On infrared divergences of gauge theory amplitudes

Giulio Falcioni,
DESY

Based on works in collaboration with **Vittorio Del Duca**, **Einan Gardi, Mark Harley**, **Lorenzo Magnea**, **Leonardo Vernazza, Chris White**

12th February 2015, DESY, Zeuthen

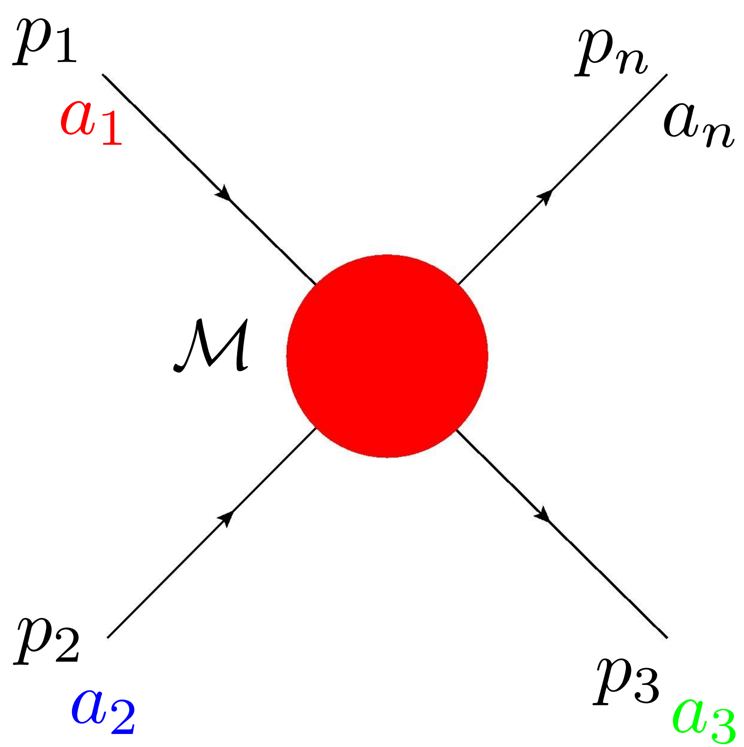
Outline

- Infrared singularities of gauge theory scattering amplitudes
  - The dipole formula
- The high-energy limit
- Beyond the dipole formula: higher order corrections

Infrared singularities of gauge theory scattering amplitudes

Gauge theory scattering amplitudes

We consider fixed-angle scattering amplidues in SU(N) gauge theory: some notation



- Fixed angle scattering amplitudes

$$p_i \cdot p_j = \mathcal{O}\left(Q^2\right)$$
$$Q^2 \gg \Lambda^2_{QCD}$$

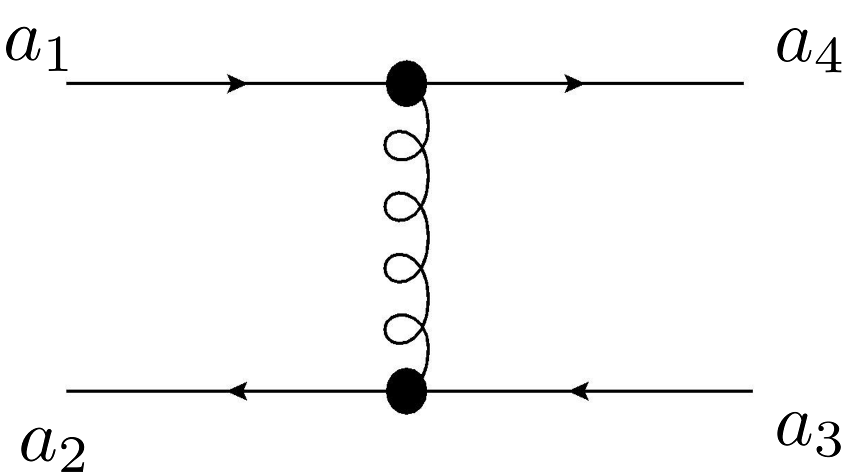
- Colour and kinematic structure

The amplitude is a vector in the space of the tensors with n indices of SU(N)

$$\mathcal{M}_{a_1,a_2,a_3,a_4} = \mathcal{M}^{[i]} \quad c^{[i]}_{a_1,a_2,a_3,a_4}$$

Gauge theory scattering amplitudes

Colour flow in quark-antiquark scattering amplitude: each diagram is decomposed on a basis, as in the following example

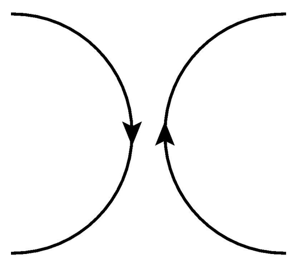
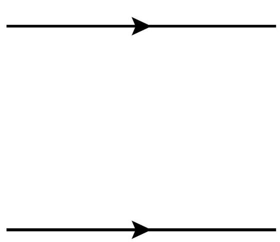


$$\sim \quad T^i_{a_4\, a_1}\, T^i_{a_2\, a_3}$$

$$= \quad \frac{1}{2}\left(\delta_{a_1\, a_2}\delta_{a_3\, a_4} - \frac{1}{N_c}\delta_{a_1\, a_4}\delta_{a_2\, a_3}\right)$$

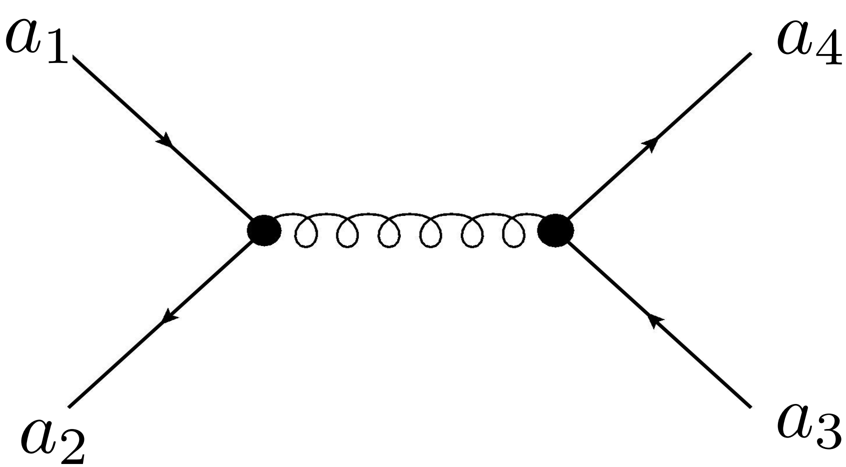
$$c^{[1]}_{a_1\, a_2\, a_3\, a_4} = \delta_{a_1\, a_2}\, \delta_{a_3\, a_4}$$

$$c^{[2]}_{a_1\, a_2\, a_3\, a_4} = \delta_{a_1\, a_4}\, \delta_{a_2\, a_3}$$

$\mathcal{M}_t \quad \simeq \quad \mathcal{M}^{[1]}_t$  $+ \quad \mathcal{M}^{[2]}_t$ 

$$\mathcal{M}^{[2]}_t = -\frac{\mathcal{M}^{[1]}_t}{N_c} \equiv -\frac{\mathcal{M}_t}{N_c}$$

Gauge theory scattering amplitudes



$$\sim \quad T^i_{a_2\,a_1}\, T^i_{a_4\,a_3}$$

$$= \quad \frac{1}{2}\left(\delta_{a_1\,a_4}\delta_{a_2\,a_3} - \frac{1}{N_c}\delta_{a_1\,a_2}\delta_{a_3\,a_4}\right)$$

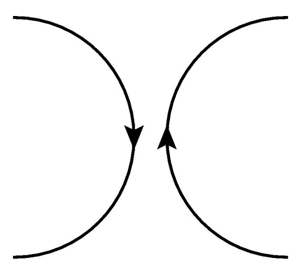
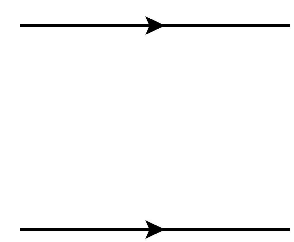
$$c^{[1]}_{a_1\,a_2\,a_3\,a_4} = \delta_{a_1\,a_2}\,\delta_{a_3\,a_4}$$

$$c^{[2]}_{a_1\,a_2\,a_3\,a_4} = \delta_{a_1\,a_4}\,\delta_{a_2\,a_3}$$

This time we write the first component as

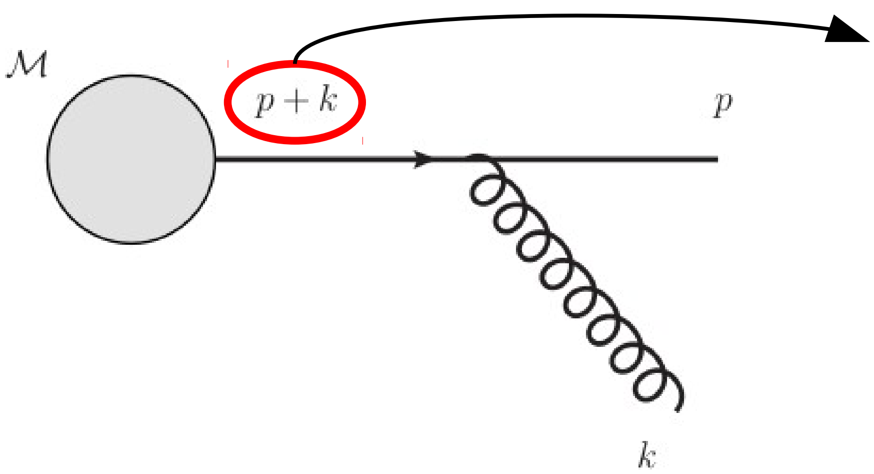
$$\mathcal{M}^{[1]}_s = -\frac{\mathcal{M}^{[2]}_s}{N_c} \equiv -\frac{\mathcal{M}_s}{N_c}$$

Finally the entire amplitude is expressed on the basis of colour flow

$$\mathcal{M} \;=\; \frac{1}{2}\left(\mathcal{M}_t - \frac{\mathcal{M}_s}{N_c}\right)$$  $$+\;\frac{1}{2}\left(\mathcal{M}_s - \frac{\mathcal{M}_t}{N_c}\right)$$ 

Soft and collinear divergences

Infrared divergences are generated by the emission of massless gauge bosons with vanishing energy or parallel to one external state.

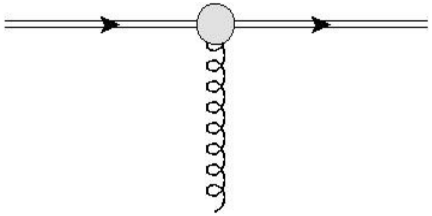


$$\frac{1}{(p+k)^2} = \frac{1}{2p^0 k^0\,(1-\cos\theta)}$$

$k^0 \to 0$ Soft divergence

$\cos\theta \to 0$ Collinear divergence

Divergences are captured by the eikonal approximation of the vertex and the propagator



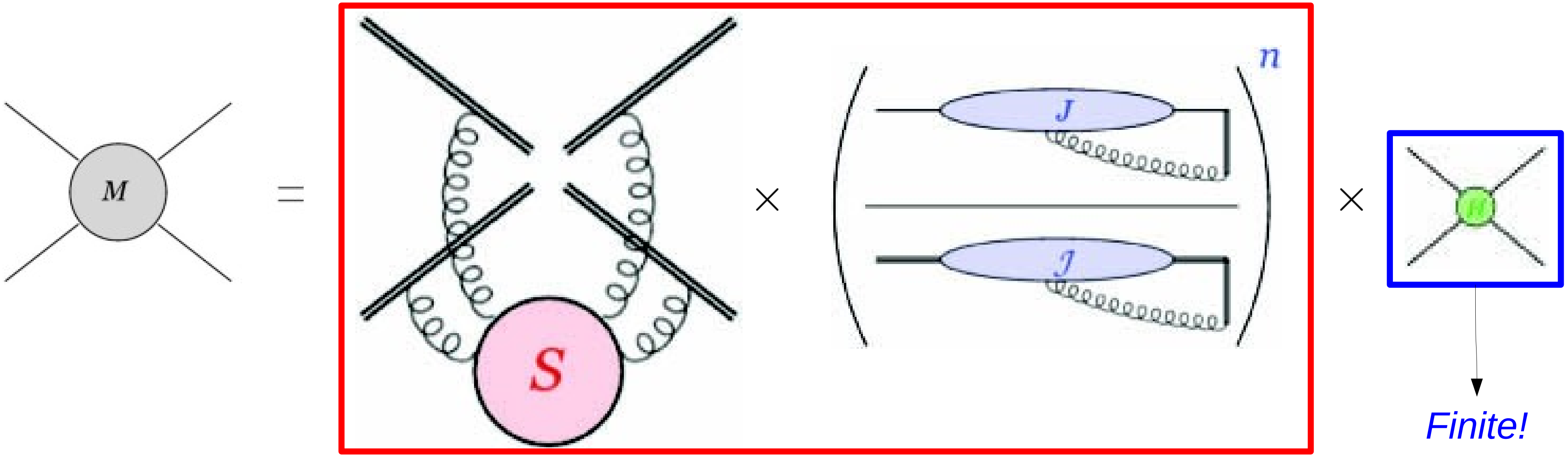
$$i g_s\, \mathbf{T}^a \beta^\mu \;\times\; \frac{i}{\beta\cdot k + i\varepsilon}$$

These rules are consistent with the replacement of external partons with Wilson lines

$$\mathcal{W}_\beta(\infty,0) = \mathcal{P}\,\exp\left[i g_s \int_0^\infty \beta_\mu\, A^\mu(\lambda\beta)\, d\lambda\right] \qquad \beta^\mu = \frac{p^\mu}{\sqrt{p^2}}$$

Soft-collinear factorisation

Infrared singularities are *factorised* in expectation values of gauge invariant operators involving Wilson lines (Sen '83, Sterman Tejeda-Yeomans '02, Aybat Dixon Sterman '06, Dixon Sterman Magnea '08)



The product of soft and jet factors gives the operator which generate infrared divergences at all orders! The factorisation formula takes the form

$$\mathcal{M}\left(\frac{p_i \cdot p_j}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \boxed{\mathcal{Z}\left(\frac{p_i \cdot p_j}{\mu_f^2}, \alpha_s(\mu_f^2), \epsilon\right)} \times \boxed{\mathcal{H}\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{\mu^2}{\mu_f^2}, \alpha_s(\mu^2)\right)}$$

Exponentiation of divergences

Once we have the factorisation formula, we use renormalisation group to exponentiate divergences:

$$\mu \frac{d}{d\mu} \mathcal{Z}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = -\mathcal{Z}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) \times \boxed{\Gamma\left(\frac{p_i}{\mu}, \alpha_s(\mu^2)\right)}$$

Soft anomalous dimension

$$\mathcal{Z}\left(\frac{p_i}{\mu}, \alpha_s(\mu^2), \epsilon\right) = \mathcal{P} \exp\left[-\int_0^{\mu^2} \frac{d\lambda}{\lambda} \Gamma\left(\frac{p_i}{\lambda}, \alpha_s(\lambda)\right)\right]$$

The soft anomalous dimension is the fundamental object!

The dipole formula is an all order ansatz for the soft anomalous dimension (Gardi, Magnea '09; Bechert, Neubert '09). It is exact at two loops, first corrections can arise at three loops.

$$\Gamma_{\rm dip}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = -\frac{1}{4}\widehat{\gamma}_K\left(\alpha_s(\lambda^2)\right) \sum_{i \neq j = 1}^{n} \boxed{\mathbf{T_i} \cdot \mathbf{T_j}} \log\left(\frac{2|p_i \cdot p_j| e^{-i\pi\sigma_{ij}}}{\lambda^2}\right) + \sum_{i=1}^{n} \gamma_{J_i}\left(\alpha_s(\lambda^2)\right)$$

$\widehat{\gamma}_K =$ cusp anomalous dimension

$\gamma_{J_i} =$ quark or gluon collinear anomalous dimension

$$\sigma_{ij} = \begin{cases} 1 & \text{both partons i,j in the initial or final state} \\ 0 & \text{otherwise} \end{cases}$$

Colour structure *fixed* as a sum over dipoles

The high-energy limit

High-energy logarithms

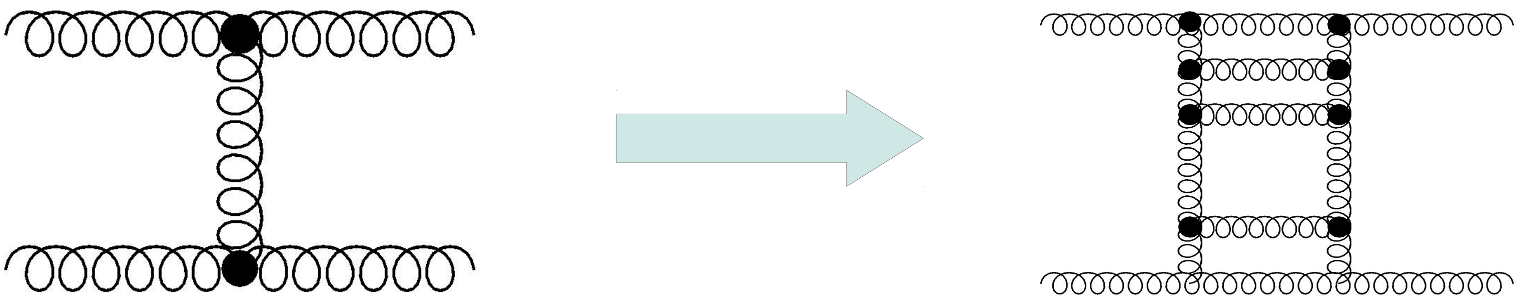
Beyond the fixed-angle regime, scattering amplitudes acquire new enhancements: we consider the high-energy regime

$$s \simeq (-u) \gg (-t)$$

This ordering of the kinematic invariants generates logarithmic enhancements at all orders in perturbation theory

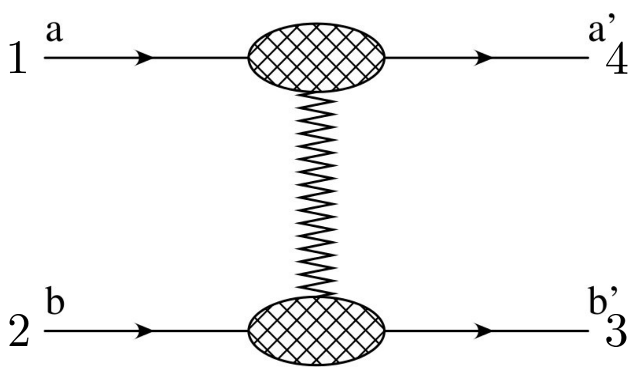
$$\mathcal{M}^{(n)} = \underbrace{M^{(n),n} \log^n \left(\frac{s}{-t}\right)}_{\text{Leading logarithm}} + \underbrace{M^{(n),n-1} \log^{n-1} \left(\frac{s}{-t}\right)}_{\text{Next-to-leading logarithm}} + \dots$$

At tree level, amplitudes in this limit are dominated by gluon exchange in the t-channel. This feature generalises to all orders with the exchange of a gluon ladder



Gluon reggeization

The **leading** contribution (LL) to **all 2-->2 QCD scattering amplitudes** is given by the exchange of a reggeized gluon in the t channel



$$\mathcal{M} = 2g_s^2 \boxed{\frac{s}{t}\left(\frac{s}{-t}\right)^{\alpha(-t)}} \boxed{(T^i)_{a'a}(T^i)_{b'b}} \times \boxed{C_{a'a}C_{b'b}}$$

(Kuraev, Fadin, Lipatov '76)

$$\alpha(-t) = \frac{\alpha_s}{\pi}\frac{N_c}{2\epsilon} + \mathcal{O}\left(\frac{\alpha_s^2}{\pi^2}\right)$$

Gluon Regge trajectory

C_{ab} (impact factors) describe the coupling of external partons to the reggeized gluon.
T^i_{ab} are SU(N) generators in the representation of the corresponding parton: for example, in quark-quark scattering

$$(T^i)_{a'a}(T^i)_{b'b} \equiv (\mathbf{T}^i_{\mathbf{R}})_{a'a}(\mathbf{T}^i_{\mathbf{R}})_{b'b}$$

The colour structure is given by the tree level amplitude and it's diagonal in the t channel

$$\mathbf{T_t^2} = (\mathbf{T}_1 + \mathbf{T}_4)^2 \Rightarrow \mathbf{T_t^2}\left[(T^i)_{a'a}(T^i)_{b'b}\right] = C_A(T^i)_{a'a}(T^i)_{b'b}$$

Regge master formula

The general formula for a scattering amplitude involving partons *r*,*s* exchanging a reggeized gluon in the t-channel is (Lipatov, Fadin '93)

$$\mathcal{M}_{rs} = \frac{g_s^2}{2} H_{rs}^{(0)} \times C_r(g_s^2) \left[A_+ \left(\frac{s}{-t} \right) + \kappa_{rs}\, A_- \left(\frac{s}{-t} \right) \right] C_s(g_s^2) \qquad (1)$$

$$A_\pm = \left(\frac{-s}{-t} \right)^{\alpha(-t)} \pm \left(\frac{s}{-t} \right)^{\alpha(-t)}$$

$$\kappa_{rs} = \begin{cases} \frac{4-N_c^2}{N_c^2} & \text{if } rs = qq \\ 0 & \text{if } rs = gg \text{ or } qg \end{cases}$$

Definite parity under crossing

$$s \leftrightarrow u \sim -s$$

This formula holds also for the real parts of the amplitude at NLL (Fadin, Fiore, Kozlov, Reznichenko '06). At NNLL, the amplitude has multiple-reggeon exchanges and Regge cuts.

Expansion of (1) at every perturbative order. At one loop:

$$\mathcal{M}_{rs}^{(1)} = \mathcal{M}_{rs}^{(0)} \times \left[\underbrace{C_{rr}^{(1)} + C_{ss}^{(1)}}_{NLL} + \underbrace{\alpha^{(1)} \log \left(\frac{s}{-t} \right)}_{LL} - i \frac{\pi}{2} (1 + \kappa_{rs}) \alpha^{(1)} \right]$$

An infrared approach to reggeization

High-energy limit of the dipole formula (Del Duca, Duhr, Gardi, Magnea, White '11)

$$\mathcal{Z}\left(\frac{s}{\mu^2},\frac{t}{\mu^2},\alpha_s\right) = \exp\left[-i\frac{\pi}{2}K(\alpha_s)\mathcal{C}_{tot}\right]\mathcal{Z}_{1,\mathbf{R}}\left(\frac{t}{\mu^2},\alpha_s\right)\widetilde{\mathcal{Z}}\left(\frac{s}{t},\alpha_s\right)$$

Each factor is defined as

$$\mathcal{Z}_{1,\mathbf{R}}\left(\frac{t}{\mu^2},\alpha_s\right) = \exp\left\{\frac{1}{2}\left[K(\alpha_s)\log\left(\frac{-t}{\mu^2}\right)+D(\alpha_s)\right]\mathcal{C}_{tot}+\sum_i B_i(\alpha_s)\right\}$$

$$\widetilde{\mathcal{Z}}\left(\frac{s}{-t},\alpha_s\right) = \exp\left\{K(\alpha_s)\left[\log\left(\frac{s}{-t}\right)\mathbf{T}_\mathbf{t}^2 + i\pi\mathbf{T}_\mathbf{s}^2\right]\right\}$$

$$\mathbf{T_s} = (\mathbf{T_1}+\mathbf{T_2}) = -(\mathbf{T_3}+\mathbf{T_4})$$
$$\mathbf{T_t} = (\mathbf{T_1}+\mathbf{T_4}) = -(\mathbf{T_2}+\mathbf{T_3})$$

Non trivial colour structure

$K(\alpha_s)$, $D(\alpha_s)$ and $B(\alpha_s)$ are integrals of cusp or collinear anomalous dimension over the scale of the coupling and contain only poles, e.g.

$$K(\alpha_s) \equiv \frac{\alpha_s}{\pi}K^{(1)} + \cdots = \frac{\alpha_s}{\pi}\frac{\widehat{\gamma}_K^{(1)}}{4\epsilon} + \mathcal{O}\left(\frac{\alpha_s^2}{\pi^2}\right)$$

Reggeization at LL

We can construct directly the scattering amplitude at LL accuracy

$$\mathcal{M}_{LL}\left(\frac{p_i\cdot p_j}{\mu^2},\alpha_s(\mu^2),\epsilon\right)=\mathcal{Z}_{LL}\left(\frac{p_i\cdot p_j}{\mu_f^2},\alpha_s(\mu_f^2),\epsilon\right)\times\mathcal{H}_{LL}\left(\frac{p_i\cdot p_j}{\mu^2},\frac{\mu^2}{\mu_f^2},\alpha_s(\mu^2)\right)$$

Where the infrared operator and the hard parts at LL are

$$Z_{LL}=\exp\left[\frac{\alpha_s}{2\pi\epsilon}\log\left(\frac{s}{-t}\right)\mathbf{T_t^2}\right]\qquad\mathcal{H}_{LL}=H^{(0)}+\frac{\alpha_s}{\pi}H^{(1),1}\log\left(\frac{s}{-t}\right)+\dots$$

Recalling that $H^{(0)}$ is an eigenstate of T_t^2, we obtain

$$\mathcal{M}_{LL}=H^{(0)}+\frac{\alpha_s}{\pi}\log\left(\frac{s}{-t}\right)\left[\frac{C_A}{2\epsilon}H^{(0)}+H^{(1),1}\right]+\frac{\alpha_s^2}{\pi^2}\log^2\left(\frac{s}{-t}\right)\left[\frac{C_A^2}{8\epsilon^2}H^{(0)}+\frac{\mathbf{T_t^2}}{2\epsilon}H^{(1),1}+H^{(2),2}\right]$$

- The leading singularity of the amplitude at all orders is already in reggeized form and we can read off the divergent part of the one loop Regge trajectory
- Because of reggeization the hard parts $H^{(n),n}$ must be proportional to the tree level

One loop amplitudes

Expanding both the high-energy and the infrared factorisation, we find the octet component

$$\mathrm{Re}M_{rs}^{(1),1,[8]} \Rightarrow \boxed{\alpha^{(1)} H_{rs}^{(0),[8]}} = \boxed{K^{(1)} C_A H_{rs}^{(0),[8]} + H_{rs}^{(1),1,[8]}}$$

$$\mathrm{Re}M_{rs}^{(1),0,[8]} \Rightarrow \boxed{\left(C_r^{(1)} + C_s^{(1)}\right) H^{(0),[8]}} = \boxed{Z_{1,\mathbf{R},rs}^{(1)} H_{rs}^{(0),[8]} + \mathrm{Re}H_{rs}^{(1),0,[8]}}$$

These equations give:

- Interpretation of the singularities of Regge trajectory and impact factors in terms of cusp and collinear anomalous dimensions.
- Constraints on the hard parts from reggeization.

For example, one loop Regge trajectory, quark and gluon impact factors are

$$\alpha^{(1)} = K^{(1)} C_A + \frac{H_{rs}^{(1),1,[8]}}{H_{rs}^{(0),[8]}} \equiv K^{(1)} C_A + \widehat{H}_{rs}^{(1),1,[8]}$$

$$\mathrm{C}_q^{(1)} = \frac{1}{2}\left[Z_{1,\mathbf{R},qq}^{(1)} + \mathrm{Re}\widehat{H}_{qq}^{(1),0,[8]}\right] \qquad C_g^{(1)} = \frac{1}{2}\left[Z_{1,\mathbf{R},gg}^{(1)} + \mathrm{Re}\widehat{H}_{gg}^{(1),0,[8]}\right]$$

- Infrared singularity of the Regge trajectory determined by the cusp anomalous dimension
- Divergences in the impact factors given by collinear anomalous dimensions

Universality constraints

Universality of Regge pole factorisation constraints the scattering amplitudes

- The LL term must be the same in all the three processes, because it's given by the gluon Regge trajectory

$$\widehat{H}^{(1),1,[8]}_{qq} = \widehat{H}^{(1),1,[8]}_{gg} = \widehat{H}^{(1),1,[8]}_{qg} \qquad \text{Direct calculation gives} \qquad \widehat{H}^{(1),1,[8]}_{rs} = \mathcal{O}(\epsilon)$$

- NLL term of the quark-gluon amplitude must be determined by

$$\widehat{M}^{(1),0,[8]}_{qg} = \boxed{C^{(1)}_q + C^{(1)}_g} = \boxed{Z^{(1)}_{1,\mathbf{R},qg} + \mathrm{Re}\widehat{H}^{(1),0,[8]}_{qg}}$$

$$\Rightarrow \boxed{\frac{1}{2}\left[Z^{(1)}_{1,\mathbf{R},qq} + Z^{(1)}_{1,\mathbf{R},gg}\right] + \frac{1}{2}\left[\mathrm{Re}\widehat{H}^{(1),0,[8]}_{qq} + \mathrm{Re}\widehat{H}^{(1),0,[8]}_{gg}\right]} = \boxed{Z^{(1)}_{1,\mathbf{R},qg} + \mathrm{Re}\widehat{H}^{(1),0,[8]}_{qg}}$$

The solutions of this equation is

$$Z^{(1)}_{1,\mathbf{R},qg} = \frac{1}{2}\left[Z^{(1)}_{1,\mathbf{R},qq} + Z^{(1)}_{1,\mathbf{R},gg}\right]$$

$$\mathrm{Re}\widehat{H}^{(1),0,[8]}_{qg} = \frac{1}{2}\left[\mathrm{Re}\widehat{H}^{(1),0,[8]}_{qq} + \mathrm{Re}\widehat{H}^{(1),0,[8]}_{gg}\right]$$

Automatically satisfied by the definition of jet function, verified by finite parts in explicit calculations.

Two loops

The structure of two-loops scattering amplitudes is more complicated, there appear NNLL contributions which are known to break the Regge pole factorisation formula.

$$\frac{\left(\alpha^{(1)}\right)^2}{2} = \frac{\left(K^{(1)}C_A\right)^2}{2} + K^{(1)}C_A\widehat{H}^{(1),1,[8]}_{rs} + \mathrm{Re}\widehat{H}^{(2),2,[8]}_{rs} \qquad \Rightarrow \mathrm{Re}\widehat{H}^{(2),2,[8]}_{rs} = \frac{\left(\widehat{H}^{(1),1,[8]}\right)^2}{2} = \mathcal{O}(\epsilon^2)$$

$$\alpha^{(2)} = K^{(2)}C_A + \mathrm{Re}\widehat{H}_{rs} \qquad \Rightarrow \mathrm{Re}\widehat{H}^{(2),1,[8]}_{qq} = \mathrm{Re}\widehat{H}^{(2),1,[8]}_{gg} = \mathrm{Re}\widehat{H}^{(2),1,[8]}_{qg}$$

- The singular part of the two-loops Regge trajectory is given by the cusp anomalous dimension
- The relations on infrared finite parts are verified by explicit calculation

$$\mathrm{Re}\widehat{H}_{qq} = \mathrm{Re}\widehat{H}_{gg} = \mathrm{Re}\widehat{H}_{qg} = \frac{C_A}{216}\left[C_A\left(202 - 27\zeta(3)\right) - 28n_f\right]$$

The analysis of NNLL terms can give some insight on the first contribution **beyond** Regge pole factorisation.

Factorisation breaking

The Regge pole factorisation formula must be corrected introducing a remainder at NNLL (Del Duca, Magnea, Vernazza, GF '13,'14):

$$\mathrm{Re} M_{rs}^{(2),0,[8]} = \left\{ C_r^{(2)} + C_s^{(2)} + C_r^{(1)} C_s^{(1)} - \frac{\pi^2}{4}\big(1+\kappa_{rs}\big)\big(\alpha^{(1)}\big)^2 + \frac{R_{rs}^{(2),0,[8]}}{2} \right\} H_{rs}^{(0),[8]}$$

Soft-collinear factorisation explains the origin of the rest function: the expansion of the amplitude at two loops is

$$\mathrm{Re} M_{rs}^{(2),0,[8]} = Z_{1,\mathbf{R},rs}^{(2)} H^{(0),[8]} + Z_{1,\mathbf{R},rs}^{(1)} \mathrm{Re} H^{(1),0,[8]} + \mathrm{Re} H^{(2),0,[8]}$$

$$- \frac{\pi^2}{8} \left(K^{(1)}\right)^2 \left((\mathbf{O_{s-u}})^2 + 2\mathbf{O_{t,s}}(1+\kappa_{rs}) - 2\mathbf{O_{t,s}}\mathbf{O_{s-u}}(1+\kappa_{rs}) + (\mathbf{T_t^2})^{\mathbf{2}}(1+\kappa_{rs})^2 \right) H_{rs}^{(0)}$$

$$- \frac{\pi}{2} K^{(1)} \left(\mathbf{O_{s-u}} - (1+\kappa_{rs}) \mathbf{T_t}^2 \right) \mathrm{Im} H^{(1),0}$$

$$\mathbf{O_{s-u}} = 2\mathbf{T_s^2} + (1+\kappa_{rs})\mathbf{T_t^2} - \mathcal{C}_{tot} \qquad \mathbf{O_{t,s}} = \left[\mathbf{T_t^2}, \mathbf{T_s^2}\right]$$

- **Not** diagonal in the t-channel as expected from Regge pole factorisation.
- **Depend** on **both** the incoming partons: this is not consistent with structure of impact factors describing the coupling of **one** state to the reggeon.

Definition of the remainder

Comparison of the expressions for $M^{(2),0,[8]}_{rr}$ give the sum of two loop impact factor and rest.

Soft-collinear factorisation suggests to include all the *non-universal* terms in the rest:

$$R^{(2),0,[8]} = -\frac{\pi^2}{4}\frac{\left(K^{(1)}\right)^2}{H^{(0),[8]}_{rs}}\left[\left((\mathbf{O_{s-u}})^2 - (1-\kappa^2_{rs})\left(T^2_t\right)^2\right)H^{(0)}_{rs}\right]^{[8]} - \frac{\pi K^{(1)}}{H^{(0),[8]}_{rs}}\left[\mathbf{O_{s-u}}\mathrm{Im}H^{(1),0}\right]^{[8]} + \frac{\pi^2}{2}K^{(1)}C_A(1-\kappa^2_{rs})\widehat{H}^{(1),1,[8]}$$

$$C^{(2)}_r = -\frac{1}{8}\left(Z^{(1)}_{1,\mathbf{R},rr}\right)^2 + \frac{1}{2}Z^{(2)}_{1,\mathbf{R},rr} + \frac{1}{4}Z^{(1)}_{1,\mathbf{R},rr}\mathrm{Re}\widehat{H}^{(1),0,[8]}_{rr} - \frac{1}{8}\left(\mathrm{Re}\widehat{H}^{(1),0,[8]}\right)^2 + \frac{1}{2}\mathrm{Re}\widehat{H}^{(2),0,[8]}_{rr}$$

- Introducing numeric values...

$$R^{(2),0,[8]}_{qq} = \frac{\pi^2}{4\epsilon^2}\left(1-\frac{3}{N_c^2}\right)\left(1-\epsilon^2\frac{\pi^2}{6}\right) \qquad R^{(2),0,[8]}_{gg} = -\frac{3\pi^2}{2\epsilon^2}\left(1-\epsilon^2\frac{\pi^2}{6}\right) \qquad R^{(2),0,[8]}_{qg} = -\frac{\pi^2}{4\epsilon^2}\left(1-\epsilon^2\frac{\pi^2}{6}\right)$$

$$\Delta \equiv \frac{1}{2}\left[R^{(2),0,[8]}_{qg} - \frac{1}{2}\left(R^{(2),0,[8]}_{qq} + R^{(2),0,[8]}_{gg}\right)\right] = \frac{3\pi^2}{16\epsilon^2}\left(\frac{N_c^2+1}{N_c^2}\right)\left(1-\epsilon^2\frac{\pi^2}{6}\right)$$

Definition of the remainder

Comparison of the expressions for $M^{(2),0,[8]}_{rr}$ give the sum of two loop impact factor and rest.
Soft-collinear factorisation suggests to include all the *non-universal* terms in the rest:

$$R^{(2),0,[8]} = -\frac{\pi^2}{4}\frac{\left(K^{(1)}\right)^2}{H^{(0),[8]}_{rs}}\left[\left((\mathbf{O_{s-u}})^2 - (1-\kappa^2_{rs})\left(T^2_t\right)^2\right)H^{(0)}_{rs}\right]^{[8]} - \frac{\pi K^{(1)}}{H^{(0),[8]}_{rs}}\left[\mathbf{O_{s-u}}\mathrm{Im}H^{(1),0}\right]^{[8]} + \frac{\pi^2}{2}K^{(1)}C_A(1-\kappa^2_{rs})\widehat{H}^{(1),1,[8]}$$

$$C^{(2)}_r = -\frac{1}{8}\left(Z^{(1)}_{1,\mathbf{R},rr}\right)^2 + \frac{1}{2}Z^{(2)}_{1,\mathbf{R},rr} + \frac{1}{4}Z^{(1)}_{1,\mathbf{R},rr}\mathrm{Re}\widehat{H}^{(1),0,[8]}_{rr} - \frac{1}{8}\left(\mathrm{Re}\widehat{H}^{(1),0,[8]}\right)^2 + \frac{1}{2}\mathrm{Re}\widehat{H}^{(2),0,[8]}_{rr}$$

- Introducing numeric values...

$$R^{(2),0,[8]}_{qq} = \frac{\pi^2}{4\epsilon^2}\left(1-\frac{3}{N_c^2}\right)\left(1-\epsilon^2\frac{\pi^2}{6}\right) \qquad R^{(2),0,[8]}_{gg} = -\frac{3\pi^2}{2\epsilon^2}\left(1-\epsilon^2\frac{\pi^2}{6}\right) \qquad R^{(2),0,[8]}_{qg} = -\frac{\pi^2}{4\epsilon^2}\left(1-\epsilon^2\frac{\pi^2}{6}\right)$$

$$\Delta \equiv \frac{1}{2}\left[R^{(2),0,[8]}_{qg} - \frac{1}{2}\left(R^{(2),0,[8]}_{qq} + R^{(2),0,[8]}_{gg}\right)\right] = \frac{3\pi^2}{16\epsilon^2}\left(\frac{N_c^2+1}{N_c^2}\right)\left(1-\epsilon^2\frac{\pi^2}{6}\right)$$

- Divergent and finite parts of impact factors satisfy again symmetric constraints

$$Z^{(2)}_{1,\mathbf{R},qg} = \frac{1}{8}\left[4Z^{(2)}_{1,\mathbf{R},qq} + 4Z^{(2)}_{1,\mathbf{R},gg} + 2Z^{(1)}_{1,\mathbf{R},qq}Z^{(1)}_{1,\mathbf{R},gg} - \left(Z^{(1)}_{1,\mathbf{R},qq}\right)^2 - \left(Z^{(1)}_{1,\mathbf{R},gg}\right)^2\right]$$

$$\mathrm{Re}\widehat{H}^{(2),0,[8]}_{1,\mathbf{R},qg} = \frac{1}{8}\left[4\mathrm{Re}\widehat{H}^{(2),0,[8]}_{1,\mathbf{R},qq} + 4\mathrm{Re}\widehat{H}^{(2),0,[8]}_{1,\mathbf{R},gg} + 2\mathrm{Re}\widehat{H}^{(1),0,[8]}_{1,\mathbf{R},qq}\mathrm{Re}\widehat{H}^{(1),0,[8]}_{1,\mathbf{R},gg} - \left(\mathrm{Re}\widehat{H}^{(1),0,[8]}_{1,\mathbf{R},qq}\right)^2 - \left(\mathrm{Re}\widehat{H}^{(1),0,[8]}_{1,\mathbf{R},gg}\right)^2\right]$$

Three loops

At three-loop order the NNLL contribution is proportional to the logarithm of *s*, so it mixes with the three loop Regge trajectory. Regge trajectory and remainder are defined similarly to impact factors in the two loops case.
(Del Duca, Magnea, Vernazza, GF, '13, '14)

$$R_{rs}^{(3),1,[8]} = \frac{\pi^2}{4}\left(K^{(1)}\right)^3\left[-\frac{4}{3}\mathbf{O_{s,t,s}} + \mathbf{O_{s-u}}\mathbf{O_{t,s}} - \frac{1}{2}\mathbf{T_t^2}\mathbf{O_{s-u}^2} + \frac{1}{2}\left(\mathbf{T_t^2}\right)^3\left(1-\kappa_{rs}^2\right)\right]\frac{2H^{(0)}}{H^{(0),[8]}}$$
$$+\left(K^{(1)}\right)^2\left[\pi\left(\mathbf{O_{t,s}} - \mathbf{T_t^2}\mathbf{O_{s-u}}\right)\mathrm{Im}\widehat{H}^{(1),0} - \frac{\pi^2}{4}\mathbf{O_{s-u}^2}\widehat{H}^{(1),1} + \frac{3\pi^2}{4}\left(\mathbf{T_t^2}\right)^2\left(1-\kappa_{rs}^2\right)\widehat{H}^{(1),1}\right] + \mathcal{O}\left(\epsilon^0\right)$$

where $\mathbf{O_{s,t,s}} \equiv \left[\mathbf{T_s^2}, \left[\mathbf{T_t^2}, \mathbf{T_s^2}\right]\right]$

Singularities of the Regge trajectory at three loops are defined as the integral of the cusp

$$\alpha^{(3)} = K^{(3)}C_A + \mathcal{O}\left(\epsilon^0\right)$$

Numerical values, up to the single pole

$$R_{qq}^{(3),1,[8]} = \frac{\pi^2}{\epsilon^3}\frac{2N_c^2-5}{12N_c}\left(1-\epsilon^2\frac{\pi^2}{4}\right) \quad R_{gg}^{(3),1,[8]} = -\frac{\pi^2}{\epsilon^3}\frac{2}{3}N_c\left(1-\epsilon^2\frac{\pi^2}{4}\right) \quad R_{qg}^{(3),1,[8]} = -\frac{\pi^2}{\epsilon^3}\frac{N_c}{24}\left(1-\epsilon^2\frac{\pi^2}{4}\right)$$

Beyond three loops

Comparison of high-energy and infrared factorisation can be extended to all orders for the leading and next-to-leading logarithms. Using the expansion of Z up to NLL (Del Duca, Duhr, Gardi, Magnea, White '11)

$$\mathcal{Z} \simeq \left(\frac{s}{-t}\right)^{\frac{\alpha_s}{\pi}K^{(1)}+\frac{\alpha_s^2}{\pi^2}K^{(2)}} \left\{1+\frac{\alpha_s}{\pi}\left[Z^{(1)}_{1,\mathbf{R}} + i\pi K^{(1)}\left(-\frac{1}{2}(1+\kappa_{rs})\mathbf{T}_\mathbf{t}^2 + \frac{\mathbf{O_{s-u}}}{2} - \frac{\alpha_s}{\pi}\frac{K^{(1)}}{2}\log\left(\frac{s}{-t}\right)\mathbf{O_{t,s}} + \ldots\right)\right]\right\}$$

we get general expression of the amplitude at LL and NLL, to be compared with the Regge formula:

$$\mathcal{M}_{LL} = \mathcal{Z}_{LL} \times \mathcal{H}_{LL}$$
$$\mathcal{M}_{NLL} = \mathcal{Z}_{LL} \times \mathcal{H}_{NLL} + \mathcal{Z}_{NLL} \times \mathcal{H}_{LL}$$

Hard parts are constrained by the lower orders (Del Duca, Magnea, Vernazza, GF '14)

$$\mathrm{Im}\widehat{H}^{(n),n,[8]} = 0,$$
$$\mathrm{Re}\widehat{H}^{(n),n,[8]} = \frac{1}{n!}\left(\mathrm{Re}\widehat{H}^{(1),1,[8]}\right)^n = \mathcal{O}\left(\epsilon^n\right)$$

Assuming the Regge formula for the imaginary part of the octet too: (Del Duca, Magnea, Vernazza, GF '14)

$$\mathrm{Im}\widehat{H}^{(n),n-1,[8]} = -\pi\frac{1+\kappa_{rs}}{2}\left(n\widehat{H}^{(n),n,[8]}\right) = \mathcal{O}\left(\epsilon^n\right)$$
$$\mathrm{Re}\widehat{H}^{(n),n-1,[8]} = \mathrm{Re}\widehat{H}^{(2),1,[8]}\widehat{H}^{(n-2),n-2,[8]} + (2-n)\mathrm{Re}\widehat{H}^{(1),0,[8]}\widehat{H}^{(n-1),n-1,[8]} = \mathcal{O}\left(\epsilon^{n-2}\right)$$

Conclusion 1

- Infrared factorisation and Regge factorisation are strictly related and we can use one to get interesting information on the other.

  - Infrared factorisation explains the reggeization of singularities in the high energy limit, including the NNLL terms which are not described in terms of Regge pole factorisation.

  - Regge factorisation constraints hard parts, which are beyond the reach of the dipole formula.

- Future work must investigate the interplay between the two factorisations and find out if it is possible to fully understand the high-energy regime in terms of long distance physics.

- Interesting phenomenological applications concern the study of quark-gluon backward scattering, which is dominated by the exchange of a reggeized quark in the u-channel and jet production in multi-Regge kinematics, which is relevant for collider searches.

Beyond the dipole formula

Structure of the infrared divergences

Infrared singularities are generated by the operator

$$\mathcal{Z}\left(\frac{p_i \cdot p_j}{\mu_F^2}, \alpha_s(\mu_F^2), \epsilon\right) = S\left(\beta_i \cdot \beta_j, \alpha_s, \epsilon\right) \prod_{i=1}^{N} \frac{J_i\left(\frac{(p_i \cdot n_i)^2}{\mu_F^2 n_i^2}, \alpha_s, \epsilon\right)}{\mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s, \epsilon\right)}$$

Where each factor has a gauge invariant definition

$$S = \langle 0 | \prod_{i=1}^{n} \mathcal{W}_{\beta_i}(\infty, 0) |0\rangle$$

Soft gluon exchanges

$$J_i = \langle 0 | \mathcal{W}_{n_i}(\infty, 0)\Psi(0) |p\rangle$$

Collinear singularities

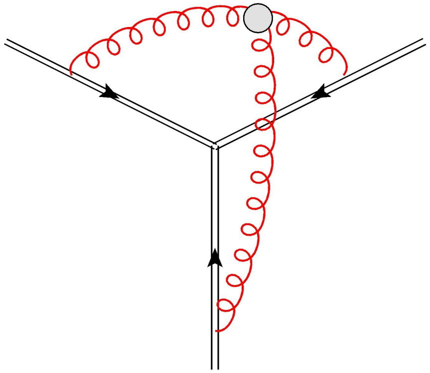
$$\mathcal{J}_i = \langle 0 | \mathcal{W}_{n_i}(\infty, 0)\mathcal{W}_{\beta_i}(0, -\infty) |0\rangle$$

Overlapping of soft and collinear

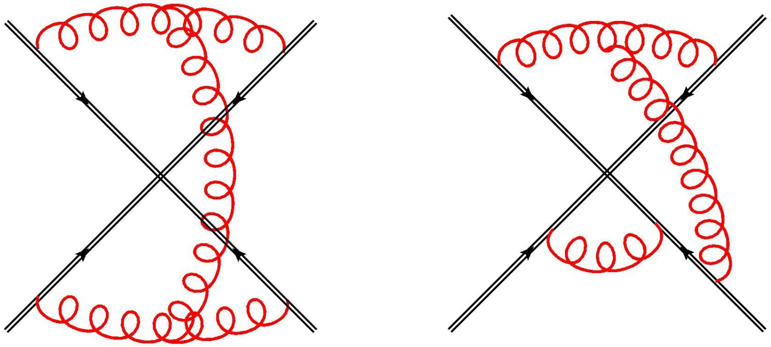
The correlator of multiple Wilson lines can include multiparton correlations

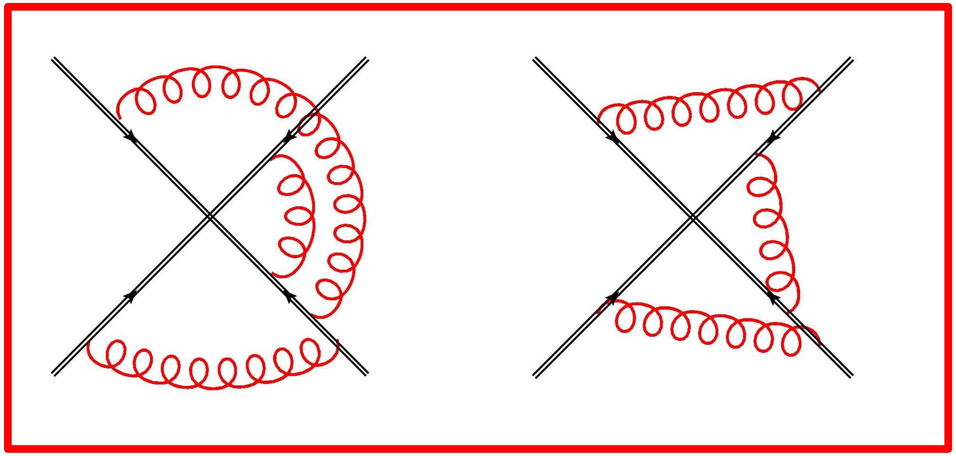
Corrections to the dipole formula

Up to two loops, the anomalous dimension contains only dipole correlations: this is a consequence of the vanishing of the correlator of three massless lines at two loops (Mert Aybat, Dixon, Sterman '06)



Starting from three loops we can expect quadrupole correlations



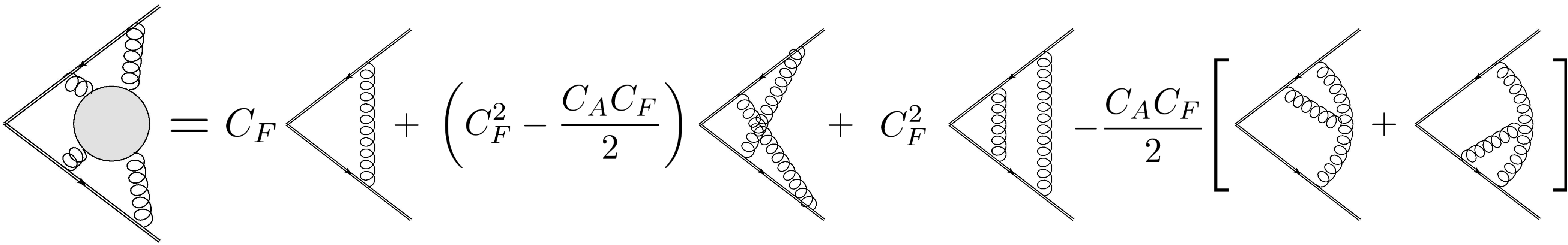


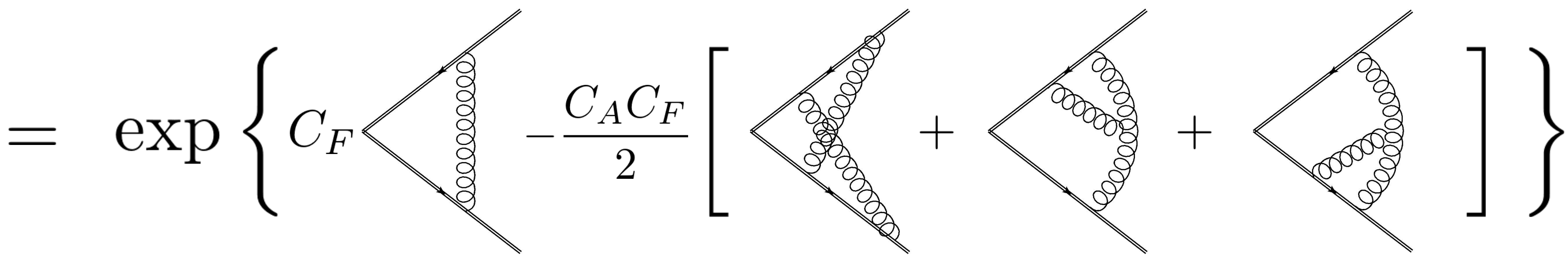
Almelid, Duhr, Gardi, in progress

Multiple gluon exchange webs (Gardi '13)

Webs and non abelian exponentiation

Correlators of Wilson lines are obtained by exponentiating a subset of Feynman diagrams with modified colour factors, called webs: the simplest example is with two lines





(Sterman '81, Gatheral, Frenkel, Taylor '83)

The proof of exponentiation in the multi line case is more recent (Gardi, Laenen, Stavenga, White '10, Mitov, Sterman, Sung '10)

$$S_{bare}(\beta_i \cdot \beta_j, \alpha_s(\mu^2)) = \langle 0 | \prod_{i=1}^{n} \mathcal{W}_{\beta_i}(\infty, 0) | 0 \rangle \equiv \exp[w] = \exp\left[\sum_{D,D'} \mathcal{F}(D)\, \mathcal{R}_{DD'}\, \mathcal{C}(D')\right]$$

Kinematic factor of diagram D

Mixing matrix of combinatoric origin

Colour factor of diagram D'

Renormalisation of webs

The multi-eikonal vertex is renormalisable (Brandt, Neri, Sato '81)

$$S_{\rm ren}\left(\gamma_{ij},\alpha_s,\mu,m\right) = S_{\rm bare}\left(\gamma_{ij},\alpha_s,\epsilon,m\right)\, Z\left(\gamma_{ij},\alpha_s,\epsilon,\mu\right) \qquad \gamma_{ij} = \frac{2\beta_i\cdot\beta_j}{\sqrt{\beta_i^2\beta_j^2}}$$

The associated renormalisation group equation involves the anomalous dimension

$$\mu\frac{dS_{\rm ren}\left(\gamma_{ij},\alpha_s\right)}{d\mu} = -S_{\rm ren}\left(\gamma_{ij},\alpha_s\right)\,\Gamma\left(\gamma_{ij},\alpha_s\right)$$

The calculation of the anomalous dimension can be done directly at the level of the exponent

$$S_{bare}^{reg} = \exp\left[\sum_n w^{(n)}\,\left(\frac{\alpha_s}{\pi}\right)^n\right], \qquad \text{with } w^{(n)} = \sum_{j=-n} \epsilon^j\, w^{(n,j)}$$

The anomalous dimension at each order is a combination of the single pole of the webs and commutators of the decompositions in lower order webs

$$\Gamma^{(1)} = -2w^{(1,-1)}$$

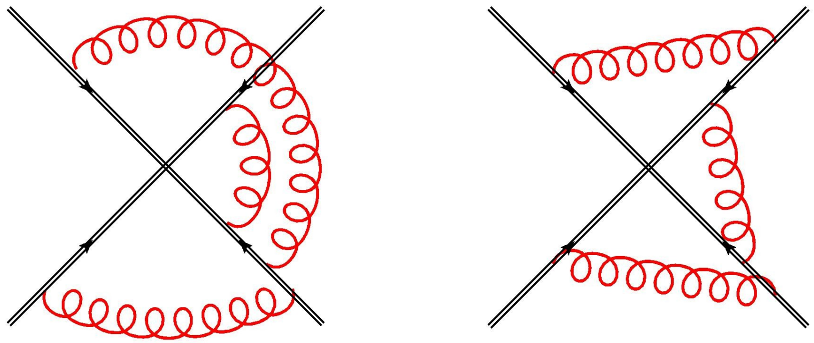
(Gardi, Smillie, White '11)

$$\Gamma^{(2)} = -4w^{(2,-1)} - 2\left[w^{(1,-1)}, w^{(1,0)}\right]$$

$$\boxed{\Gamma^{(n)} \equiv -2n\,\overline{w}^{(n,-1)}}$$

Definition of **subtracted web**

Multiple gluon exchange webs



(Gardi '13)

Expansion of the subtracted web on a colour basis

$$\overline{w}^{(n)} = \sum_j F^{(n)}_{w,j}\,(\gamma_{ij})\;c^{[j]}$$

Kinematic coefficients have the integral representation

$$F^{(n)}_{w,j}\,(\gamma_{ij}) = \int_0^1 \left[\prod_{k=1}^{n} dx_k\, p_0(x_k,\alpha_k)\right]\,\mathcal{G}^{(n)}_{W,\,j}\Big(x_i, q(x_i,\alpha_i)\Big)$$

$$p_0(x,\alpha) \equiv \left(q(x,\alpha)\right)^{-1} = \left[x^2 + (1-x)^2 + \left(\alpha + \frac{1}{\alpha}\right)\right]^{-1} \qquad \gamma_k = -\left(\alpha_k + \frac{1}{\alpha_k}\right)$$

The subtracted web kernel has a surprisingly simple structure

$$\Big\{\log x_k, \log\left(1 - x_k\right), \log\left(q(x_k,\alpha_k)\right)\Big\} \in \mathcal{G}(x_k,\alpha_k)$$

Structure of MGEWs

(Gardi '13)

- Factorisation conjecture: the result is a sum of products of polylogarithms involving a single cusp angle.

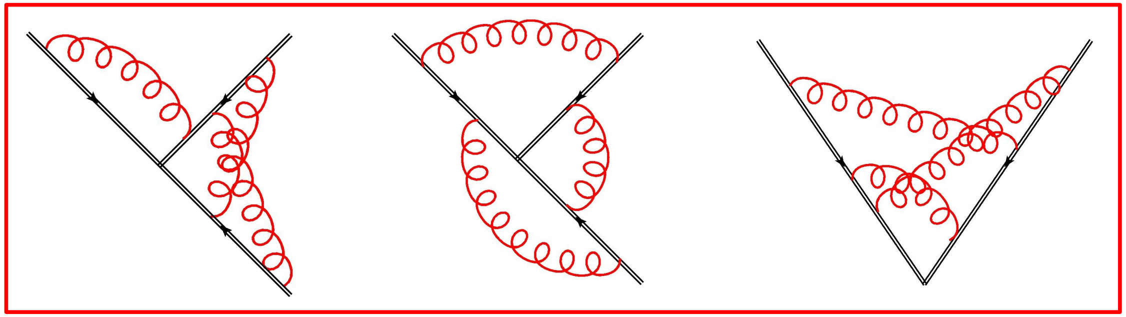
- Alphabet conjecture: the symbol of each function appearing in the subtracted web have the alphabet $\left\{\alpha_k, \eta_k \equiv 1-\alpha_k^2\right\}$

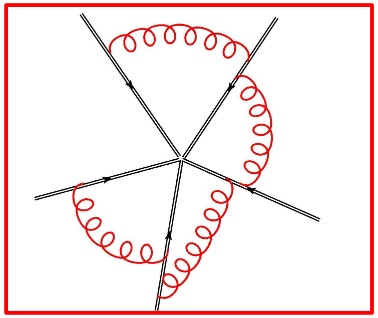
(Gardi, Harley, Magnea, White, GF '14)

Basis of functions with these properties, describing all the MGEWs at 3 loops

$$M_{k,l,n}(\alpha) = \frac{1}{r(\alpha)} \int_0^1 dx\, p_0(x,\alpha)\, \log^k\left(\frac{q(x,\alpha)}{x^2}\right) \log^l\left(\frac{x}{1-x}\right) \log^n\left(\tilde{q}(x,\alpha)\right)$$

$$r(\alpha) = \frac{1+\alpha^2}{1-\alpha^2}\,,\; \tilde{q}(x,\alpha) = \log\frac{1-(1-\alpha)x}{1+\frac{1-\alpha}{\alpha}x}$$





Conclusion 2

- The structure of infrared singularities at two loops is determined by dipole correlations. Quadrupole interactions can arise at three loops and they can be tested via direct calculations of Wilson lines correlators.

- The non-Abelian exponentiation theorem organizes the calculation in terms of web diagrams. The particular class of webs (MGEWs), which doesn't include gluon self interaction has been studied recently.

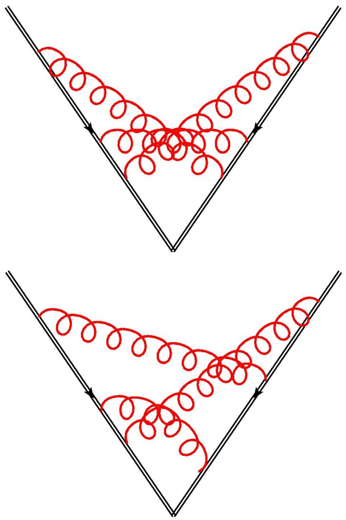
- The kinematic coefficients of MGEWs have a factorised structure in the different cusp angles, which seems consistent with a dipole description. All the functions appearing in the anomalous dimension are written in terms of a basis, characterised by a one-dimensional integral representation.

- This particular structure of MGEWs contribution has been tested in non-trivial cases, but it is still necessary to prove it in general.

- The calculation of the three-loop soft anomalous dimension must be completed, in order to test four partons correlations at three loops.

Thank you!

Results for three-loop webs



$$\mathcal{G}^{(3)}_{(3,3),\,a}(x,y,z) \;=\; -\frac{4}{3}\ln^2\left(\frac{x}{1-x}\,\frac{1-z}{z}\right)\theta(z-x)\,\theta(y-z)\,,$$

$$\mathcal{G}^{(3)}_{(3,3),\,b}(x,y,z) \;=\; -\frac{4}{3}\ln\left(\frac{x}{1-x}\,\frac{1-y}{y}\right)\ln\left(\frac{y}{1-y}\,\frac{1-z}{z}\right)\theta(y-x)\,\theta(z-y)\,.$$

It's convenient to group these diagrams according to a particular colour basis

$$F^{(3)}_{(V_1V_2)_+(V_1V_2)_+}(\alpha) \;= 2\,F^{(3)}_{(3,3),a}(\alpha) + F^{(3)}_{(3,3),b}(\alpha)\,,$$

$$F^{(3)}_{V_3V_3}(\alpha) \;= F^{(3)}_{(3,3),a}(\alpha) + \tfrac{3}{2}F^{(3)}_{(3,3),b}(\alpha)\,,$$

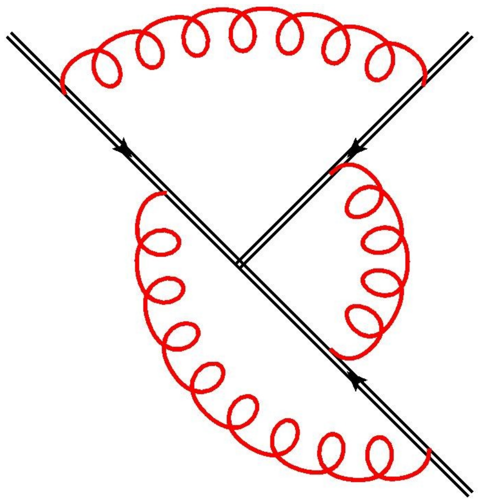
Giving the final result

$$F^{(3)}_{(V_1V_2)_+(V_1V_2)_+}(\alpha) \;=\; -\frac{2}{3}\,r^3(\alpha)\,M_{0,2,0}(\alpha)\,M^2_{0,0,0}(\alpha)$$

$$F^{(3)}_{V_3V_3}(\alpha) \;=\; -\frac{4}{3}\,r^3(\alpha)\left[\frac{1}{4}M^2_{0,0,0}(\alpha)M_{2,0,0}(\alpha) - \frac{1}{4}M_{0,0,0}(\alpha)M^2_{1,0,0}(\alpha) + M_{0,0,0}(\alpha)M_{1,1,1}(\alpha)\right.$$

$$\left. - M_{0,1,1}(\alpha)M_{1,0,0}(\alpha) + \frac{3}{2}M_{0,2,2}(\alpha) - \frac{1}{4}M^2_{0,0,0}(\alpha)M_{0,2,0}(\alpha) + \frac{1}{48}M^5_{0,0,0}(\alpha)\right].$$

Results for three-loop webs



Three colour structures are present in this web. The web kernels are written in terms of

$$L_{ij} \equiv \log\left(\frac{q(x_i,\alpha_{ij})}{x_i^2}\right) \; ; \qquad R_i \equiv \log\left(\frac{x_i}{1-x_i}\right) \, .$$

$$\mathcal{G}^{(3)}_{(2,2,2),\,1} = \frac{1}{3}\left[R_2^2 - \frac{1}{4}L_{23}^2 + \frac{1}{8}L_{12}^2 + \frac{1}{8}L_{31}^2 + \frac{1}{4}L_{12}L_{23} - \frac{1}{2}L_{31}L_{12} + \frac{1}{4}L_{23}L_{31}\right],$$

$$\mathcal{G}^{(3)}_{(2,2,2),\,2} = \frac{1}{3}\left[R_3^2 - \frac{1}{4}L_{31}^2 + \frac{1}{8}L_{23}^2 + \frac{1}{8}L_{12}^2 + \frac{1}{4}L_{23}L_{31} - \frac{1}{2}L_{12}L_{23} + \frac{1}{4}L_{31}L_{12}\right],$$

$$\mathcal{G}^{(3)}_{(2,2,2),\,3} = -\frac{1}{3}\left[R_1^2 - \frac{1}{4}L_{12}^2 + \frac{1}{8}L_{23}^2 + \frac{1}{8}L_{31}^2 + \frac{1}{4}L_{31}L_{12} - \frac{1}{2}L_{23}L_{31} + \frac{1}{4}L_{12}L_{23}\right].$$

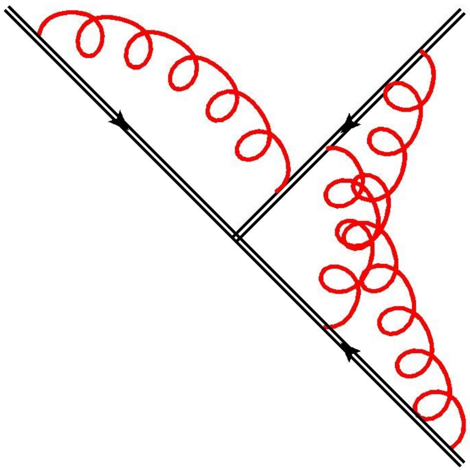
The integrated result is

$$
\begin{aligned}
F^{(3)}_{(2,2,2),\,1}(\alpha_{12},\alpha_{23},\alpha_{13}) \quad = \quad & \frac{1}{3}\, r(\alpha_{12}) r(\alpha_{23}) r(\alpha_{13}) \; \times \\
& \Bigg[- \, M_{0,0,0}(\alpha_{12}) M_{0,0,0}(\alpha_{13}) \left(\frac{1}{4}\, M_{2,0,0}(\alpha_{23}) - \, M_{0,2,0}(\alpha_{23}) \right) \\
& + \frac{1}{8}\, M_{0,0,0}(\alpha_{13}) M_{0,0,0}(\alpha_{23}) M_{2,0,0}(\alpha_{12}) \\
& + \frac{1}{8}\, M_{0,0,0}(\alpha_{12}) M_{0,0,0}(\alpha_{23}) M_{2,0,0}(\alpha_{13}) - \frac{1}{2}\, M_{0,0,0}(\alpha_{23}) M_{1,0,0}(\alpha_{12}) M_{1,0,0}(\alpha_{13}) \\
& + \frac{1}{4}\, M_{0,0,0}(\alpha_{13}) M_{1,0,0}(\alpha_{12}) M_{1,0,0}(\alpha_{23}) + \frac{1}{4}\, M_{0,0,0}(\alpha_{12}) M_{1,0,0}(\alpha_{13}) M_{1,0,0}(\alpha_{23}) \Bigg] .
\end{aligned}
$$

Where the remaining colour structures are obtained by symmetry

$$
\begin{aligned}
F^{(3)}_{(2,2,2),\,2}(\alpha_{12},\alpha_{23},\alpha_{13}) \quad &= \quad F^{(3)}_{(2,2,2),\,1}(\alpha_{23},\alpha_{13},\alpha_{12}) \; , \\
F^{(3)}_{(2,2,2),\,3}(\alpha_{12},\alpha_{23},\alpha_{13}) \quad &= \quad - F^{(3)}_{(2,2,2),\,1}(\alpha_{13},\alpha_{12},\alpha_{23}) \; ,
\end{aligned}
$$

Results for three-loop webs



Web kernels for the three colour structures.

$$\mathcal{G}^{(3)}_{(123),\,2} = \frac{1}{3}\left[\frac{1}{8}L_{13}^2 - \frac{1}{8}L_{23}^2 + \frac{1}{4}L_{23}L_{32} - \frac{1}{4}L_{13}L_{23}\right],$$

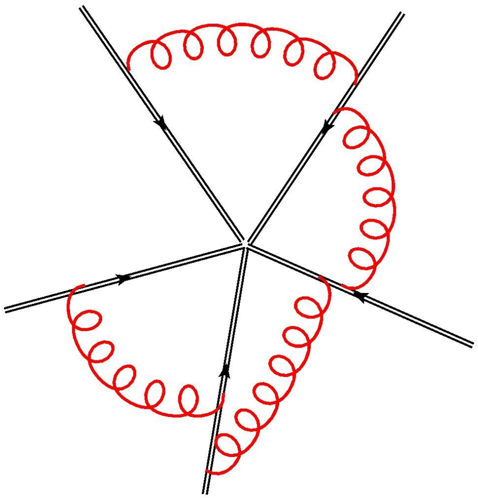
$$\mathcal{G}^{(3)}_{(123),\,3} = -\frac{1}{3}\left[\frac{1}{4}L_{23}L_{13} - \frac{1}{4}L_{23}L_{32} + \frac{1}{8}L_{23}^2 - \frac{1}{8}L_{13}^2 - R_2^2\right],$$

$$\mathcal{G}^{(3)}_{(1,2,3),4} = \frac{2}{3}\,\theta(x_2 - x_3)\Bigg[2\,L_{13}R_2 + L_{23}\,(R_3 - R_2) - \log^2\left(\frac{x_2}{x_3}\right) + \log\left(\frac{x_2}{x_3}\right)\log\left(\frac{1-x_2}{1-x_3}\right)\Bigg],$$

Again, the result is expressed in terms of the basis

$$
\begin{aligned}
F^{(3)}_{(1,2,3),\,2}(\alpha_{13},\alpha_{23}) \quad &= \quad \frac{1}{12}\, r(\alpha_{13}) r^2(\alpha_{23}) \Bigg[\frac{1}{2}\, M_{2,0,0}(\alpha_{13}) M^2_{0,0,0}(\alpha_{23}) \\
&- \frac{1}{2}\, M_{2,0,0}(\alpha_{23}) M_{0,0,0}(\alpha_{13}) M_{0,0,0}(\alpha_{23}) + M_{0,0,0}(\alpha_{13}) M^2_{1,0,0}(\alpha_{23}) \\
&- M_{0,0,0}(\alpha_{23}) M_{1,0,0}(\alpha_{13}) M_{1,0,0}(\alpha_{23}) \Bigg]\,, \\
F^{(3)}_{(1,2,3),\,3}(\alpha_{13},\alpha_{23}) \quad &= \quad -\frac{1}{12}\, r(\alpha_{13}) r^2(\alpha_{23}) \Bigg[-\frac{1}{2}\, M_{2,0,0}(\alpha_{13}) M^2_{0,0,0}(\alpha_{23}) \\
&+ \frac{1}{2} \Big(M_{2,0,0}(\alpha_{23}) - 8 M_{0,2,0}(\alpha_{23}) \Big) M_{0,0,0}(\alpha_{13}) M_{0,0,0}(\alpha_{23}) \\
&- M_{0,0,0}(\alpha_{13}) M^2_{1,0,0}(\alpha_{23}) + M_{0,0,0}(\alpha_{23}) M_{1,0,0}(\alpha_{13}) M_{1,0,0}(\alpha_{23}) \Bigg]\,, \\
F^{(3)}_{(1,2,3),\,4}(\alpha_{13},\alpha_{23}) \quad &= \quad \frac{4}{3}\, r(\alpha_{13}) r^2(\alpha_{23}) \Bigg[M_{0,1,1}(\alpha_{23}) M_{1,0,0}(\alpha_{13}) \\
&+ \frac{1}{8} \Big(M^2_{1,0,0}(\alpha_{23}) - M_{0,0,0}(\alpha_{23}) M_{2,0,0}(\alpha_{23}) - \frac{1}{12}\, M^4_{0,0,0}(\alpha_{23}) \\
&+ 2\, M_{0,0,0}(\alpha_{23}) M_{0,2,0}(\alpha_{23}) \Big) M_{0,0,0}(\alpha_{13}) \Bigg]\,.
\end{aligned}
$$

Result of a four-loop web



Only one colour structure: the corresponding web kernel is

$$\mathcal{G}^{(4)}_{(1,2,2,2,1),\,1}\big(x_i, q(x_i,\alpha_i)\big) = -\frac{1}{144}\Big\{L_{12}^3 - 3L_{23}^3 + 3L_{34}^3 - L_{45}^3$$
$$+ 3L_{12}^2\Big[L_{23} + L_{34} - 3L_{45}\Big] - 3L_{45}^2\Big[L_{23} + L_{34} - 3L_{12}\Big]$$
$$+ 3L_{23}^2\Big[L_{12} - 3L_{34} + 5L_{45}\Big] - 3L_{34}^2\Big[L_{45} - 3L_{23} + 5L_{12}\Big]$$
$$+ 6\Big[L_{12}L_{23}L_{34} - 3L_{12}L_{23}L_{45} + 3L_{12}L_{34}L_{45} - L_{23}L_{34}L_{45}\Big]$$
$$+ 24\Big[R_2^2\Big(L_{12} + L_{23} + L_{34} - 3L_{45}\Big) - R_3^2\Big(L_{23} + L_{34} + L_{45} - 3L_{12}\Big)\Big]\Big\}.$$

Four-loop web result

$$
\begin{aligned}
&F^{(4)}_{(1,2,2,2,1),1}(\alpha_{ij}) = -\frac{1}{144}\, r(\alpha_{12})r(\alpha_{23})r(\alpha_{34})r(\alpha_{45}) \times \\
&\times\Bigg\{\Bigg[6\Bigg(M_{1,0,0}(\alpha_{12})M_{1,0,0}(\alpha_{23})M_{1,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{45}) \\
&\quad - 3M_{1,0,0}(\alpha_{12})M_{1,0,0}(\alpha_{23})M_{1,0,0}(\alpha_{45})M_{0,0,0}(\alpha_{34})\Bigg) \\
&\quad + \Bigg(M_{3,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45}) - 9M_{2,0,0}(\alpha_{12})M_{1,0,0}(\alpha_{45})\Bigg)M_{0,0,0}(\alpha_{23})M_{0,0,0}(\alpha_{34}) \\
&\quad - 3\Bigg(M_{3,0,0}(\alpha_{23})M_{0,0,0}(\alpha_{34}) + 3M_{2,0,0}(\alpha_{23})M_{1,0,0}(\alpha_{34})\Bigg)M_{0,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45}) \\
&\quad + 3M_{2,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45})\Bigg(M_{1,0,0}(\alpha_{23})M_{0,0,0}(\alpha_{34}) + M_{1,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{23})\Bigg) \\
&\quad + 3M_{2,0,0}(\alpha_{23})M_{0,0,0}(\alpha_{34})\Bigg(M_{1,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45}) + 5M_{1,0,0}(\alpha_{45})M_{0,0,0}(\alpha_{12})\Bigg) \\
&\quad + 24M_{0,2,0}(\alpha_{23})\Bigg(M_{1,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{45}) \\
&\quad + M_{1,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{45}) - 3M_{1,0,0}(\alpha_{45})M_{0,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{34})\Bigg) + \\
&\quad 24M_{1,2,0}(\alpha_{23})M_{0,0,0}(\alpha_{12})M_{0,0,0}(\alpha_{34})M_{0,0,0}(\alpha_{45})\Bigg] \\
&\quad - \Bigg[(\alpha_{12} \leftrightarrow \alpha_{45}), (\alpha_{23} \leftrightarrow \alpha_{34})\Bigg]\Bigg\}.
\end{aligned}
$$