{+0.53}_{-0.52}$ |
| | $7.01\,^{+1.66\,+2.27\,+0.36\,+\ \ 9.91}_{-1.65\,-2.59\,-0.18\,-11.13}$ | $6.15$ | $-8.2\,^{+0.6}_{-0.6}$ |
| $\bar{B}_d^0 \to \pi^0 \bar{K}^0$ | $5.82\,^{+0.31\,+4.05\,+0.07\,+5.58}_{-0.31\,-2.72\,-0.16\,-2.26}$ | $5.31$ | $9.93\,^{+0.49}_{-0.49}$ |
| | $-4.47\,^{+1.05\,+3.39\,+0.23\,+4.09}_{-0.98\,-2.08\,-0.17\,-3.83}$ | $-4.47$ | $-1\,^{+10}_{-10}$ |

# The decays $B \to D^{(*)}\pi\,/\,\rho$

[Kränkl,TH'15 and in progress]

- Only colour-allowed tree amplitude
  - No colour-suppressed tree amplitude, no penguins
  - Spectator scattering power suppressed
- Applications
  - Ratios of decay widths

$$\frac{\Gamma(\bar{B}_d \to D^+\pi^-)}{\Gamma(\bar{B}_d \to D^{*+}\pi^-)} = \frac{(m_B^2 - m_D^2)^2 |\vec{q}|_{D\pi}}{4m_B^2 |\vec{q}|^3_{D^*\pi}} \left(\frac{F_0(m_\pi^2)}{A_0(m_\pi^2)}\right)^2 \left|\frac{a_1(D\pi)}{a_1(D^*\pi)}\right|^2$$

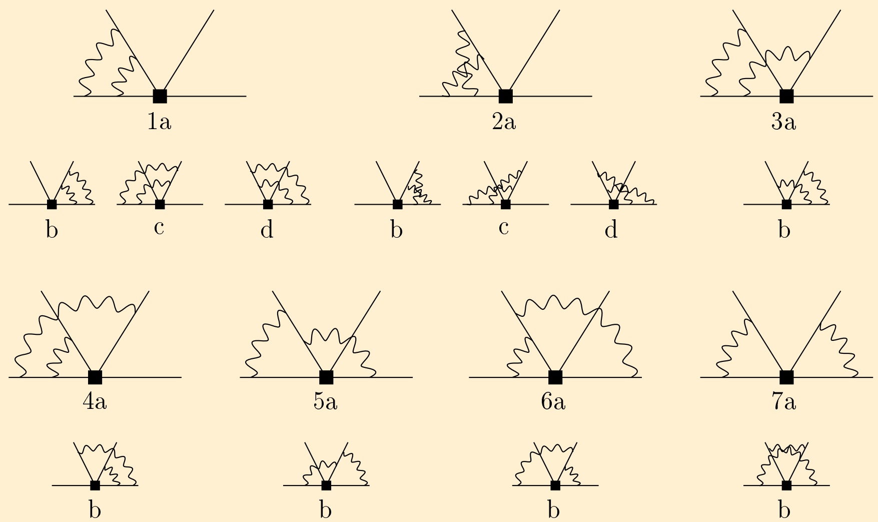
$$\frac{\Gamma(\bar{B}_d \to D^+\rho^-)}{\Gamma(\bar{B}_d \to D^+\pi^-)} = \frac{4m_B^2 |\vec{q}|^3_{D\rho}}{(m_B^2 - m_D^2)^2 |\vec{q}|_{D\pi}} \frac{f_\rho^2}{f_\pi^2} \left(\frac{F_+(m_\rho^2)}{F_0(m_\pi^2)}\right)^2 \left|\frac{a_1(D\rho)}{a_1(D\pi)}\right|^2$$

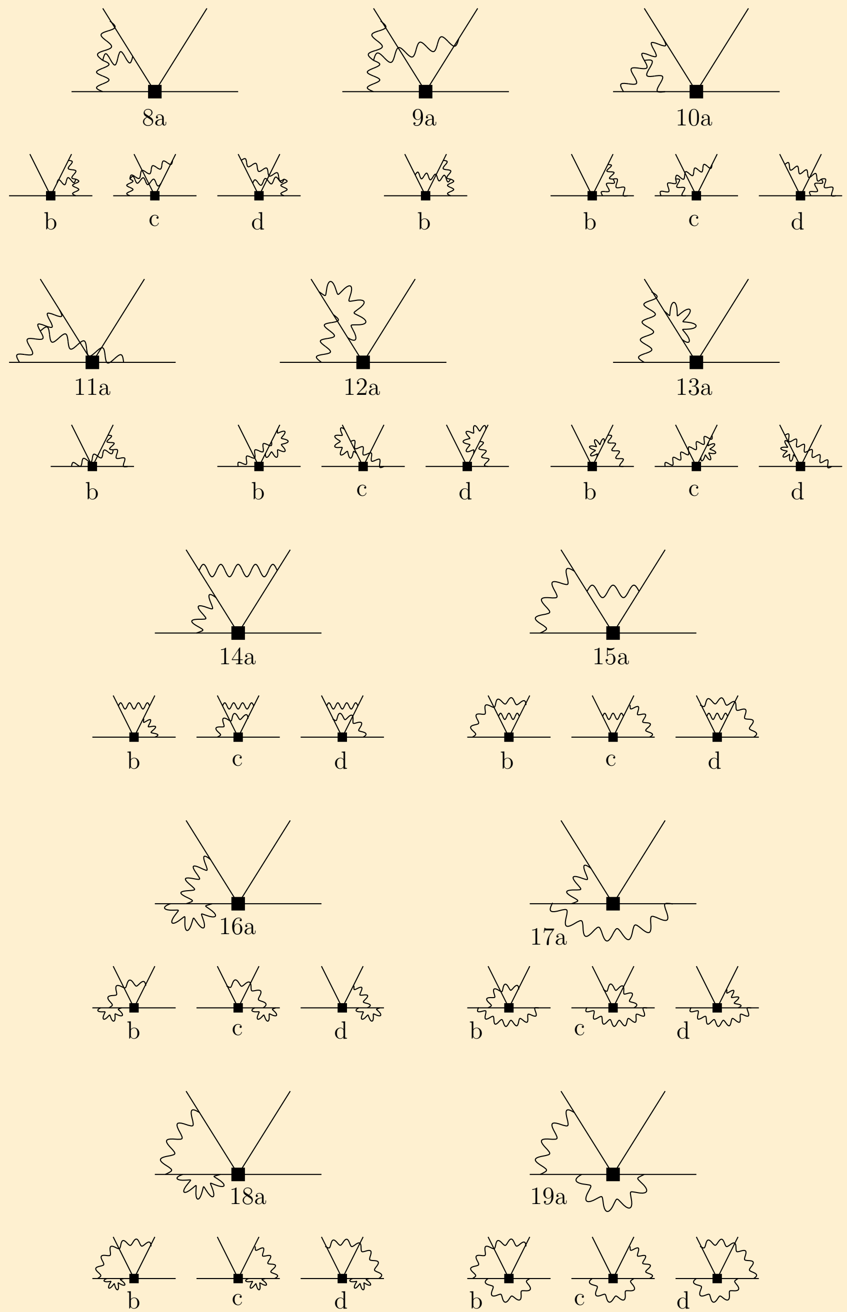
  - Test of factorisation

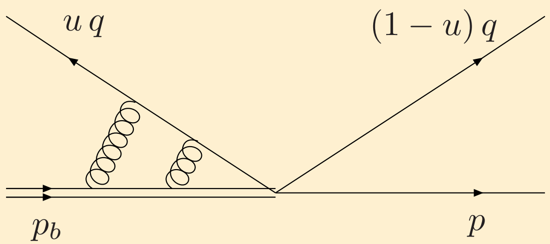
$$\frac{\Gamma(\bar{B}_d \to D^{(*)+}\pi^-)}{d\Gamma(\bar{B}_d \to D^{(*)+}l^-\bar{\nu})/dq^2\big|_{q^2=m_\pi^2}} = 6\pi^2 |V_{ud}|^2 \, f_\pi^2 \, |a_1(D^{(*)}\pi)|^2$$

  - Angular analysis in case of $D^*\rho$
  - Estimate size of power corrections

# Two-loop diagrams for $B \to D\pi$



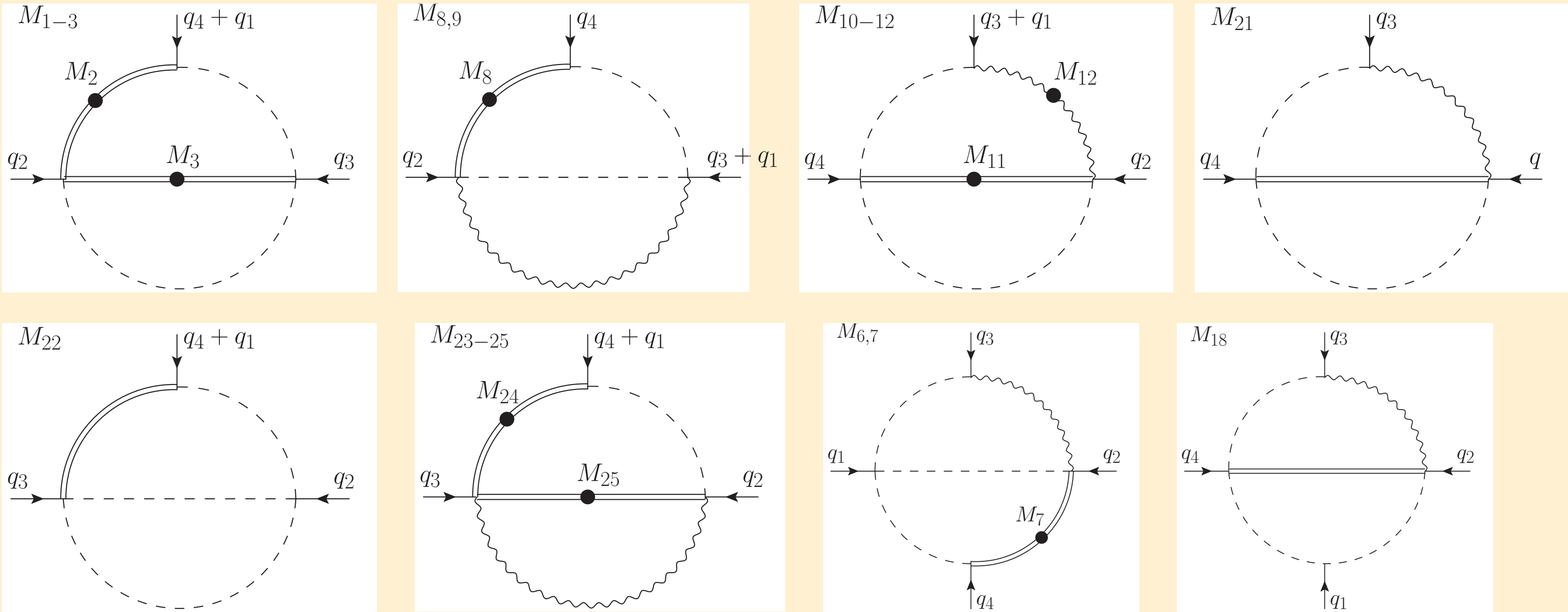




- Kinematics: $p_b^2 = m_b^2$

$$q^2 = 0$$

$$p^2 = m_c^2$$

# Master integrals for $B \to D\pi$




- Also here: Genuine two-scale problem: $u$ , $z_c \equiv m_c^2/m_b^2$. Poles up to $1/\epsilon^4$
- Use Goncharov polylogs with weights $0, \pm 1$ and

$$a_1 = \frac{1}{1 - z_c}\,, \qquad a_3 = \frac{1}{1 - \sqrt{z_c}}\,, \qquad a_5 = \frac{\sqrt{z_c}}{\sqrt{z_c} - 1}\,,$$

$$a_2 = \frac{z_c}{z_c - 1}\,, \qquad a_4 = \frac{1}{1 + \sqrt{z_c}}\,, \qquad a_6 = \frac{\sqrt{z_c}}{\sqrt{z_c} + 1}\,.$$

- Simpler kinematics (can use $u, z_c$) since both $m_b$ and $m_c$ appear in external states.
- Simpler linear combinations to form canonical basis, but more five-liners and $\mathcal{O}(\epsilon^4)$

# Conclusion and Outlook

- $Q_{1,2}$-contribution to penguin amplitudes $\alpha_4^u$ and $\alpha_4^c$ at NNLO ready
  - Solution to master integrals in canonical basis enables results which are almost completely analytical and numerically accurate to high precision.
- Preliminary results
  - NNLO shift in amplitudes is rather sizable
  - Shift in amplitude ratios, BRs, CP asymmetries is moderate
- Need also precise non-perturbative input (Sum Rules, Lattice QCD)
- Future plans
  - Include also penguin operators
  - Phenomenology based on NNLO results
  - Connect QCDF with flavour symmetries *[c.f. Descotes-Genon et. al.; Ciuchini et. al. ]*
  - Power suppressed amplitude $a_6$ at NNLO
  - QED corrections?

# Backup slides

# Some definitions

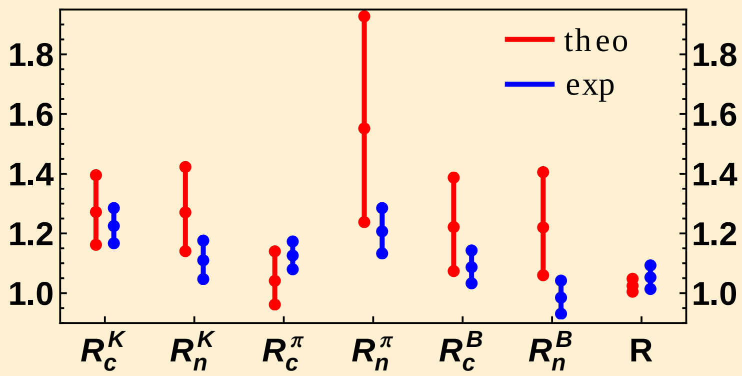
$$A_{\pi\pi} = i\,\frac{G_F}{\sqrt{2}}\,m_B^2 F_+^{B\to\pi}(0) f_\pi$$

$$r_{\rm sp} = \frac{9 f_\pi \hat{f}_B}{m_b \lambda_B F_+^{B\to\pi}(0)}$$

$$\lambda_B^{-1} = \int_0^\infty \frac{d\omega}{\omega}\,\phi_B(\omega,\mu)$$

$$\Delta A_{\rm CP}^-(\pi K) = A_{\rm CP}(B^- \to \pi^0 K^-) - A_{\rm CP}(\bar{B}^0 \to \pi^+ K^-) = \Delta A_{\rm CP}(\pi K)$$

$$\Delta A_{\rm CP}^0(\pi K) = A_{\rm CP}(B^- \to \pi^- \bar{K}^0) - A_{\rm CP}(\bar{B}^0 \to \pi^0 \bar{K}^0)$$



# Results: Tree Amplitudes

*[Beneke,Li,TH'09;Bell'09]*

$$\alpha_1(\pi\pi) = 1.009 + [0.023 + 0.010\,i]_{\rm NLO} + [0.026 + 0.028\,i]_{\rm NNLO}$$
$$- \left[\frac{r_{\rm sp}}{0.445}\right] \Big\{[0.014]_{\rm LOsp} + [0.034 + 0.027\,i]_{\rm NLOsp} + [0.008]_{\rm tw3}\Big\}$$
$$= (1.000^{+0.029}_{-0.069}) + (0.011^{+0.023}_{-0.050})\,i$$

$$r_{\rm sp} = \frac{9 f_{M_1} \hat{f}_B}{m_b \lambda_B f_+^{B\pi}(0)}$$

$$\alpha_2(\pi\pi) = 0.220 - [0.179 + 0.077\,i]_{\rm NLO} - [0.031 + 0.050\,i]_{\rm NNLO}$$
$$+ \left[\frac{r_{\rm sp}}{0.445}\right] \Big\{[0.114]_{\rm LOsp} + [0.049 + 0.051\,i]_{\rm NLOsp} + [0.067]_{\rm tw3}\Big\}$$
$$= (0.240^{+0.217}_{-0.125}) + (-0.077^{+0.115}_{-0.078})\,i$$

- Significant cancellation of NNLO vertex and spectator terms