Will a physicist prove the Riemann Hypothesis?

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Georg Friedrich Bernard Riemann

(1826 - 1866)

Riemann: the greatest mathematician ever.

1900: Hilbert's list of problems for the XX centaury:
the first part of the eighth problem: The Riemann Hypothesis

2000: Clay Institute: problems for millennium.

Millennium Problems

In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) has named seven *Prize Problems*. The Scientific Advisory Board of CMI selected these problems, focusing on important classic questions that have resisted solution over the years. The Board of Directors of CMI designated a $7 million prize fund for the solution to these problems, with $1 million allocated to each. During the Millennium Meeting held on May 24, 2000 at the Collège de France, Timothy Gowers presented a lecture entitled *The Importance of Mathematics*, aimed for the general public, while John Tate and Michael Atiyah spoke on the problems. The CMI invited specialists to formulate each problem.

One hundred years earlier, on August 8, 1900, David Hilbert delivered his famous lecture about open mathematical problems at the second International Congress of Mathematicians in Paris. This influenced our decision to announce the millennium problems as the central theme of a Paris meeting.

The rules for the award of the prize have the endorsement of the CMI Scientific Advisory Board and the approval of the Directors. The members of these boards have the responsibility to preserve the nature, the integrity, and the spirit of this prize.

Paris, May 24, 2000

Please send inquiries regarding the Millennium Prize Problems to prize.problems@claymath.org.

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equations
- P vs NP
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills Theory

- Rules
- Millennium Meeting Videos

15 years old Gauss conjectured:

$$\pi(x) := \sharp\{p - prime, \quad p < x\}$$

$$\pi(x) \sim \int_2^x \frac{du}{\ln(u)} := \mathrm{Li}(x)$$

Remark:

$$\mathrm{Li}(x) = \gamma + \ln(\ln(x)) + \sum_{n=1}^{\infty} \frac{(\ln(x))^n}{nn!}$$

$\gamma \approx 0.577216$ is the Euler–Mascheroni constant
$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right)$.

Euler's Gold Formula

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

This formula is valid only for $\Re[s] > 1$.
There are no zeros of $\zeta(s)$ for $\Re[s] > 1$.
1 is neither prime nor composite.

Euler has derived expressions giving values of zeta for even arguments (B_n are Bernoulli numbers):

$$\zeta(2m) = \frac{|B_{2m}|\pi^{2m}}{2(2m)!}$$

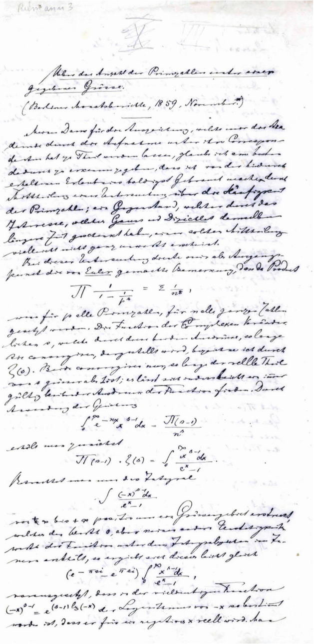
$$\zeta(-m) = -\frac{B_{m+1}}{m+1}$$

R.Apery (1998): $\zeta(3)$ is irrational.
W.I. Zudilin (2001): One of the numbers:

$$\zeta(5),\ \zeta(7)\ \zeta(9),\ \zeta(11)$$

is irrational.

The path leading to the proof of the Gauss conjecture was outlined by Riemann in the paper: "Ueber die Anzal der Primzahlen unter einer gegebenen Grөsse" (Monatsberichte der Berliner Akademie, November 1859). First Riemann has continued analytically $\zeta(s)$ to the whole complex plane with exception of $s = 1$: at $s = 1 \quad \zeta(s)$ has a pole.



Riemann 3

XX

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.*)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\cdot\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

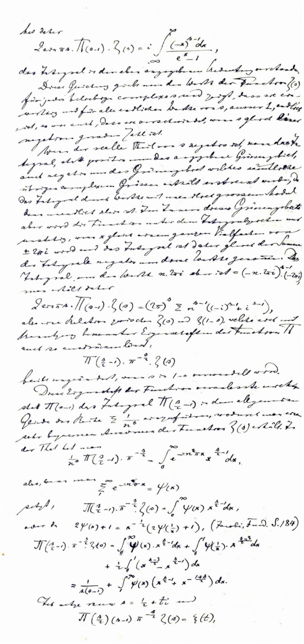
Betrachtet man nun das Integral

$$\int \frac{(-x)^{s-1}\,dx}{e^x-1}$$

von $+\infty$ bis $+\infty$ positiv um ein Grössengebiet erstreckt, welches den Werth 0, aber keinen andern Unstetigkeitswerth der Function unter dem Integralzeichen im Innern enthält, so ergiebt sich dieses leicht gleich

$$(e^{-\pi s i} - e^{\pi s i})\int_0^\infty \frac{x^{s-1}\,dx}{e^x-1},$$

vorausgesetzt, dass in der vieldeutigen Function $(-x)^{s-1} = e^{(s-1)\log(-x)}$ der Logarithmus von $-x$ so bestimmt worden ist, dass er für ein negatives x reell wird. Man



hat daher

$$2\sin\pi s\cdot\Pi(s-1)\cdot\zeta(s) = i\int_\infty^\infty \frac{(-x)^{s-1}\,dx}{e^x-1},$$

das Integral in der eben angegebenen Bedeutung verstanden.

Diese Gleichung giebt nun den Werth der Function $\zeta(s)$ für jeden beliebigen complexen Werth von s und zeigt, dass sie einwerthig und für alle endlichen Werthe von s, ausser 1, endlich ist, so wie auch, dass sie verschwindet, wenn s gleich einer negativen geraden Zahl ist.

Wenn der reelle Theil von s negativ ist, kann das Integral, statt positiv um das angegebene Grössengebiet, auch negativ um das Grössengebiet, welches sämmtliche übrigen complexen Grössen enthält, erstreckt werden, da das Integral durch Werthe mit unendlich grossem Modul dann unendlich klein ist. Im Innern dieses Grössengebiets aber wird die Function unter dem Integralzeichen nur unstetig, wenn x gleich einem ganzen Vielfachen von $\pm 2\pi i$ wird, und das Integral ist daher gleich der Summe der Integrale negativ um diese Werthe genommen. Das Integral um den Werth $n\,2\pi i$ aber ist $= (-n\,2\pi i)^{s-1}(-2\pi i)$, man erhält daher

$$2\sin\pi s\cdot\Pi(s-1)\,\zeta(s) = (2\pi)^s\sum n^{s-1}\left((-i)^{s-1} + i^{s-1}\right),$$

also eine Relation zwischen $\zeta(s)$ und $\zeta(1-s)$, welche sich mit Benutzung bekannter Eigenschaften der Function Π auch so ausdrücken lässt:

$$\Pi\left(\frac{s}{2}-1\right)\cdot\pi^{-\frac{s}{2}}\cdot\zeta(s)$$

bleibt ungeändert, wenn s in $1-s$ verwandelt wird.

Diese Eigenschaft der Function veranlasste mich statt $\Pi(s-1)$ das Integral $\Pi\left(\frac{s}{2}-1\right)$ in dem allgemeinen Gliede der Reihe $\sum\frac{1}{n^s}$ einzuführen, wodurch man einen sehr bequemen Ausdruck der Function $\zeta(s)$ erhält. In der That hat man

$$\frac{1}{n^s}\,\Pi\left(\frac{s}{2}-1\right)\cdot\pi^{-\frac{s}{2}} = \int_0^\infty e^{-nn\pi x}\,x^{\frac{s}{2}-1}\,dx,$$

also, wenn man

$$\sum_1^\infty e^{-nn\pi x} = \psi(x)$$

setzt,

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_0^\infty \psi(x)\,x^{\frac{s}{2}-1}\,dx,$$

oder da $2\psi(x)+1 = x^{-\frac{1}{2}}\left(2\psi\left(\frac{1}{x}\right)+1\right)$, (Jacobi, Fund. S. 184)

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty \psi(x)\,x^{\frac{s}{2}-1}\,dx + \int_0^1 \psi\left(\frac{1}{x}\right)x^{\frac{s-3}{2}}\,dx$$

$$+\frac{1}{2}\int_0^1\left(x^{\frac{s-3}{2}} - x^{\frac{s}{2}-1}\right)dx$$

$$= \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right)dx.$$

Ich setze nun $s = \frac{1}{2} + ti$ und

$$\Pi\left(\frac{s}{2}\right)(s-1)\,\pi^{-\frac{s}{2}}\,\zeta(s) = \xi(t),$$

so dass

$$\xi(t) = \frac{1}{2} - \left(tt + \frac{1}{4}\right)\int_1^\infty \Psi(x)\, x^{-\frac{3}{4}} \cos\left(\frac{1}{2} t \log x\right) dx$$

oder auch

$$\xi(t) = 4\int_1^\infty \frac{d\left(x^{\frac{3}{2}}\Psi'(x)\right)}{dx}\, x^{-\frac{1}{4}} \cos\left(\frac{1}{2} t \log x\right) dx.$$

Diese Function ist für alle endlichen Werthe von t endlich, und lässt sich nach Potenzen von tt in eine sehr schnell convergirende Reihe entwickeln. Da für einen Werth von s, dessen reeller Bestandtheil grösser als 1 ist, $\log \zeta(s) = -\Sigma \log(1-p^{-s})$ endlich bleibt und von den Logarithmen der übrigen Factoren von $\xi(t)$ dasselbe gilt, so kann die Function $\xi(t)$ nur verschwinden, wenn der imaginäre Theil von t zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ liegt. Die Anzahl der Wurzeln von $\xi(t)=0$, deren reeller Theil zwischen 0 und T liegt, ist etwa $= \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}$; denn das Integral $\int d\log\xi(t)$ positiv um den Inbegriff der Werthe von t erstreckt, deren imaginärer Theil zwischen $\frac{1}{2}i$ und $-\frac{1}{2}i$ und deren reeller Theil zwischen 0 und T liegt, ist (bis auf einen Bruchtheil von der Ordnung der Grösse $\frac{1}{T}$) gleich $\left(T\log\frac{T}{2\pi} - T\right)i$; dieses Integral aber ist gleich der Anzahl der in diesem Gebiet liegenden Wurzeln von $\xi(t)=0$, multiplicirt mit $2\pi i$. Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien.

Bezeichnet man durch α jede Wurzel der Gleichung $\xi(\alpha)=0$, so kann man $\log\xi(t)$ durch

$$\Sigma \log\left(1 - \frac{tt}{\alpha\alpha}\right) + \log\xi(0)$$

ausdrücken; denn da die Dichtigkeit der Wurzeln von der Grösse t mit t nur wie $\log\frac{t}{2\pi}$ wächst, so convergirt dieser Ausdruck und wird für ein unendliches t nur unendlich wie $t\log t$; er unterscheidet sich also von $\log\xi(t)$ um eine Function von tt, die für ein endliches t stetig und endlich bleibt und mit tt dividirt für ein unendliches t unendlich klein wird. Dieser Unterschied ist folglich eine Constante, deren Werth durch Einsetzung von $t=0$ bestimmt werden kann.

Mit diesen Hülfsmitteln lässt sich nun die Anzahl der Primzahlen, die kleiner als x sind, bestimmen.

Es sei $F(x)$, wenn x nicht gerade einer Primzahl gleich ist, gleich dieser Anzahl, wenn aber x eine Primzahl ist, um $\frac{1}{2}$ grösser, so dass für ein x, bei welchem $F(x)$ sich sprungweise ändert,

$$F(x) = \frac{F(x+0)+F(x-0)}{2}.$$

Ersetzt man nun in

$$\log\zeta(s) = -\Sigma\log(1-p^{-s}) = \Sigma p^{-s} + \frac{1}{2}\Sigma p^{-2s} + \frac{1}{3}\Sigma p^{-3s} + \cdots$$

p^{-s} durch $s\int_p^\infty x^{-s-1}dx$, p^{-2s} durch $s\int_{p^2}^\infty x^{-s-1}dx$, $\ldots$, so erhält man

$$\frac{\log\zeta(s)}{s} = \int_1^\infty f(x)\, x^{-s-1}dx,$$

wenn man

$$F(x) + \frac{1}{2}F(x^{\frac{1}{2}}) + \frac{1}{3}F(x^{\frac{1}{3}}) + \cdots$$

durch $f(x)$ bezeichnet.

Diese Gleichung ist gültig für jeden complexen Werth $a+bi$ von s, wenn $a>1$. Wenn aber in diesem Umfange die Gleichung

$$g(s) = \int_0^\infty h(x)\, x^{-s}\, d\log x$$

gilt, so kann man mit Hülfe des Fourier'schen Satzes die Function h durch die Function g ausdrücken. Die Gleichung zerfällt, wenn $h(x)$ reell ist und

$$g(a+bi) = g_1(b) + ig_2(b),$$

in die beiden folgenden:

$$g_1(b) = \int_0^\infty h(x)\, x^{-a}\cos(b\log x)\, d\log x,$$

$$ig_2(b) = -i\int_0^\infty h(x)\, x^{-a}\sin(b\log x)\, d\log x.$$

Wenn man beide Gleichungen mit $(\cos(b\log y) + i\sin(b\log y))\,db$ multiplicirt und von $-\infty$ bis $+\infty$ integrirt, so erhält man in beiden auf der rechten Seite nach dem Fourier'schen Satze $\pi h(y)\, y^{-a}$, also, wenn man beide Gleichungen addirt und mit iy^a multiplicirt,

$$2\pi i\, h(y) = \int_{a-\infty i}^{a+\infty i} g(s)\, y^s ds,$$

worin die Integration so auszuführen ist, dass der reelle Theil von s constant bleibt.

Das Integral stellt für einen Werth von y, bei welchem eine sprungweise Änderung der Function $h(y)$ stattfindet, den Mittelwerth aus den Werthen der Function h zu beiden Seiten des Sprunges dar. Bei der hier vorausgesetzten Bestimmungsweise der Function $f(x)$ besitzt diese dieselbe Eigenschaft, und man hat daher völlig allgemein

$$f(y) = \frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i} \frac{\log\zeta(s)}{s}\, y^s ds.$$

Für $\log\zeta$ kann man nun den früher gefundenen Ausdruck

$$\frac{s}{2}\log\pi - \log(s-1) - \log\Pi\left(\frac{s}{2}\right) + \Sigma^\alpha \log\left(1 + \frac{(s-\frac{1}{2})^2}{\alpha\alpha}\right) + \log\xi(0)$$

substituiren; die Integrale der einzelnen Glieder dieses Ausdrucks würden aber dann ins Unendliche ausgedehnt nicht convergiren, weshalb es zweckmässig ist, die Gleichung vorher durch partielle Integration in

$$f(x) = -\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i} \frac{d\frac{\log\zeta(s)}{s}}{ds}\, x^s ds$$

umzuformen.

Da

$$-\log\Pi\left(\frac{s}{2}\right) = \lim\left(\sum_{n=1}^{n=m}\log\left(1+\frac{s}{2n}\right) - \frac{s}{2}\log m\right), \text{ für } m=\infty,$$

also

$$-\frac{d\frac{1}{s}\log\Pi\left(\frac{s}{2}\right)}{ds} = \sum_1^\infty \frac{d\frac{1}{s}\log\left(1+\frac{s}{2n}\right)}{ds},$$

so erhalten dann sämmtliche Glieder des Ausdrucks für $f(x)$ mit Ausnahme von

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i}\frac{1}{ss}\log\xi(0)\, x^s ds = \log\xi(0)$$

die Form
$$\pm \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i} \frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{ds}\, x^s\, ds.$$
Nun ist aber
$$\frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{d\beta} = \frac{1}{(\beta-s)\beta}$$
und, wenn der reelle Theil von s grösser als der reelle Theil von β ist,
$$-\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{x^s\, ds}{(\beta-s)\beta} = \frac{x^\beta}{\beta} = \int_\infty^x x^{\beta-1}dx,$$
$$\text{oder } = \int_0^x x^{\beta-1}dx,$$
je nachdem der reelle Theil von β negativ oder positiv ist. Man hat daher
$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-\infty i}^{a+\infty i}\frac{d\left(\frac{1}{s}\log\left(1-\frac{s}{\beta}\right)\right)}{ds}\, x^s\, ds$$
$$= -\frac{1}{2\pi i}\int_{a-\infty i}^{a+\infty i}\frac{1}{s}\log\left(1-\frac{s}{\beta}\right) x^s\, ds$$
$$= \int_\infty^x \frac{x^{\beta-1}}{\log x}dx + \text{const. im ersten}$$
und
$$= \int_0^x \frac{x^{\beta-1}}{\log x}dx + \text{const. im zweiten Falle.}$$
Im ersten Falle bestimmt sich die Integrationsconstante, wenn man den reellen Theil von β negativ unendlich werden lässt; im zweiten Falle erhält das Integral von 0 bis x um $2\pi i$ verschiedene Werthe, je nachdem die Integration durch complexe Werthe mit positivem oder negativem Arcus geschieht, und wird, auf jenem Wege genommen, unendlich klein, wenn der Coefficient von i in dem Werthe von β positiv unendlich wird, auf letzterem aber, wenn dieser Coefficient negativ unendlich wird. Hieraus ergiebt sich, wie auf der linken Seite $\log\left(1-\frac{s}{\beta}\right)$ zu bestimmen ist, damit die Integrationsconstante wegfällt.

Durch Einsetzung dieser Werthe in den Ausdruck von $f(x)$ erhält man
$$f(x) = Li(x) - \Sigma^\alpha\left(Li\left(x^{\frac12+\alpha i}\right) + Li\left(x^{\frac12-\alpha i}\right)\right) + \int_x^\infty \frac{1}{x^2-1}\frac{dx}{x\log x} + \log\xi(0),$$
wenn in Σ^α für α sämmtliche positiven (oder einen positiven reellen Theil enthaltenden) Wurzeln der Gleichung $\xi(\alpha)=0$, ihrer Grösse nach geordnet, gesetzt werden. Es lässt sich, mit Hülfe einer genaueren Discussion der Function ξ, leicht zeigen, dass bei dieser Anordnung der Werth der Reihe
$$\Sigma\left(Li\left(x^{\frac12+\alpha i}\right) + Li\left(x^{\frac12-\alpha i}\right)\right)\log x$$
mit dem Grenzwerth, gegen welchen
$$\frac{1}{2\pi i}\int_{a-bi}^{a+bi}\frac{d\,\frac{1}{s}\,\Sigma\log\left(1+\frac{\left(s-\frac12\right)^2}{\alpha^2}\right)}{ds}\, x^s\, ds$$
bei unaufhörlichem Wachsen der Grösse b convergirt, übereinstimmt; durch veränderte Anordnung aber würde sie jeden beliebigen reellen Werth erhalten können.

Aus $f(x)$ findet sich $F(x)$ mittelst der durch Umkehrung der Relation
$$f(x) = \Sigma\frac{1}{n}F\left(x^{\frac1n}\right)$$
sich ergebenden Gleichung
$$F(x) = \Sigma(-1)^\mu\frac{1}{m}f\left(x^{\frac1m}\right),$$
worin für m der Reihe nach die durch kein Quadrat ausser 1 theilbaren Zahlen zu setzen sind und μ die Anzahl der Primfactoren von m bezeichnet.

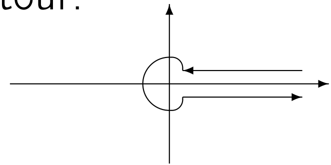
Beschränkt man Σ^α auf eine endliche Zahl von Gliedern, so giebt die Derivirte des Ausdrucks für $f(x)$ oder, bis auf einen mit wachsendem x sehr schnell abnehmenden Theil,
$$\frac{1}{\log x} - 2\Sigma^\alpha\frac{\cos(\alpha\log x)\,x^{-\frac12}}{\log x}$$
einen angenäherten Ausdruck für die Dichtigkeit der Primzahlen + der halben Dichtigkeit der Primzahlquadrate + $\frac13$ von der Dichtigkeit der Primzahlcuben u.s.w. von der Grösse x.

Die bekannte Näherungsformel $F(x) = Li(x)$ ist also nur bis auf Grössen von der Ordnung $x^{\frac12}$ richtig und giebt einen etwas zu grossen Werth; denn die nicht periodischen Glieder in dem Ausdrucke von $F(x)$ sind, von Grössen, die mit x nicht ins Unendliche wachsen, abgesehen:
$$Li(x) - \frac12 Li\left(x^{\frac12}\right) - \frac13 Li\left(x^{\frac13}\right) - \frac15 Li\left(x^{\frac15}\right) + \frac16 Li\left(x^{\frac16}\right) - \frac17 Li\left(x^{\frac17}\right) + \dots$$
In der That hat sich bei der von Gauss und Goldschmidt vorgenommenen und bis zu x = drei Millionen fortgesetzten Vergleichung von $Li(x)$ mit der Anzahl der Primzahlen unter x diese Anzahl schon vom ersten Hunderttausend an stets kleiner als $Li(x)$ ergeben, und zwar wächst die Differenz unter manchen Schwankungen allmählig mit x. Aber auch die von den periodischen Gliedern abhängige stellenweise Verdichtung und Verdünnung der Primzahlen hat schon bei den Zählungen die Aufmerksamkeit erregt, ohne dass jedoch hierin eine Gesetzmässigkeit bemerkt worden wäre. Bei einer etwaigen neuen Zählung würde es interessant sein, den Einfluss der einzelnen in dem Ausdrucke für die Dichtigkeit der Primzahlen enthaltenen periodischen Glieder zu verfolgen. Einen regelmässigeren Gang als $F(x)$ würde die Function $f(x)$ zeigen, welche sich schon im ersten Hundert sehr deutlich als mit $Li(x) + \log\xi(0)$ im Mittel übereinstimmend erkennen lässt.

Riemann has shown that the integral ($s \neq 1$):

$$\zeta(s) = \frac{\Gamma(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$$

where the contour:



is equal to $\sum_{n=1}^{\infty} 1/n^s$ on the right of the line $\Re[s] = 1$.

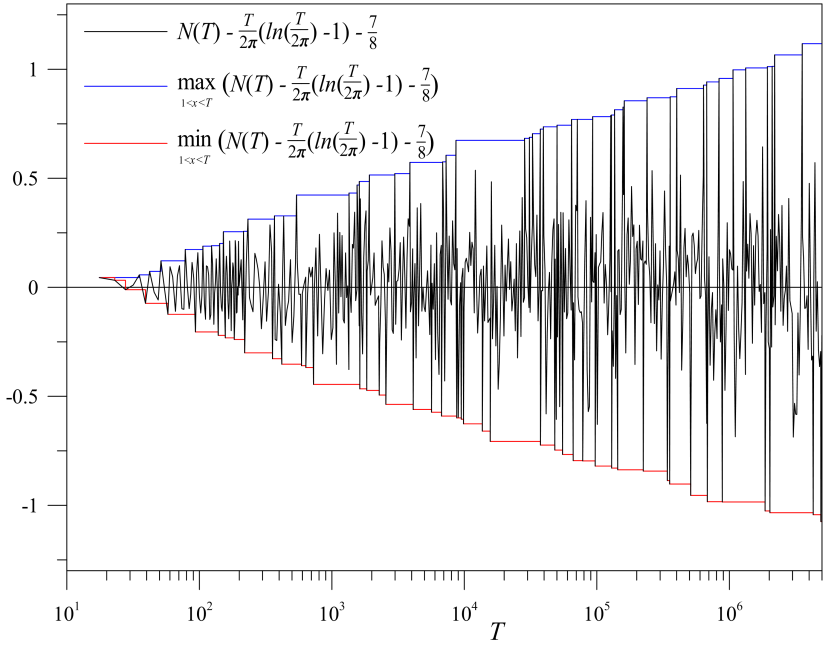
Trivial zeros: $s = -2, -4, -6, \ldots$ i.e. $\zeta(-2n) = 0$. Besides that there exist infinity of zeros $\rho = \sigma + it$ in the *critical strip* $0 \le \Re[\rho] = \sigma \le 1$. If ρ is zero, then also $\overline{\rho}$ and $1 - \rho$ are zeros. Zeros are located symmetrically around the *critical line* $\Re[s] = \frac{1}{2}$.

The Riemann hypothesis

All non-trivial zeros are lying on the critical line

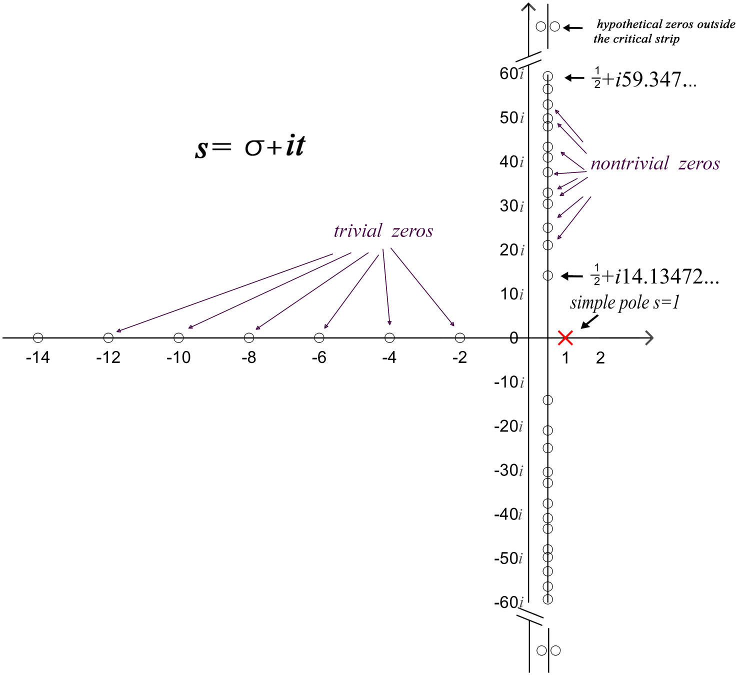
$$s = \frac{1}{2} + it$$

i.e. are of the form $\rho = \frac{1}{2} + i\gamma$
von Mangoldt(1905):
$N(T) = \frac{T}{2\pi} \ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + \mathcal{O}(\ln(T))$
Hardy (1914) There is infinitely many zeros of $\zeta(s)$ on the critical line.


$N(T) - \frac{T}{2\pi}(ln(\frac{T}{2\pi}) - 1) - \frac{7}{8}$
$\max_{1<x<T} (N(T) - \frac{T}{2\pi}(ln(\frac{T}{2\pi}) - 1) - \frac{7}{8})$
$\min_{1<x<T} (N(T) - \frac{T}{2\pi}(ln(\frac{T}{2\pi}) - 1) - \frac{7}{8})$
1
0.5
0
-0.5
-1
10^1
10^2
10^3
10^4
10^5
10^6
T

$$\zeta(\rho_i) = 0 \qquad i = 1, 2, \ldots, 26$$

1	$\frac{1}{2} + i14.134725142$	14	$\frac{1}{2} + i60.831778525$
2	$\frac{1}{2} + i21.022039639$	15	$\frac{1}{2} + i65.112544048$
3	$\frac{1}{2} + i25.010857580$	16	$\frac{1}{2} + i67.079810529$
4	$\frac{1}{2} + i30.424876126$	17	$\frac{1}{2} + i69.546401711$
5	$\frac{1}{2} + i32.935061588$	18	$\frac{1}{2} + i72.067157674$
6	$\frac{1}{2} + i37.586178159$	19	$\frac{1}{2} + i75.704690699$
7	$\frac{1}{2} + i40.918719012$	20	$\frac{1}{2} + i77.144840069$
8	$\frac{1}{2} + i43.327073281$	21	$\frac{1}{2} + i79.337375020$
9	$\frac{1}{2} + i48.005150881$	22	$\frac{1}{2} + i82.910380854$
10	$\frac{1}{2} + i49.773832478$	23	$\frac{1}{2} + i84.735492981$
11	$\frac{1}{2} + i52.970321478$	24	$\frac{1}{2} + i87.425274613$
12	$\frac{1}{2} + i56.446247697$	25	$\frac{1}{2} + i88.809111208$
13	$\frac{1}{2} + i59.347044003$	26	$\frac{1}{2} + i92.491899271$


hypothetical zeros outside the critical strip
$\frac{1}{2}+i59.347...$
$s = \sigma + it$
nontrivial zeros
trivial zeros
$\frac{1}{2}+i14.13472...$
simple pole s=1
60i
50i
40i
30i
20i
10i
0
-10i
-20i
-30i
-40i
-50i
-60i
-14
-12
-10
-8
-6
-4
-2
1
2

Next Riemann obtained the **exact** formula for $\pi(x)$. Let

$$R(x) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \mathrm{Li}(x^{1/m})$$

where $\mu(n)$ is the Möbius function:

$$\mu(n) = \begin{cases} 1 & \text{for } n = 1 \\ 0 & \text{when } p^2 | n \\ (-1)^r & \text{when } n = p_1 p_2 \ldots p_r \end{cases}$$

Then

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}) \qquad (*)$$

where the sum runs over zeros of $\zeta(s)$, i.e. $\zeta(\rho) = 0$.

$$\pi(x) \approx \mathrm{Li}(x)$$

Littlewood has proved that $\mathrm{Li}(x) - \pi(x)$ changes infinitely many times sign. Here is the computer illustration of the formula (*):

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}) \qquad (*)$$

gdzie suma przebiega po wszystkich zerach $\zeta(s)$, tzn. $\zeta(\rho) = 0$. Pochodna tego wzoru to suma δ zlokalizowanych na liczbach pierwszych.

Wząr (*) moňna sprawdziŁ na komputerze

$$\pi(x) \approx \mathrm{Li}(x) \qquad \pi(x) = \mathrm{Li}(x) + \mathcal{O}(\ln(x)\sqrt{x})$$

Z grubsza n-ta liczba pierwsza i $\mathrm{Li}^{-1}(n)$ maję $\frac{1}{2}$ tych samych cyfr.

x	$\pi(x)$	$\mathrm{Li}(x)$	$\mathrm{Li}(x) - \pi(x)$
10^2	25	30	5
10^3	168	178	10
10^4	1229	1246	17
10^5	9592	9630	38
10^6	78498	78628	130
10^7	664579	664918	339
10^8	5761455	5762209	754
10^9	50847534	50849235	1701
10^{10}	455052511	455055615	3104
10^{11}	4118054813	4118066401	11588
10^{12}	37607912018	37607950281	38263
10^{13}	346065536839	346065645810	108971
10^{14}	3204941750802	3204942065692	314890

x	$\pi(x)$	$\mathrm{Li}(x) - \pi(x)$
10^{15}	29844570422669	1052619
10^{16}	279238341033925	3214632
10^{17}	2623557157654233	7956589
10^{18}	24739954287740860	21949555
10^{19}	234057667276344607	99877775
10^{20}	2220819602560918840	222744647
10^{21}	21127269486018731928	597394250
10^{22}	201467286689315906290	1932355520
10^{23}	1925320391606803968923	7250186752
10^{24}	18435599767349200867866	17146872832
10^{25}	176846309399143769411680	55160980939

Jacques Salomon Hadamard (1865 – 1963) and Charles-Jean -tienne Gustave Nicolas, Baron de la VallÚe Poussin (1866 – 1962) proved in 1896 the PNT: there are no zeros of $\zeta(s)$ on the line $1 + it$.

J.P. Gram(1903): 15 zeros are on the critical line

$\vdots$

A.Turing (1953): 1104 zeros are on the critical line
("in an optimistic hope that a zero would be found off critical line")

$\vdots$

D.H. Lehmer (1956): 25000 zeros are on the critical line

$\vdots$

S. Wedeniwski: 2.5×10^{11} zeros are on the critical line: $s = \frac{1}{2} + it$ $|t| < 29,538,618,432.236$
X. Gourdon(2004): 10^{13} zeros are on the critical line.

How to prove the Riemann Hypothesis?

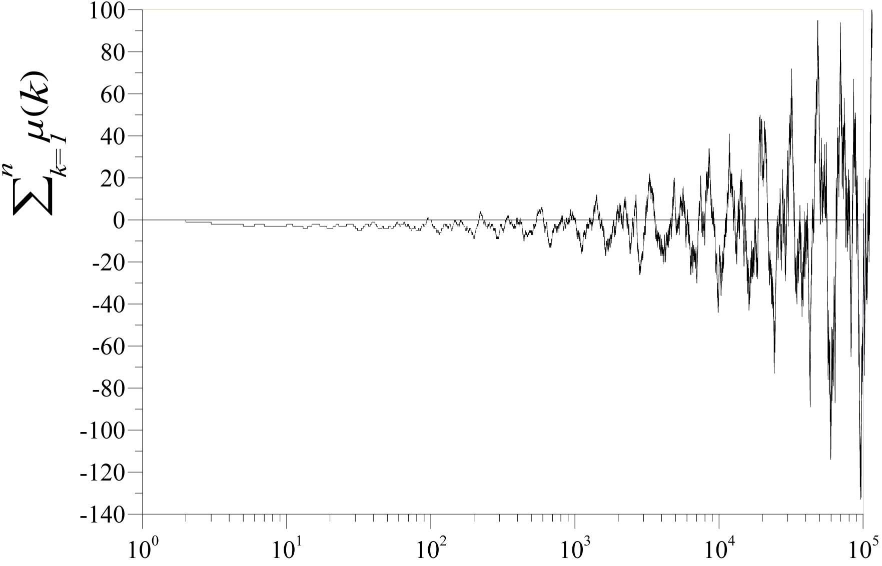
Mertens hypothesis:

$$M(x) = \sum_{n<x} \mu(n)$$

If $|M(x)| < \sqrt{x}$ then Riemann hypothesis is fulfilled.

Littlewood: RH is equivalent to the:

$$M(x) = \mathcal{O}(x^{1/2})$$



$\sum_{k=1}^{n} \mu(k)$
100
80
60
40
20
0
-20
-40
-60
-80
-100
-120
-140
10^0
10^1
10^2
10^3
10^4
10^5

A.M. Odłyżko and H.J.J. te Riele in 1985 have disproved Mertens hypothesis. They have calculated first 2000 zeros of $\zeta(s)$ with accuracy 100-105 digits – It took 40 hours on the CDC CYBER 750 + 10 hours on the Cray-1.

INTERNATIONAL Herald Tribune

Published With The New York Times and The Washington Post

** THE HAGUE, WEDNESDAY, FEBRUARY 15, 1984

r President Suleiman leader of the Syrian-

flicted a major defeat when they

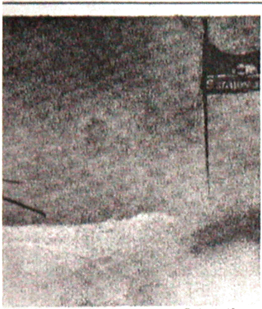
(Continued on Page 2, Col. 3)

that "we should begin to talk and negotiate on the problems that at the moment, lie between us."

cials, however, suggested that it would not take much substantive progress or even much promise of

ever, that "you'll have to wait and see what the position's going to be."

peaceful coexistence is needed more than ever in the age of nuclear missiles.



The Associated Press

'rofile at Winter Olympics

and his skis went up after he took a spill Tuesday in pics at Sarajevo. He was disqualified. His teammate ant slalom, won the event. Meanwhile, the British ice in won the gold medal. Stories on Pages 6 and 7.

Math Theory Gets a Nay After 10 Billion Yeas

By Lee Dembart

Los Angeles Times Service

LOS ANGELES — A mathematical conjecture, first proposed almost a 100 years ago, that was known to be true for the first 10 billion numbers has been proved false.

Besides being a reminder that in mathematics nothing is true until it is proved true, the finding that the so-called Mertens conjecture is false has important consequences in several fields of study, including number theory and algebra.

"It just shows you again that you have to be very careful," Andrew Odlyzko of Bell Laboratories, one of the disprovers of the Mertens conjecture, said Monday. "Empirical evidence can very often be misleading."

The most significant consequence is that the Riemann hypothesis, considered the most important unsolved problem in mathematics today, remains unsolved. If the Mertens conjecture were true, it would have directly implied the truth of the Riemann hypothesis, which is in itself the linchpin of a sheaf of unsolved problems in mathematics.

However, the falsity of the Mertens conjecture does not imply that the Riemann hypothesis is false. That remains an open question.

Mr. Odlyzko and Herman te Riele of the Center for Mathematics and Computer Science in Amsterdam disproved the Mertens conjecture using fast computers and an improved method of testing. The conjecture, first proposed by T.J. Stieltjes in 1885 and later by F. Mertens in 1897, is a statement about the behavior of a function derived from the number of prime factors in each whole number from 1 to infinity.

Mr. Odlyzko and Mr. te Riele have not found a single counterexample to the conjecture, and it is not clear that they or anyone ever will. They believe that a counterexample will be found around 10 to the 10th power to the 70th power, which is a number larger than the number of atoms in the universe and well beyond the ability of any computer to calculate.

"No one has an inkling of how you might compute it," Ronald Graham of Bell Laboratories said. "It might conceivably be a problem which you know has to fail but you'll never find a value for which it fails."

Nonetheless, Mr. Odlyzko said Monday, his and Mr. te Riele's work shows that "there are infinitely many counterexamples" to the Mertens conjecture — even if none is ever found.

Using pencil and paper, Mertens himself showed that his hypothesis was true for the first 10,000 integers, which are the whole numbers 1, 2, 3, 4 and so on. In 1913, another mathematician calculated that it was true for the first 5 million integers. In 1963, a computer was used to show that it was true for the first 10 billion integers.

"It would be hard to disprove by computation," he said.

The Mertens conjecture says that the special summation function derived from the number of prime factors in a number is always less than the number's square root. As the numbers increase, the summation function shows no particular form.

Those who had hoped that the Mertens conjecture was true because of its implications for the Riemann hypothesis will be disappointed. A proof of the Riemann hypothesis would result in important improvements in work on prime numbers, one of the most significant elements of contemporary work in number theory.

So important is the Riemann hypothesis that an entire chapter of a 1927 book was titled "Under the Assumption of the Riemann Hypothesis," and it listed many theorems that depend on it. The hypothesis, proposed by Bernhard Riemann in 1859, involves the places where the Riemann zeta function equals 0.

Using computers, the hypothesis has been tested for the first 320 million 0s. Similarly, Fermat's last theorem, another important outstanding problem, has been tested and shown to be true for the first 125,000 numbers.

But, as is shown by the disproof of the Mertens conjecture, just because the Riemann hypothesis is true for the first 320 million 0s does not mean it is always true. It is a similar case for Fermat's last theorem and the first 125,000 exponents.

The problem is that the number of integers is infinite, so no matter how many individual cases are studied, there will always be infinitely many that have not been.

RH is equivalent to

$$|\pi(x) - \mathrm{Li}(x)| \leq \frac{1}{8\pi}\sqrt{x}\ln(x) \text{ for all } x \geq 2657.$$

Criteria for RH involving integrals

RH is true $\Leftrightarrow$

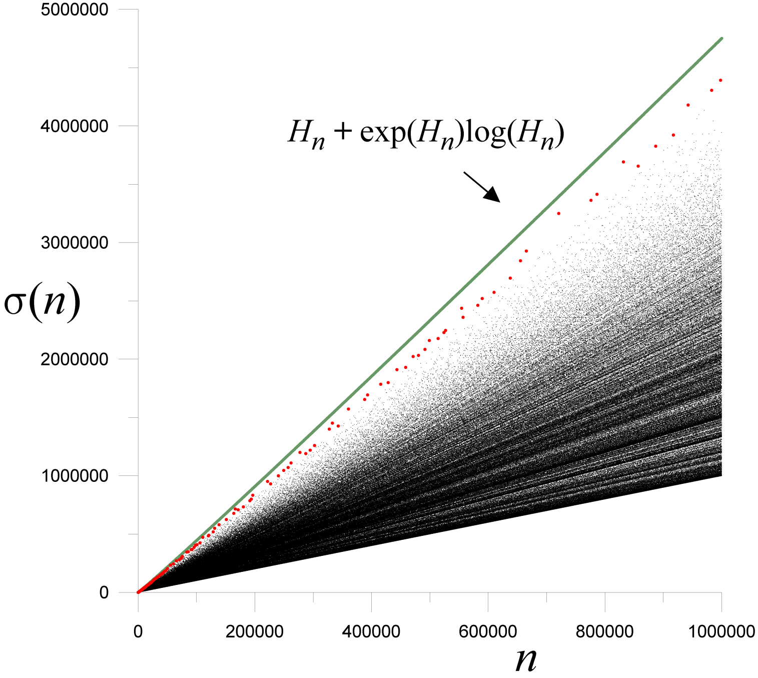
$$\int_0^\infty \frac{1-12t^2}{(1+4t^2)^3}\int_{\frac{1}{2}}^\infty \ln(|\zeta(\sigma+it)|d\sigma\, dt = \pi\frac{3-\gamma}{32}.$$

$$\text{RH is true} \Leftrightarrow \int_{\Re(s)=\frac{1}{2}} \frac{\ln(|\zeta(s)|)}{|s|^2}|ds| = 0.$$

Lagarias (2000): An Elementary Problem Equivalent to the Riemann Hypothesis: RH is equivalent to the inequalities:

$$\sigma(n) \equiv \sum_{d|n} d \leq H_n + \exp(H_n) \log(H_n)$$

for each $n = 1, 2, \ldots$. Here H_n are n-th harmonic numbers $H_n = \sum_{j=1}^{n} \frac{1}{j}$.


$H_n + \exp(H_n)\log(H_n)$
$\sigma(n)$
n
5000000
4000000
3000000
2000000
1000000
0
0
200000
400000
600000
800000
1000000

The de Bruijn-Newmana constant (1950, 1976):

$$\xi(iz) = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right)$$

RH $\Leftrightarrow$ all zeros of $\xi(iz)$ are real.

$$\Phi(t) = \sum_{n=1}^{\infty}(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t})e^{-\pi n^2 e^{4t}}$$

$t \geq 0$. Then ξ is a Fourier transform of Φ:

$$\frac{1}{8}\xi\left(\frac{z}{2}\right) = \int_0^{\infty}\Phi(t)\cos(zt)dt$$

Larger class: $H(z,\lambda)$ is the Fourier transform of $\Phi(t)e^{\lambda t^2}$ and $H(z,0)=\frac{1}{8}\xi(\frac{1}{2}z)$ N. G. De Bruijn proved that (1950):

1. $H(z,\lambda)$ has only real zeros for $\lambda \geq \frac{1}{2}$
2. If $H(z,\lambda)$ has only real zeros for some λ', then $H(z,\lambda)$ has only real zeros for each $\lambda' > \lambda$.

Ch. Newman (1976) has proved that there exists such parameter λ_1, that $H(z, \lambda_1)$ has at least one non-real zero. Thus there exists such constant Λ in the interval $-\infty < \Lambda < \frac{1}{2}$, that $H(z, \lambda)$ has real zeros $\Leftrightarrow \lambda > \Lambda$. Riemann Hypothesis is equivalent to $\Lambda \leq 0$.

Newman believes $\Lambda \geq 0$.

Csordas *et al* (360 digits) (1988) $-50 < \Lambda$

$$\vdots$$

te Riele (250 digitsr) (1991) $-5 < \Lambda$

$$\vdots$$

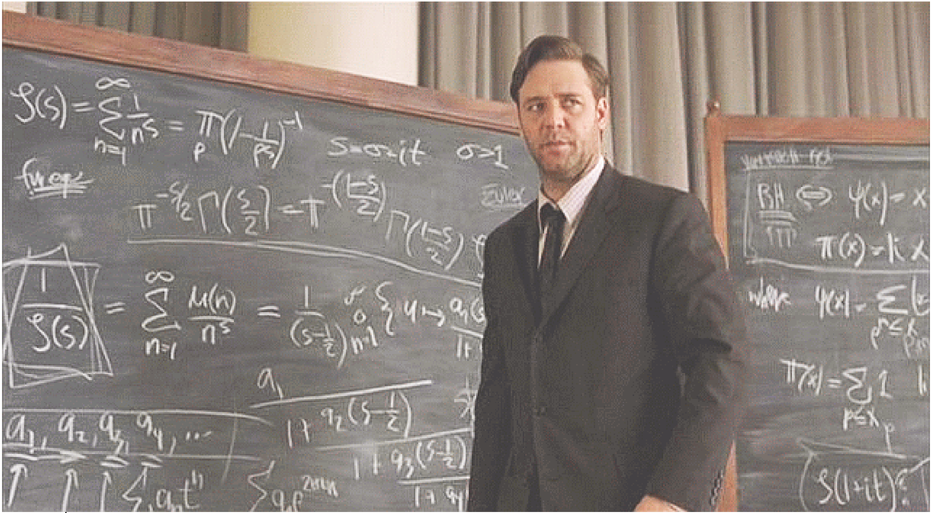
Odyzko (2000) $-2.7 \cdot 10^{-9} < \Lambda$

$$k = 10^{20} + 71810732, \quad \gamma_{k+1} - \gamma_k < 0.000145$$

" if RH is true, it is barely true"

Maybe RH is undecidable

(P. Cohen, Fields medallist 1966)

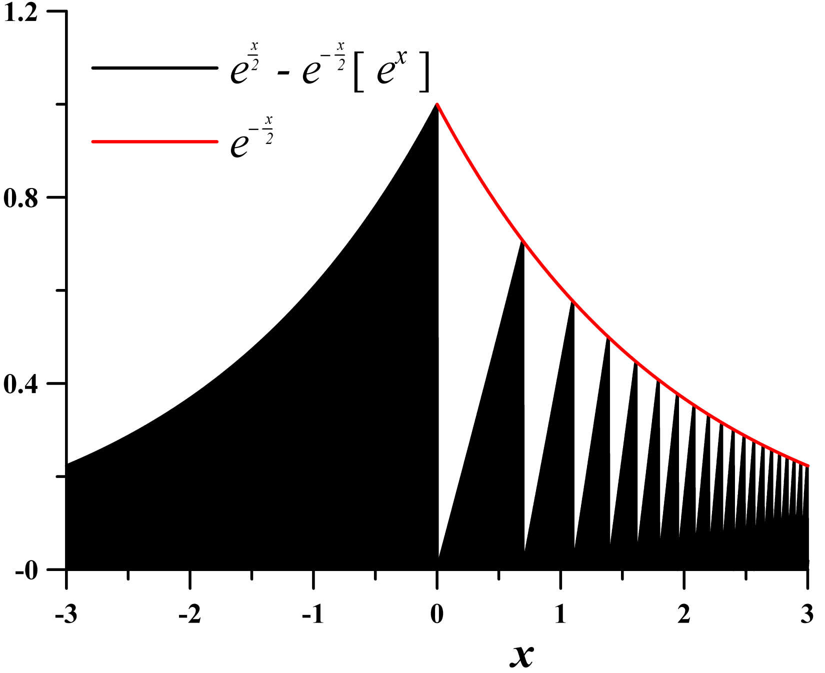


B. van der Pol

Bulletin of the AMS **53** (1947), pp. 976-981

$$\frac{\zeta(\frac{1}{2}+it)}{\frac{1}{2}+it} = \int_{-\infty}^{\infty} \left(e^{-x/2} \lfloor e^x \rfloor - e^{x/2} \right) e^{-ixt} dx$$

paper, scissors, rotor, source of light, photocell, sinusoidal current of variable frequency.



1.2
0.8
0.4
-0
-3
-2
-1
0
1
2
3
x
$e^{\frac{x}{2}} - e^{-\frac{x}{2}}[e^x]$
$e^{-\frac{x}{2}}$

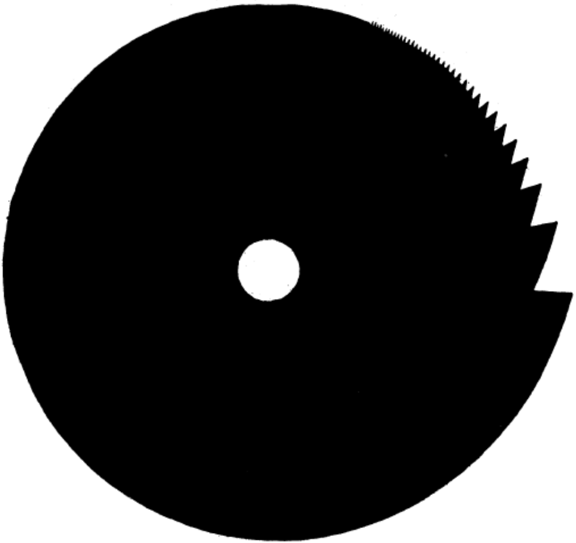


FIG. 2. Paper disc, showing the sawtooth function $y(x)$ for the range $-9<x<+9$.

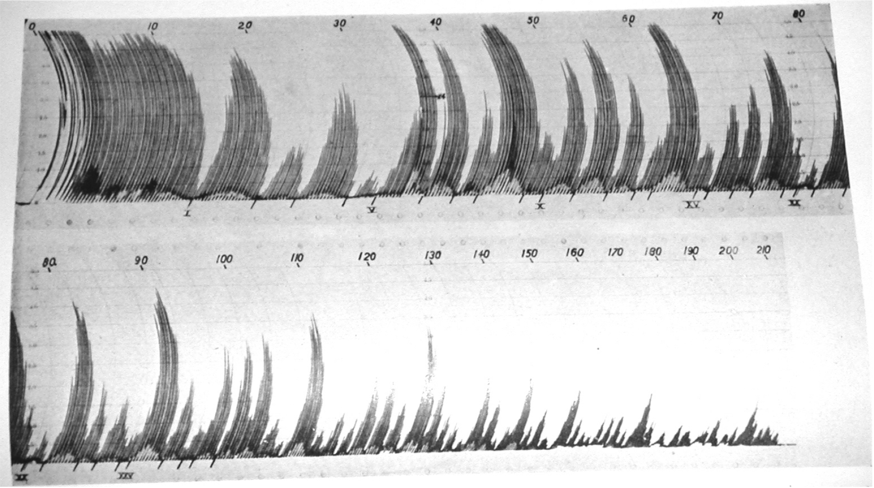


Fig. 3. Record of $|\zeta(1/2+it)/(1/2+it)|$ as produced electromechanically showing o.a. minima, the first 29 of which (marked /) correspond to the 29 known zeros of the zeta-function on the critical line.

The Poly'a– Hilbert Conjecture: "$\zeta(\frac{1}{2} + i\hat{H}) = 0$"

RH is true, because complex parts of the non-trivial zeros correspond to eigenvalues of the positive self-adjoint operator. Was proposed around 1910, first published in 1973.

letter of G. Polya (1887-1985) to Odlyzko from January 1982:

STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

DEPARTMENT OF MATHEMATICS

Jan. 3, 82

Dear Mr. Odlyzko,

Many thanks for your letter of Dec. 8, I can only tell you what happened to me.

I spent 2 years in Göttingen ending around the begin of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann hypothesis should be true."

This would be the case, I answered, if the nontrivial zeros of the ξ-function were so connected with the physical problem that the Riemann hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.

I never published this remark, but somehow it became known and it is still remembered.

With best regards
Yours sincerely
George Polya

Montgomery (1973):Assume RH: $\rho = \frac{1}{2} + i\gamma$.

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ \frac{2\pi\alpha}{\ln T}\leq\gamma-\gamma'\leq\frac{2\pi\beta}{\ln T}}} 1 = \int_{\alpha}^{\beta} \left(1 - \left(\frac{sin\pi u}{\pi u}\right)^2\right) du$$

F. Dyson recognized in the above formula the same dependence as in the behavior of the differences between pairs of eigenvalues of random Hermitian matrices.

A. Odlyzko performed computer experiments

The 10^{20}*-th zero of the Riemann zeta function and 70 million of its neighbors*,
thousands of hours on Cray-1 i Cray X-MP

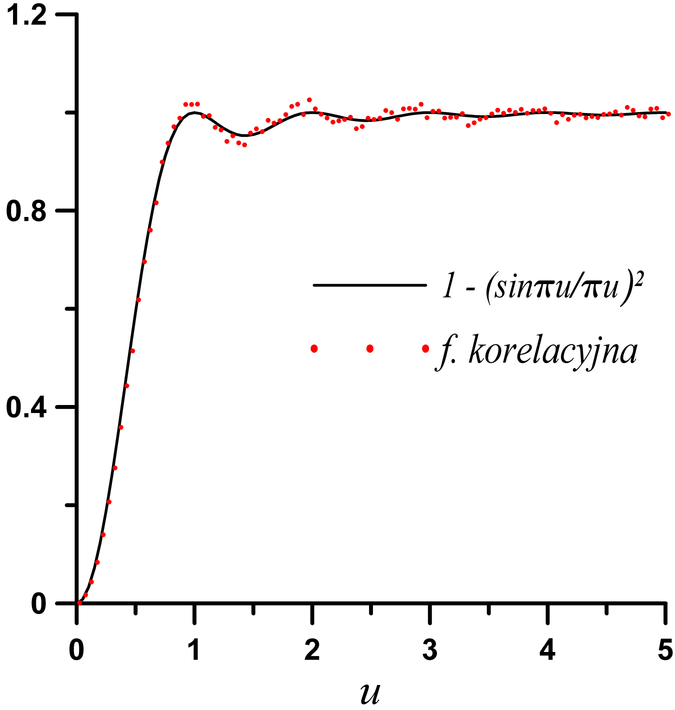
The 10^{20}*-th zero of the Riemann zeta function and 175 million of its neighbors*,
1992 revision of 1989

The 10^{21}*-st zero of the Riemann zeta function*,

The 10^{22}*-nd zero of the Riemann zeta function*, ($\sim 10^9$ zeros)

Odlyzko (1987): $\delta_n = (\gamma_{n+1} - \gamma_n)\frac{\ln(\gamma_n/(2\pi))}{2\pi}$

$$\frac{1}{N} \sum_{\substack{1 \le n \le N, \\ k \ge 0 \\ \delta_n + \delta_{n+1} + \ldots + \delta_{n+k} \in [\alpha, \beta]}} 1 \sim \int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u} \right)^2 \right) du$$


1.2
0.8
0.4
0
0
1
2
3
4
5
u
1 - (sinπu/πu)²
f. korelacyjna

The results confirmed the GUE (Gaussian unitary ensemble) distribution: the gaps between imaginary parts of consecutive nontrivial zeros of $\zeta(s)$ display the same behavior as the differences between pairs of eigenvalues of random Hermitian matrices. P. Sarnak wrote: "At the phenomenological level this is perhaps the most striking discovery about the zeta function since Riemann."

$H=xp$ AND THE RIEMANN ZEROS

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1. INTRODUCTION

The Riemann hypothesis [1, 2] states that the complex zeros of $\zeta(s)$ lie on the critical line Re $s=1/2$; that is, the nonimaginary solutions E_n of

$$\zeta\left(\tfrac{1}{2}+\mathrm{i}E_n\right)=0 \tag{1}$$

are all real. Here we will present some evidence that the E_n are energy levels, that is eigenvalues of a hermitian quantum operator (the 'Riemann operator'), associated with the classical hamiltonian

$$H_{\mathrm{cl}}(x,p)=xp \tag{2}$$

$$\hat{H} = \frac{1}{2}(xp + px)$$

- $\hat{H}$ has the classical counterpart describing a chaotic, unstable and bounded dynamics.
- The dynamics of the Riemann does not have reversal time symmetry.
- And the Riemann dynamics is one-dimensional

The number of levels of $\hat{H} < E$:

$$N(E) = \frac{E}{2\pi}\left(\ln\left(\frac{E}{2\pi}\right) - 1\right) + \frac{7}{8} + \dots$$

$$N(T) = \frac{T}{2\pi}\ln\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + \mathcal{O}(\ln(T))$$

$$\psi_E(x) \sim \frac{const}{|x|^{1/2-iE}}\zeta\left(\frac{1}{2} - iE\right)$$

American Scientist, Volume 91 2003 July–August

The Spectrum of Riemannium

Acknowledgments
I owe the title of this column to Oriol Bohigas, who mentioned "the spectrum of Riemannium" in a talk at the Mathematical Sciences Research Institute in Berkeley, where I was an embedded journalist in 1999.

Brian Hayes

The year: 1972. The scene: Afternoon tea in Fuld Hall at the Institute for Advanced Study. The camera pans around the Common Room, passing by several Princetonians in tweeds and corduroys, then zooms in on Hugh Montgomery, boyish Midwestern number theorist with sideburns. He has just been introduced to Freeman Dyson, dapper British physicist.

Dyson: So tell me, Montgomery, what have you been up to?

Montgomery: Well, lately I've been looking into the distribution of the zeros of the Riemann zeta function.

Dyson: Yes? And?

Montgomery: It seems the two-point correlations go as.... (*turning to write on a nearby blackboard*):

$$1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$

Dyson: Extraordinary! Do you realize that's the games of solitaire, one-dimensional gases and chaotic quantum systems. Is it all just a cosmic coincidence, or is there something going on behind the scenes?

The Spectrum of Interstatium

How things distribute themselves in space or time or along some more abstract dimension is a question that comes up in all the sciences. An astronomer wants to know how galaxies are scattered around the universe; a biologist might study the distribution of genes along a strand of chromatin; a seismologist records the temporal pattern of earthquakes; a mathematician ponders the sprinkling of prime numbers among the integers. Here I shall consider only discrete, one-dimensional distributions, where the positions of items can be plotted along a line.

Figure 1 shows samples of several such distributions, some of them mathematically defined

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Lorentz-invariant Hamiltonian and Riemann hypothesis

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Abstract. We have given some arguments that a two-dimensional Lorentz-invariant Hamiltonian may be relevant to the Riemann hypothesis concerning zero points of the Riemann zeta function. Some eigenfunction of the Hamiltonian corresponding to infinite-dimensional representation of the Lorentz group have many interesting properties. Especially, a relationship exists between the zero zeta-function condition and the absence of trivial representations in the wavefunction. We also give a heuristic argument for the validity of the hypothesis.

The Riemann hypothesis (RH) [1–3] is one of the long-standing problems in the number theory. The Riemann's zeta function $\zeta(z)$ for a complex variable z is defined for $\mathrm{Re}\, z > 1$ by

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **44** (2011) 145203 (9pp) doi:10.1088/1751-8113/44/14/145203

Riemann hypothesis and quantum mechanics

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North-Holland, Amsterdam

ON THE RIEMANN HYPOTHESIS: A FRACTAL RANDOM WALK APPROACH

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In his investigation of the distribution of prime numbers Riemann, in 1859, introduced the zeta function with a complex argument. His analysis led him to hypothesize that all the complex zeros of the zeta function lie on a vertical line in the complex plane. The proof or disproof of this hypothesis has been a famous outstanding problem in mathematics. We are able to recast Riemann's Hypothesis into a probabilistic framework connected to the fractal behavior of a lattice random walk. Fractal random walks were introduced by P. Levy, and in the continuum are called Levy flights. For one particular lattice version of a Levy flight we show the connection to Weierstrass' continuous but nowhere differentiable function. For a different lattice version, using a Mellin transform analysis, we show how the zeroes of the zeta function

M. Shlesinger has investigated a very special one-dimensional random walk which can be linked with the RH. The probability of jumping to other sites with steps having a displacement of $\pm l$ sites involves directly the Möbius function:

$$p(\pm l) = \frac{1}{2} C \left(\frac{1}{l^{\,1+\beta}} \pm \frac{\mu(l)}{l^{\,1+\beta-\epsilon}} \right), \qquad \beta > 0,$$

where $C = \frac{1}{\zeta(1+\beta) + \frac{1}{\zeta(1+\beta)}}$ is a normalization factor.

Some general properties of the "structure function" $\lambda(k)$ being the Fourier of the probabilities $p(l)$: $\lambda(k) = \sum_l e^{ikl} p(l)$, enabled Shlesinger to locate the complex zeros inside the critical strip, however the result of J. Hadamard and Ch. J. de la Vallée–Poussin that $\zeta(1+it) \neq 0$ can not be recovered by this method. What is interesting the existence of off critical line zeros is not in contradiction with behavior of $\lambda(k)$ following from the laws of probability.

Open Circular Billiards and the Riemann Hypothesis

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A comparison of escape rates from one and from two holes in an experimental container (e.g., a laser trap) can be used to obtain information about the dynamics inside the container. If this dynamics is simple enough one can hope to obtain exact formulas. Here we obtain exact formulas for escape from a circular billiard with one and with two holes. The corresponding quantities are expressed as sums over zeros of the Riemann zeta function. Thus we demonstrate a direct connection between recent experiments and a major unsolved problem in mathematics, the Riemann hypothesis.

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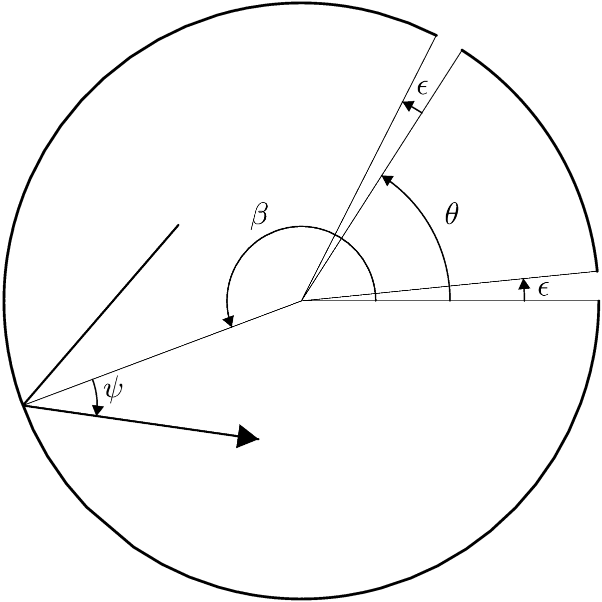
Billiard systems, in which a point particle moves freely except for specular reflections from rigid walls, permit close connections between rigorous mathematics and experimental physics. Very general physical situations, in which particles or waves are confined to cavities or other homogeneous regions, are related to well understood billiard dynamical systems, directly for particles and via semiclassical (short wavelength) theories for waves. Precise billiard experiments have used microwaves in metal [1,2] and superconducting [3] cavities and with wave guides [4], visible light reflected from mirrors [5], phonons in quartz blocks [6], electrons in semiconductors

At long times, the probability of a particle remaining in an integrable billiard with a hole is well-known to exhibit power law decay, in contrast to exponential decay from strongly chaotic billiards [19]; however the coefficient of the power ("escape rate") in the integrable case has not been computed exactly to our knowledge. Numerical simulations can be misleading; for example, a power law decay at long times can be masked by an exponential term at short times.

Here we consider the circle billiard, which is integrable due to angular momentum conservation. Some three dimensional cases, namely, the cylinder and sphere, can be



ϵ
β
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Dynamics of point particle bouncing inside the circular billiard

$$T(\beta, \psi) = (\beta + \pi - 2\psi, \psi)$$

Survival probability $P(t)$ that the particle will not escape till time t:

$$tP(t) = \frac{1}{8\pi} \sum_{n=1}^{\infty} n(\phi(n) - \mu(n))$$

$$\times \left[g\left(\frac{2\pi}{n} - \theta' - \epsilon \right) + g\left(\theta' - \epsilon\right) \right]$$

$\phi(n)$ Euler's totient function: the number of positive integers $m \leq n$ that are relatively prime to n: $gcd(m, n) = 1$

$$g(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Then they proved that RH is equivalent to

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\epsilon^{\delta}(tP_1(t)-2/\epsilon)=0$$

be true for every $\delta > -1/2$.

The functional equation can be written in non–symmetrical form:

$$2\Gamma(s)\cos\left(\frac{\pi}{2}s\right)\zeta(s) = (2\pi)^s\zeta(1-s).$$

The Kramers–Wannier duality relation for the partition function $Z(J)$ of the two dimensional Ising model with parameter J expressed in units of kT

$$Z(J) = 2^N(\cosh(J))^{2N}(\tanh(J))^N Z(\widetilde{J}),$$

where N denotes the number of spins and $\widetilde{J}$ is related to J via $e^{-2\widetilde{J}} = \tanh(J)$,

The Lee–Yang circle theorem on the zeros of the partition function. Let $Z(\beta, z)$ denote the grand – canonical partition function. Phase transitions are connected with the singularities of the derivatives of $Z(\beta, z)$, and they appear when $Z(\beta, z) = 0$. The Lee–Yang theorem asserts that in the thermodynamical limit, when the sum for partition function involves infinite number of terms, all zeros of $Z(\beta, z)$ for a class of spin models are pure imaginary and lie in the complex plane of the magnetic field z on the unit circle: $|z| = 1$.
$s = \frac{1}{2} + it$ can be mapped into the unit circle
$s \to u = s/(1-s) = (\frac{1}{2} + it)/(\frac{1}{2} - it) : |u| = 1$.

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The Number-Theoretical Spin Chain and the Riemann Zeroes

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Abstract: It is an empirical observation that the Riemann zeta function can be well approximated in its critical strip using the Number-Theoretical Spin Chain. A proof of this would imply the Riemann Hypothesis. Here we relate that question to the one of spectral radii of a family of Markov chains. This in turn leads to the question whether certain graphs are Ramanujan.

The general idea is to explain the pseudorandom features of certain number-theoretical functions by considering them as observables of a spin chain of statistical mechanics. In an appendix we relate the free energy of that chain to the Lewis Equation of modular theory.

Birds and Frogs

Freeman Dyson

Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time. I happen to be a frog, but many of my best friends are birds. The main theme of my talk tonight is this. Mathematics needs both birds and frogs. Mathematics is rich and beautiful because birds give it broad visions and frogs give it intricate details. Mathematics is both great art and important science, because it combines generality of concepts with depth of structures. It is stupid to claim that birds are better than frogs because they see farther, or that frogs are better than birds because they see deeper. The world of mathematics is both broad and deep, and we need birds and frogs working together to explore it.

This talk is called the Einstein lecture, and I am grateful to the American Mathematical Society for inviting me to do honor to Albert Einstein. Einstein was not a mathematician, but a physicist who had mixed feelings about mathematics. On the one hand, he had enormous respect for the power of mathematics to describe the workings of nature, and he had an instinct for mathematical beauty which led him onto the right track to find nature's laws. On the other hand, he had no interest in pure mathematics, and he had no technical skill as a mathematician. In his later years he hired younger colleagues with the title of assistants to do mathematical calculations for him. His way of thinking was physical rather than mathematical. He was supreme among physicists as a bird who saw further than others. I will not talk about Einstein since I have nothing new to say.

Francis Bacon and René Descartes

At the beginning of the seventeenth century, two great philosophers, Francis Bacon in England and René Descartes in France, proclaimed the birth of modern science. Descartes was a bird, and Bacon was a frog. Each of them described his vision of the future. Their visions were very different. Bacon said, "All depends on keeping the eye steadily fixed on the facts of nature." Descartes said, "I think, therefore I am." According to Bacon, scientists should travel over the earth collecting facts, until the accumulated facts reveal how Nature works. The scientists will then induce from the facts the laws that Nature obeys. According to Descartes, scientists should stay at home and deduce the laws of Nature by pure thought. In order to deduce the laws correctly, the scientists will need only the rules of logic and knowledge of the existence of God. For four hundred years since Bacon and Descartes led the way, science has raced ahead by following both paths simultaneously. Neither Baconian empiricism nor Cartesian dogmatism has the power to elucidate Nature's secrets by itself, but both together have been amazingly successful. For four hundred years English scientists have tended to be Baconian and French scientists Cartesian. Faraday and Darwin and Rutherford were Baconians; Pascal and Laplace and Poincaré were Cartesians. Science was greatly enriched by the cross-fertilization of the two contrasting cultures. Both cultures were always at work in both countries. Newton was at heart a Cartesian, using

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crystals. That is a major program of research which is still in progress.

A fourth joke of nature is a similarity in behavior between quasi-crystals and the zeros of the Riemann Zeta function. The zeros of the zeta-function are exciting to mathematicians because they are found to lie on a straight line and nobody understands why. The statement that with trivial exceptions they all lie on a straight line is the famous Riemann Hypothesis. To prove the Riemann Hypothesis has been the dream of young mathematicians for more than a hundred years. I am now making the outrageous suggestion that we might use quasi-crystals to prove the Riemann Hypothesis. Those of you who are mathematicians may consider the suggestion frivolous. Those who are not mathematicians may consider it uninteresting. Nevertheless I am putting it forward for your serious consideration. When the physicist Leo Szilard was young, he became dissatisfied with the ten commandments of Moses and wrote a new set of ten commandments to replace them. Szilard's

I had some vague ideas that I thought might lead to a proof. In recent years, after the discovery of quasi-crystals, my ideas became a little less vague. I offer them here for the consideration of any young mathematician who has ambitions to win a Fields Medal.

Quasi-crystals can exist in spaces of one, two, or three dimensions. From the point of view of physics, the three-dimensional quasi-crystals are the most interesting, since they inhabit our three-dimensional world and can be studied experimentally. From the point of view of a mathematician, one-dimensional quasi-crystals are much more interesting than two-dimensional or three-dimensional quasi-crystals because they exist in far greater variety. The mathematical definition of a quasi-crystal is as follows. A quasi-crystal is a distribution of discrete point masses whose Fourier transform is a distribution of discrete point frequencies. Or to say it more briefly, a quasi-crystal is a pure point distribution that has a pure point spectrum. This definition includes

In 1975 S.M. Voronin proved theorem on the universality of the Riemann $\zeta(s)$ function: Let $0 < r < 1/4$ and $f(s)$ be a complex function continuous for $|s| \leq r$ and analytical in the interior of the disk. If $f(s) \neq 0$, then for every $\epsilon > 0$ there exists real number $T = T(\epsilon, f)$ such that:

$$\max_{|s| \leq r} \left| f(s) - \zeta\left(s + \left(\frac{3}{4} + i\,T\right)\right)\right| < \epsilon.$$

Riemann zeta function is a fractal

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Abstract – Voronin's theorem on the "Universality" of Riemann zeta function is shown to imply that Riemann zeta function is a fractal (in the sense that Mandelbrot set is a fractal) and a concrete "representation" of the "giant book of theorems" that Paul Halmos referred to.

Keywords: Riemann zeta function, Voronin, fractal, Paul Halmos, Info. theory

Voronin's theorem [1] on the "Universality" of Riemann zeta function [2] states that

Theorem Let $0 < r < 1/4$ and let $f(s)$ be a complex function analytic and continuous for $|s| \leq r$. If $f(s) \neq 0$, then for every $\epsilon > 0$, there exists a real number $T = T(\epsilon, f)$ such that

$$\max_{|s|\leq r} \left| f(s) - \zeta(s + (\frac{3}{4} + i\,T)) \right| < \epsilon$$

Let's infer 3 Corollaries from Voronin's theorem. The 1st is interesting, the 2nd is a strange and amusing consequence, and the 3rd is ludicrous and shocking (but a consequence nevertheless).

Corollary 1 *Riemann zeta function is a fractal.*

J. Derbyshire "Prime Obsession" (2001) pp. 357-358:
JD: Andrew, you have gazed on more non-trivial zeros of the Riemann zeta function than any person alive. What do you think about this darn Hypothesis? Is it true, or not?
AO: Either it's true, or else it isn't.
JD: Oh, come on, Andrew. You must have some feeling for an answer. Give me a probability. Eighty percent it's true, twenty percent it's false? Or what?
AO: Either it's true, or else it isn't.