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# Double-logarithmic endpoint resummations for (semi-) inclusive lepton-hadron processes

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with G. Soar, A. Lo Presti, C.H. Kom (UoL), A. Almasy (UoL, now DESY),  
K. Yeats (Simon-Frazer U), S. Moch (DESY) and J. Vermaseren (NIKHEF)

- Splitting and coefficient functions and their endpoint behaviour
- Generalized threshold resummation of  $1/\text{Mellin-}N$  contributions
- Small- $x$  resummation of  $x^{-1} \ln^\ell x$  (SIA) and  $x^0 \ln^\ell x$  (DIS) terms

# Conventions and references

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**Double-log enhancement:** two additional logs  $L$  per additional order in  $\alpha_s$

$$Q|_{\alpha_s^{n+n_a}} \sim \underbrace{\#L^{2n}}_{\text{LL}} + \underbrace{\#L^{2n-1}}_{\text{NLL}} + \underbrace{\#L^{2n-2}}_{\text{NNLL}} + \dots$$

LL, NLL, ...: leading logarithms, next-to-leading logarithms, ...

Counting of a resummation, cf. small- $x$ , not of a (stronger) exponentiation,  
cf. soft gluons: **NNLL resummation  $\Leftrightarrow$  (re-expanded) NLL exponentiation**

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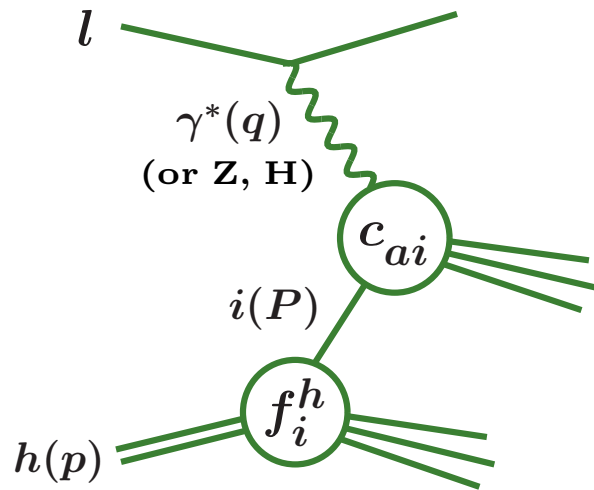
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- **Non-singlet NNLL (NLL for DY) resummation from physical kernels**  
MV, arXiv:0902.2342 (JHEP), 0909.2124 (JHEP)
- **Singlet NNLL for fourth-order splitting functions and  $F_L$  in DIS**  
SMVV, 0912.0369 (NPB), 1008.0952 (Loops & Legs)
- **Generalized threshold resummation in inclusive DIS and SIA**  
A.V., 1005.1606 (PLB); ASV, 1012.3352 (JHEP); ALPV, 1202.5224 (Radcor), to app.
- **Small- $x$  resummation of splitting & coefficient funct's in SIA and DIS**  
A.V., arXiv:1108.2993 (JHEP); KVY, arXiv:1207.5631 (JHEP); KV, to appear

# Hard lepton-hadron processes in pQCD

Inclusive deep-inelastic scattering (DIS), semi-incl.  $l^+l^-$  annihilation (SIA)



Left  $\rightarrow$  right: DIS,  $q$  spacelike,  $Q^2 = -q^2$

$$P = \xi p, f_i^h = \text{parton distributions}$$

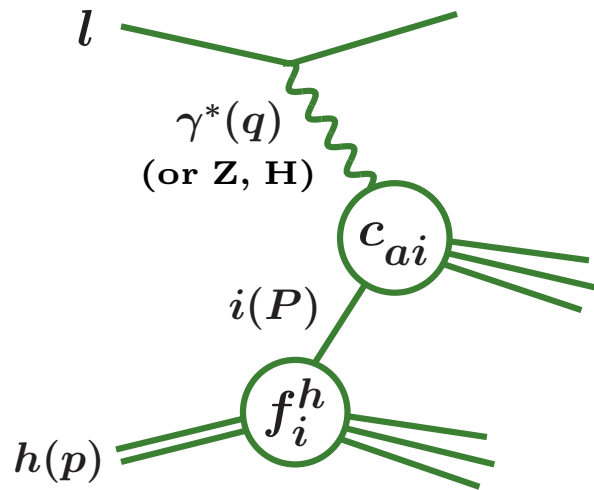
Top  $\rightarrow$  bottom:  $l^+l^-$ ,  $q$  timelike,  $Q^2 = q^2$

$$p = \xi P, \text{fragmentation distributions}$$

Drell-Yan (DY)  $l^+l^-$  production: bottom  $\rightarrow$  top, 2<sup>nd</sup> hadron from right ( $\{...\}$ )

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Structure functions, fragmentation functions etc  $F_a$ : **coefficient functions**

$$F_a(x, Q^2) = \left[ C_{a,i\{j\}}(\alpha_s(\mu^2), \mu^2/Q^2) \otimes f_i^h(\mu^2) \{ \otimes f_j^{h'}(\mu^2) \} \right](x) + \mathcal{O}(1/Q^{(2)})$$

Scaling variables:  $x = Q^2/(2p \cdot q)$  in DIS etc.  $\mu$ : renorm./mass-fact. scale

# Splitting and coefficient functions in pQCD

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Parton/fragmentation distributions  $f_i$  : (renorm. group) evolution equations

$$\frac{d}{d \ln \mu^2} f_i(\xi, \mu^2) = \left[ P_{ik/ki}^{S/T}(\alpha_s(\mu^2)) \otimes f_k(\mu^2) \right](\xi), \quad \otimes : \text{Mellin convolution}$$

Initial conditions: incalculable, fit-analyses of reference processes

Expansion in  $\alpha_s$ : **splitting functions  $P$** , **coefficient fct's  $c_a$**  of observables

$$\begin{aligned} P &= \alpha_s P^{(0)} + \alpha_s^2 P^{(1)} + \alpha_s^3 P^{(2)} + \alpha_s^4 P^{(3)} + \dots \\ C_a &= \underbrace{\alpha_s^{n_a} \left[ c_a^{(0)} + \alpha_s c_a^{(1)} \right]} + \alpha_s^2 c_a^{(2)} + \alpha_s^3 c_a^{(3)} + \dots \end{aligned}$$

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**NLO: first real prediction of size of cross sections**

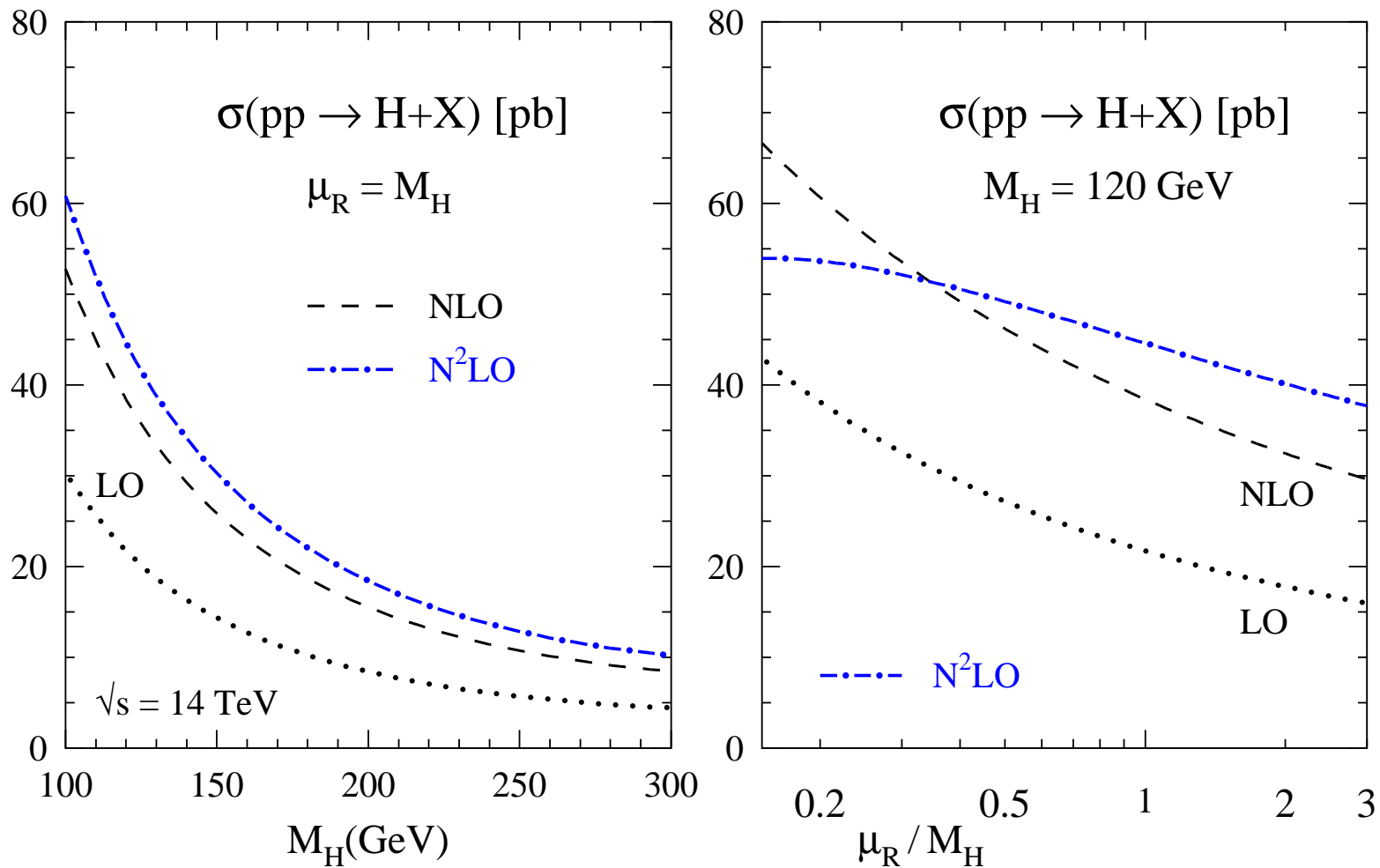
**NNLO,  $P^{(2)}$ ,  $c_a^{(2)}$ : first serious error estimate of pQCD predictions**

**N<sup>3</sup>LO: for high precision ( $\alpha_s$  from DIS), slow convergence (Higgs in  $pp/p\bar{p}$ )**

$F_2/F_3$  : **MVV (2005/8)**,  $P_{\text{ns}, N=2}^{(3)}$  : **Baikov, Chetyrkin (06), ...**;  $\sigma_{\text{H,soft}}$  : **MV (05), ...**

**Endpoint logarithms for  $x \rightarrow 0, 1$ : resummation can be useful or necessary**

# Higgs boson production at the LHC

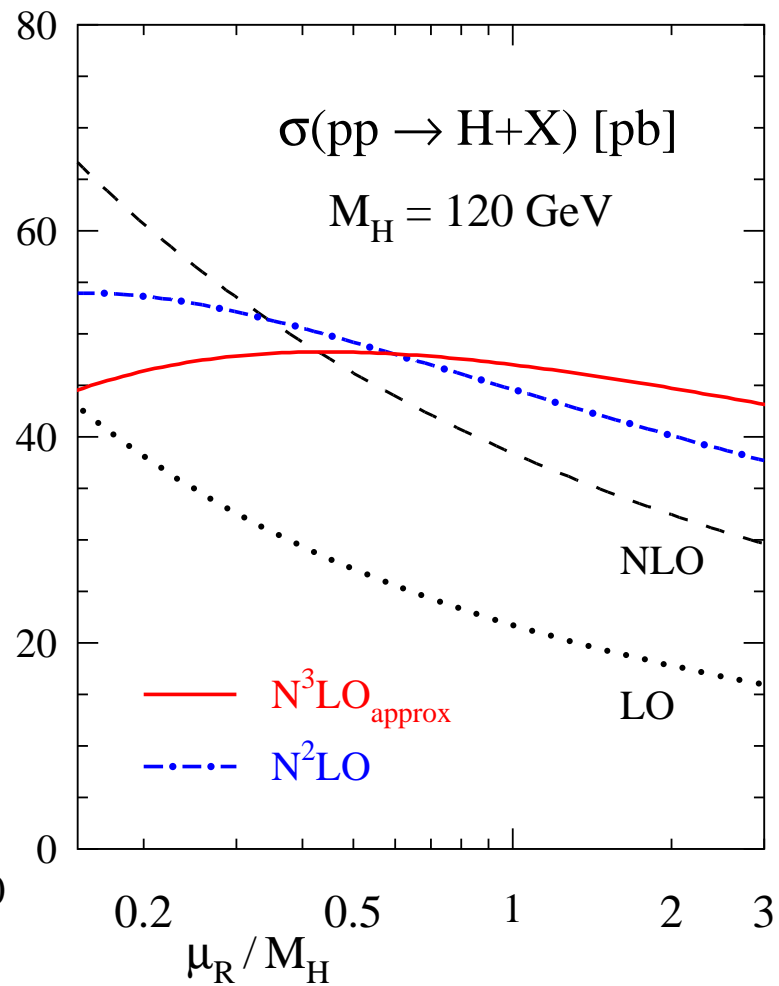
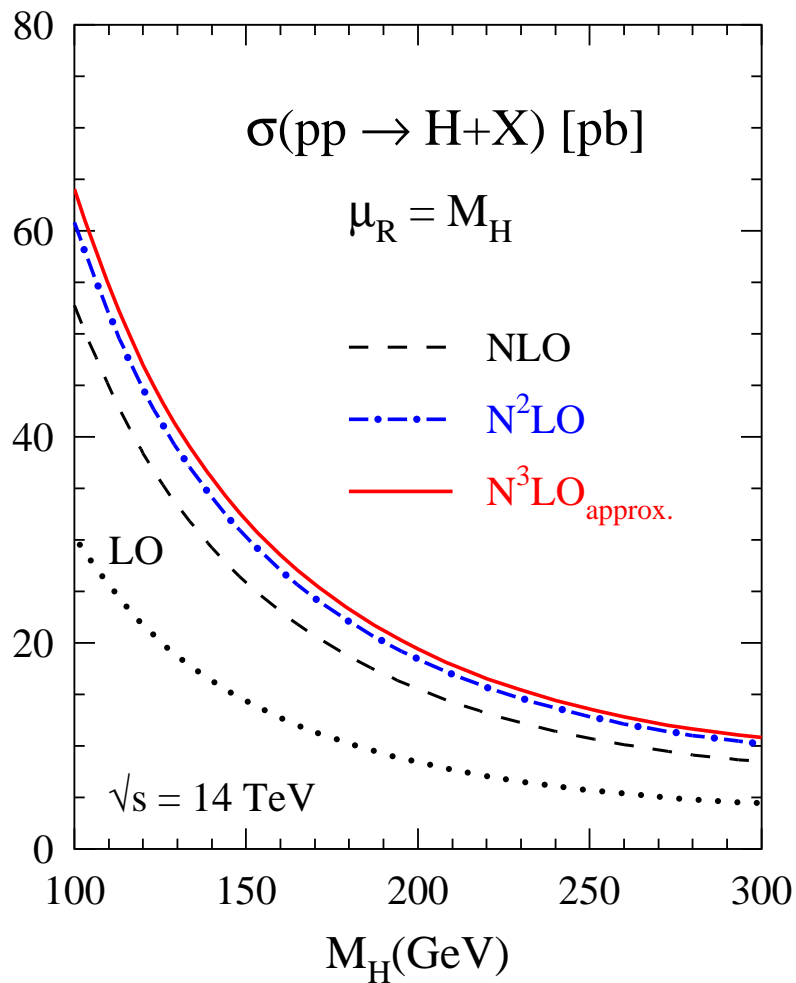


**NNLO (heavy top-quark limit):**

**Harlander, Kilgore; Anastasiou, Melnikov (02); Ravindran, Smith, van Neerven (03)**



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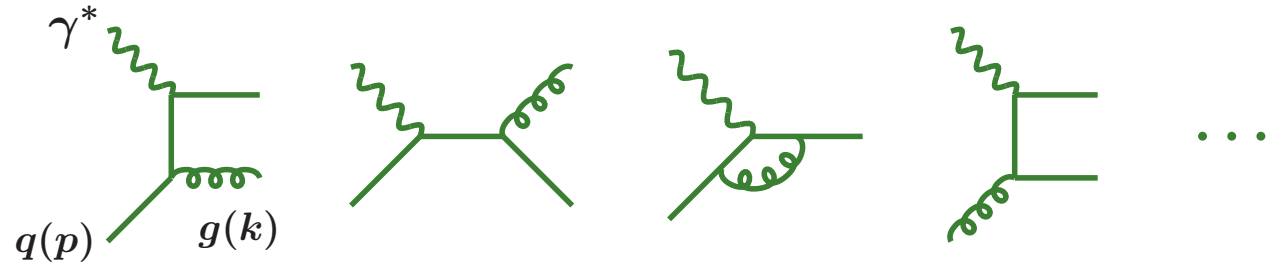
N<sup>3</sup>LO increase at  $\mu_r = M_H$ : 5% (NNLO pdf's).  $\mu_r$  variation: 4%

⇒ 5% accuracy reached by approx. N<sup>3</sup>LO

Moch, A.V. (2005)

# From mass singularities to pQCD functions

**First-order DIS:**  
 (inclusive,  $\int d^4k$ )

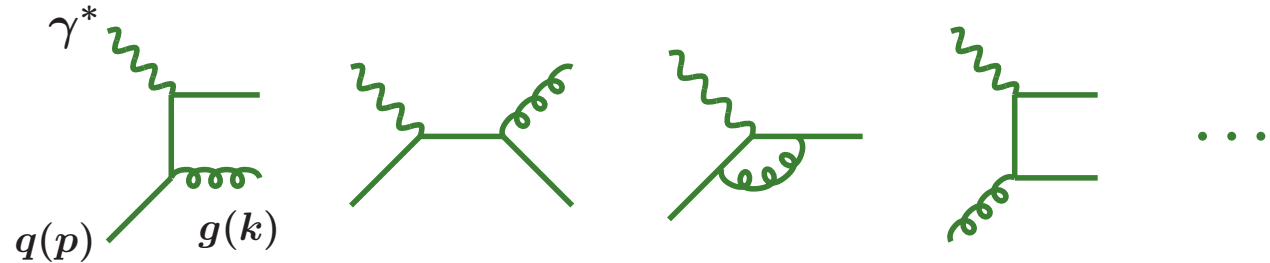


**Emissions collinear to the incoming partons ( $m_{q,g} = 0$ ): denominators**

$$(p - k)^2 = -2|\vec{p}||\vec{k}|(1 - \cos \vartheta) \xrightarrow{\vartheta \rightarrow 0} -|\vec{p}||\vec{k}| \vartheta^2 \xrightarrow{\int d\vartheta} \text{mass singularities}$$

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**Regularization (dim. =  $4 - 2\varepsilon$ , singularities  $\sim 1/\varepsilon$ ) and mass factorization**

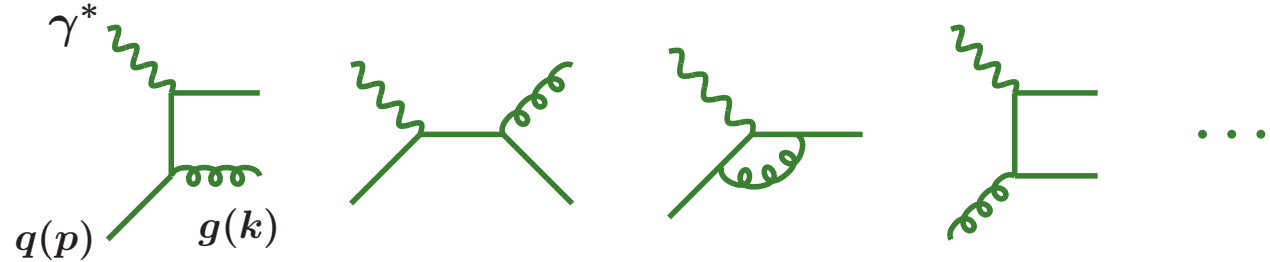
$$F_a(Q^2) = \hat{F}_{a,k}(\alpha_s(Q^2), \varepsilon) \otimes \hat{f}_k = C_{a,i}(\alpha_s(Q^2)) \otimes \Gamma_{ik}(\alpha_s(Q^2), \varepsilon) \otimes \hat{f}_k$$

$C_{a,i}$ : coefficient functions of observable  $a$

$\Gamma_{ik}$ : universal  $1/\varepsilon$ -poles + ... (fact. scheme). Usual:  $\overline{\text{MS}}$

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**Renormalized parton distributions  $f_i$ : splitting functions  $P_{ij}$**

$$\frac{\partial}{\partial \ln Q^2} f_i = \frac{\partial \Gamma_{ik}}{\partial \ln Q^2} \otimes \hat{f}_k = \frac{\partial \Gamma_{ik}}{\partial \ln Q^2} \otimes \Gamma_{kj}^{-1} \otimes f_j \equiv P_{ij} \otimes f_j$$

# $\overline{\text{MS}}$ splitting functions at large $x$ / large $N$

---

Mellin trf.  $f(N) = \int_0^1 dx (x^{N-1} \{-1\}) f(x)_{\{+\}}$ : M-convolutions  $\rightarrow$  products

$$\frac{\ln^n(1-x)}{(1-x)_+} \stackrel{\text{M}}{=} \frac{(-1)^{n+1}}{n+1} \ln^{n+1} N + \dots, \quad \ln^n(1-x) \stackrel{\text{M}}{=} \frac{(-1)^n}{N} \ln^n N + \dots$$

Diagonal splitting functions: no higher-order enhancement at  $N^0, N^{-1}$

$$P_{\text{qq/gg}}^{(\ell-1)}(N) = A_{\text{q/g}}^{(\ell)} \ln N + B_{\text{q/g}}^{(\ell)} + C_{\text{q/g}}^{(\ell)} \frac{1}{N} \ln N + \dots, \quad A_{\text{g}} = C_A/C_F A_{\text{q}}$$

...; Korchemsky (89); MVV(04); Dokshitzer, Marchesini, Salam (05)

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Off-diagonal: double-log behaviour, colour structure with  $C_{AF} = C_A - C_F$

$$C_F^{-1} P_{\text{gq}}^{(\ell)} / n_f^{-1} P_{\text{qg}}^{(\ell)} = \frac{1}{N} \ln^{2\ell} N \# C_{AF}^{\ell} + \frac{1}{N} \ln^{2\ell-1} N (\# C_{AF} + \# C_F + \# n_f) C_{AF}^{\ell-1} + \dots$$

Double logs  $\ln^n N$ ,  $\ell+1 \leq n \leq 2\ell$  vanish for  $C_F = C_A$  ( $\rightarrow$  SUSY case)

Leading logarithms: maximally non-supersymmetric contributions

# MS coefficient functions at large $x$ / large $N$

---

'Diagonal' [ $\mathcal{O}(1)$ ] coeff. fct's for  $F_{2,3,\phi}$  in DIS,  $F_{T,A,\phi}$  in SIA,  $F_{\text{DY}} = \frac{1}{\sigma_0} \frac{d\sigma_{q\bar{q}}}{dQ^2}$

$$C_{2,q/\phi,g/\dots}^{(\ell)} = \# \ln^{2\ell} N + \dots + N^{-1} (\# \ln^{2\ell-1} N + \dots) + \dots$$

$N^0$  parts: threshold exponentiation      Serman (87); Catani, Trentadue (89); ...

Exponents known to next-to-next-to-next-to-leading log ( $N^3$ LL) accuracy - mod.  $A^{(4)}$

$\Rightarrow$  highest seven (DIS, SIA), six (DY, Higgs prod.) coefficients known to all orders

DIS: MVV (05), DY/Higgs prod.: MV (05); Laenen, Magnea (05); Idilbi, Ji, Ma, Yuan (05)  
(+ SCET papers, from 06), SIA: Blümlein, Ravindran (06); MV, arXiv:0908.2746 (PLB)

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‘Off-diagonal’ [ $\mathcal{O}(\alpha_s)$ ] quantities: leading  $N^{-1}$  double logarithms

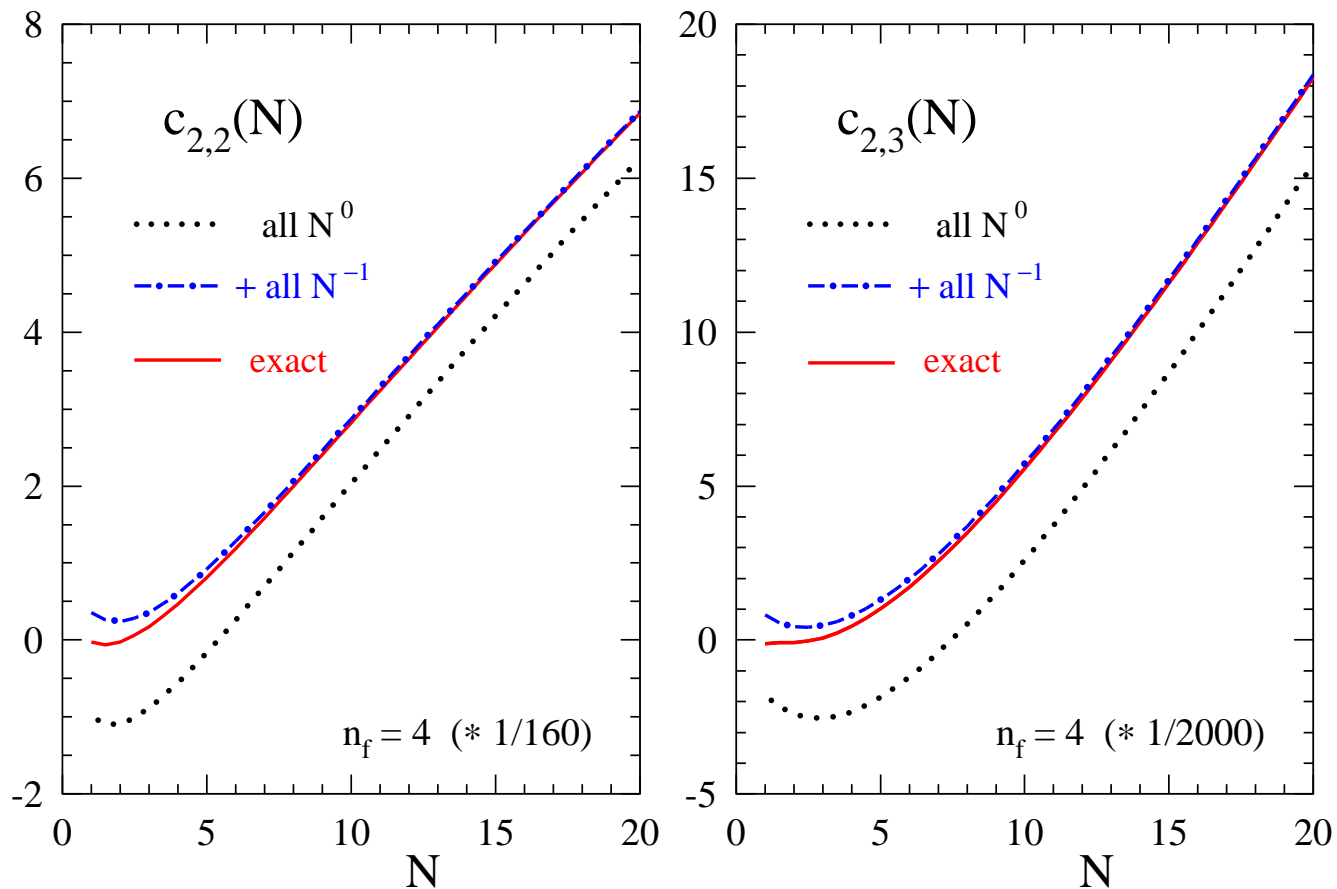
$$C_{\phi,q/2,g/\dots}^{(\ell)} = N^{-1} (\# \ln^{2\ell-1} N + \dots) + \dots$$

Longitudinal DIS/SIA structure functions [convention:  $\ell = \text{order in } \alpha_s - 1$ ]

$$C_{L,q}^{(\ell)} = N^{-1} (\# \ln^{2\ell} N + \dots) + \dots, \quad C_{L,g}^{(\ell)} = N^{-2} (\# \ln^{2\ell} N + \dots) + \dots$$



# Second- and third-order $N$ -space $C_{2,ns}$ in DIS



- $N^{-1}$  terms relevant over full range shown,  $\mathcal{O}(N^{-2})$  sizeable only at  $N < 5$
- Sum of  $N^{-1} \ln^k N$  looks almost constant: half of maximum only at  $N \simeq 150$
- DIS  $\rightarrow$  SIA  $\rightarrow$  DY : increase of the  $N^0$  terms,  $N^{-1}$  corrections less important

# $\overline{\text{MS}}$ splitting functions at small $x / N \rightarrow 1$ or 0

---

Logs in  $x$ -space  $\Leftrightarrow$  poles in  $N$ -space,  $x^a \ln^n x \stackrel{\text{M}}{=} \frac{(-1)^n n!}{(N+a)^{n+1}}$

Space-like case, **non-singlet**: no  $x^{-1}$  terms, leading  $x^0$  double logarithms :  
LL: Kirschner, Lipatov (83); Blümlein, A.V. (95)

Singlet quantities: dominant  $x^{-1}$  terms single-log enhanced

$$P_{ij}^{(\ell)S} = x^{-1} (\# \ln^{\ell - \delta_{iq}} x + \dots) + (\# \ln^{2\ell} x + \dots) + \dots$$

$x^{-1}$  part: BFKL (77/78); Jaroszewicz (82); Catani, Fiorani, Marchesini (89);  
Catani, Hautmann (94); ..., Fadin, Lipatov; Camici, Ciafaloni (98)

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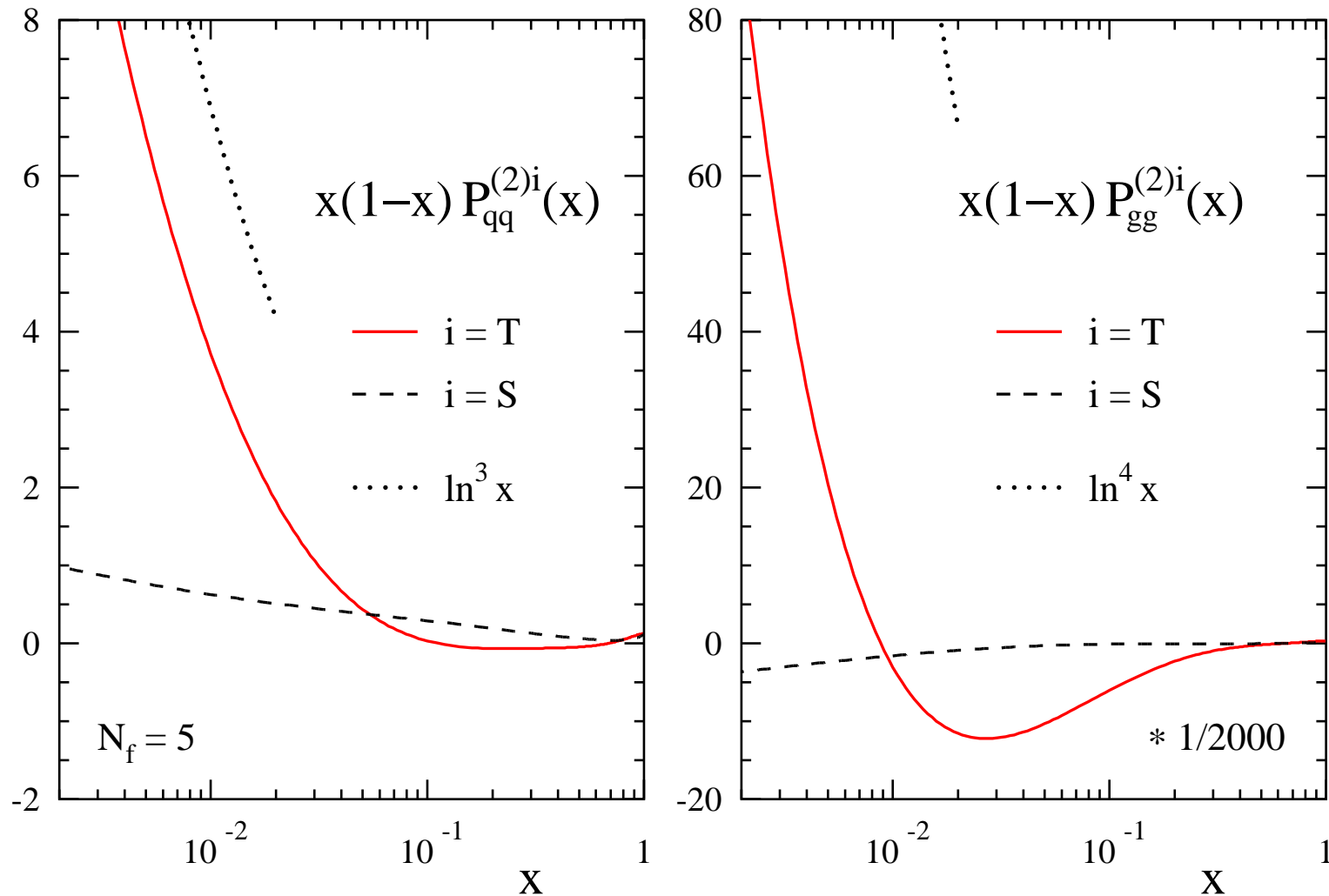
Timelike case: huge  $x^{-1}$  double logarithms

$$P_{ij}^{(\ell)T} = x^{-1} (\# \ln^{2\ell - \delta_{iq}} x + \dots) + (\# \ln^{2\ell} x + \dots) + \dots$$

LL: Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82). NLL: Mueller (83)  
– but latter not in  $\overline{\text{MS}}$ , see Albino, Bolzoni, Kniehl, Kotikov (11)

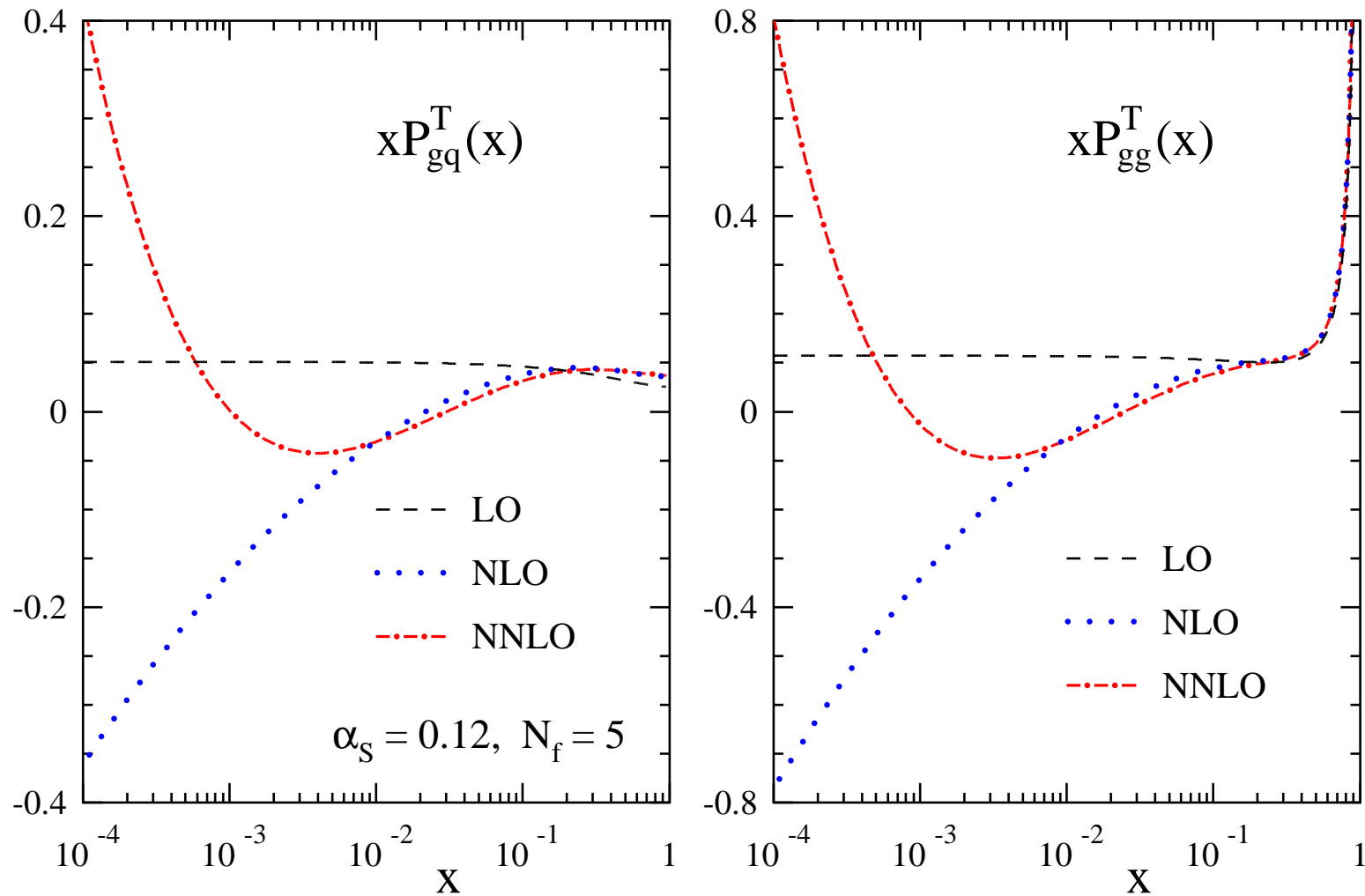
Behaviour of gauge-boson exchange coefficient functions analogous

# Third-order diagonal splitting functions



**T: extreme small- $x$  rise from  $x \gtrsim 10^{-2}$ , in gg despite huge cancellations**

# NNLO approximations for $P_{gi}^T(x, \alpha_s)$



**NLO/NNLO: terms up to  $x^{-1} \ln^2 x / x^{-1} \ln^4 x$ . Unstable at  $x \lesssim 0.005$**

# Non-singlet (NS) physical evolution kernels

---

Eliminate quark densities from **scaling violations of observables** ( $\mu = Q$ )

$$\begin{aligned}\frac{dF_a}{d\ln Q^2} &= \frac{dC_a}{d\ln Q^2} q + C_a P q = \left( \beta(a_s) \frac{dC_a}{da_s} + C_a P \right) C_a^{-1} F_a \\ &= \left( P_a + \beta(a_s) \frac{d\ln C_a}{da_s} \right) F_a = K_a F_a \equiv \sum_{\ell=0} a_s^{\ell+1} K_{a,\ell} F_a\end{aligned}$$

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**$K_a$ : physical kernel for the NS observable  $F_a$  in  $N$ -space. For  $c_{a,0} = 1$ :**

$$K_a = a_s P_{a,0} + \sum_{l=1} a_s^{\ell+1} \left( P_{a,\ell} - \sum_{k=0}^{\ell-1} \beta_k \tilde{c}_{a,\ell-k} \right), \quad a_s \equiv \alpha_s / (4\pi)$$

with

$$\begin{aligned} \tilde{c}_{a,1} &= c_{a,1} \quad , \quad \tilde{c}_{a,2} = 2 c_{a,2} - c_{a,1}^2 \\ \tilde{c}_{a,3} &= 3 c_{a,3} - 3 c_{a,2} c_{a,1} + c_{a,1}^3 \\ \tilde{c}_{a,4} &= 4 c_{a,4} - 4 c_{a,3} c_{a,1} - 2 c_{a,2}^2 + 4 c_{a,2} c_{a,1}^2 - c_{a,1}^4 \quad , \quad \dots \end{aligned}$$

**Manipulations of harmonic sums/polylogarithms, (inverse) Mellin transform**

**FORM3 + packages: Vermaseren (00); TFORM: Tentyukov, Vermaseren (07); ...**

# Large- $x$ logarithms in the physical kernels

---

Soft limit  $1 - x \ll 1 \Leftrightarrow$  large  $L \equiv \ln N$ : threshold exponentiation

$$C_a(N) = g_0 \exp\{Lg_1(a_s L) + g_2(a_s L) + \dots\} + \mathcal{O}(1/N)$$

$\Rightarrow$  single-logarithmic (SL) enhancement of physical evolution kernels  $K_a$

$$K_a(N) = -\sum_{\ell=1} A_\ell a_s^\ell L + \beta(a_s) \frac{d}{da_s} \{Lg_1(a_s L) + g_2(a_s L) + \dots\} + \dots$$



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Crucial observation: all  $K_a$  singly enhanced to all orders in  $N^{-1}$  or  $(1-x)$

DIS/SIA  $a \neq L$  leading-logarithmic kernels, with  $p_{qq}(x) = 2/(1-x)_+ - 1-x$

$$K_{a,0}(x) = 2 C_F p_{qq}(x)$$

$$K_{a,1}(x) = \ln(1-x) p_{qq}(x) [-2 C_F \beta_0 \mp 8 C_F^2 \ln x]$$

$$K_{a,2}(x) = \ln^2(1-x) p_{qq}(x) [2 C_F \beta_0^2 \pm 12 C_F^2 \beta_0 \ln x + \mathcal{O}(\ln^2 x)]$$

$$K_{a,3}(x) = \ln^3(1-x) p_{qq}(x) [-2 C_F \beta_0^3 \mp 44/3 C_F^2 \beta_0^2 \ln x + \mathcal{O}(\ln^2 x)]$$

$$K_{a,4}(x) = \ln^4(1-x) p_{qq}(x) [2 C_F \beta_0^4 \pm \xi_{K_4} C_F^2 \beta_0^3 \ln x + \mathcal{O}(\ln^2 x)]$$

First term: leading large  $n_f$ , all orders via  $C_2$  of Mankiewicz, Maul, Stein (97)

# Higher-order non-singlet predictions

---

Conjecture: Single-log behaviour of  $K_a$  persists to (all) higher orders in  $\alpha_s$   
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**Recall** 
$$\underbrace{\tilde{c}_{a,4}}_{\text{SL}} = \underbrace{4c_{a,4}}_{\text{DL, new}} \underbrace{- 4c_{a,3}c_{a,1} - 2c_{a,2}^2 + 4c_{a,2}c_{a,1} - c_{a,1}^4}_{\text{DL, known for DIS/SIA}}$$

$\Rightarrow$  coefficients of highest three powers of  $\ln(1-x)$  from fourth order in  $\alpha_s$ ,  
 i.e.,  $\ln^{7,6,5}(1-x)$  at order  $\alpha_s^4$ ,  
 $\ln^{9,8,7}(1-x)$  at order  $\alpha_s^5$ , ... for  $F_{1,2,3}$  in DIS and  $F_{T,I,A}$  in SIA

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Leading terms:  $K_1 = K_2, K_T = K_I$  [total ('integrated') fragmentation fct.]

$\Rightarrow$  also three logs for space- and timelike  $F_L$ :  $\ln^{6,5,4}(1-x)$  at  $\alpha_s^4$  etc

Alternative derivation: physical kernels for  $F_L$ , agreement non-trivial check

Drell-Yan: only NNLO known  $\Rightarrow$  only two logarithms fully predicted from  $\alpha_s^3$

# Example: $\alpha_s^4$ coefficient function for $F_1$ in DIS

$$\begin{aligned}
 c_{1,\text{ns}}^{(4)}(x) = & \left( \ln^7(1-x) \frac{8}{3} C_F^4 - \ln^6(1-x) \frac{14}{3} C_F^3 \beta_0 + \ln^5(1-x) \frac{8}{3} C_F^2 \beta_0^2 \right) p_{\text{qq}}(x) \\
 & + \ln^6(1-x) \left[ C_F^4 \left\{ p_{\text{qq}}(x) (-14 - 68/3 H_0) + 4 + 8 H_0 - (1-x)(6 + 4 H_0) \right\} \right] \\
 & + \ln^5(1-x) \left[ C_F^4 \left\{ p_{\text{qq}}(x) (-9 - 8 \tilde{H}_{1,0} + 448/3 H_{0,0} + 84 H_0 - 64 \zeta_2) + 48 \tilde{H}_{1,0} \right. \right. \\
 & \quad \left. \left. - 22 - 96 H_{0,0} - 104 H_0 - (1-x)(13 + 24 \tilde{H}_{1,0} - 48 H_{0,0} - 84 H_0 - 16 \zeta_2) \right\} \right. \\
 & + C_F^3 \beta_0 \left\{ p_{\text{qq}}(x) (41 + 316/9 H_0) - 10 - 32/3 H_0 + (1-x)(41/3 + 16/3 H_0) \right\} \\
 & + C_F^3 C_A \left\{ p_{\text{qq}}(x) (16 + 8 \tilde{H}_{1,0} + 8 H_{0,0} - 24 \zeta_2) + 4 + (1-x)(28 - 8 \zeta_2) \right\} \\
 & \left. + C_F^3 (C_A - 2 C_F) p_{\text{qq}}(-x) (16 \tilde{H}_{-1,0} - 8 H_{0,0}) \right] + \mathcal{O}(\ln^4(1-x))
 \end{aligned}$$

First line includes identity of coefficients of leading  $\ln^k(1-x)$  and  $\frac{\ln^k(x-1)}{x-1}$  terms

Conjectured by Krämer, Laenen, Spira (97)

Modified basis  $\tilde{H}_{m_1, m_2, \dots} \equiv \tilde{H}_{m_1, m_2, \dots}(x)$  of harmonic polylogarithms, e.g.,

$$\tilde{H}_{1,0} = H_{1,0} + \zeta_2, \quad \tilde{H}_{1,1,0} = H_{1,1,0} - \zeta_2 \ln(1-x) - \zeta_3$$

All  $\ln(1-x)$  terms and  $\zeta$ -functions taken out of expansions to all orders in  $1-x$

# All-order resummation of the $1/N$ terms (I)

For  $F_{1,2,3}$ ,  $F_{T,I,A}$  and  $F_{DY}$ , up to terms of order  $1/N^2$ , with  $L \equiv \ln N$

$$C_a(N) - C_a \Big|_{N^0 L^k} = \frac{1}{N} \left( \left[ d_{a,1}^{(1)} L + d_{a,0}^{(1)} \right] a_s + \left[ \tilde{d}_{a,1}^{(2)} L + d_{a,0}^{(2)} \right] a_s^2 + \dots \right) \exp \{ L h_1(a_s L) + h_2(a_s L) + a_s h_3(a_s L) + \dots \}$$

Exponentiation functions defined by expansions  $h_k(a_s L) \equiv \sum_{n=1} h_{kn}(a_s L)^n$

Coefficients for DIS/SIA (upper/lower sign) relative to  $N^0 L^k$  resummation

$$h_{1k} = g_{1k} \quad g_{lk} = \text{coefficients in soft-gluon exponentiation}$$

$$h_{21} = g_{21} + \frac{1}{2} \beta_0 \pm 6 C_F$$

$$h_{22} = g_{22} + \frac{5}{24} \beta_0^2 \pm \frac{17}{9} \beta_0 C_F - 18 C_F^2$$

$$h_{23} = g_{23} + \frac{1}{8} \beta_0^3 \pm \left( \frac{\xi_{K4}}{8} - \frac{53}{18} \right) \beta_0^2 C_F - \frac{34}{3} \beta_0 C_F^2 \pm 72 C_F^3$$

First term of  $h_3$  also known, but non-universal within DIS and SIA ( $\Leftrightarrow F_L$ )

# All-order resummation of the $1/N$ terms (II)

For space-like (-) and time-like (+) structure/fragmentation functions  $F_L$

$$C_L^{(\pm)}(N) = N^{-1} (d_1^{(\pm)} a_s + d_2^{(\pm)} a_s^2 + \dots) \exp \{ L h_1(a_s L) + h_2(a_s L) + \dots \}$$

with

$$h_{11} = 2 C_F, \quad h_{12} = \frac{2}{3} \beta_0 C_F, \quad h_{13} = \frac{1}{3} \beta_0^2 C_F$$

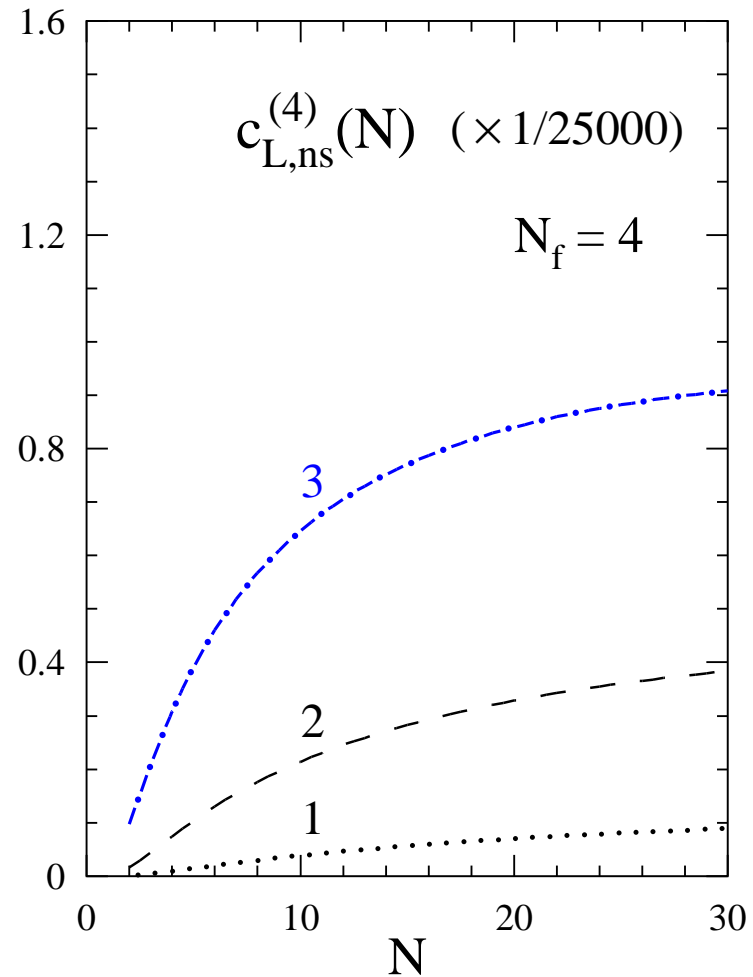
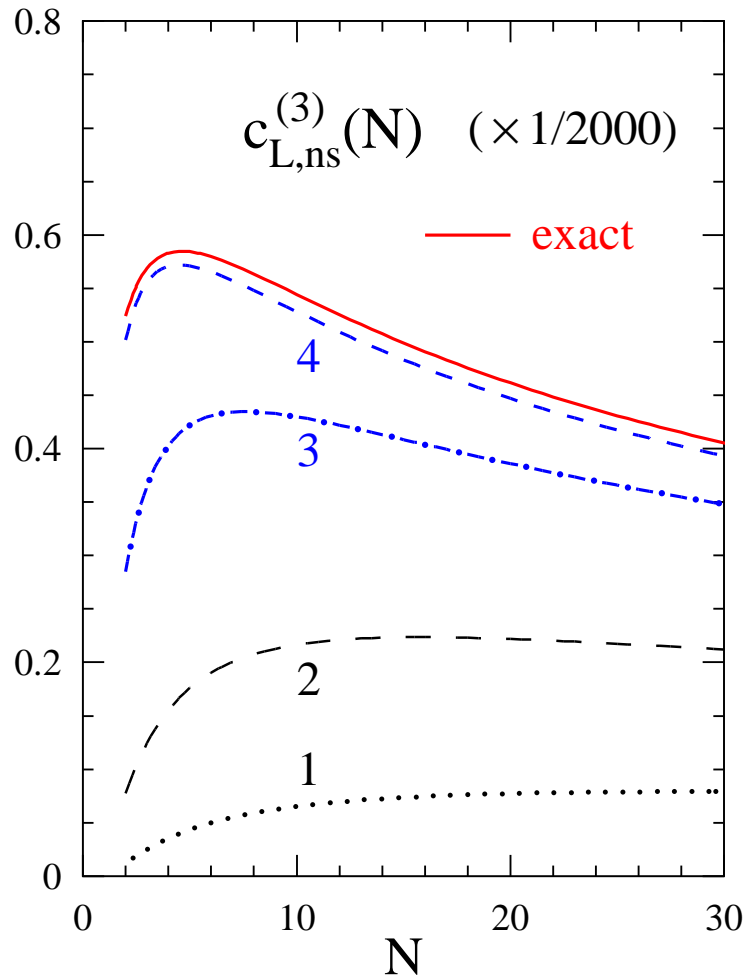
$$h_{21} = \beta_0 - C_F + (4 - 4 \zeta_2)(C_A - 2C_F)$$

$$h_{22} = \underbrace{\frac{1}{2} (\beta_0 h_{21} + A_2)}_{\text{as } g_{22} \text{ in soft-gluon exp.}} - \underbrace{8 (C_A - 2C_F)^2 (1 - 3 \zeta_2 + \zeta_3 + \zeta_2^2)}_{\text{Who ordered THIS?}}$$

## Remarks/questions

- Less predictive than  $N^0 L^k$  exponentiation: nothing new, but  $A_2$ , in  $g_{22}$
- NLL exponentiation – complete  $h_2(a_s L)$  – could be feasible for  $F_{a \neq L}$
- Full NNLL for  $F_{1,2,3}$  etc, NLL for  $F_L$ : a log too far?  $h_{23}$  for  $F_L$ , anyone?

# Third- and fourth-order $C_L$ in DIS in $N$ -space



**1 = leading log etc. Good  $\alpha_s^3$  approximation by all four  $N^{-1}$  logarithms**  
**As usual, cf. small- $x$ : leading logs do not lead. Padé:  $\approx 2.0$  at  $N = 20$**



# Large- $N$ cross sections before factorization

Unfactorized partonic structure functions in  $D = 4 - 2\varepsilon$  dimensions

$$T_{a,j} = \tilde{C}_{a,i} Z_{ij}, \quad -\gamma \equiv P = \frac{dZ}{d \ln Q^2} Z^{-1}, \quad \frac{da_s}{d \ln Q^2} = -\varepsilon a_s + \beta_{D=4}$$

$N^0$  and  $N^{-1}$  transition functions,  $D$ -dimensional coefficient functions

$$\begin{aligned} Z \Big|_{a_s^n} &= \frac{1}{\varepsilon^n} \frac{\gamma_0^{n-1}}{n!} \left[ \gamma_0 - \frac{\beta_0}{2} n(n-1) \right] + \sum_{\ell=1}^{n-1} \frac{1}{\varepsilon^{n-\ell}} \sum_{k=1}^{n-\ell-1} \gamma_0^{n-\ell-k-1} \gamma_\ell \gamma_0^k \frac{(\ell+k)!}{n! \ell!} \\ &\quad - \frac{\beta_0}{2} \sum_{\ell=1}^{n-2} \frac{1}{\varepsilon^{n-\ell}} \sum_{k=1}^{n-\ell-2} \gamma_0^{n-\ell-k-2} \gamma_\ell \gamma_0^k \frac{(\ell+k)!}{n! \ell!} (n(n-1) - \ell(\ell+k+1)) \\ &\quad + \text{NNLL contributions (explicit expressions)} + \dots \end{aligned}$$

$$\tilde{C}_{a,i} = 1_{(\text{diagonal cases})} + \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} a_s^n \varepsilon^\ell c_{a,i}^{(n,\ell)}, \quad \ell \text{ additional logs at order } \varepsilon^\ell$$

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$\alpha_s^n \varepsilon^{-n+\ell}$  off-diagonal entries: contributions up to  $N^{-1} \ln^{n+\ell-1} N$

Full  $N^m$  LO calc. of  $T_{a,j}$ : highest  $m+1$  powers of  $\varepsilon^{-1}$  to all orders in  $\alpha_s$

Extension to all  $\varepsilon$  for highest  $n+1$  logarithms:  $N^n$  LL all-order resummation

# All-order off-diagonal leading-log amplitudes

---

**Example: Leading-log (LL)  $1/N$  terms of  $T_{\phi,q}^{(n)}$  and  $T_{2,g}^{(n)}$ , with  $L \equiv \ln N$**

$$\frac{1}{C_F} T_{\phi,q}^{(n)} = \frac{1}{n_f} T_{2,g}^{(n)} = \frac{L^{n-1}}{N \epsilon^n} \sum_{k=0}^{\infty} (\epsilon L)^k \mathcal{L}_{n,k} \left( C_F^{n-1} + C_F^{n-2} C_A + \dots + C_A^{n-1} \right)$$

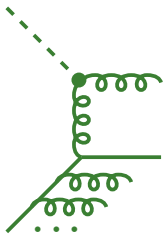
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$\Rightarrow$  all-order relation for one colour structure of either amplitude sufficient



$$T_{\phi,q}^{(n)} \Big|_{C_F \text{ only}} \stackrel{\text{LL}}{=} \frac{1}{n} T_{\phi,q}^{(1)} \underbrace{T_{2,q}^{(n-1)}}_{\stackrel{\text{LL}}{=} \frac{1}{n!} T_{\phi,q}^{(1)} (T_{2,q}^{(1)})^{n-1}} \stackrel{\text{LL}}{=} \frac{1}{(n-1)!} (T_{2,q}^{(1)})^{n-1}$$

$$\Rightarrow T_{\phi,q} \Big|_{C_F \text{ only}} \stackrel{\text{LL}}{=} T_{\phi,q}^{(1)} \frac{\exp(a_s T_{2,q}^{(1)}) - 1}{T_{2,q}^{(1)}}$$

**Exact  $D$ -dimensional leading-log expressions for the one-loop amplitudes**

$$T_{\phi,q}^{(1)} \stackrel{\text{LL}}{=} -2C_F \frac{1}{\epsilon} (1-x)^{-\epsilon} \stackrel{\text{M}}{=} -\frac{2C_F}{N} \frac{1}{\epsilon} \exp(\epsilon \ln N)$$

$$T_{2,q}^{(1)} \stackrel{\text{LL}}{=} -4C_F \frac{1}{\epsilon} (1-x)^{-1-\epsilon} + \text{virtual} \stackrel{\text{M}}{=} 4C_F \frac{1}{\epsilon^2} (\exp(\epsilon \ln N) - 1)$$

# Leading-log splitting and coefficient functions

---

Expansions and iterative mass factorization to 'any' order [done in **FORM**]

⇒ **All-order expressions for LL off-diagonal splitting and coefficient fct's**

$$P_{\text{qg}}^{\text{LL}}(N, \alpha_s) = \frac{n_f}{N} \frac{\alpha_s}{2\pi} \sum_{n=0}^{\infty} \frac{B_n}{(n!)^2} \tilde{a}_s^n, \quad \tilde{a}_s = \frac{\alpha_s}{\pi} (C_A - C_F) \ln^2 N$$

**Bernoulli numbers  $B_n$ : zero for odd  $n \geq 3$  ⇒  $P_{\text{gq}}^{(3)}(N) \stackrel{\text{LL}}{=} 0$  understood**

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad \dots, \quad B_{12} = -\frac{691}{2730}, \quad \dots$$

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$$C_{2,\text{g}}^{\text{LL}} = \frac{1}{2N \ln N} \frac{n_f}{C_A - C_F} \left\{ \exp(2C_F a_s \ln^2 N) \mathcal{B}_0(\tilde{a}_s) - \exp(2C_A a_s \ln^2 N) \right\}$$

exp(...): LL soft-gluon exponentials      Parisi; Curci, Greco; Amati et al. (80)

$$\mathcal{B}_0(x) = \sum_{n=0}^{\infty} \frac{B_n}{(n!)^2} x^n$$

$P_{\text{gq}}^{\text{LL}}, C_{\phi,\text{q}}^{\text{LL}}$ : same functions but with  
 $C_F \leftrightarrow C_A$  (also in  $\tilde{a}_s$ ), then  $n_f \rightarrow C_F$

# Next-to-leading logarithmic iteration for $T_{H,q}^{(n)}$

---

Ansatz for  $T_{\phi,q}^{(n)}$  in terms of first-order quantity and diagonal amplitudes

$$T_{\phi,q}^{(n)} \stackrel{\text{NL}}{=} \frac{1}{n} T_{\phi,q}^{(1)} \left\{ \sum_{i=0}^{n-1} T_{\phi,g}^{(i)} T_{2,q}^{(n-i-1)} f(n, i) - \frac{\beta_0}{\varepsilon} \sum_{i=0}^{n-2} T_{\phi,g}^{(i)} T_{2,q}^{(n-i-2)} g(n, i) \right\}$$

All-order agreement with known highest four powers of  $\varepsilon^{-1}$  for

$$f(n, i) = \binom{n-1}{i}^{-1} \left[ 1 + \varepsilon \left( \frac{\beta_0}{8C_A} (i+1)(n-i) \theta_{i1} - \frac{3}{2} (1 - n \delta_{i0}) \right) \right]$$

$$g(n, i) = \binom{n}{i+1}^{-1}$$

LL: A.V. (2010)

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Soft-gluon exponentiation: also  $T_{\phi,g}^{(n)}$  and  $T_{2,q}^{(n)}$  known at all powers of  $\varepsilon$

$\Rightarrow$  next-to-leading logarithmic expression for  $T_{\phi,q}$  completely predicted

Mass factorization  $\Rightarrow P_{gq}^{\text{NLL}}, c_{\phi,q}^{\text{NLL}}$  to all orders.  $P_{qg}^{\text{NLL}}, c_{2,g}^{\text{NLL}}$  analogous

Extension of this approach to higher-log accuracy (at least) cumbersome



# $D$ -dim. structure of unfactorized observables

---

## Maximal phase space for deep-inelastic scattering/semi-incl. annihilation

NLO :  $2 \rightarrow 2 / 1 \rightarrow 1 + 2$   $(1-x)^{-\epsilon} x \cdots \int_0^1$  one other variable

$N^2$ LO :  $2 \rightarrow 3 / 1 \rightarrow 1 + 3$   $(1-x)^{-2\epsilon} x \cdots \int_0^1$  four other variables

$N^3$ LO :  $2 \rightarrow 4 / 1 \rightarrow 1 + 4$   $(1-x)^{-3\epsilon} x \cdots \int_0^1$  seven other variables

...

$N^2$ LO: Matsuura, van Neerven (88), Rijken, vN (95),  $N^{n \geq 3}$ LO, indirectly: MV[V] (05)

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$\text{N}^2\text{LO}$ : Matsuura, van Neerven (88), Rijken, vN (95),  $\text{N}^{n \geq 3}\text{LO}$ , indirectly: MV[V] (05)

## Purely real contributions to unfactorized structure functions

$$T_{a,j}^{(n)\text{R}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\xi=0} (1-x)^{-1+\xi-n\varepsilon} \left\{ R_{a,j,\xi}^{(n)\text{LL}} + \varepsilon R_{a,j,\xi}^{(n)\text{NLL}} + \dots \right\}$$

## Mixed contributions ( $2 \rightarrow r+1$ with $n-r$ loops in DIS)

$$T_{a,j}^{(n)\text{M}} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=r}^n \sum_{\xi=0} (1-x)^{-1+\xi-\ell\varepsilon} \left\{ M_{a,j,\ell,\xi}^{(n)\text{LL}} + \varepsilon M_{a,j,\ell,\xi}^{(n)\text{NLL}} + \dots \right\}$$

## Purely virtual part (diagonal cases, $\xi = 0$ present): $\gamma^* \text{qq}$ , $H\text{gg}$ form factors

$$T_{a,j}^{(n)\text{V}} = \delta(1-x) \frac{1}{\varepsilon^{2n}} \left\{ V_{a,j}^{(n)\text{LL}} + \varepsilon V_{a,j}^{(n)\text{NLL}} + \dots \right\}$$

# Resulting resummation of large- $x$ double logs

---

**KLN cancellation between purely real, mixed and purely virtual contributions**

$$\mathbf{T}_{a,j}^{(n)} = \mathbf{T}_{a,j}^{(n)\text{R}} + \mathbf{T}_{a,j}^{(n)\text{M}} \left( + \mathbf{T}_{a,j}^{(n)\text{V}} \right) = \frac{1}{\epsilon^n} \left\{ \mathbf{T}_{a,j}^{(n)0} + \epsilon \mathbf{T}_{a,j}^{(n)1} + \dots \right\}$$

$\Rightarrow$  **Up to  $n - 1$  relations between the coeff's of  $(1 - x)^{-\ell\epsilon}$ ,  $\ell = 1, \dots, n$**

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$\Rightarrow$  Up to  $n-1$  relations between the coeff's of  $(1-x)^{-\ell\varepsilon}$ ,  $\ell = 1, \dots, n$

Log expansion:  $N^k$  LL higher-order coefficients completely fixed, if first  $k+1$  powers of  $\varepsilon$  known to all orders – provided by  $N^k$  LO calculation, see above

Present situation: (a)  $N^3$  LO for non-singlet  $F_{a \neq L}$  in DIS – recall **DMS (05)**  
(b)  $N^2$  LO for SIA, non-singlet  $F_L$  in DIS, and singlet DIS

$\Rightarrow$  **resummation of the (a) four and (b) three highest  $N^{-1} \ln^k N$  terms to all orders in  $\alpha_s$ : consistent with, and extending, our previous results**

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KLN cancellation between purely real, mixed and purely virtual contributions

$$T_{a,j}^{(n)} = T_{a,j}^{(n)\text{R}} + T_{a,j}^{(n)\text{M}} \left( + T_{a,j}^{(n)\text{V}} \right) = \frac{1}{\varepsilon^n} \left\{ T_{a,j}^{(n)0} + \varepsilon T_{a,j}^{(n)1} + \dots \right\}$$

$\Rightarrow$  Up to  $n-1$  relations between the coeff's of  $(1-x)^{-\ell\varepsilon}$ ,  $\ell = 1, \dots, n$

Log expansion:  $N^k$  LL higher-order coefficients completely fixed, if first  $k+1$  powers of  $\varepsilon$  known to all orders – provided by  $N^k$  LO calculation, see above

Present situation: (a)  $N^3$  LO for non-singlet  $F_{a \neq L}$  in DIS – recall **DMS (05)**  
 (b)  $N^2$  LO for SIA, non-singlet  $F_L$  in DIS, and singlet DIS

$\Rightarrow$  **resummation of the (a) four and (b) three highest  $N^{-1} \ln^k N$  terms to all orders in  $\alpha_s$ : consistent with, and extending, our previous results**

Soft-gluon exponentiation of  $(1-x)^{-1}/N^0$  diagonal coefficient functions:

$(1-x)^{-1-\varepsilon}, \dots, (1-x)^{-1-(n-1)\varepsilon}$  at order  $n$ : products of lower-order quantities

$\Rightarrow N^n$  LO [ $+A^{(n+1)}$ ]  $\rightarrow N^n$  LL exponentiation;  $2n[+1]$  highest logs predicted

# NS results, off-diagonal splitting fct's and $C_{L,g}$

---

NS cases:  $K_{a,4}(x)$  of p. 13 confirmed with  $\xi_{K_4} = \frac{100}{3}$ : fourth log for  $c_{a,ns}^{(n \geq 4)}$

also: Grunberg (2010)

## Off-diagonal splitting functions

$$NP_{\text{qg}}^{\text{NL}}(N, \alpha_s) = 2a_s n_f \mathcal{B}_0(\tilde{a}_s) + a_s^2 \ln \tilde{N} n_f \left\{ (6C_F - \beta_0) \left( \frac{2}{\tilde{a}_s} \mathcal{B}_{-1}(\tilde{a}_s) + \mathcal{B}_1(\tilde{a}_s) \right) + \frac{\beta_0}{\tilde{a}_s} \mathcal{B}_{-2}(\tilde{a}_s) \right\}$$

$$\tilde{a}_s = \frac{\alpha_s}{\pi} (C_A - C_F) \ln^2 \tilde{N}$$

$$NP_{\text{gq}}^{\text{NL}}(N, \alpha_s) = 2a_s C_F \mathcal{B}_0(-\tilde{a}_s) + a_s^2 \ln \tilde{N} C_F \left\{ (12C_F - 6\beta_0) \frac{1}{\tilde{a}_s} \mathcal{B}_{-1}(-\tilde{a}_s) - \frac{\beta_0}{\tilde{a}_s} \mathcal{B}_{-2}(-\tilde{a}_s) + (14C_F - 8C_A - \beta_0) \mathcal{B}_1(-\tilde{a}_s) \right\}$$

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Gluon contribution to  $F_L$  – ‘non-singlet’  $C_F = 0$  part done before **MV (09)**

$$N^2 C_{L,g}^{NL}(N, \alpha_s) = 8a_s n_f \exp(2C_A a_s \ln^2 \tilde{N}) + 4a_s C_F N C_{2,g}^{LL}(N, \alpha_s) + 16a_s^2 \ln \tilde{N} n_f \left\{ 4C_A - C_F + \frac{1}{3} a_s \ln^2 \tilde{N} C_A \beta_0 \right\} \exp(2C_A a_s \ln^2 \tilde{N})$$

New: also NNLL terms now in closed form ( $\mathcal{B}_{-4} \dots \mathcal{B}_2$ ) **A. Almasy, A.V.**

# Resummed gluon coefficient function for $F_2$

$$\begin{aligned}
 NC_{2,g}(N, \alpha_s) = & \\
 & \frac{1}{2 \ln \tilde{N}} \frac{n_f}{C_A - C_F} \left[ \exp(2a_s C_F \ln^2 \tilde{N}) \mathcal{B}_0(a_s^3) - \exp(2a_s C_A \ln^2 \tilde{N}) \right] \\
 & - \frac{1}{8 \ln^2 \tilde{N}} \frac{n_f (3C_F - \beta_0)}{(C_A - C_F)^2} \left[ \exp(2a_s C_F \ln^2 \tilde{N}) \mathcal{B}_0(a_s^3) - \exp(2a_s C_A \ln^2 \tilde{N}) \right] \\
 & - \frac{a_s}{4} \frac{n_f}{C_A - C_F} \exp(2a_s C_A \ln^2 \tilde{N}) (8C_A + 4C_F - \beta_0) \\
 & - \frac{a_s}{4} \frac{n_f}{C_A - C_F} \exp(2a_s C_F \ln^2 \tilde{N}) \left[ -6C_F \mathcal{B}_0(a_s^3) - (6C_F - \beta_0) \mathcal{B}_1(a_s^3) \right. \\
 & \quad \left. - (12C_F - 4\beta_0) \frac{1}{a_s^3} \mathcal{B}_{-1}(a_s^3) - \frac{\beta_0}{a_s^3} \mathcal{B}_{-2}(a_s^3) \right] \\
 & - \frac{a_s^2}{3} \beta_0 \ln^2 \tilde{N} \frac{n_f}{C_A - C_F} \left[ C_A \exp(2a_s C_A \ln^2 \tilde{N}) - C_F \exp(2a_s C_F \ln^2 \tilde{N}) \mathcal{B}_0(a_s^3) \right] \\
 & + \text{known NNLL contributions (now in closed form)} + \dots
 \end{aligned}$$

$C_{H,q}$  analogous. Analytic form identified via the physical kernel for  $(F_2, F_H)$

Resummed timelike splitting and coefficient functions: same structure



# $\mathcal{B}$ -functions: $\mathcal{B}_0$ and general definition

---

Relation between even- $n$  Bernoulli numbers and the Riemann  $\zeta$ -function

$$\mathcal{B}_0(x) \equiv \sum_{n=0}^{\infty} \frac{B_n}{(n!)^2} x^n = 1 - \frac{x}{2} - 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \zeta_{2n} \left( \frac{x}{2\pi} \right)^{2n}$$

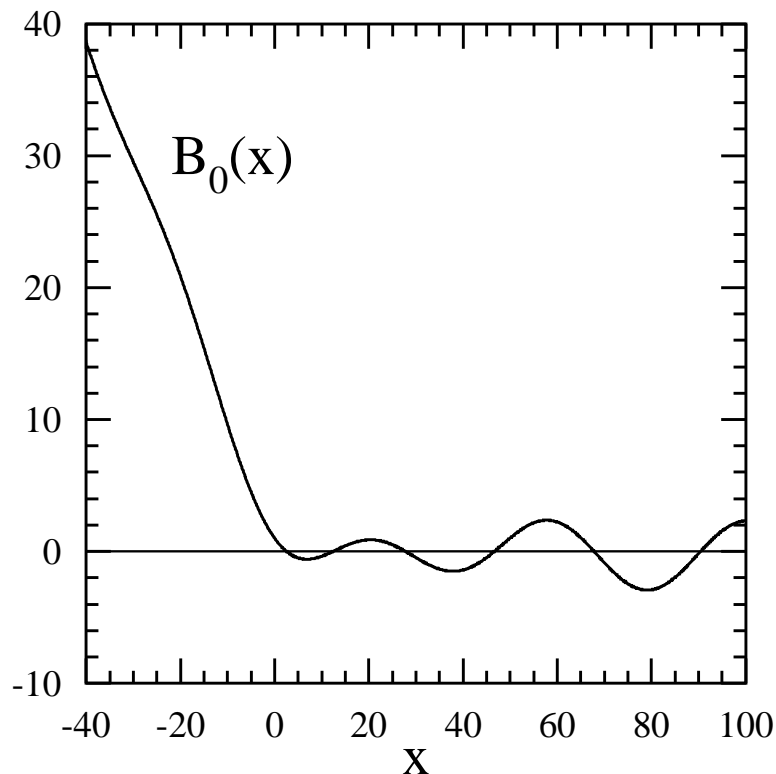
$\mathcal{B}_0(2\pi i)$  numerically known (Wolfram MathWorld, Sloane's A093721), no closed form

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## Further $\mathcal{B}$ -functions

$$\mathcal{B}_k(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!(n+k)!} x^n$$

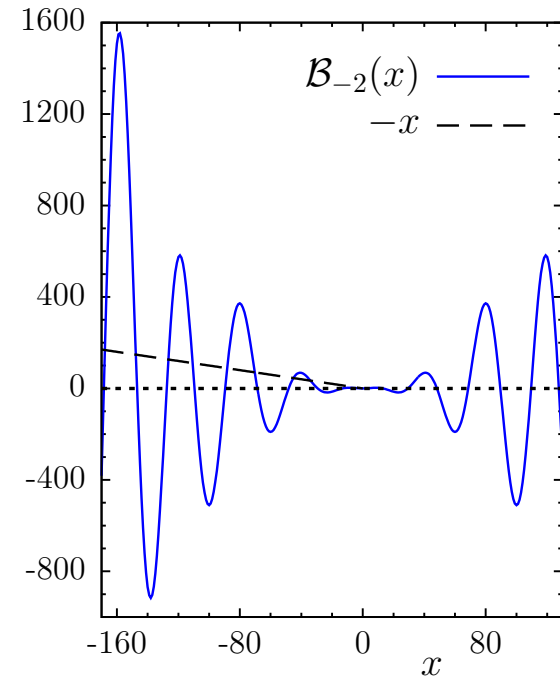
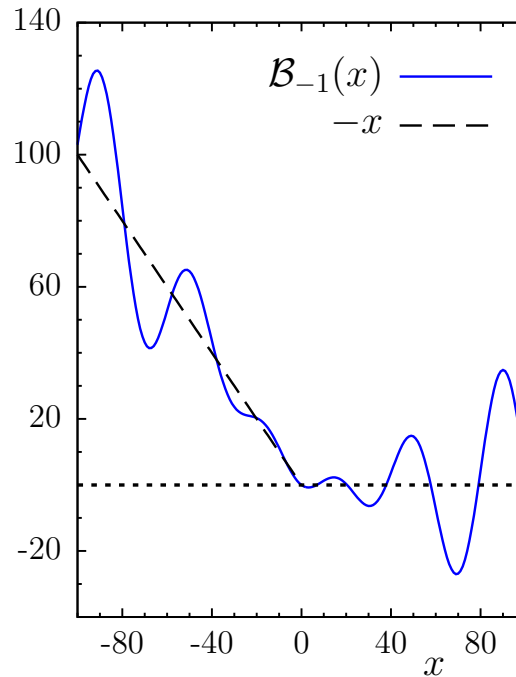
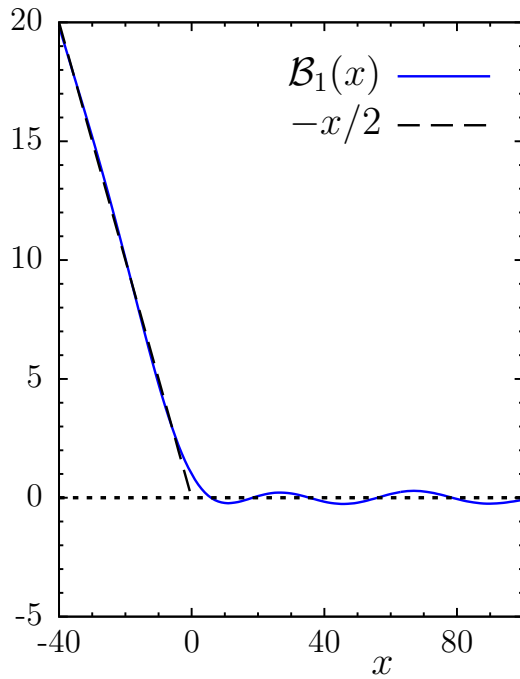
$$\mathcal{B}_{-k}(x) = \sum_{n=k}^{\infty} \frac{B_n}{n!(n-k)!} x^n$$

## Relations to $\mathcal{B}_0(x)$

$$\frac{d^k}{dx^k} (x^k \mathcal{B}_k) = \mathcal{B}_0, \quad x^k \frac{d^k}{dx^k} \mathcal{B}_0 = \mathcal{B}_{-k}$$

A.V. (2010)

# $\mathcal{B}$ -functions with index unequal zero



$x > 0$ : all functions  $\mathcal{B}_k(x)$  oscillate about  $y = 0$

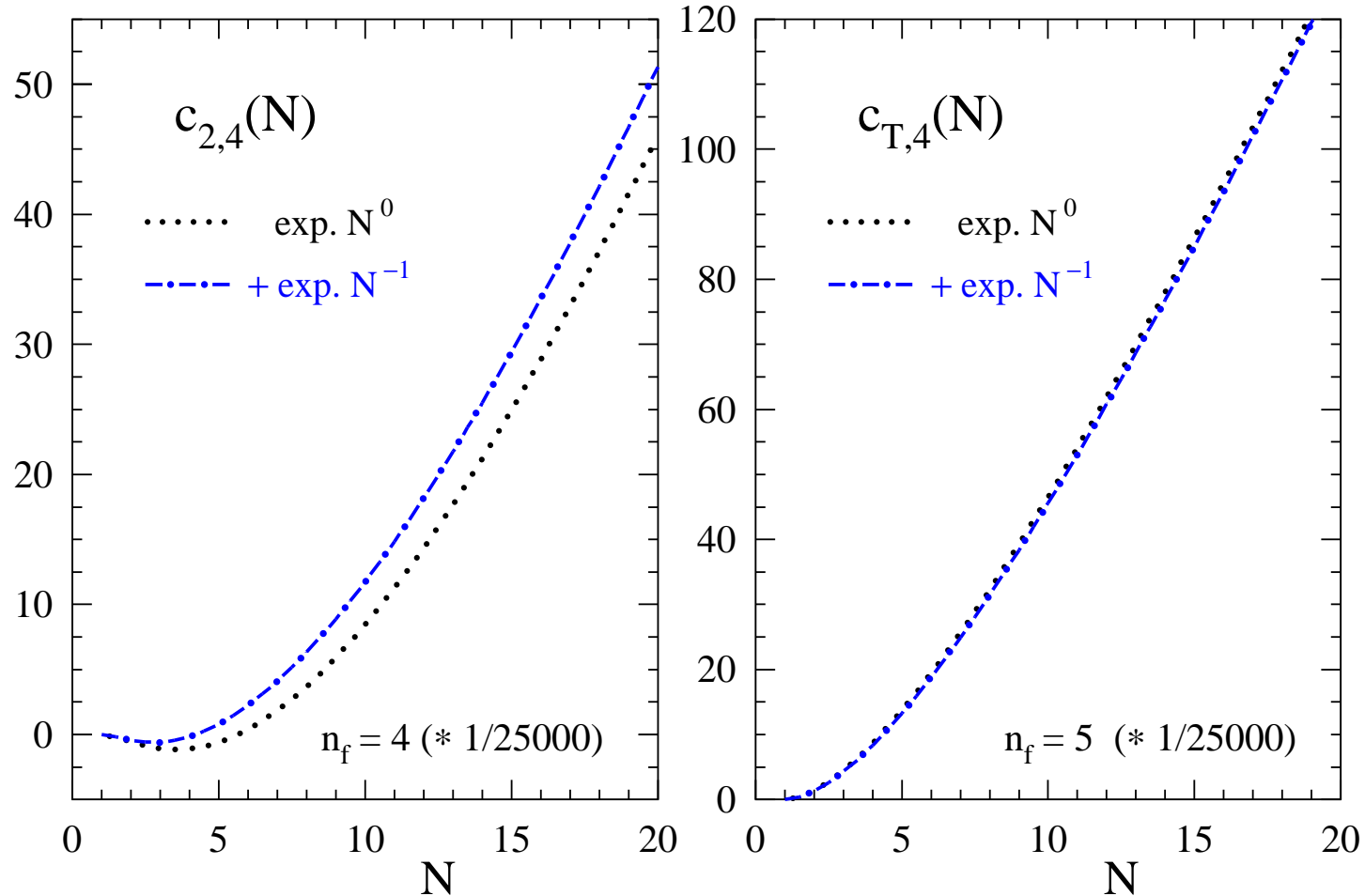
$x < 0$ : oscillations about  $y = -\frac{x}{(k+1)!}$  for  $k \geq 0$  and  $y = -x$  for  $k < 0$

Amplitudes increase very rapidly with decreasing  $k$

Oscillation of  $\mathcal{B}_0$  continues (much more irregularly) to very large  $x$

D. Broadhurst, private communication

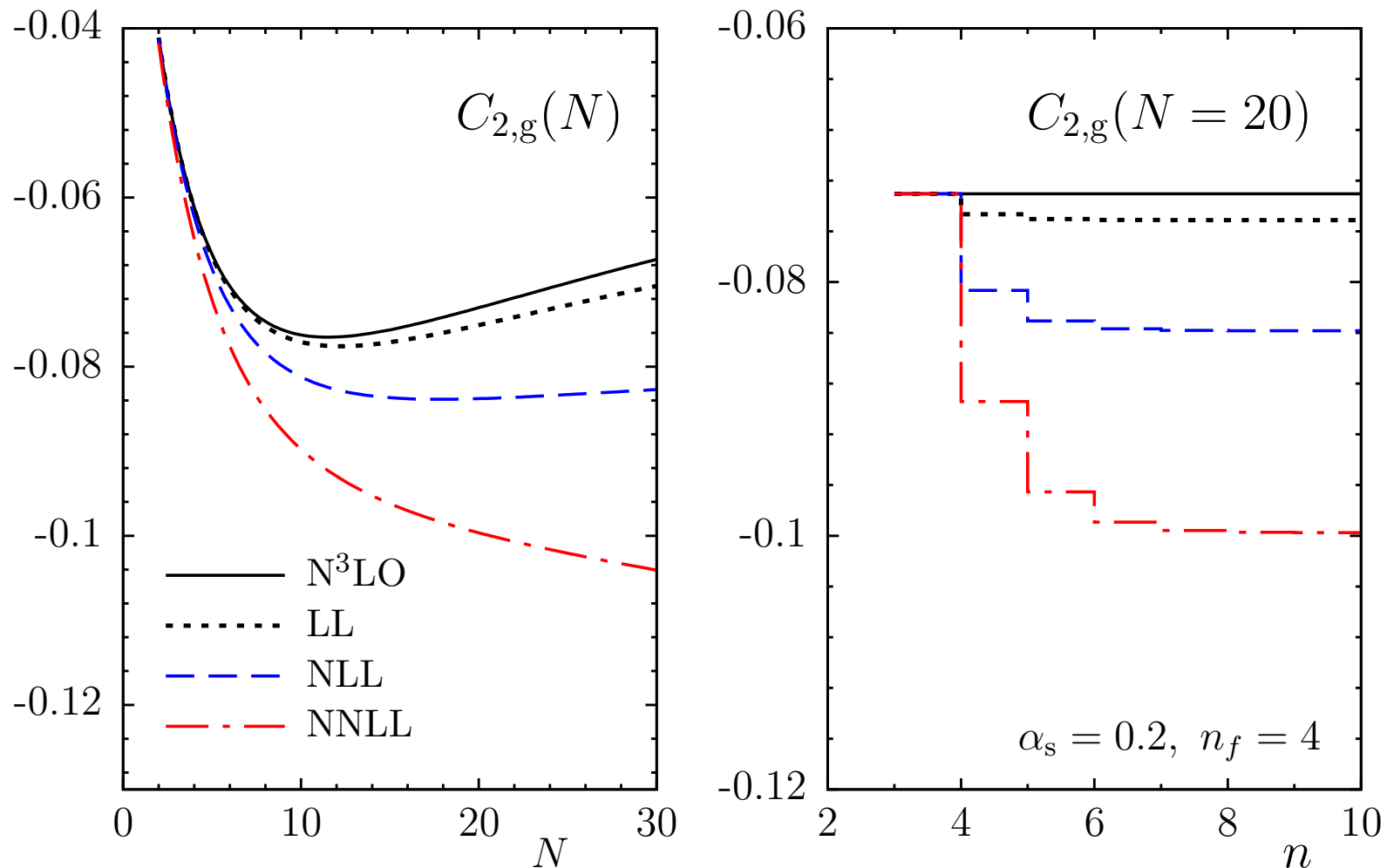
# Fourth-order $C_2$ (DIS) and $C_T$ (SIA) at large $N$



**Exp.  $N^0$ : 7 of 8 logs, exp.  $N^{-1}$ : 4 of 7 logs  $\Rightarrow$  large- $x$  higher-twist analyses**

$N^{-1}$  contributions again relevant for  $F_2$ , but small for  $F_T$  at least at  $N > 5$

# Numerical illustration of $C_{2,g}$



**NNLL terms dominate  $\Rightarrow$  impact of high orders presumably underestimated**  
**About 35% correction at  $N = 20$ , 4<sup>th</sup>-order coefficient  $\approx$  Padé estimate**

# Small- $x$ resummation via unfactorized SIA

---

Phase-space integrations:  $x^{a\varepsilon}$  terms analogous to  $(1-x)^{b\varepsilon}$  large- $x$  factors

2<sup>nd</sup> order: Matsuura, van Neerven (88), Rijken, vN (95)

Decomposition of the  $D$ -dim. partonic fragmentation functions for  $a = T, \phi$

$$\widehat{F}_{a,g}^{(n)} = \frac{1}{\varepsilon^{2n-1}} \sum_{\ell=0}^{n-1} x^{-1-2(n-\ell)\varepsilon} \left\{ A_{a,g}^{(\ell,n)} + \varepsilon B_{a,g}^{(\ell,n)} + \varepsilon^2 C_{a,g}^{(\ell,n)} + \dots \right\}$$

Leading log: terms of the form  $x^{-1} \ln^{n+\delta-1} x$  at all orders  $\varepsilon^{-n+\delta}$  with  $\delta = 0, 1, 2, \dots$ , and  $\widehat{F}_{a,g}^{(n)}$  is decomposed into  $n$  contributions of the form

$$\varepsilon^{-2n+1} x^{-1-k\varepsilon} = \varepsilon^{-2n+1} x^{-1} \left[ 1 - k\varepsilon \ln x + \frac{1}{2} (k\varepsilon)^2 \ln^2 x + \dots \right],$$

$k = 2, 4, \dots, 2n$

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$k = 2, 4, \dots, 2n$

$n-1$  KLN-type cancellations –  $\widehat{F}_{a,g}^{(n)}$  starts at order  $1/\varepsilon^n$  – plus 3 constraints from the NNLO results  $\Rightarrow n+2$  linear equations for  $n$  coefficients  $A_{a,g}^{(\ell,n)}$

Thus: N<sup>n</sup>LO known  $\Rightarrow$  highest  $n+1$  (N<sup>n</sup>LL) double logs fixed at all orders

‘All-order’ mass factorization: NNLL timelike splitting & coefficient functions

# Splitting & coefficient functions, status 2011

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$$\frac{C_A}{C_F} P_{gq}^T(N, \alpha_s) \stackrel{\text{LL}}{=} P_{gg}^T(N, \alpha_s) \stackrel{\text{LL}}{=} \frac{1}{4} (N-1) \left\{ (1-4\xi)^{1/2} - 1 \right\}, \quad \xi = -\frac{8C_A a_s}{(N-1)^2}$$

**Mueller (81); Bassetto, Ciafaloni, Marchesini, Mueller (82)**

**NLL contributions to the  $\overline{\text{MS}}$  splitting functions: only partially in closed form**

$$\begin{aligned} \left[ P_{gg}^T \right]_{C_F=0}^{\text{NLL}} &= \left\{ (1-4\xi)^{-1/2} + 1 \right\} a_s \left( \frac{11}{6} C_A + \frac{1}{3} n_f \right) \\ \left[ \frac{C_A}{C_F} P_{gq}^T \right]_{C_F=0}^{\text{NLL}} &= \left[ P_{gg}^T \right]_{C_F=0}^{\text{NLL}} + \left\{ (1-4\xi)^{1/2} - 1 \right\} \frac{1}{24} (N-1)^2 (1 + n_f/C_A) \end{aligned}$$



# Splitting & coefficient functions, status 2011

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LL coefficient functions for  $F_T$  &  $F_\phi$  [also: Albino, Bolzoni, Kniehl, Kotikov (11)]

$$C_{T,g}^{\text{LL}} = \frac{C_F}{C_A} \left( C_{\phi,g}^{T,\text{LL}} - 1 \right) = \frac{C_F}{C_A} \left\{ (1-4\xi)^{-1/4} - 1 \right\} \quad \text{in } \overline{\text{MS}}$$

‘Everything else’, including all of  $P_{\text{qq}}^T, P_{\text{qg}}^T$ , the quark coefficient fct’s,  $C_{L,i}$ :  
Tables of coefficients to order  $\alpha_s^{16}$  – numerically sufficient for  $x \gtrsim 10^{-4}$  – e.g.

$$P_{\text{gg},\text{NLL}}^{(n)T}(N) = -\frac{(-8)^n C_A^{n-1}}{3(N-1)^{2n}} \left[ (11C_A^2 + 2C_A n_f) B_{\text{gg},1}^{(n)} - 2C_F n_f B_{\text{gg},2}^{(n)} \right]$$

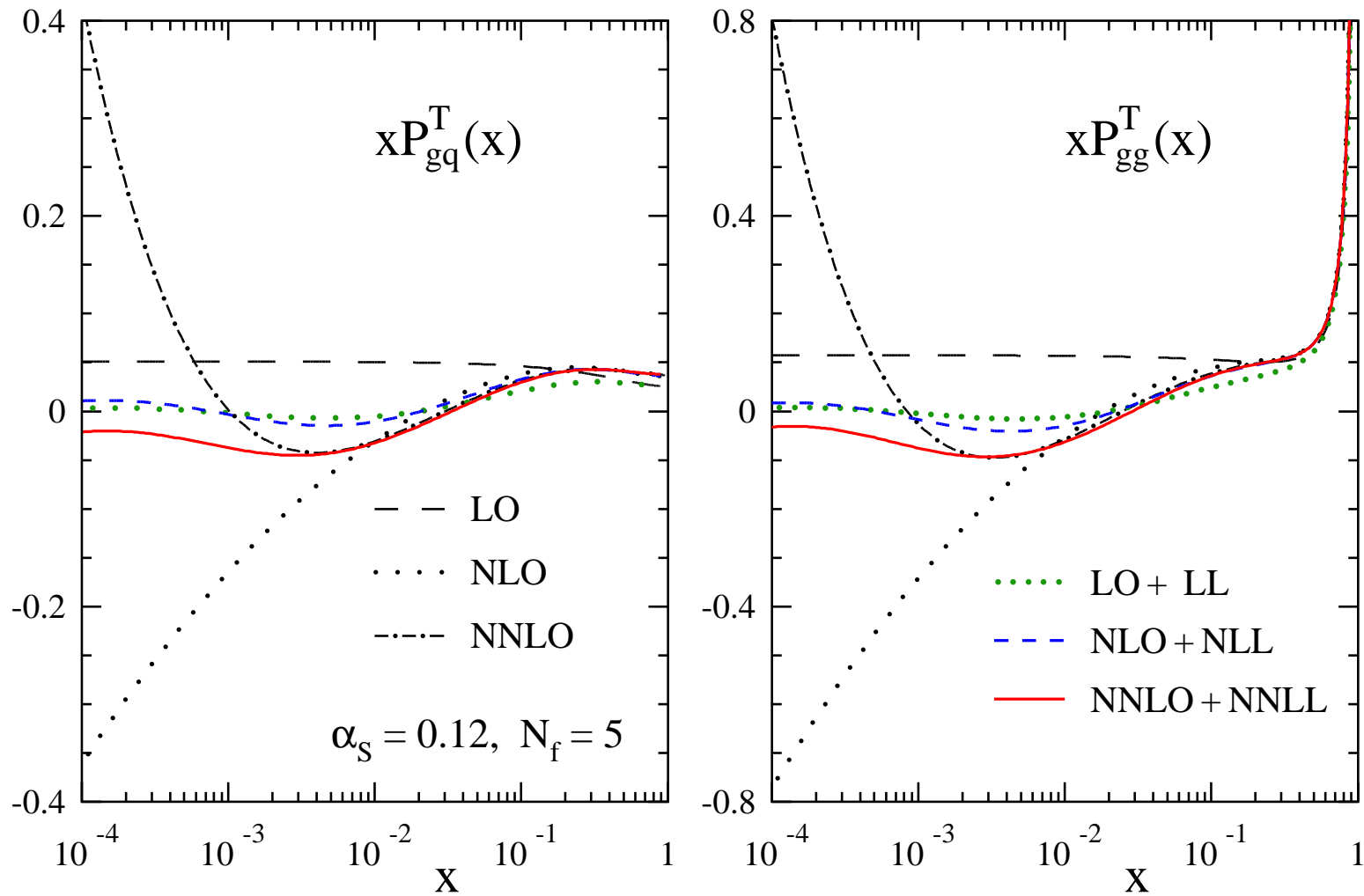
# Normalized LL, NLL splitting-fct. coefficients

$n$	$A_{gi}^{(n)}$	$B_{gg,1}^{(n)}$	$B_{gg,2}^{(n)}$	$B_{gq,1}^{(n)}$	$B_{gq,2}^{(n)}$	$B_{gq,3}^{(n)}$	$A_{qi}^{(n)}$
0	1	1	—	9	—	—	—
1	1	1	2	9	—	—	—
2	2	3	5	29	1	1	1
3	5	10	$\frac{49}{3}$	100	5	$\frac{19}{3}$	$\frac{11}{3}$
4	14	35	$\frac{347}{6}$	357	21	$\frac{179}{6}$	$\frac{73}{6}$
5	42	126	$\frac{6353}{30}$	1302	84	$\frac{3833}{30}$	$\frac{1207}{30}$
6	132	462	$\frac{11839}{15}$	4818	330	$\frac{7879}{15}$	$\frac{2021}{15}$
7	429	1716	$\frac{624557}{210}$	18018	1287	$\frac{444377}{210}$	$\frac{96163}{210}$
8	1430	6435	$\frac{316175}{28}$	67925	5005	$\frac{236095}{28}$	$\frac{44185}{28}$
9	4862	24310	$\frac{54324719}{1260}$	257686	19448	$\frac{42072479}{1260}$	$\frac{6936481}{1260}$

All integer series known,  $B_{gg,2}^{(n)} - B_{gq,3}^{(n)} = 2A_{gi}^{(n)}$ ,  $A_{qi}^{(n)} + B_{gg,2}^{(n)} = 2B_{gg,1}^{(n)}$

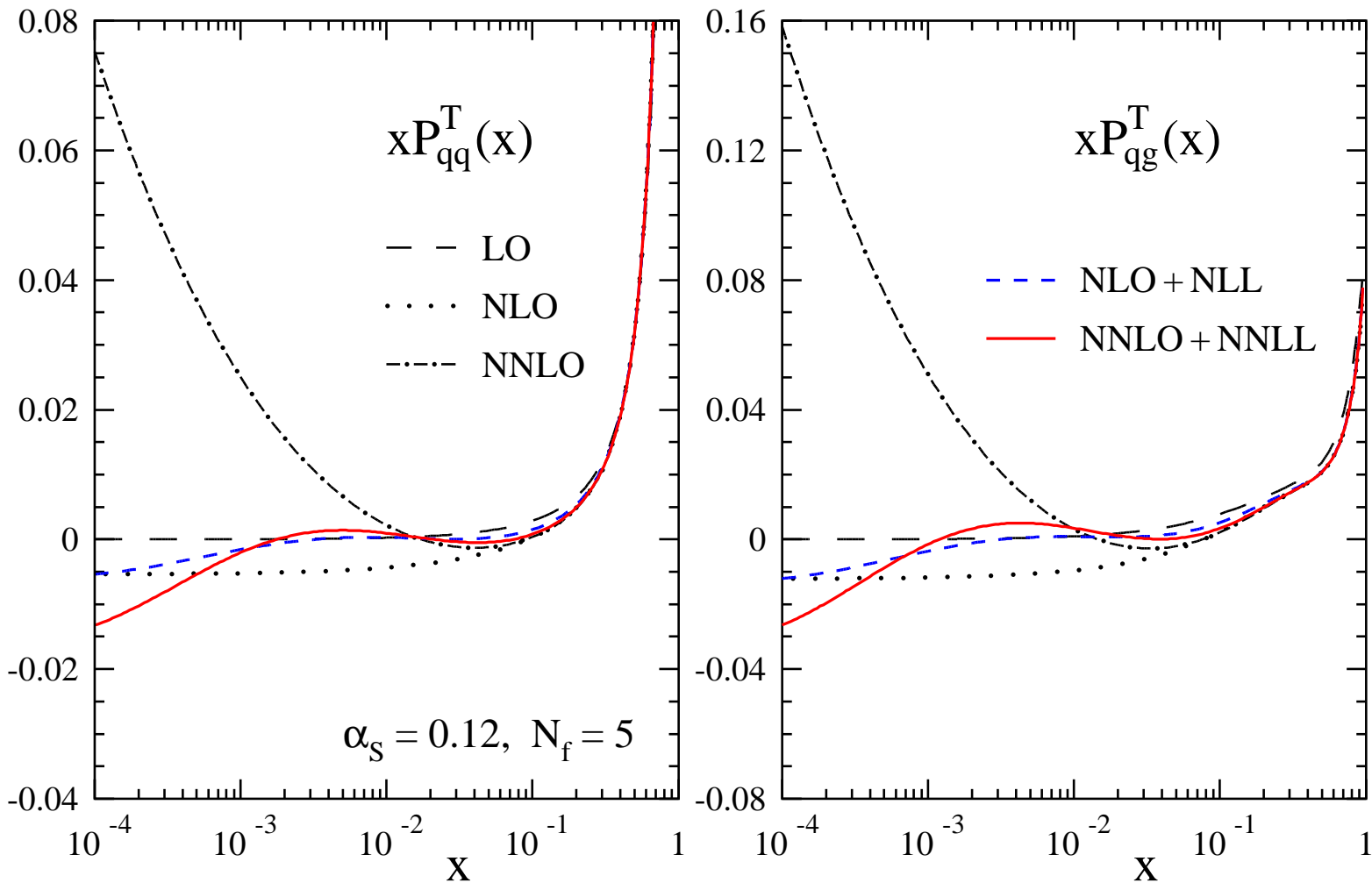
Solution of one non-integer series: analytic structure of all NLL contributions

# Small- $x$ gluon-parton splitting functions



Approximation sequence LO+LL, NLO+NLL, NNLO+NNLL rather stable to very small  $x$

# Small- $x$ quark-parton splitting functions



Also consistent with  $xP_{ji}^T \approx 0$  at  $x < 10^{-2}$  ( $N^3LL$  corr's known and positive)

# 2012 progress: solution of the $A_{qi}^{(n)}$ series

Denominators  $\leftrightarrow$  triangular numbers (A025555 in OEIS) + 'playing around'

$$\begin{aligned}
 A_{qi}^{(2)} &= 1 = \frac{1}{1}, & A_{qi}^{(5)} &= \frac{1207}{30} = \frac{14}{10} + \frac{19}{6} + \frac{23}{3} + \frac{28}{1}, \\
 A_{qi}^{(3)} &= \frac{11}{3} = \frac{2}{3} + \frac{3}{1}, & A_{qi}^{(6)} &= \frac{2015}{15} = \frac{42}{15} = \frac{56}{10} + \frac{66}{6} + \frac{76}{3} + \frac{90}{1}, \\
 A_{qi}^{(4)} &= \frac{73}{6} = \frac{5}{6} + \frac{7}{3} + \frac{9}{1}, & & \dots
 \end{aligned}$$

Numerators: sequence A028364, sums of products of Catalan numbers

$$\Rightarrow A_{qi}^{(n)} = \frac{2(2n-2)!}{(n-1)!(n+1)!} \left( \frac{1}{n-1} + \frac{1}{n} + \frac{6}{n+1} - 2 \right) + \frac{2(2n)!}{n!(n+1)!} \sum_{k=n}^{2n-3} \frac{1}{k}$$

K. Yeats

Checked to 136<sup>th</sup> entries by 'brute-force' determination of  $A_{qi}^{(n \leq 17)}$

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K. Yeats

Checked to 136<sup>th</sup> entries by 'brute-force' determination of  $A_{qi}^{(n \leq 17)}$

Generating function

$$A_{qi}(\xi) = \left( \sqrt{1-4\xi} - 1 \right) \left( 1 + \ln \left( \frac{1}{2} [\sqrt{1-4\xi} + 1] \right) \right) + 2\xi$$

C.H. Kom

The logarithm is the key for solving all sequences in the 2011 article

# Diagonal splitting functions at NNLL accuracy

**Notation:**  $S = (1 - 4\xi)^{1/2}$ ,  $\mathcal{L} = \ln\left(\frac{1}{2}(1 + S)\right)$  with  $\xi = -8C_A a_s/\bar{N}^2$ ,  $\bar{N} \equiv N - 1$

$$\begin{aligned}
 P_{\text{qq}}^T(N) &= \frac{4}{3} \frac{C_F n_f}{C_A} a_s \left\{ \frac{1}{2\xi} (S - 1)(\mathcal{L} + 1) + 1 \right\} \\
 &+ \frac{1}{18} \frac{C_F n_f}{C_A^3} a_s \bar{N} \left\{ (-11 C_A^2 + 6 C_A n_f - 20 C_F n_f) \frac{1}{2\xi} (S - 1 + 2\xi) + 10 C_A^2 \frac{1}{\xi} (S - 1)\mathcal{L} \right. \\
 &\quad - (51 C_A^2 - 6 C_A n_f + 12 C_F n_f) \frac{1}{2} (S - 1) + (11 C_A^2 + 2 C_A n_f - 4 C_F n_f) S^{-1} \mathcal{L} \\
 &\quad \left. + (5 C_A^2 - 2 C_A n_f + 6 C_F n_f) \frac{1}{\xi} (S - 1)\mathcal{L}^2 + (51 C_A^2 - 14 C_A n_f + 36 C_F n_f) \mathcal{L} \right\} \\
 P_{\text{gg}}^T(N) &= \frac{1}{4} \bar{N} (S - 1) - \frac{1}{6 C_A} a_s (11 C_A^2 + 2 C_A n_f - 4 C_F n_f) (S^{-1} - 1) - P_{\text{qq}}^T(N) \\
 &+ \frac{1}{576 C_A^3} a_s \bar{N} \left\{ ([1193 - 576 \zeta_2] C_A^4 - 140 C_A^3 n_f + 4 C_A^2 n_f^2 - 56 C_A^2 C_F n_f - 48 C_F^2 n_f^2 \right. \\
 &\quad \left. + 16 C_A C_F n_f^2) (S - 1) + ([830 - 576 \zeta_2] C_A^4 + 96 C_A^3 n_f - 8 C_A^2 n_f^2 - 208 C_A^2 C_F n_f \right. \\
 &\quad \left. + 64 C_A C_F n_f^2 - 96 C_F^2 n_f^2) (S^{-1} - 1) + (11 C_A^2 + 2 C_A n_f - 4 C_F n_f)^2 (S^{-3} - 1) \right\}
 \end{aligned}$$

**First lines: LL (for  $P_{\text{gg}}^T$ ) and NLL contributions. Rest: NNLL corrections**

**Off-diagonal splitting functions: similar. For  $P_{\text{qi}}^T$  also N<sup>3</sup>LL terms known**

# NLO + resummed first moments

---

Fixed-order  $N = 1$  poles removed by the resummation (NLO requires NNLL)

$$P_{\text{qg}}^T(N=1) = \frac{8}{3} n_f a_s - \frac{1}{3C_A^2} \left( 17 C_A^2 n_f - 2 C_A n_f^2 + 4 C_F n_f^2 \right) (2C_A a_s^3)^{1/2} + \mathcal{O}(a_s^2)$$

$$P_{\text{qq}}^T(N=1) = \frac{C_F}{C_A} \left( P_{\text{qg}}^T(N=1) - \frac{4}{3} n_f a_s \right) + \mathcal{O}(a_s^2)$$

$$\begin{aligned} P_{\text{gg}}^T(N=1) &= (2C_A a_s)^{1/2} - \frac{1}{6C_A} (11 C_A^2 + 2 C_A n_f + 12 C_F n_f) a_s \\ &+ \frac{1}{144 C_A^3} \left( [1193 - 576 \zeta_2] C_A^4 - 140 C_A^3 n_f + 4 C_A^2 n_f^2 + 760 C_A^2 C_F n_f \right. \\ &\quad \left. - 80 C_A C_F n_f^2 + 144 C_F^2 n_f^2 \right) (2C_A a_s^3)^{1/2} + \mathcal{O}(a_s^2) \end{aligned}$$

$$P_{\text{gq}}^T(N=1) = \frac{C_F}{C_A} \left( P_{\text{gg}}^T(N=1) + \frac{4}{3} \frac{C_F n_f}{C_A} a_s \right) + \mathcal{O}(a_s^2) .$$



# NLO + resummed first moments

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$$P_{\text{gg}}^T(N=1) = (2C_A a_s)^{1/2} - \frac{1}{6C_A} (11 C_A^2 + 2 C_A n_f + 12 C_F n_f) a_s \\ + \frac{1}{144 C_A^3} \left( [1193 - 576 \zeta_2] C_A^4 - 140 C_A^3 n_f + 4 C_A^2 n_f^2 + 760 C_A^2 C_F n_f \right. \\ \left. - 80 C_A C_F n_f^2 + 144 C_F^2 n_f^2 \right) (2C_A a_s^3)^{1/2} + \mathcal{O}(a_s^2)$$

$$P_{\text{gq}}^T(N=1) = \frac{C_F}{C_A} \left( P_{\text{gg}}^T(N=1) + \frac{4}{3} \frac{C_F n_f}{C_A} a_s \right) + \mathcal{O}(a_s^2).$$

Numerically for QCD with  $n_f = 5$ , including N<sup>3</sup>LL for  $P_{\text{qi}}^T$  ( $\alpha_s^2$  contributions)

$$P_{\text{qq}}^T(N=1) \cong 0.2358 \alpha_s - 0.6773 \alpha_s^{3/2} + 0.5880 \alpha_s^2$$

$$P_{\text{qg}}^T(N=1) \cong 1.0610 \alpha_s - 1.5240 \alpha_s^{3/2} + 1.8089 \alpha_s^2$$

$$P_{\text{gq}}^T(N=1) \cong 0.3071 \alpha_s^{1/2} - 0.3059 \alpha_s + 0.2884 \alpha_s^{3/2}$$

$$P_{\text{gg}}^T(N=1) \cong 0.6910 \alpha_s^{1/2} - 0.9240 \alpha_s + 0.6490 \alpha_s^{3/2}$$

First application to multiplicities: **Bolzoni, Kniehl, Kotikov (Sept. 12)**

# Partial $x$ -space expressions

**Non-log parts (integer series): Bessel functions in  $z = (32 C_A a_s)^{1/2} \ln \frac{1}{x}$**

$$\begin{aligned}
 x P_{\text{gg}}^T + x P_{\text{qq}}^T \Big|_{\text{NNLL}} &= \left\{ 4 C_A a_s + \frac{8}{3} (11 C_A^2 + 2 C_A n_f - 4 C_F n_f) a_s^2 \ln \frac{1}{x} \right\} \frac{2}{z} J_1(z) \\
 &+ \left\{ \frac{4}{9} (26 C_F n_f - 23 C_A n_f) a_s^2 + \frac{8}{9 C_A} (11 C_A^2 + 2 C_A n_f - 4 C_F n_f)^2 a_s^3 \ln^2 \frac{1}{x} \right\} \frac{2}{z} J_1(z) \\
 &+ \frac{32}{9 C_A} \left( [134 - 72 \zeta_2] C_A^4 + 23 C_A^3 n_f - 48 C_A^2 C_F n_f + 4 C_A C_F n_f^2 - 8 C_F^2 n_f^2 \right) a_s^3 \ln^2 \frac{1}{x} \frac{4}{z^2} J_2(z)
 \end{aligned}$$

$$x P_{\text{gq}}^T(N) - \frac{C_F}{C_A} x P_{\text{gg}}^T \Big|_{\text{NLL}} = - \frac{32}{3} \frac{C_F}{C_A} (C_A^2 + C_A n_f - 2 C_F n_f) a_s^2 \ln \frac{1}{x} \frac{4}{z^2} J_2(z)$$

**Single-logarithmic enhancement of the oscillations at extremely small  $x$**

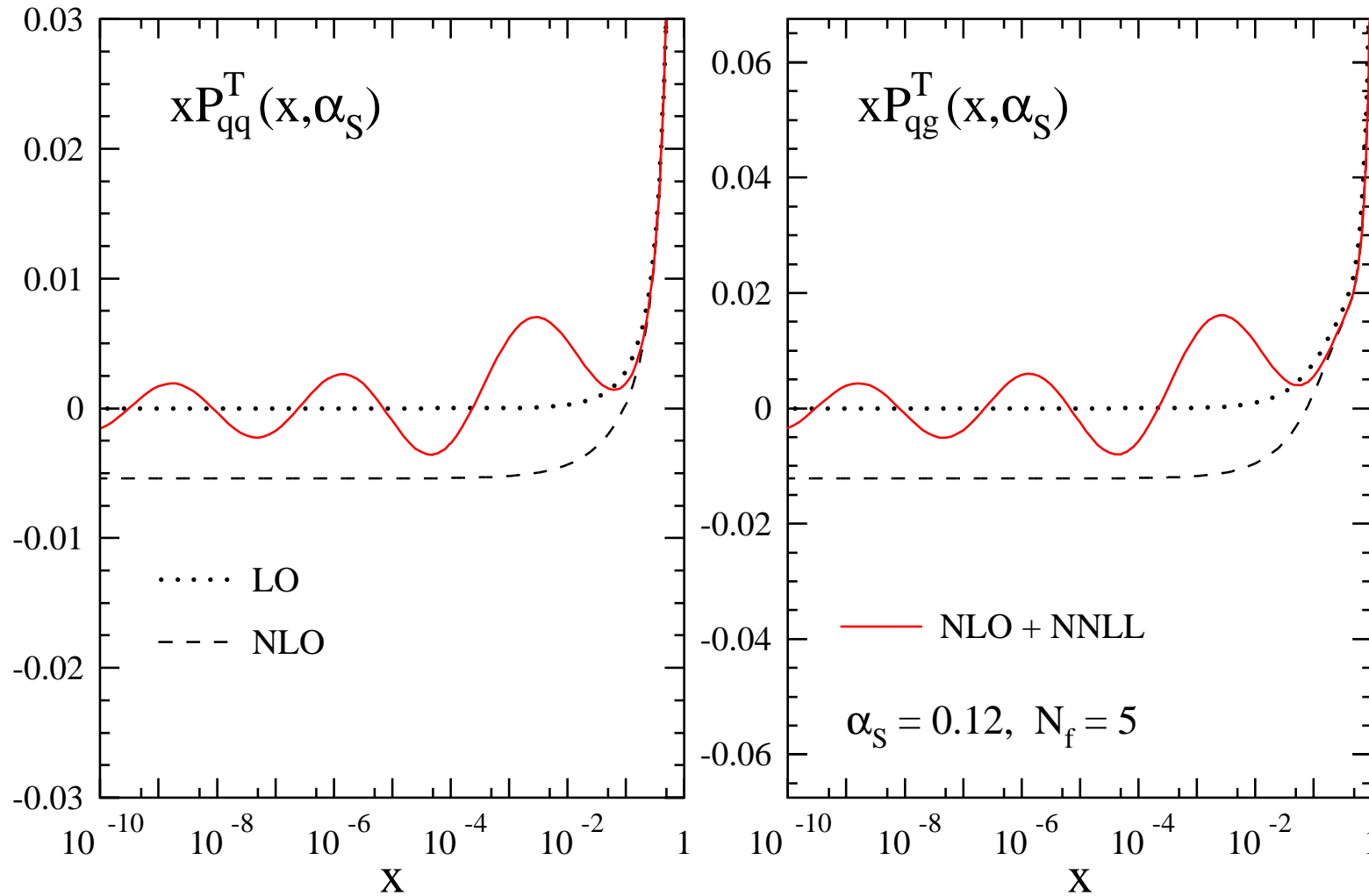
$x \rightarrow 0$  **dominant**  $a_s (a_s \ln \frac{1}{x})^\ell \frac{2}{z} J_1(z)$  **terms**  $\propto (11 C_A^2 - 2 C_A n_f + 4 C_F n_f)^\ell$

**(non- $C_F$  known to  $\ell = 4$ , see below). Possibility of a ‘second resummation’?**

**For the basic logarithm  $\mathcal{L}$ , unlike  $A_{\text{qi}}(\xi)$ , we found a simple Mellin inverse**

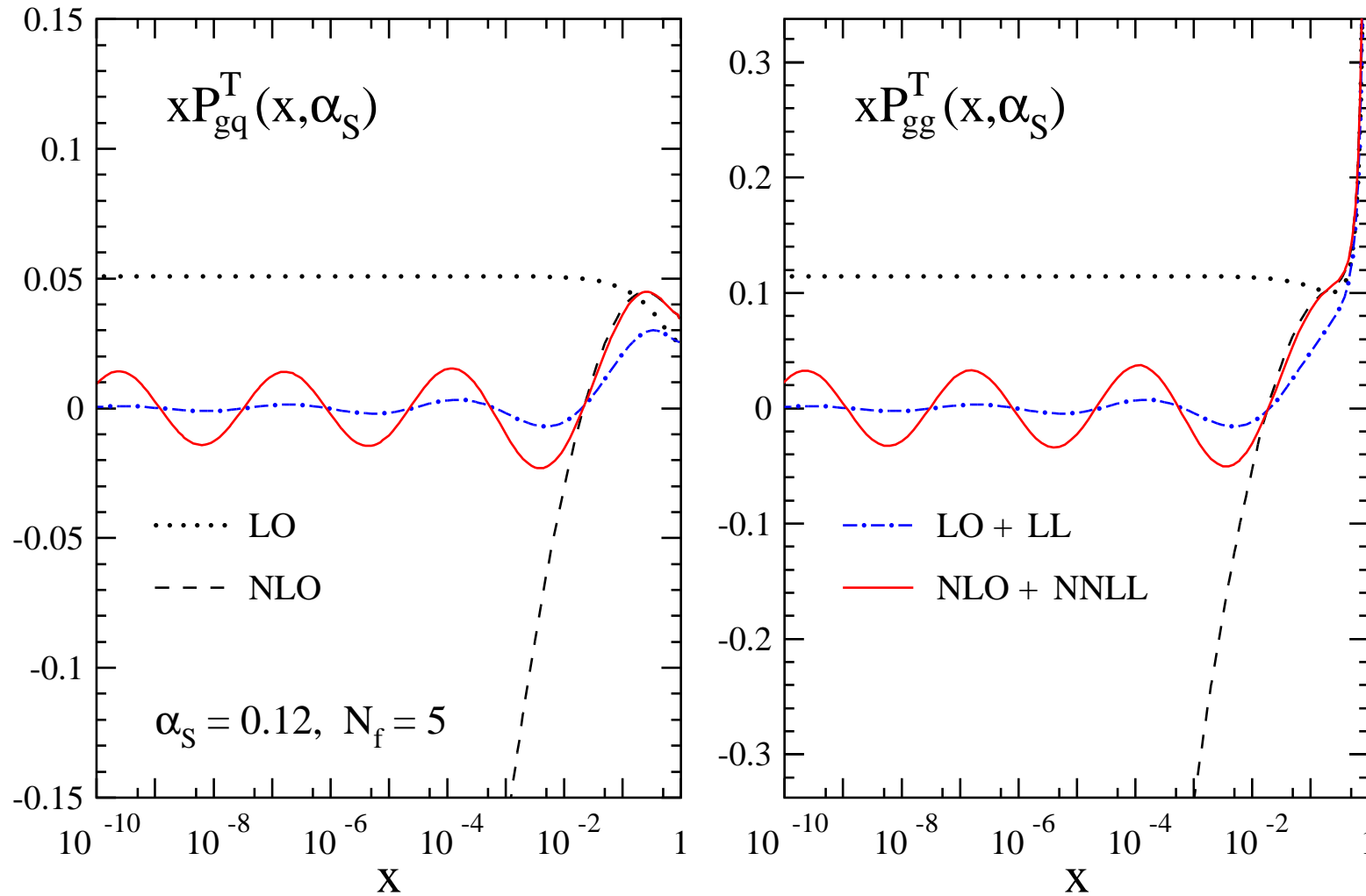
$$\int_0^1 dx x^{N-2} \frac{1}{\ln x} (J_0(2\sqrt{a} \ln x) - 1) = \ln \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{4a}{(N-1)^2}} \right)$$

# Resummed quark-parton splitting functions



Small- $x$   $P_{ig}^T$  vs.  $P_{iq}^T$ : approximate 'Casimir scaling' by factor  $C_A/C_F = 9/4$

# Resummed gluon-parton splitting functions



LL contributions numerically small; expect large corrections beyond NNLL

# Towards higher logarithmic accuracy

---

$N = 1$  finite NNLO + small- $x$  resummed evolution requires N<sup>4</sup>LL accuracy

Relation between SIA ( $\sigma = 1, P^T$ ) and DIS ( $\sigma = -1, P^S$ ) parton evolution

$$\frac{\partial}{\partial \ln Q^2} f_\sigma(x, Q^2) = [P_u(\alpha_s(Q^2)) \otimes f_\sigma(z^\sigma Q^2)](x)$$

with  $P_u$  independent of  $\sigma$

Dokshitzer, Marchesini, Salam (2005)

Non-singlet relation ( $n_f = 0$  for  $P_{gg}$ ), but found to hold for all non- $C_F$  terms

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In  $N$ -space :

$$\partial_{\ln Q^2} f_\sigma(N, Q^2) = P_\sigma(N) f_\sigma(N, Q^2) = P_u(N + \sigma \partial_{\ln Q^2}) f_\sigma(N, Q^2)$$

$$\Rightarrow P_\sigma(N) = P_u(N) + \sum_{n=1}^{\infty} \frac{\sigma^n}{n!} \frac{\partial^{n-1}}{\partial N^{n-1}} \left( \frac{\partial P_u}{\partial N} [P_u(N)]^n \right)$$

Difference  $\delta P_{gg} = P_{gg}^T - P_{gg}^S$  given by lower-order quantities at any order

$P_{gg}^S$  single-log enhanced (BFKL)  $\rightarrow$  resummation of non- $C_F$  double logs in  $P_{gg}^T$

To NNLL as above, plus  $N^3$ LL and (mod. one BFKL coeff.)  $N^4$ LL corrections

# Results for $\ln^\ell x$ contributions in DIS

Analogous to SIA: highest three  $x^0 \ln^\ell x$  double logarithms derived for non-singlet<sup>+</sup> (even- $N$ ) and flavour-singlet splitting & coefficient functions

**Splitting functions  $P_{NS^+}^{(n \geq 1)}$  : all-order expressions for coefficients to NNLL**

**LL:**  $2^{n+1} C(n) C_F^{n+1}$  as in Blümlein, A.V. (95)

**NLL:**  $2^n C(n) (n + 1) C_F^n (C_F - \frac{1}{2} \beta_0)$

**NNLL:**  $2^n C(n-1) \left\{ [n(n+1) - 4 - \zeta_2(48n - 44)] C_F^{n+1} \right.$   
 $\left. + \left[ \frac{10}{3} n + 48 \zeta_2(n-1) \right] C_F^n C_A + n \left( n - \frac{14}{3} \right) C_F^n \beta_0 \right.$   
 $\left. - 15 \zeta_2(n-1) C_F^{n-1} C_A^2 + \frac{1}{4} n(n-1) C_F^{n-1} \beta_0^2 \right\}$

in terms of the **Catalan numbers**  $C(n) = (2n)! [n! (n+1)!]^{-1}$

Also other quantities now in closed all- $n$  form beyond leading logarithms

**NNLL expressions alone insufficient for stable results – details another time**

**Combine with future fixed Mellin- $N$  fourth-order calculations, ...**

# Large- $x$ summary and outlook

---

- **Non-singlet physical kernels for nine observables in DIS, SIA and DY:**  
**Single-log behaviour  $\Rightarrow$  leading three (DY: two) logs of higher-order  $C_\alpha$**
- **Singlet kernels for  $(F_2, F_\phi)$  and  $(F_2, F_L)$  in DIS also single-logarithmic**  
 **$\Rightarrow$  Prediction of three logs in N<sup>3</sup>LO  $\alpha_s^4$  splitting and  $F_L$  coefficient fct's**



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- **$D$ -dimensional structure of unfactorized DIS/SIA structure functions**  
**Verification, extension of above results to N<sup>3</sup>LL or N<sup>2</sup>LL for  $N^{-1}$  terms**

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- **Complementary: Grunberg; Laenen, Gardi, Magnea, Stavenga, White**
- **Applications, now: assess relevance of NS  $1/N$  terms, large- $x$  DIS fits**
- **Near/mid future: combine with other results, esp. fixed- $N$  calculations**  
**(close to) feasible now: 4-loop sum rules Baikov, Chetyrkin, Kühn (10)**
- **Beyond-LL extension to Drell-Yan, Higgs prod'n needs more insights**

# Small- $x$ summary and outlook

---

- $D$ -dimensional structure of unfactorized SIA/DIS structure functions  
⇒ NNLL small- $x$  resummation of timelike splitting & coefficient fct's  
Required for using NNLO results in SIA below  $x \approx 10^{-2} \dots 10^{-3}$
- Analogous for (singlet: subdominant)  $x^0 \ln^\ell x$  terms for  $F_{2,L,\phi}$  in DIS  
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Formally similar, numerically very different: diff. sign in roots,  $(1 - \dots)^r$
- Unlike large- $x$  case: no direct generalization to all higher  $a$  in  $x^a \ln^\ell x$   
But works for even/odd  $a$  for even- $N$ /odd- $N$  based DIS, SIA quantities
- Does not work at  $a = 0$  for the odd- $N$  based  $F_3$  and  $g_1$  in DIS,  $F_A$  in SIA  
E.g., leading logs with group factor  $d_{abc} d^{abc}$  at third order in  $F_3$  and  $F_A$

**All large- $x$  and many, but not all, small- $x$  double logarithms in SIA and DIS appear to be 'inherited' from lower-order results.**

# Reserve slides

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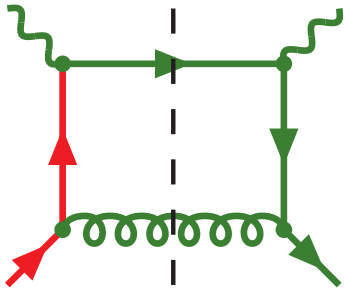
# Flavour singlet – non-singlet decomposition

---

Quark-quark splitting functions:

$$P_{q_i q_k} = P_{\bar{q}_i \bar{q}_k} = \delta_{ik} P_{qq}^V + P_{qq}^S$$

$$P_{q_i \bar{q}_k} = P_{\bar{q}_i q_k} = \delta_{ik} P_{q\bar{q}}^V + P_{q\bar{q}}^S$$



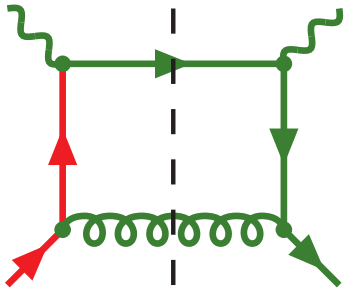
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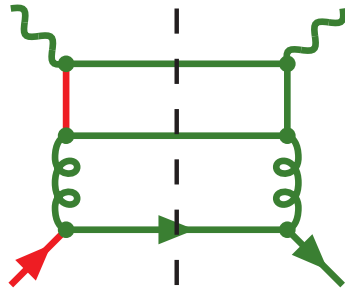
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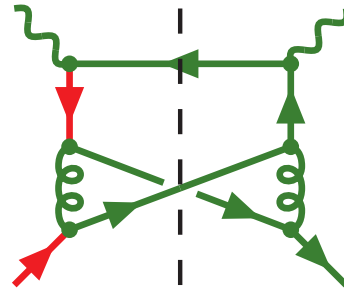
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$$P_{qq}^V = \mathcal{O}(\alpha_s)$$



$$P_{qq}^S, P_{q\bar{q}}^S : \alpha_s^2$$



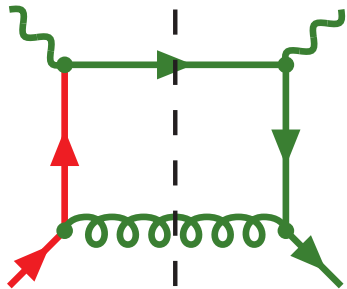
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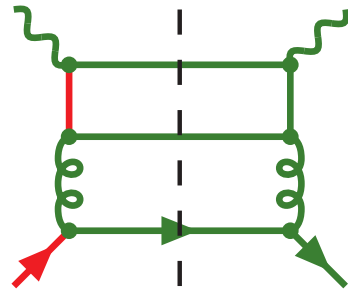
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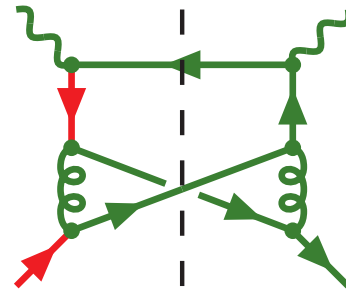
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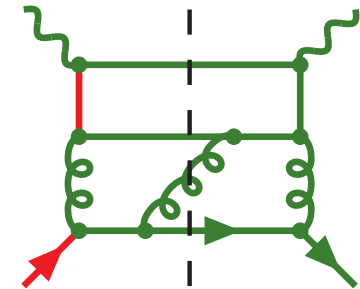
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$$P_{q\bar{q}}^S \neq P_{qq}^S : \alpha_s^3$$

Three types of difference (non-singlet) combinations:  $P_{ns}^\pm = P_{qq}^V \pm P_{q\bar{q}}^V, P_{ns}^V$

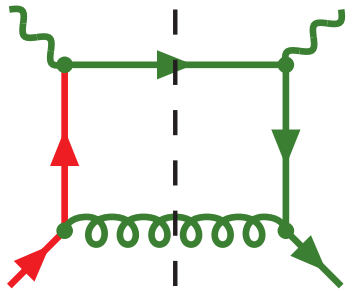


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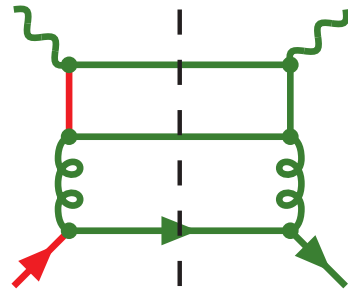
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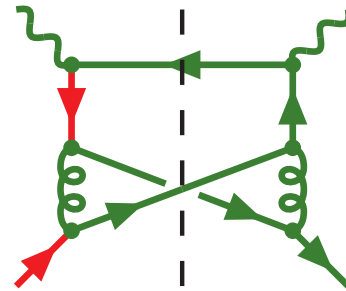
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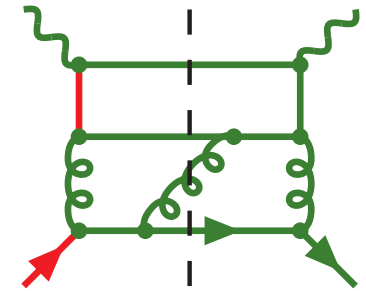
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Three types of difference (non-singlet) combinations:  $P_{ns}^\pm = P_{qq}^V \pm P_{q\bar{q}}^V$ ,  $P_{ns}^V$

Evolution of gluon and flavour-singlet quark distributions  $g$  and  $q_s$

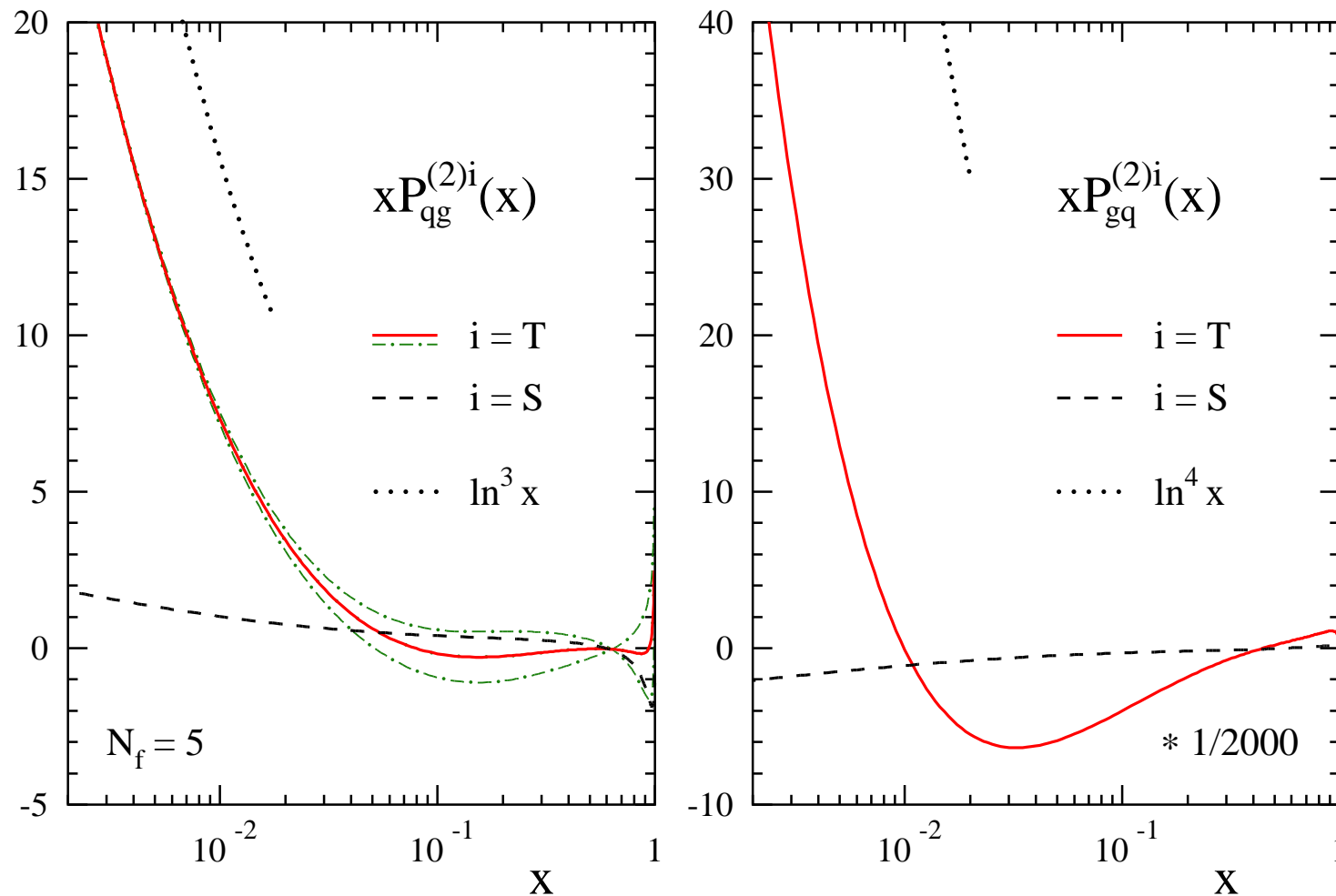
$$q_s = \sum_{r=1}^{n_f} (q_r + \bar{q}_r), \quad \frac{d}{d \ln \mu^2} \begin{pmatrix} q_s \\ g \end{pmatrix} = \begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q_s \\ g \end{pmatrix}$$

with (ps = 'pure singlet')

$$P_{qq} = P_{ns}^+ + n_f (P_{qq}^S + P_{q\bar{q}}^S) \equiv P_{ns}^+ + P_{ps}$$

Quark coefficient fct's: analogous decomposition  $C_{a,q\{\bar{q}\}} = C_{a,ns} + C_{a,ps}$

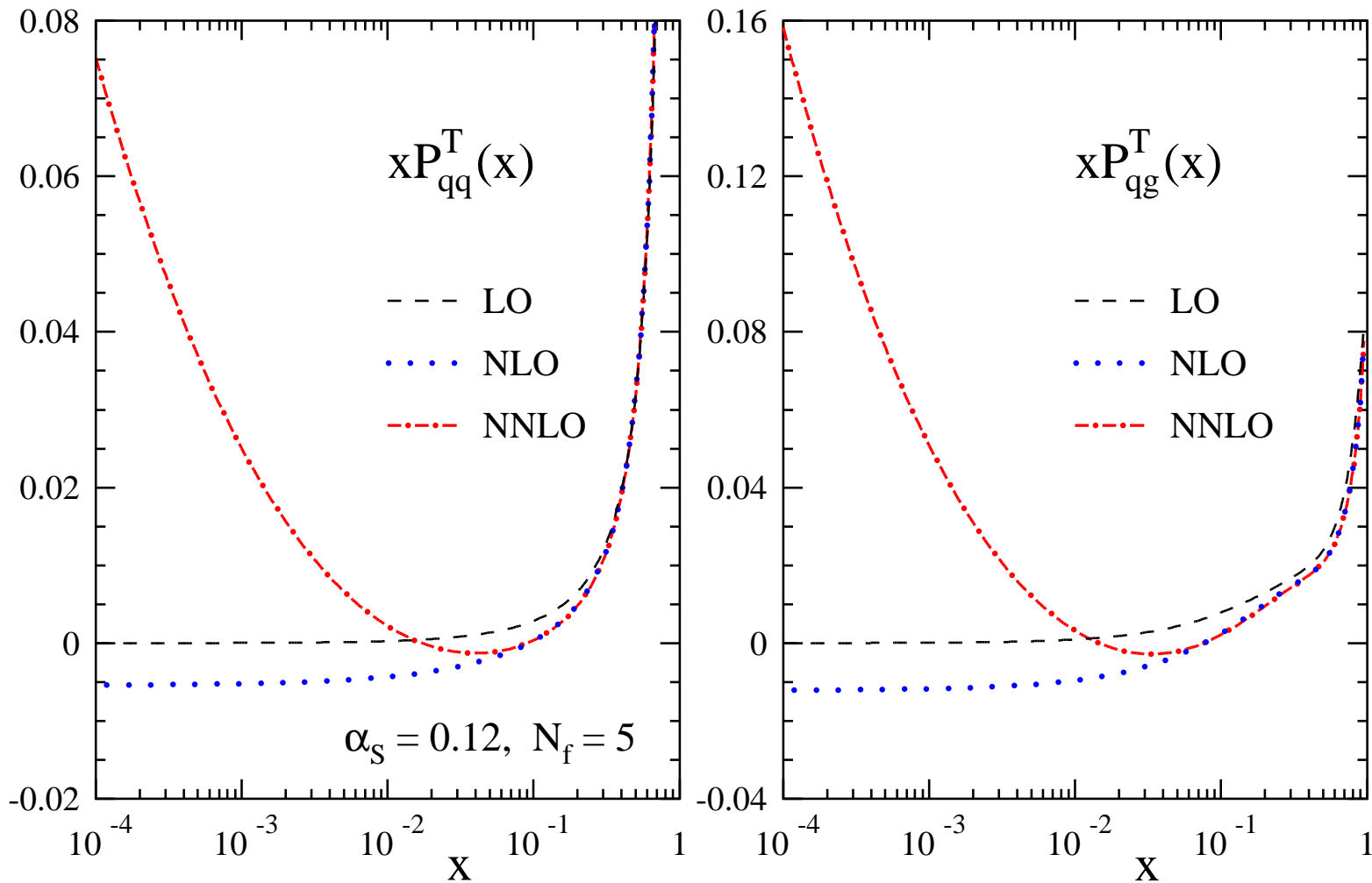
# Third-order off-diagonal splitting functions



$q \rightarrow g$ : not entirely fixed by Crewther-like  $ST$ -relation,  $N = 2$ , SUSY limit

**Dash-dotted:**  $\delta P_{qg}^{(2)T}(x) = \pm 2\zeta_2\beta_0 (C_A - C_F) (11 + 24 \ln x) P_{qg}^{(0)T}(x)$

# NNLO approximations for $P_{qi}^T(x, \alpha_s)$



**NLO: no  $x^{-1} \ln x$  terms. NNLO: up to  $x^{-1} \ln^3 x$ . Unstable at  $x \lesssim 0.02$**

# Singlet physical evolution kernel for $(F_2, F_\phi)$

---

$F_\phi$ : Higgs-exchange DIS in heavy-top limit, to order  $\alpha_s^2$  also by

Daleo, Gehrmann-De Ridder, Gehrmann, Luisoni (09)

As in the non-singlet case above, but with 2-vectors/ $2 \times 2$  matrices  $P_{ij}$  and

$$F = \begin{pmatrix} F_2 \\ F_\phi \end{pmatrix}, \quad C = \begin{pmatrix} C_{2,q} & C_{2,g} \\ C_{\phi,q} & C_{\phi,g} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2\phi} \\ K_{\phi 2} & K_{\phi\phi} \end{pmatrix}$$

Furmanski, Petronzio (81); ...

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Furmanski, Petronzio (81); ...

$$\begin{aligned} \frac{dF}{d \ln Q^2} &= \frac{dC}{d \ln Q^2} q + CPq = \left( \beta(a_s) \frac{dC}{da_s} + CP \right) C^{-1} F \\ &= \underbrace{\left( \beta(a_s) \frac{d \ln C}{da_s} \right)}_{\text{DL (ns + ps)}} + \underbrace{[C, P] C^{-1} + P}_{\text{DL (singlet only)}} F = KF \end{aligned}$$

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Furmanski, Petronzio (81); ...

$$\begin{aligned} \frac{dF}{d \ln Q^2} &= \frac{dC}{d \ln Q^2} q + CPq = \left( \beta(a_s) \frac{dC}{da_s} + CP \right) C^{-1} F \\ &= \underbrace{\left( \beta(a_s) \frac{d \ln C}{da_s} \right)}_{\text{DL (ns + ps)}} + \underbrace{[C, P] C^{-1} + P}_{\text{DL (singlet only)}} F = KF \end{aligned}$$

Observation at NLO, NNLO: **single-log enhancement** to all powers of  $1-x$

$$K_{ab}^{(n)} \sim \ln^n(1-x) + \dots, \quad \text{leading } K_{22/\phi\phi}^{(n)} \text{ same as NS}/C_F = 0$$

**Conjecture: this behaviour persists to N<sup>3</sup>LO**

$\Rightarrow$  prediction of  $\ln^{6,5,4}(1-x)$  of  $P_{qg,gq}^{(3)}$  [and  $\ln^{5,4,3}(1-x)$  of  $P_{ps,gg|C_F}^{(3)}$ ]

# Example: $\alpha_s^4$ splitting function $P_{\text{qg}}^{(3)}(x)$

---

For brevity: only  $(1-x)^0$  part shown – known to all powers,  $C_{AF} \equiv C_A - C_F$

$$\begin{aligned}
 P_{\text{qg}}^{(3)}(x) &= \ln^6(1-x) \cdot 0 \\
 &+ \ln^5(1-x) \left[ \frac{22}{27} C_{AF}^3 n_f - \frac{14}{27} C_{AF}^2 C_F n_f + \frac{4}{27} C_{AF}^2 n_f^2 \right] \\
 &+ \ln^4(1-x) \left[ \left( \frac{293}{27} - \frac{80}{9} \zeta_2 \right) C_{AF}^3 n_f + \left( \frac{4477}{16} - 8\zeta_2 \right) C_{AF}^2 C_F n_f \right. \\
 &\quad \left. - \frac{13}{81} C_{AF} C_F^2 n_f - \frac{116}{81} C_{AF}^2 n_f^2 + \frac{17}{81} C_{AF} C_F n_f^2 - \frac{4}{81} C_{AF} n_f^3 \right] \\
 &+ \mathcal{O}(\ln^3(1-x))
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 \end{aligned}$$

- Vanishing of the coefficient of the leading term at order  $\alpha_s^4$ :  
accidental (??) cancellation of contributions, for all four splitting fct's
- Remaining terms vanish in the supersymmetric case  $C_A = C_F (= n_f)$   
Nontrivial check: same as for  $P_{\text{qg}}^{(2)}$ , not obvious from above construction



# Singlet physical evolution kernel for $(F_2, F_L)$

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As above, but with  $F_\phi \rightarrow \hat{F}_L = F_L/a_s c_{L,q}^{(0)}$ , hence  $\hat{c}_{L,q/g}^{(n)} \sim \{1/\frac{1}{N}\} \ln^{2n} N$

$$F = \begin{pmatrix} F_2 \\ \hat{F}_L \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 1 & \hat{c}_{L,g}^{(0)} \end{pmatrix} + \sum_{n=1} a_s^n \begin{pmatrix} c_{2,q}^{(n)} & c_{2,g}^{(n)} \\ \hat{c}_{L,q}^{(n)} & \hat{c}_{L,g}^{(n)} \end{pmatrix}, \quad K = \begin{pmatrix} K_{22} & K_{2L} \\ K_{L2} & K_{LL} \end{pmatrix}$$

Catani (96), Blümlein, Ravindran, van Neerven (00) [different normalization]

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Observation: single-log enhancement of  $N^0$  part of  $K$  at NLO and NNLO

**N<sup>3</sup>LO conjecture + above  $P_{qg}^{(3)}$** : prediction of three double logs in  $c_{L,q/g}^{(3)}$ , e.g.

$$\begin{aligned} N^2 c_{L,g}^{(3)}(N) &= \ln^6 N \frac{32}{3} C_A^3 n_f \\ &+ \ln^5 N \left[ \frac{1504}{9} C_A^3 n_f - \frac{64}{9} C_A^2 n_f^2 - \frac{104}{3} C_A^2 n_f C_F - \frac{40}{3} n_f C_F^2 \right] \\ &+ \ln^4 N \left[ \text{known coefficients} \right] + \mathcal{O}(\ln^3 N) \end{aligned}$$

Agrees with/extends results [NS-like  $C_F = 0$  part of  $C_{L,g}$  only] of MV (09)

# Reminder: soft limits of $q\bar{q} \rightarrow \gamma^*$ , $gg \rightarrow H$

$a_s^n$  expansion coefficients of bare partonic cross sections to  $n = 3$

$$W_0^b = \delta(1 - x)$$

cf. Matsuura, van Neerven (88)

$$W_1^b = 2 \operatorname{Re} \mathcal{F}_1 \delta(1 - x) + \mathcal{S}_1$$

$$W_2^b = (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \delta(1 - x) + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{S}_1 + \mathcal{S}_2$$

$$W_3^b = (2 \operatorname{Re} \mathcal{F}_3 + 2 |\mathcal{F}_1 \mathcal{F}_2|) \delta(1 - x) + (2 \operatorname{Re} \mathcal{F}_2 + |\mathcal{F}_1|^2) \mathcal{S}_1 + 2 \operatorname{Re} \mathcal{F}_1 \mathcal{S}_2 + \mathcal{S}_3$$

$\mathcal{F}_\ell$ : bare  $\ell$ -loop time-like  $q$  or  $g$  form factor,  $\mathcal{S}_\ell$  includes soft real emissions

$$\mathcal{S}_k = \mathbf{S}_k(\varepsilon) \cdot \varepsilon [(1 - x)^{-1 - 2k\varepsilon}]_+ = \mathbf{S}_k(\varepsilon) \left[ -\frac{1}{2k} \delta(1 - x) + \sum_{i=0} \frac{(-2k\varepsilon)^i}{i!} \varepsilon \mathcal{D}_i \right]$$

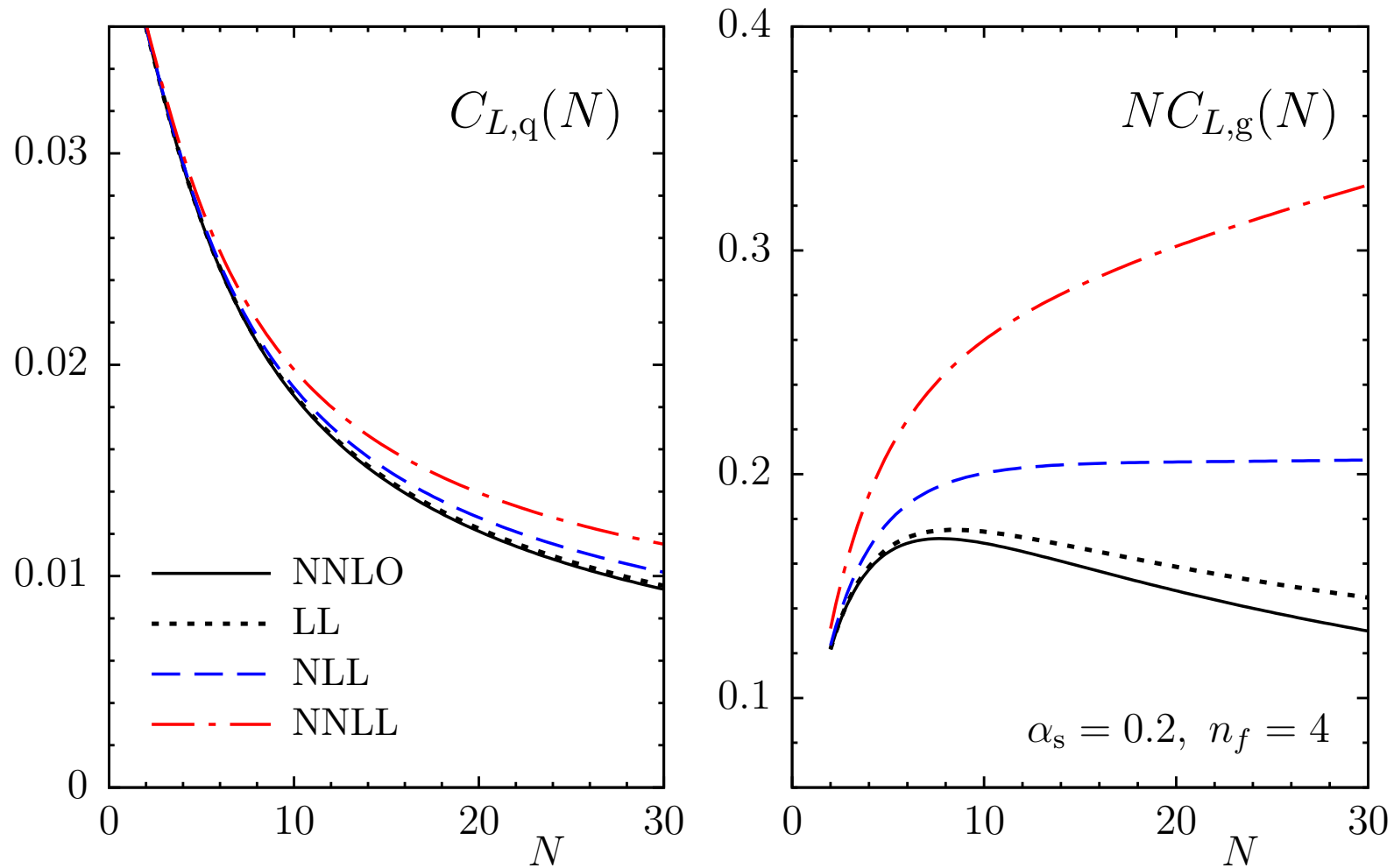
Poles in  $\varepsilon = 2 - D/2$ : KLN, renormalization, mass factorization

$1/\varepsilon$  pieces of  $\mathcal{F}_n$  +  $n$ -loop splitting functions  $\rightarrow 1/\varepsilon$  coefficients of  $\mathbf{S}_n$

$\rightarrow \mathcal{D}_{2n, \dots, 0}$  terms of coefficient fct's  $c_n \rightarrow \mathbf{N}^n$  LL resummation coeff's  $D_n$

$n = 3$ : Moch, A.V. (2005)

# Numerical illustration of $C_{L,q}$ and $C_{L,g}$



**Corrections smaller and convergence with order  $n$  faster in quark case(s)**  
 $\simeq 15\%$  NNLL correction at  $N = 20$  for  $C_{L,q}$  vs.  $100\%$  for  $C_{L,g}$  ( $\approx$  Padé)