

Operator Product Expansion Algebra

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based on joint work with S. Hollands and Ch. Kopper

DESY Zeuthen
January 15, 2015



European Research Council

Established by the European Commission

Commun.Math.Phys.313 (2012) 257–290, J.Math.Phys.54 (2013) 072302,
arXiv:1401.3144, arXiv:1411.1785

Introduction

Operator Product Expansion [Wilson '69]

Products of composite fields (e.g. φ^2 , $\varphi^3 \partial_\mu \varphi$ or $\bar{\psi} \gamma_\mu \psi$) can be expanded as

$$\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \underbrace{\cdots}_{\text{Spectators}} \rangle \sim \sum_B \underbrace{\mathcal{C}_{A_1 \dots A_N}^B(x_1, \dots, x_N)}_{\text{OPE coefficients}} \langle \mathcal{O}_B(x_N) \dots \rangle$$

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- ▶ **Asymptotic short distance expansion:**
Difference vanishes in the limit $x_i \rightarrow x_N$ for all $i \leq N$
- ▶ Verified in various approaches, even on curved spacetimes [Brandt, Zimmermann, Kopper-Keller, Hollands, ...]
- ▶ Practical application e.g. in deep-inelastic scattering
- ▶ Plays fundamental role in conformal field theory (Conformal bootstrap, "Vertex operator algebras", ...)

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2. What are algebraic relations between OPE coefficients?
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Toy model: Perturbative, Euclidean φ_4^4 -theory

The model: Euclidean φ^4 -theory

- ▶ Correlation functions are defined via the path integral

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle := \mathcal{N} \int \mathcal{D}\varphi \exp[-S(\varphi)] \varphi(x_1) \cdots \varphi(x_n),$$

where the action is given by

$$S(\varphi) := \int d^4x \left(\frac{1}{2} (\partial_\mu \varphi)^2(x) + \frac{m^2}{2} \varphi^2(x) + g\varphi(x)^4 - \text{counterterms} \right)$$

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- ▶ **Composite operator insertions** $\langle \mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) \rangle$:
 - ▶ Define $\mathcal{O}_A(x) = (\partial^{v_1} \varphi \cdots \partial^{v_n} \varphi)(x)$
 - ▶ Add sources $\int_x f_A(x) \mathcal{O}_A(x)$ (and counterterms) to action
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- ▶ We use a “renormalization group flow equation” approach
[Wilson, Polchinski, Kopper-Keller-Salmhofer]

Outline

- 1 OPE convergence
- 2 OPE factorisation
- 3 Recursive construction of OPE

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Bound on OPE remainder I

Theorem (Hollands-Kopper-JH)

At any perturbation order r and for any $D \in \mathbb{N}$,

$$\left| \left\langle \left(\mathcal{O}_{A_1}(x_1) \cdots \mathcal{O}_{A_N}(x_N) - \underbrace{\sum_{\dim[B] \leq D} \mathcal{C}_{A_1 \dots A_N}^B(x_1, \dots, x_N) \mathcal{O}_B(x_N)}_{\text{OPE-Remainder}} \right) \underbrace{\hat{\varphi}(p_1) \cdots \hat{\varphi}(p_n)}_{\text{Spectator fields}} \right\rangle \right|$$

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► $M = \begin{cases} m & \text{for } m > 0 \\ \mu & \text{for } m = 0 \end{cases}$ mass or renormalization scale

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- ▶ $|P| = \sup_i |p_i|$: maximal momentum of spectators
- ▶ $\kappa := \inf(\mu, \varepsilon)$, where $\varepsilon = \min_{I \subset \{1, \dots, n\}} |\sum_I p_i|$
 ε : distance of (p_1, \dots, p_n) to “exceptional” configurations

Conclusions from bound on OPE remainder

$$\text{"OPE remainder"} \leq \frac{M^{n-1}}{\sqrt{D!}} \frac{\left(KM \max_{1 \leq i \leq N} |x_i - x_N|\right)^{D+1}}{\min_{1 \leq i < j \leq N} |x_i - x_j|^{\sum_i \dim[A_i] + 1}} \cdot \sup \left(1, \frac{|P|}{\sup(m, \kappa)}\right)^{(D+2)(r+5)}$$

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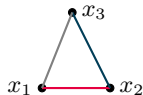
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 - ▶ ratio of max. and min. distances is large, e.g. for $N = 3$



Slow convergence



Fast convergence

Bound on OPE remainder II

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M : mass for $m > 0$ or renormalization scale μ for massless fields

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Heuristic idea: Points in OPE can be scaled in different ways

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x_3
•

•
 x_1

•
 x_2

Successive expansions and "associativity"

Algebraic analogy: Associativity puts constraints on structure constants

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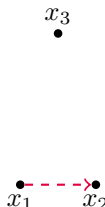
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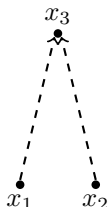
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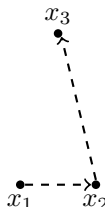
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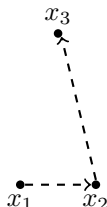
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Consistency of expansions puts algebraic restrictions on coefficients

The OPE factorises

Theorem (Hollands-JH)

At any perturbation order r the relation

$$\mathcal{C}_{A_1 \dots A_N}^B(x_1, \dots, x_N) = \sum_C \mathcal{C}_{A_1 \dots A_M}^C(x_1, \dots, x_M) \mathcal{C}_{C A_{M+1} \dots A_N}^B(x_M, \dots, x_N)$$

holds on the domain $\frac{\max_{1 \leq i \leq M} |x_i - x_M|}{\min_{M < j \leq N} |x_j - x_M|} < \varepsilon(r)$, where $0 < \varepsilon(r) \leq 1$.

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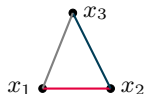
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For $N = 3$: $\frac{|x_1 - x_2|}{|x_2 - x_3|} < \varepsilon$



for $\varepsilon \ll 1$



for $\varepsilon \approx 1$

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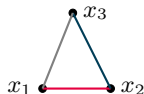
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Particularly strong if we could show $\varepsilon = 1$ [Hollands-Olbermann]:

- ▶ Related to “vertex operator algebras”
- ▶ $\mathcal{C}_{A_1 \dots A_N}^B$ uniquely determined in terms of $\mathcal{C}_{A_1 A_2}^B$
- ▶ ”Bootstrap construction” of OPE coefficients possible

Outline

1 OPE convergence

2 OPE factorisation

3 Recursive construction of OPE

Motivation for a new construction method

Textbook method (roughly):

- ▶ Write down correlation function with operator insertions
- ▶ Perform short distance/large momentum expansion (in some clever way)
- ▶ Argue that the coefficients obtained this way are state independent

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Not entirely satisfying:

- ▶ Relies on correlation functions \Rightarrow OPE not 'fundamental'
- ▶ State independence not obvious
- ▶ Hard to study general properties of OPE

Recursion formula (for mass $m > 0$)

Theorem (Hollands-JH)

Coupling constant derivatives of OPE coefficients in $g\varphi^4$ -theory can be expressed as

$$\begin{aligned} \partial_g \mathcal{C}_{A_1 \dots A_N}^B(x_1, \dots, x_N) = & - \int d^4 y \left[\mathcal{C}_{\varphi^4 A_1 \dots A_N}^B(y, x_1, \dots, x_N) \right. \\ & - \sum_{i=1}^N \sum_{[C] \leq [A_i]} \mathcal{C}_{\varphi^4 A_i}^C(y, x_i) \mathcal{C}_{A_1 \dots \widehat{A}_i \dots A_N}^B(x_1, \dots, x_N) \\ & \left. - \sum_{[C] < [B]} \mathcal{C}_{A_1 \dots A_N}^C(x_1, \dots, x_N) \mathcal{C}_{\varphi^4 C}^B(y, x_N) \right]. \end{aligned}$$

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- ▶ Compute OPE coefficients to any perturbation order by iteration.
Initial data: Coefficients of free theory.

Recursion formula (for mass $m > 0$)

Theorem (Hollands-JH)

OPE coefficients at perturbation order $(r + 1)$ can be expressed as

$$\begin{aligned} (\mathcal{C}_{r+1})_{A_1 \dots A_N}^B(x_1, \dots, x_N) = & - \int d^4 y \left[(\mathcal{C}_r)_{\varphi^4 A_1 \dots A_N}^B(y, x_1, \dots, x_N) \right. \\ & - \sum_{s=0}^r \sum_{i=1}^N \sum_{[C] \leq [A_i]} (\mathcal{C}_s)_{\varphi^4 A_i}^C(y, x_i) (\mathcal{C}_{r-s})_{A_1 \dots \widehat{A}_i C \dots A_N}^B(x_1, \dots, x_N) \\ & \left. - \sum_{s=0}^r \sum_{[C] < [B]} (\mathcal{C}_s)_{A_1 \dots A_N}^C(x_1, \dots, x_N) (\mathcal{C}_{r-s})_{\varphi^4 C}^B(y, x_N) \right]. \end{aligned}$$

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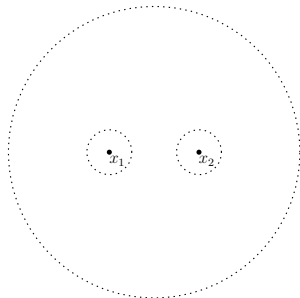
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- ▶ Compute OPE coefficients to any perturbation order by iteration.
Initial data: Coefficients of free theory.
- ▶ State independence obvious.
No other objects enter the construction.
- ▶ The formula depends on the renormalisation conditions.
(Here BPHZ)

Built-in renormalisation (Example: $N = 2$)

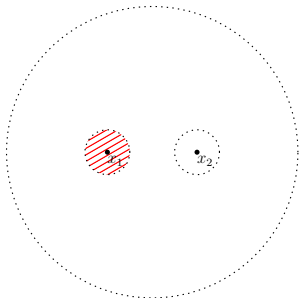
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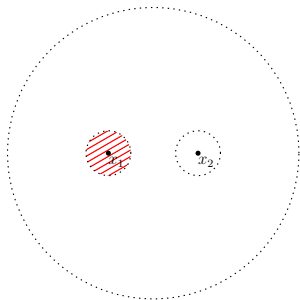
UV-region I ($y \approx x_1$): $\mathcal{C}_{\varphi^4 A_1 A_2}^B$ factorises



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$$\int d^4y \left[\sum_{[C]=0}^{\infty} \mathcal{C}_{\varphi^4 A_1}^C(y, x_1) \mathcal{C}_{CA_2}^B(x_1, x_2) - \sum_{[C] \leq [A_1]} \mathcal{C}_{\varphi^4 A_1}^C(y, x_1) \mathcal{C}_{CA_2}^B(x_1, x_2) \right. \\ \left. - \sum_{[C] \leq [A_2]} \mathcal{C}_{\varphi^4 A_2}^C(y, x_2) \mathcal{C}_{A_1 C}^B(x_1, x_2) - \sum_{[C] < [B]} \mathcal{C}_{A_1 A_2}^C(x_1, x_2) \mathcal{C}_{\varphi^4 C}^B(y, x_2) \right]$$

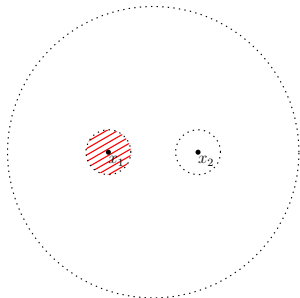
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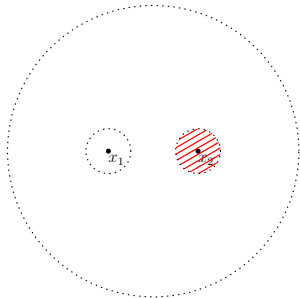
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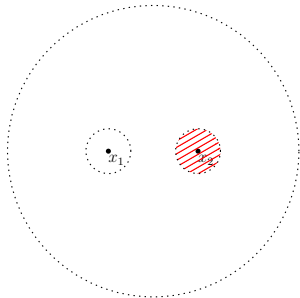
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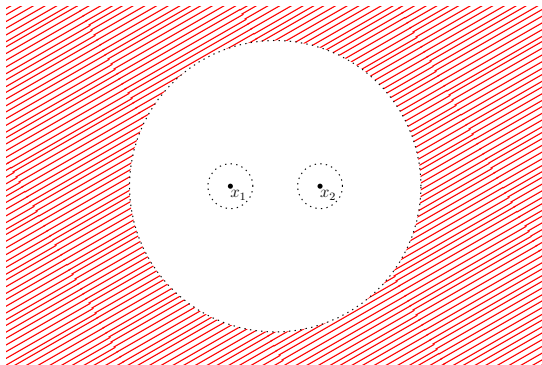
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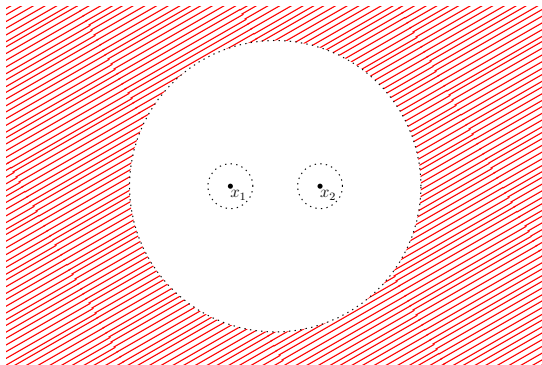
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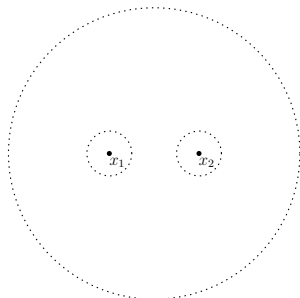
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The integral is absolutely convergent due to the factorisation property.



Conclusions & Outlook

In Euclidean perturbation theory, we found that:

1. The OPE converges at finite distances.
2. The OPE factorises at large distances.
3. The OPE satisfies a recursion formula.

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Possible Generalisations

- ▶ Gauge theories (in progress)
- ▶ Minkowski space
- ▶ Curved manifolds
- ▶ ...

Applications of the Recursion Formula

- ▶ Can one improve the factorisation result ($\varepsilon = 1$)?
- ▶ Does the algorithm facilitate computations?
- ▶ Does the perturbation series for OPE coefficients converge?