Hadronic and leptonic corrections to the anomalous magnetic moment of the muon at four-loop order

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Outline

Anomalous magnetic moment

$\vec{\mu} = g \, \frac{e}{2m} \, \vec{s} \qquad g = 2 \cdot (1 + a)$

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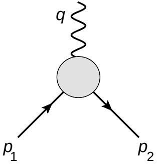

$$= \; -ie \, \bar{\psi}(p_2) \, \left(\gamma^\mu F_E(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2m} F_M(q^2)\right) \, \psi(p_1)$$

$$F_E(0) = 1, \qquad F_M(0) = a$$

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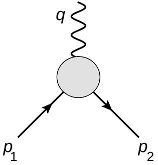

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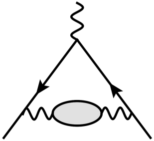
[PDG] $a_\mu^{\text{exp}} = 11659209.1(6.3) \cdot 10^{-10}$

$a_\mu^{\text{th}} = 11659180.4(5.9) \cdot 10^{-10}$

$\Rightarrow \; \Delta a_\mu \; \approx \; 3 \; \sigma$

Hadronic contribution

LO



$\approx 695(4) \cdot 10^{-10}$

[Davier et al 2010] [Hagiwara et al 2011]

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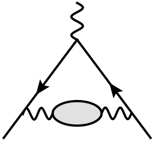

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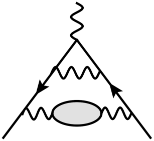
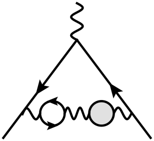
NLO





$-9.84(6) \cdot 10^{-10}$

[Barbieri, Remiddi 1975] [Krause 1997]

Hadronic contribution

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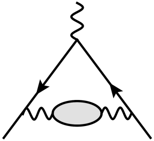

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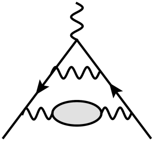

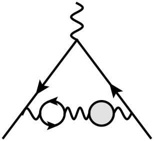

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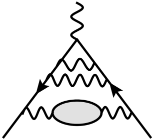
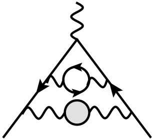
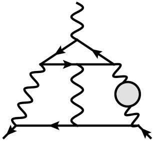
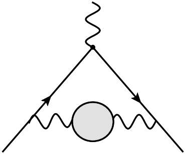


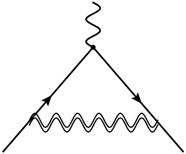




Hadronic calculation



$\longrightarrow$



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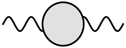

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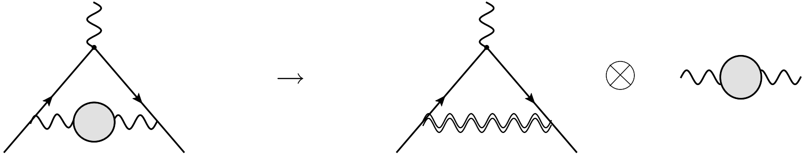

$$a_\mu^{(1)} = \int d\ell \left[P^\omega e\gamma_\rho \frac{1}{\not{p}+\not{q}+\not{\ell}-m_\mu} \gamma_\omega \frac{1}{\not{p}+\not{\ell}-m_\mu} e\gamma_\lambda \frac{g^{\rho\mu}}{\ell^2} \frac{g^{\nu\lambda}}{\ell^2} \right] \left(\ell^2 g_{\mu\nu} - \ell_\mu \ell_\nu\right) \Pi(\ell^2)$$

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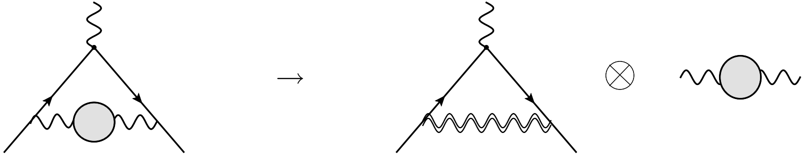

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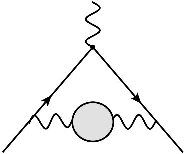
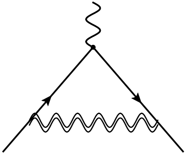


$\rightarrow$



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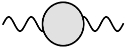

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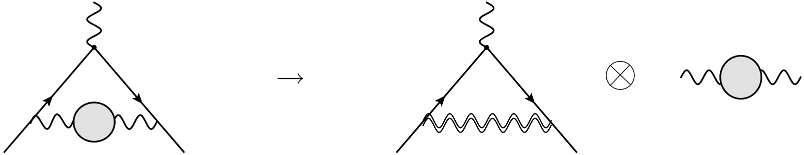

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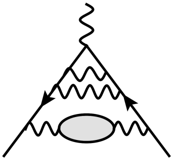
$$= \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^\infty K^{(1)}(s) \; R(s) \, \frac{ds}{s}$$

Hadronic calculation

$$a_\mu^{(3)} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^4 \int_{m_\pi^2}^{\infty} ds\ K^{(3)}(s)\ R(s)\ /\ s$$



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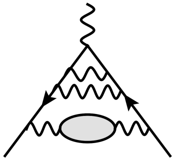

- $K^{(3)}(s)$: diagrammatic calculation with massive photon $(s \gg m_\mu^2 \gg m_e^2)$

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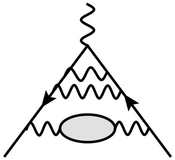

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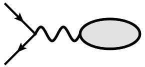
  compilation from [Nomura, Teubner]



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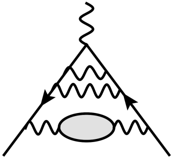

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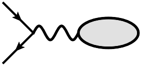
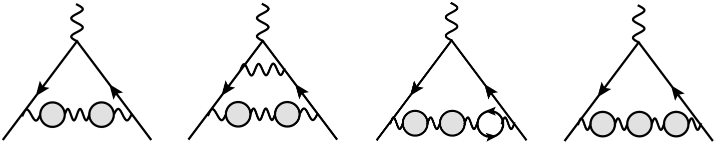


- resonances $J/\Psi, \Psi'$ and $\Upsilon(nS)$ in narrow-width approximation

$$R(s) \sim \Gamma_{ee}\ M_R \cdot \delta(s - M_R^2)$$

Hadronic calculation



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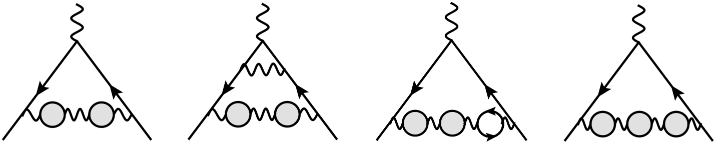

- asymptotic expansions in $s \gg s'$, $s \approx s'$ and $s \ll s'$
- construct interpolating function

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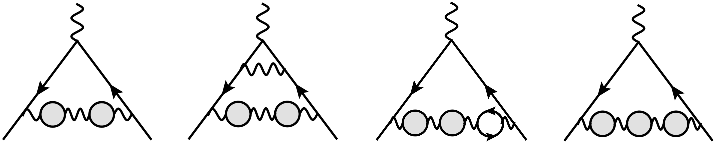

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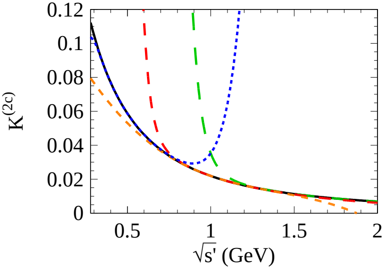
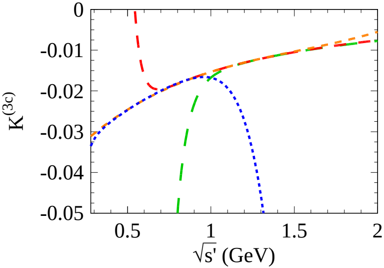
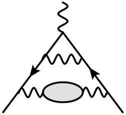
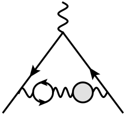
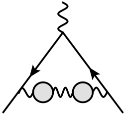




Hadronic result







$$a_\mu^{\mathrm{NLO}} \cdot 10^{10} \quad = \quad -20.90 \quad +10.68 \quad +0.35 \quad = \quad -9.87(9)$$

[Hagiwara, Liao, Martin, Nomura, Teubner 2011] $-9.84(6)$

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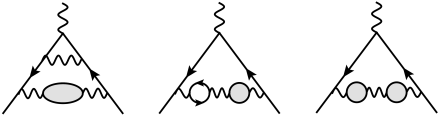

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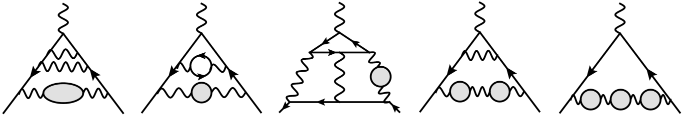

$$a_\mu^{\mathrm{NNLO}} \cdot 10^{10} = \quad 0.80 \quad -0.41 \quad +0.91 \quad -0.06 \quad +0.0005$$

$$= \quad 1.24(1)$$

[Kurz, Liu, Marquard, Steinhauser 2014]

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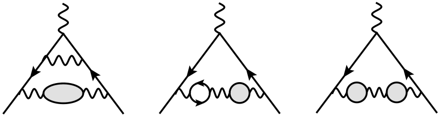

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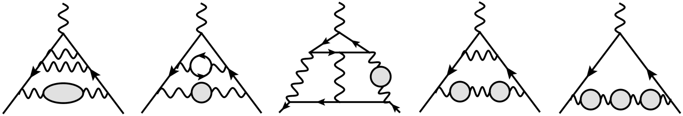

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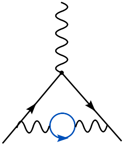
[Kurz, Liu, Marquard, Steinhauser 2014]

$$\Delta a_\mu^{\mathrm{old}} \quad \Rightarrow \quad \Delta a_\mu^{\mathrm{new}} = a_\mu^{\mathrm{exp}} - a_\mu^{\mathrm{th}} - a_\mu^{\mathrm{NNLO}} = 23.7(8.6) \cdot 10^{-10}$$

$$2.9\ \sigma \quad \Rightarrow \quad 2.7\ \sigma$$

Leptonic correction



$$a_\mu(\tau) \sim \mathcal{O}\left(\frac{m_\mu^2}{m_\tau^2}\right)$$

$$a_\mu(e) \sim \mathcal{O}\left(\ln \frac{m_\mu}{m_e}\right)$$

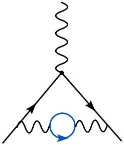
2ℓ [Elend 1966]

3ℓ [Laporta, Remiddi 1993; Laporta 1993]

4ℓ [Kinoshita, Nio 2003] [Lee et al 2013]

5ℓ [Aoyama et al 2011]

Leptonic correction



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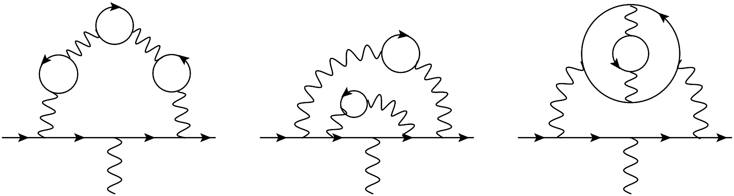
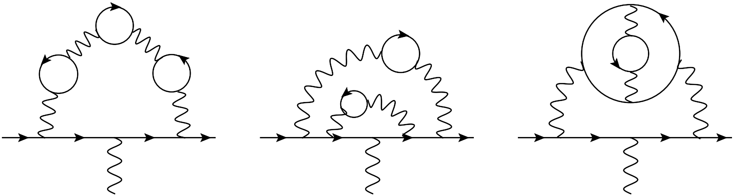
Calculation procedure

- generate Feynman diagrams, `QGRAF` [Nogueira]
- asymptotic expansion with $m_e \ll m_\mu \ll m_\tau$,
  `asy.m` [Pak, Smirnov; Jantzen, Smirnov, Smirnov] / own Mathematica package
- treatment of complicated structure, `FORM` [Vermaseren]
- reduction to master integrals,
  `FIRE` [A.V.Smirnov] / `Crusher` [Marquard, Seidel]
- calculation of master integrals, `FIESTA` [A.V.Smirnov]



[Lee, Marquard, Smirnov, Smirnov, Steinhauser 2013]



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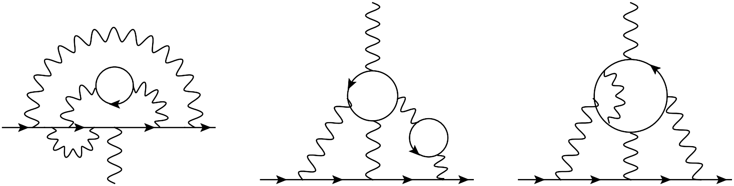
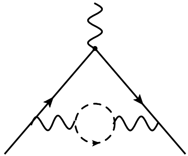


[work in progress]

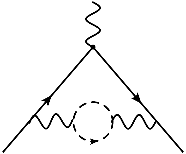
Asymptotic expansion in m_e/m_μ

- 

$$\ell_1^2 \approx \ell_2^2 \approx m_\mu^2 = q^2$$

$$(m_e = 0,\ \ 4\ell \text{ [Marquard, Smirnov, Smirnov, Steinhauser]})$$

Asymptotic expansion in m_e/m_μ

- 

  $\ell_1^2 \approx \ell_2^2 \approx m_\mu^2 = q^2$

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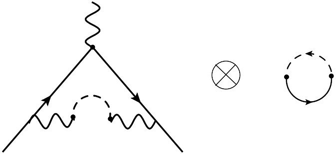
- 

  $\ell_1^2 \approx m_\mu^2 \;\gg\; \ell_2^2 \approx m_e^2$

  $(\ell_1 \ell_2)^2 \rightarrow \frac{\ell_1^2 \ell_2^2}{d}$

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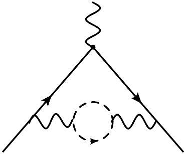
- 

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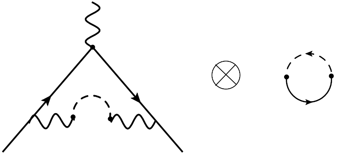
- 

  $\ell_1^2 \approx m_\mu^2 \gg \ell_2^2 \approx m_e^2$

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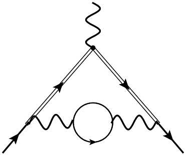
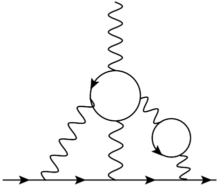
- 

  $\ell_1^2 \approx \ell_2^2 \approx m_e^2$

  $\frac{1}{(\ell+q)^2 - m_\mu^2} = \frac{1}{\ell^2 + 2\ell q} = \frac{1}{2\ell q} \sum_n \left(\frac{-\ell^2}{2\ell q} \right)^n$

Preliminary results



$$117.4(17)\ n_e^2 \left(\frac{\alpha}{\pi}\right)^4 + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right)$$

$\leftrightarrow$ 117.4(5) [Chlouber, Samuel 1977]

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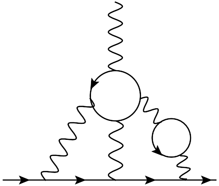

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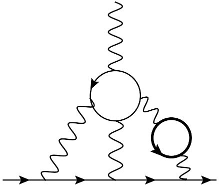
$+$



$$123.9(19)\ n_f^2 \left(\frac{\alpha}{\pi}\right)^4 + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right)$$

$\leftrightarrow$ 123.78551(44) [Aoyama, Hayakawa, Kinoshita, Nio 2012]

$[4\ell, \text{QED}: \ 130.8796(63)]$

Heavy-lepton result

- asymptotic expansion leads to analytically known 4-loop vacuum and 3-loop on-shell integrals

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$$A_\mu^{(8)}(\tau) = \left(\frac{m_\mu}{m_\tau}\right)^2 \left(\frac{4\pi^2\zeta_3}{15} - \frac{52\ln^5(2)}{675} - \frac{3851\pi^2}{3600} + \dots \right.$$
$$\left. + \ln\frac{m_\mu^2}{m_\tau^2}\left(-\frac{38891}{12150} + \frac{19\pi^2}{135} + \frac{3\zeta_3}{2}\right) + \frac{359}{1080}\ln^2\frac{m_\mu^2}{m_\tau^2}\right)$$
$$+ \left(\frac{m_\mu}{m_\tau}\right)^3 \frac{\pi^2}{90} + \dots + \mathcal{O}\left(\left(\frac{m_\mu}{m_\tau}\right)^8\right)$$

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$$= (\ 4.21670 + 0.03257 + 0.00015\ ) \cdot 10^{-2}$$

$$= 4.24941(2)(53) \cdot 10^{-2} \qquad \leftrightarrow \qquad 4.234(12) \cdot 10^{-2}$$

[Kurz, Liu, Marquard, Steinhauser 2014] [Aoyama, Hayakawa, Kinoshita, Nio 2012]

Heavy-lepton result

diagram group	$10^2 \cdot A_\mu^{(8)}(\tau)$	
	our work	[Aoyama et al 2012]
I(a)	0.00324281(2)	0.0032(0)
I(b) + I(c) + II(b) + II(c)	-0.6292808(6)	-0.6293(1)
I(d)	0.0367796(4)	0.0368(0)
III	4.5208986(6)	4.504(14)
II(a) + IV(d)	-2.316756(5)	-2.3197(37)
IV(a)	3.851967(3)	3.8513(11)
IV(b)*	0.612661(5)	0.6106(31)
IV(c)	-1.83010(1)	-1.823(11)

* analytical check of order $\left(\frac{m_\mu}{m_\tau}\right)^2$ [Boughezal, Melnikov 2011] [Kataev 2012]

Conclusions

- New result for a_μ^{had} at NNLO
  $\Rightarrow$ significant change of Δa_μ by 0.2 σ
- Work on $a_\mu(e)$ at $\mathcal{O}\left(\alpha^4\right)$ in progress,
  first preliminary result of n_e^2- and n_f^2-term
  in agreement with literature

BACKUP

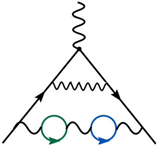
$g-2$ of electron

$$a_e^{\text{QED}} = a_{\text{uni}} + a_e(\mu) + a_e(\tau) + a_e(\mu,\tau)$$

$$a_e^{(8)}(\mu) = 9.161970703(2)(372)\cdot 10^{-4}\cdot\left(\frac{\alpha}{\pi}\right)^4 \leftrightarrow \quad 9.222(66)\cdot 10^{-4}\cdot\left(\frac{\alpha}{\pi}\right)^4$$
$$= 2.7\cdot 10^{-14} \;\approx\; \delta a_e^{\text{exp}}/10$$

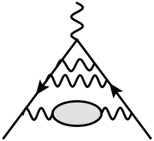
[Aoyama, Hayakawa, Kinoshita, Nio 2012]

$$A_e^{(8)}(\tau) = 7.42924(0)(118)\cdot 10^{-6} \;\leftrightarrow\; 7.38(12)\cdot 10^{-6}$$



$$A_e^{(8)}(\mu,\tau) = \frac{m_e^2}{m_\tau^2}\left(\frac{89}{810}\ln^2\frac{m_\mu^2}{m_\tau^2} + \ldots\right) + \mathcal{O}\left(\frac{1}{M^6}\right)$$
$$\approx 7.4687(26)(10)\cdot 10^{-7} \leftrightarrow\; 7.465(18)\cdot 10^{-7}$$

[Aoyama, Hayakawa, Kinoshita, Nio 2012]



$$a_e^{\text{had,NNLO}} = 2.8(1)\cdot 10^{-14}$$